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A new upper bound for total colourings of graphs

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Abstract

We give a new upper bound on the total chromatic number of a graph. This bound improves the results known for some classes of graphs. The bound is stated as follows: $\chi_T \leqslant \chi_e + \lfloor \frac{1}{3}\chi \rfloor + 2$, where χ is the chromatic number, χ_e is the edge chromatic number (chromatic index) and χ_T is the total chromatic number.

1. Introduction

Let $G=(V,E)$ be a graph without loops. A *k-colouring* of a finite set C is a map $\phi: C \mapsto \{1,2,\ldots,k\}$. When we consider a colouring of a subset of $V \cup E$ we shall always assume that it is *proper*; that is no two adjacent or incident elements receive the same colour. A *vertex colouring*, *edge colouring*, *total colouring* of a graph is a proper colouring of $V, E, V \cup E$ respectively. The *chromatic number* $\chi(G)$, *edge chromatic number* $\chi_e(G)$, *total chromatic number* $\chi_T(G)$ is the least number of colours in a vertex, edge, total colouring of G, respectively.

Let $\Delta(G)$ be the maximum degree of a vertex in G. It is clear that $\chi_e \geqslant \Delta$, and that $\chi_T \geqslant \Delta + 1$. For simple graphs (without multiple edges) it is well known that $\chi \leqslant \Delta + 1$, *and* $\chi_e \leqslant \Delta + 1$. The *total colouring conjecture* states that $\chi_T \leqslant \Delta + 2$ [1, 8].

The total colouring conjecture has been verified for several special classes of graphs (see [2, 3] for surveys). It has also been shown that if k is an integer with $k!$ at least the number of vertices then $\chi_T \leqslant \chi_e + k + 1$; and that 'most' graphs satisfy $\chi_T = \Delta + 1$ [5, 6].

Recently McDiarmid and Sánchez-Arroyo [7] have shown that $\chi_T \leqslant \chi_e + \frac{2}{5}\chi + \frac{8}{5}$. Our main result is an upper bound that improves the above bound, and it is stated as follows.

Theorem. *For any graph* G, $\chi_T \leqslant \chi_e + \lfloor \frac{1}{3}\chi \rfloor + 2$.

The best upper bound known to the author was obtained by Hind [4] who proved that $\chi_T \leqslant \chi_e + 2\lceil \sqrt{\chi} \rceil$. Our theorem improves Hind's bound for graphs having small chromatic number.

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2. The proof

The proof relies upon the following result.

Lemma 1. *Let $G=(V,E)$ be a graph with $\Delta(G)\leqslant 2$ and let $W\subseteq V$ be the union of three stable sets $W=V_1\cup V_2\cup V_3$, then there exists a total colouring $\phi: W\cup E\mapsto\{1,2,3,\alpha\}$ such that*

1. *if $x\in V_i$, then $\phi(x)=i$ for $i=1,2,3$, and*
2. *if $a=\{x,y\}$ and $\phi(a)=\alpha$, then $\{x,y\}\subseteq W$.*

This Lemma is essentially Lemma 2.1 of [7], and does not need proof here.

Our theorem is an easy consequence of our final Lemma.

Lemma 2. *Let $G=(V,E)$ be a multigraph with chromatic number $\chi(G)$, edge chromatic number $\chi_e(G)$, and total chromatic number $\chi_T(G)$. Then*

1. *if $\chi(G)=0,1 \bmod 3$, then $\chi_T(G)\leqslant\chi_e(G)+\lfloor\frac{1}{3}\chi(G)\rfloor+1$, and*
2. *if $\chi(G)=2 \bmod 3$, then $\chi_T(G)\leqslant\chi_e(G)+\lfloor\frac{1}{3}\chi(G)\rfloor+2$.*

Proof. For complete graphs or cycles the result holds. Thus we may assume that $\chi(G)\leqslant\chi_e(G)$. Consider a vertex and an edge colouring of G with $\chi(G)$ and $\chi_e(G)$ colours, respectively. Let $V_1,\ldots,V_{\chi(G)}$ and $M_1,\ldots,M_{\chi_e(G)}$ be the chromatic classes. We have three cases:

Case 1: $\chi(G)=3k$, Set $p=\chi(G)$, and for each $j=1,\ldots,k$, define the set of vertices:

$$W_j=\bigcup_{i=3(j-1)+1}^{3j} V_i,$$

and the set of edges

$$E_j=\bigcup_{i=3(j-1)+1}^{3j-1} M_i.$$

Now consider the subgraph $H_j=(V,E_j)$. We now apply Lemma 1 to each H_j with the subset W_j, to obtain a colouring ϕ_j of $E_j\cup W_j$ using colours from $\{3(j-1)+1,3(j-1)+2,3j,\alpha\}$, respectively. It is clear from Lemma 1 that the union of these colourings form a total colouring. Now for each $j=1,\ldots,k$ use a new colour β_j to recolour the matching M_{3j}. Thus the set of colours used on the vertex set and the first p matchings is $\{1,2,\ldots,p\}\cup\{\alpha\}\cup\{\beta_1,\ldots,\beta_k\}$, so we have used $p+\frac{1}{3}p+1$ colours already. It remains to add the remaining $\chi_e(G)-p$ colours used on the matchings $M_{p+1},\ldots,M_{\chi_e(G)}$ to obtain that $\chi_T(G)$ is bounded by

$$(\chi_e(G)-p)+p+\tfrac{1}{3}p+1=\chi_e(G)+\lfloor\tfrac{1}{3}\chi(G)\rfloor+1$$

as stated.

Case 2: $\chi(G)=3k+1$. Set $p=\chi(G)-1$, and apply case 1 above. Now recolour the vertices in $V_{\chi(G)}$ with colour α. By Lemma 1.2 this is a proper total colouring of G.

Thus the number of colours used is

$$(\chi_e(G)-p)+p+\tfrac{1}{3}p+1=\chi_e(G)+\lfloor\tfrac{1}{3}\chi(G)\rfloor+1,$$

as claimed.

Case 3: $\chi(G)=3k+2$. Set $p=\chi(G)-2$, and apply case 1 above. Now recolour the vertices in $V_{\chi(G)}$, and $V_{\chi(G)-1}$ with colours α and β_{k+1}, respectively. Thus the number of colours used is

$$(\chi_e(G)-p)+p+\tfrac{1}{3}p+2=\chi_e(G)+\lfloor\tfrac{1}{3}\chi(G)\rfloor+2,$$

as required.

This completes the proof of the Lemma. □

References

[1] M. Behzad, Graphs and their chromatic numbers, Ph. D. Thesis, Michigan State University, 1965.

[2] M. Behzad, The total chromatic number of a graph: a survey, in: D.J.A. Welsh, ed., Combinatorial Mathematics and its Applications (Academic Press, London, 1971) 1–10.

[3] A.G. Chetwynd, Total colourings, in: R. Nelson & R.J. Wilson, eds., Graphs Colourings, Pitman Research Notes in Mathematics No. 218 (London, 1990) 65–77.

[4] H. Hind, An improved bound for the total chromatic number of a graph, Graphs Combin. 6 (1990) 153–159.

[5] C.J.H. McDiarmid, Colourings of random graphs, in: R. Nelson and R.J. Wilson, eds., Graphs Colourings, Pitman Research Notes in Mathematics No. 218 (London, 1990) 79–86.

[6] C.J.H. McDiarmid and B. Reed, On total colourings of graphs, J. Combin. Theory Ser. B 57 (1993) 122–130.

[7] C.J.H. McDiarmid and A. Sánchez-Arroyo, An upper bound for total colourings of graphs, Discrete Math. 111 (1993) 389–392.

[8] V.G. Vizing, Some unsolved problems in graph theory (in Russian), Uspekhi Math. Nauk. 23 (1968) 117–134. Russian Math. Surveys 23 (1968) 125–142.