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Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: An Orlicz–Sobolev space setting

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Abstract

We study the boundary value problem $-\operatorname{div}(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = f(u)$ in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. We distinguish the cases where either $f(u) = -\lambda|u|^{p-2}u + |u|^{r-2}u$ or $f(u) = \lambda|u|^{p-2}u - |u|^{r-2}u$, with $p, q > 1$, $p+q < \min\{N, r\}$, and $r < (Np - N + p)/(N-p)$. In the first case we show the existence of infinitely many weak solutions for any $\lambda > 0$. In the second case we prove the existence of a nontrivial weak solution if λ is sufficiently large. Our approach relies on adequate variational methods in Orlicz–Sobolev spaces.

Keywords: Nonhomogeneous operator; Orlicz–Sobolev space; Critical point; Weak solution

1. Introduction and main results

Classical Sobolev and Orlicz–Sobolev spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, nonlinear potential theory, the theory of quasiconformal mappings, differential geometry, geometric function theory, and probability theory. These spaces consists of functions that have weak derivatives and satisfy certain integrability conditions. The study of nonlinear elliptic equations

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involving quasilinear homogeneous type operators is based on the theory of Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. In the case of nonhomogeneous differential operators, the natural setting for this approach is the use of Orlicz–Sobolev spaces. The basic idea is to replace the Lebesgue spaces $L^p(\Omega)$ by more general spaces $L_\Phi(\Omega)$, called *Orlicz spaces*. The spaces $L_\Phi(\Omega)$ were thoroughly studied in the monograph by Krasnosel'skii and Rutickii [19] and also in the doctoral thesis of Luxemburg [18]. If the role played by $L^p(\Omega)$ in the definition of the Sobolev spaces $W^{m,p}(\Omega)$ is assigned instead to an Orlicz space $L_\Phi(\Omega)$ the resulting space is denoted by $W^m L_\Phi(\Omega)$ and called an *Orlicz–Sobolev space*. Many properties of Sobolev spaces have been extended to Orlicz–Sobolev spaces, mainly by Dankert [9], Donaldson and Trudinger [11], and O'Neill [22] (see also Adams [2] for an excellent account of those works). Orlicz–Sobolev spaces have been used in the last decades to model various phenomena. Chen et al. [6] proposed a framework for image restoration based on a variable exponent Laplacian. A second application which uses variable exponent type Laplace operators is modeling electrorheological fluids [1,5,12,13,21,25].

This paper is devoted to the study of weak solutions for problems of the type

$$\begin{cases} -\operatorname{div}(a(|\nabla u(x)|)\nabla u(x)) = f(u(x)), & \text{for } x \in \Omega, \\ u(x) = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) is a bounded domain with smooth boundary.

The first general existence result using the theory of monotone operators in Orlicz–Sobolev spaces were obtained in [10] and in [15,16]. Other recent work that puts the problem into this framework is contained in [7,8,14,17]. In these papers, the existence results are obtained using variational techniques, monotone operator methods or fixed point and degree theory arguments.

The case where $a(t) = t^{p-2}$ ($p > 1$, $t \geqslant 0$) is fairly understood and a great variety of existence results are available. In this paper we focus on the case where $a : [0, \infty) \to \mathbb{R}$ is defined by $a(t) = \log(1 + t^q) \cdot t^p$, where $p, q > 1$. We treat separately the cases where either $f(t) = -\lambda|t|^{p-2}t + |t|^{r-2}t$ or $f(t) = \lambda|t|^{p-2}t - |t|^{r-2}t$, where $r < (Np - N + p)/(N - p)$ and λ is a positive parameter.

We remark that we deal with a nonhomogeneous operator in the divergence form. Thus, we introduce an Orlicz–Sobolev space setting for problems of type (1).

Define

$$\varphi(t) := \log\bigl(1 + |t|^q\bigr) \cdot |t|^{p-2}t, \quad \text{for all } t \in \mathbb{R},$$

and

$$\Phi(t) := \int_0^t \varphi(s), \quad \text{for all } t \in \mathbb{R}.$$

A straightforward computation yields

$$\Phi(t) = \frac{1}{p}\log\bigl(1 + |t|^q\bigr) \cdot |t|^p - \frac{q}{p}\int_0^{|t|} \frac{s^{p+q-1}}{1 + s^q}\,ds,$$

for all $t \in \mathbb{R}$. We point out that φ is an odd, increasing homeomorphism of $\mathbb{R}$ into $\mathbb{R}$, while Φ is convex and even on $\mathbb{R}$ and increasing from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Set

$$\Phi^\star(t) := \int_0^t \varphi^{-1}(s)\,ds, \quad \text{for all } t \in \mathbb{R}.$$

The functions Φ and $\Phi^\star$ are complementary N-functions (see [2,19,20]).

Define the Orlicz class

$$K_\Phi(\Omega) := \left\{u:\Omega \to \mathbb{R},\ \text{measurable};\ \int_\Omega \Phi\big(|u(x)|\big)\,dx < \infty\right\}$$

and the Orlicz space

$$L_\Phi(\Omega) := \text{the linear hull of } K_\Phi(\Omega).$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)dx \leqslant 1\right\}$$

or the equivalent norm (the Orlicz norm)

$$\|u\|_{(\Phi)} := \sup\left\{\left|\int_\Omega uv\,dx\right|;\ v \in K_{\bar\Phi}(\Omega),\ \int_\Omega \bar\Phi\big(|v|\big)\,dx \leqslant 1\right\},$$

where $\bar\Phi$ denotes the conjugate Young function of Φ, that is,

$$\bar\Phi(t) = \sup\big\{ts - \Phi(s);\ s \in \mathbb{R}\big\}.$$

By Lemma 2.4 and Example 2 in [8, p. 243] we have

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leqslant \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty. \tag{2}$$

The above inequalities imply that Φ satisfies the Δ_2-condition. By Lemma C.4 in [8] it follows that $\Phi^\star$ also satisfies the Δ_2-condition. Then, according to [2, p. 234], it follows that $L_\Phi(\Omega) = K_\Phi(\Omega)$. Moreover, by Theorem 8.19 in [2] $L_\Phi(\Omega)$ is reflexive.

We denote by $W^1L_\Phi(\Omega)$ the Orlicz–Sobolev space defined by

$$W^1L_\Phi(\Omega) := \left\{u \in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1,\dots,N\right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \big\||\nabla u|\big\|_\Phi.$$

We also define the Orlicz–Sobolev space $W_0^1L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$. By Lemma 5.7 in [15] we obtain that on $W_0^1L_\Phi(\Omega)$ we may consider an equivalent norm

$$\|u\| := \big\||\nabla u|\big\|_\Phi.$$

The space $W_0^1L_\Phi(\Omega)$ is also a reflexive Banach space.

In the first part of the present paper we study the boundary value problem

$$\begin{cases} -\operatorname{div}(\log(1+|\nabla u(x)|^q)|\nabla u(x)|^{p-2}\nabla u(x)) = -\lambda|u(x)|^{p-2}u(x) + |u(x)|^{r-2}u(x), \\ \quad \text{for } x \in \Omega, \\ u(x) = 0, \quad \text{for } x \in \partial\Omega. \end{cases} \tag{3}$$

We say that $u \in W_0^1 L_\Phi(\Omega)$ is a *weak solution* of problem (3) if

$$\int_\Omega \log\bigl(1 + \bigl|\nabla u(x)\bigr|^q\bigr)\bigl|\nabla u(x)\bigr|^{p-2}\nabla u \nabla v\,dx + \lambda\int_\Omega \bigl|u(x)\bigr|^{p-2}u(x)v(x)\,dx - \int_\Omega \bigl|u(x)\bigr|^{r-2}u(x)v(x)\,dx = 0,$$

for all $v \in W_0^1 L_\Phi(\Omega)$.

We prove the following multiplicity result.

Theorem 1. *Assume that $p, q > 1$, $p+q < N$, $p+q < r$ and $r < (Np - N + p)/(N - p)$. Then for every $\lambda > 0$ problem* (3) *has infinitely many weak solutions.*

Next, we consider the problem

$$\begin{cases} -\operatorname{div}(\log(1 + |\nabla u(x)|^q)|\nabla u(x)|^{p-2}\nabla u(x)) = \lambda|u(x)|^{p-2}u(x) - |u(x)|^{r-2}u(x), \\ \quad \text{for } x \in \Omega, \\ u(x) = 0, \quad \text{for } x \in \partial\Omega. \end{cases} \tag{4}$$

We say that $u \in W_0^1 L_\Phi(\Omega)$ is a *weak solution* of problem (4) if

$$\int_\Omega \log\bigl(1 + \bigl|\nabla u(x)\bigr|^q\bigr)\bigl|\nabla u(x)\bigr|^{p-2}\nabla u \nabla v\,dx - \lambda\int_\Omega \bigl|u(x)\bigr|^{p-2}u(x)v(x)\,dx + \int_\Omega \bigl|u(x)\bigr|^{r-2}u(x)v(x)\,dx = 0,$$

for all $v \in W_0^1 L_\Phi(\Omega)$.

We prove

Theorem 2. *Assume that the hypotheses of Theorem* 1 *are fulfilled. Then there exists $\lambda_\star > 0$ such that for any $\lambda \geqslant \lambda_\star$, problem* (4) *has a nontrivial weak solution.*

A careful analysis of the proofs shows that Theorems 1 and 2 still remain valid for more general classes of differential operators. Indeed, we can replace $\operatorname{div}(\log(1 + |\nabla u(x)|^q)|\nabla u(x)|^{p-2} \times \nabla u(x))$ by $\operatorname{div}(a(|\nabla u(x)|)\nabla u(x))$, where $a(t)$ is so that the assumption (2) is fulfilled. Some potentials $a(t)$ satisfying this hypothesis are $a(t) = |t|^{\alpha-1}$ ($\alpha > 0$) and $a(t) = |t|^\alpha/\log(1 + |t|^\beta)$ ($0 < \beta < \alpha$).

We remark that in the particular case corresponding to $q = 1$, $\lambda = 0$, $1 < p < N - 1$, and $p < r \leqslant [N(p-1) + p]/(N - p)$, problem (3) has a nontrivial weak solution, by means of Theorem 1.2 in [7]. On the other hand, Theorem 1.2 in [7] also applies for solving equations involving more general differential operators $\operatorname{div}(a(|\nabla u(x)|)\nabla u(x))$. We also remark that if $a(t) = 1$ and $f(u) = -\lambda u + |u|^{r-2}u$, then problem (1) becomes

$$\begin{cases} -\Delta u = -\lambda u + |u|^{r-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

This problem has been studied by Ambrosetti and Rabinowitz [3] provided $2 < r < 2^* = 2N/(N-2)$. Using the Mountain Pass theorem combined with the remark that the operator

$-\Delta + \lambda I$ ($\lambda > 0$) is coercive in $H_0^1(\Omega)$, Ambrosetti and Rabinowitz showed that problem (5) has a positive solution for any $\lambda > 0$. The result we establish in Theorem 1 establishes the existence of *infinitely many solutions* (not necessarily positive) for a related class of boundary value problems, but involving another differential operator in the class of Orlicz–Sobolev spaces.

The strong difference between the results of Theorems 1 and 2 should be understood by the following elementary arguments. Indeed, consider the corresponding problems

$$\begin{cases} -\Delta u = -\lambda u + u^{r-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ u > 0, & \text{in } \Omega \end{cases} \tag{6}$$

and

$$\begin{cases} -\Delta u = \lambda u - u^{r-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ u > 0, & \text{in } \Omega. \end{cases} \tag{7}$$

As we have seen, the Mountain Pass theorem implies that problem (6) has at least one solution for any $\lambda > 0$, provided $2 < r < 2^* = 2N/(N-2)$. Problem (7) corresponds to the case studied in Theorem 2. A simple multiplication with the first eigenfunction $\varphi_1 > 0$ in (7) implies

$$\lambda_1 \int_\Omega u\varphi_1\, dx = \lambda \int_\Omega u\varphi_1\, dx - \int_\Omega u^{r-1}\varphi_1\, dx.$$

Thus, a necessary condition that problem (7) has a solution is that λ is sufficiently large. The same arguments apply in the general cases studied in the main results of this paper. Indeed, under the assumptions of Theorem 1, the nonlinear term $f_1(u) := -\lambda|u|^{p-2}u + |u|^{r-2}u$ satisfies the Ambrosetti–Rabinowitz condition $0 \leqslant r \int_0^t f_1(s)\, ds \leqslant t f_1(t)$, for all $t \geqslant 0$. This condition fails if $f_2(u) := \lambda|u|^{p-2}u - |u|^{r-2}u$ but, in this case, we show that the corresponding energy functional is coercive and lower semicontinuous.

2. Auxiliary results on Orlicz–Sobolev embeddings

In many applications of Orlicz–Sobolev spaces to boundary value problems for nonlinear partial differential equations, the compactness of the embeddings plays a central role. Compact embedding theorems for Sobolev or Orlicz–Sobolev spaces are also intimately connected with the problem of discreteness of spectra of Schrödinger operators (see Benci and Fortunato [4] and Reed and Simon [24]).

While the Banach spaces $W^1L_\Phi(\Omega)$ and $W_0^1L_\Phi(\Omega)$ can be defined from fairly general convex properties of Φ, it is also well known that the specific functional-analytic and topological properties of these spaces depend very sensitively on the rate of growth of Φ at infinity. Compactness is not an exception and, using standard notions traditionally used to describe convex functions, we recall in this section a compact embedding theorem for a class of Orlicz–Sobolev spaces.

Define the Orlicz–Sobolev conjugate $\Phi_\star$ of Φ by

$$\Phi_\star^{-1}(t) := \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds.$$

Proposition 1. *Assume that the hypotheses of Theorems* 1 *or* 2 *are fulfilled. Then the following properties hold true*:

(a) $\lim_{t\to 0}\int_t^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds<\infty$;

(b) $\lim_{t\to\infty}\int_1^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds=\infty$;

(c) $\lim_{t\to\infty}\frac{|t|^{\gamma+1}}{\Phi_\star(kt)}=0$, *for all* $k>0$ *and all* $1\leqslant\gamma<\frac{Np-N+p}{N-p}$.

Proof. (a) By L'Hôpital's rule we have

$$\lim_{t\searrow 0}\frac{\Phi(t)}{t^{p+q}}=\lim_{t\searrow 0}\frac{\varphi(t)}{(p+q)t^{p+q-1}}=\frac{1}{p+q}\lim_{t\searrow 0}\frac{\log(1+t^q)}{t^q}=\frac{1}{p+q}\lim_{t\searrow 0}\frac{\frac{qt^{q-1}}{1+t^q}}{qt^{q-1}}=\frac{1}{p+q}.$$

We deduce that Φ is equivalent to t^{p+q} near zero. Using that fact and the remarks [2, p. 248] we infer that (a) holds true if and only if

$$\lim_{t\to 0}\int_t^1 \frac{s^{\frac{1}{p+q}}}{s^{\frac{N+1}{N}}}\,ds<\infty$$

or

$$p+q<N.$$

The last inequality holds since the hypotheses of Theorems 1 or 2 are fulfilled.

(b) By the change of variable $s=\Phi(\tau)$ we obtain

$$\int_1^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds=\int_{\Phi^{-1}(1)}^{\Phi^{-1}(t)}\frac{\tau\varphi(\tau)}{\Phi(\tau)}\big(\Phi(\tau)\big)^{-1/N}\,d\tau. \tag{8}$$

A simple calculation yields

$$0\leqslant\lim_{\tau\to\infty}\frac{\int_0^\tau \frac{s^{p+q-1}}{1+s^q}\,ds}{\tau^p\log(1+\tau^q)}\leqslant\lim_{\tau\to\infty}\frac{\int_0^\tau \frac{s^{p+q-1}}{s^q}\,ds}{\tau^p\log(1+\tau^q)}=\lim_{\tau\to\infty}\frac{\frac{1}{p}\tau^p}{\tau^p\log(1+\tau^q)}=0.$$

Thus

$$\lim_{\tau\to\infty}\frac{\int_0^\tau \frac{s^{p+q-1}}{1+s^q}\,ds}{\tau^p\log(1+\tau^q)}=0. \tag{9}$$

A first consequence of the above relation is that

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^p\log(1+t^q)}=\frac{1}{p}. \tag{10}$$

On the other hand, by (9),

$$\lim_{\tau\to\infty}\frac{\tau\varphi(\tau)}{\Phi(\tau)}=\lim_{\tau\to\infty}\frac{\tau^p\log(1+\tau^q)}{\frac{1}{p}\tau^p\log(1+\tau^q)-\frac{q}{p}\int_0^\tau\frac{s^{p+q-1}}{1+s^q}\,ds}$$
$$=p\lim_{\tau\to\infty}\left(1-q\cdot\frac{\int_0^\tau \frac{s^{p+q-1}}{1+s^q}\,ds}{\tau^p\log(1+\tau^q)}\right)^{-1}=p \tag{11}$$

and

$$\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\frac{1}{p}t^p\log\bigl(1+t^q\bigr)\left[1-q\cdot\frac{\int_0^{|t|}\frac{s^{p+q-1}}{1+s^q}\,ds}{t^p\log(1+t^q)}\right]=\infty. \tag{12}$$

Relations (8), (11) and (12) yield

$$\lim_{t\to\infty}\int_1^t\frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds=\infty.$$

Equivalently, we can write

$$\int_{\Phi^{-1}(1)}^{\infty}\frac{d\tau}{[\Phi(\tau)]^{1/N}}=\infty$$

or, by (10),

$$\int_{\Phi^{-1}(1)}^{\infty}\frac{d\tau}{\tau^{p/N}[\log(1+\tau^q)]^{1/N}}=\infty. \tag{13}$$

Since

$$\log(1+\theta)\leqslant\theta,\quad \forall\theta>0,$$

we deduce that

$$\frac{1}{\tau^{p/N}[\log(1+\tau^q)]^{1/N}}\geqslant\frac{1}{\tau^{(p+q)/N}},\quad \forall\tau>0.$$

Since $p+q<N$, we find

$$\int_{\Phi^{-1}(1)}^{\infty}\tau^{-(p+q)/N}\,d\tau=\infty$$

and thus relation (13) holds true. We conclude that

$$\lim_{t\to\infty}\int_1^t\frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds=\infty.$$

(c) Let γ be fixed such that $1\leqslant\gamma<(Np-N+p)/(N-p)$.

By Adams [2, p. 231], we have

$$\lim_{t\to\infty}\frac{|t|^{\gamma+1}}{\Phi_\star(kt)}=0,\quad \forall k>0,$$

if and only if

$$\lim_{t\to\infty}\frac{\Phi_\star^{-1}(t)}{t^{1/(\gamma+1)}}=0. \tag{14}$$

Using again L'Hôpital's rule we deduce that

$$\limsup_{t\to\infty}\frac{\Phi_\star^{-1}(t)}{t^{1/(\gamma+1)}}\leqslant(\gamma+1)\limsup_{t\to\infty}\frac{\Phi^{-1}(t)}{t^{\frac{1}{\gamma+1}+\frac{1}{N}}}.$$

Setting $\tau = \Phi(t)$ we obtain

$$\limsup_{t\to\infty} \frac{\Phi_\star^{-1}(t)}{t^{1/(\gamma+1)}} \leqslant (\gamma+1) \limsup_{\tau\to\infty} \frac{\tau}{[\Phi(\tau)]^{\frac{1}{\gamma+1}+\frac{1}{N}}}.$$

Since $\gamma < (Np - N + p)/(N - p)$ we have

$$p > \frac{N(\gamma+1)}{N+\gamma+1}.$$

Using the above inequality and (9) we get

$$\limsup_{\tau\to\infty} \frac{\tau^{\frac{N(\gamma+1)}{N+\gamma+1}}}{\Phi(\tau)} = 0.$$

We conclude that (c) holds true.

Thus the proof of Proposition 1 is complete. □

Remark 1. Proposition 1 enables us to apply Theorem 2.2 in [14] (see also Theorem 8.33 in [2]) in order to obtain that $W_0^1 L_\Phi(\Omega)$ is compactly embedded in $L^{\gamma+1}(\Omega)$ provided that $1 \leqslant \gamma < (Np - N + p)/(N - p)$.

An important role in what follows will be played by

$$p^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}.$$

Remark 2. By Example 2 [8, p. 243] it follows that

$$p^0 = p + q.$$

3. Proof of Theorem 1

The key argument in the proof of Theorem 1 is the following $\mathbb{Z}_2$-symmetric version (for even functionals) of the Mountain Pass lemma (see [23, Theorem 9.12]).

Mountain Pass lemma. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the Palais–Smale condition (that is, any sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0$ in $X^\star$ has a convergent subsequence) and $I(0) = 0$. Suppose that*

(I1) *There exist two constants $\rho, b > 0$ such that $I(x) \geqslant b$ if $\|x\| = \rho$.*
(I2) *For each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1;\ I(x) \geqslant 0\}$ is bounded.*

Then I has an unbounded sequence of critical values.

Let E denote the Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$. Let $\lambda > 0$ be arbitrary but fixed.

The energy functional associated to problem (3) is $J_\lambda : E \to \mathbb{R}$ defined by

$$J_\lambda(u) := \int_\Omega \Phi(|\nabla u(x)|)\, dx + \frac{\lambda}{p} \int_\Omega |u(x)|^p\, dx - \frac{1}{r} \int_\Omega |u(x)|^r\, dx.$$

By Remark 1, J_λ is well defined on E.

Let us denote by $J_{\lambda,1}, J_{\lambda,2} : E \to \mathbb{R}$ the functionals

$$J_{\lambda,1}(u) := \int_\Omega \Phi\big(\big|\nabla u(x)\big|\big)\,dx \quad \text{and} \quad J_{\lambda,2}(u) := \frac{\lambda}{p}\int_\Omega \big|u(x)\big|^p\,dx - \frac{1}{r}\int_\Omega \big|u(x)\big|^r\,dx.$$

Therefore

$$J_\lambda(u) = J_{\lambda,1}(u) + J_{\lambda,2}(u), \quad \forall u \in E.$$

By Lemma 3.4 in [14] it follows that $J_{\lambda,1}$ is a C^1 functional, with the Fréchet derivative given by

$$\langle J'_{\lambda,1}(u), v\rangle = \int_\Omega \log\big(1 + \big|\nabla u(x)\big|^q\big)\big|\nabla u(x)\big|^{p-2}\nabla u(x)\nabla v(x)\,dx,$$

for all $u, v \in E$.

Similar arguments as those used in the proof of Lemma 2.1 in [7] imply that $J_{\lambda,2}$ is of class C^1 with the Fréchet derivative given by

$$\langle J'_{\lambda,2}(u), v\rangle = \lambda\int_\Omega \big|u(x)\big|^{p-2}u(x)v(x)\,dx - \int_\Omega \big|u(x)\big|^{r-2}u(x)v(x)\,dx,$$

for all $u, v \in E$.

The above information shows that $J_\lambda \in C^1(E, \mathbb{R})$ and

$$\langle J'_\lambda(u), v\rangle = \int_\Omega \log\big(1 + \big|\nabla u(x)\big|^q\big)\big|\nabla u(x)\big|^{p-2}\nabla u(x)\nabla v(x)\,dx + \lambda\int_\Omega \big|u(x)\big|^{p-2}u(x)v(x)\,dx - \int_\Omega \big|u(x)\big|^{r-2}u(x)v(x)\,dx,$$

for all $u, v \in E$. Thus, the weak solutions of (3) coincide with the critical points of J_λ.

Lemma 1. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\| = \eta$.*

Proof. In order to prove Lemma 1 we first show that

$$\Phi(t) \geqslant \tau^{p^0}\Phi(t/\tau), \quad \forall t > 0 \text{ and } \tau \in (0, 1], \tag{15}$$

where p^0 is defined in the previous section.

Indeed, since

$$p^0 = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}$$

we have

$$\frac{t\varphi(t)}{\Phi(t)} \leqslant p^0, \quad \forall t > 0.$$

Let $\tau \in (0, 1]$ be fixed. We have

$$\log\big(\Phi(t/\tau)\big) - \log\big(\Phi(t)\big) = \int_t^{t/\tau} \frac{\varphi(s)}{\Phi(s)}\,ds \leqslant \int_t^{t/\tau} \frac{p^0}{s}\,ds = \log\big(\tau^{-p^0}\big)$$

and it follows that (15) holds true.

Fix $u \in E$ with $\|u\| < 1$ and $\xi \in (0, \|u\|)$. Using relation (15) we have

$$\int_\Omega \Phi\big(|\nabla u(x)|\big)\,dx \geqslant \xi^{p^0} \int_\Omega \Phi\left(\frac{|\nabla u(x)|}{\xi}\right) dx. \tag{16}$$

Defining $v(x) = |\nabla u(x)|/\xi$, for all $x \in \Omega$, we have $\|v\|_\Phi = \|u\|/\xi > 1$. Since $\Phi(t) \leqslant \frac{t\varphi(t)}{p}$, for all $t \in \mathbb{R}$, by [8, Lemma C.9] we deduce that

$$\int_\Omega \Phi\big(v(x)\big)\,dx \geqslant \|v\|_\Phi^p > 1. \tag{17}$$

Relations (16) and (17) show that

$$\int_\Omega \Phi\big(|\nabla u(x)|\big)\,dx \geqslant \xi^{p^0}.$$

Letting $\xi \nearrow \|u\|$ in the above inequality we obtain

$$\int_\Omega \Phi\big(|\nabla u(x)|\big)\,dx \geqslant \|u\|^{p^0}, \quad \forall u \in E \text{ with } \|u\| < 1. \tag{18}$$

On the other hand, since E is continuously embedded in $L^r(\Omega)$, it follows that there exists a positive constant $C_1 > 0$ such that

$$\int_\Omega |u(x)|^r\,dx \leqslant C_1 \cdot \|u\|^r, \quad \forall u \in E. \tag{19}$$

Using relations (18) and (19) we deduce that for all $u \in E$ with $\|u\| \leqslant 1$ we have

$$J_\lambda(u) \geqslant \int_\Omega \Phi\big(|\nabla u(u)|\big)\,dx - \frac{1}{r}\int_\Omega |u(x)|^r\,dx \geqslant \|u\|^{p^0} - \frac{C_1}{r}\cdot\|u\|^r$$
$$= \left(1 - \frac{C_1}{r}\cdot\|u\|^{r-p^0}\right)\|u\|^{p^0}.$$

But, by Remark 2 and the hypotheses of Theorem 1, we have $p^0 = p + q < r$. We conclude that Lemma 1 holds true. □

Lemma 2. *Assume that E_1 is a finite dimensional subspace of E. Then the set $S = \{u \in E_1;\ J_\lambda(u) \geqslant 0\}$ is bounded.*

Proof. With the same arguments as those used in the proof of relation (15) we have

$$\frac{\Phi(\sigma t)}{\Phi(t)} \leqslant \sigma^{p^0}, \quad \forall t > 0 \text{ and } \sigma > 1. \tag{20}$$

Then, for all $u \in E$ with $\|u\| > 1$, relation (20) implies

$$\int_\Omega \Phi\big(|\nabla u(x)|\big)\,dx = \int_\Omega \Phi\left(\|u\|\frac{|\nabla u(x)|}{\|u\|}\right) dx \leqslant \|u\|^{p^0}\int_\Omega \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right) dx \leqslant \|u\|^{p^0}. \tag{21}$$

On the other hand, since E is continuously embedded in $L^p(\Omega)$ it follows that there exists a positive constant $C_2 > 0$ such that

$$\int_\Omega \left|u(x)\right|^p dx \leqslant C_2 \cdot \|u\|^p, \quad \forall u \in E. \tag{22}$$

Relations (21) and (22) yield

$$J_\lambda(u) \leqslant \|u\|^{p^0} + \frac{\lambda}{p} \cdot C_2 \cdot \|u\|^p - \frac{1}{r}\int_\Omega \left|u(x)\right|^r dx, \tag{23}$$

for all $u \in E$ with $\|u\| > 1$.

We point out that the functional $|\cdot|_r : E \to \mathbb{R}$ defined by

$$|u|_r = \left(\int_\Omega \left|u(x)\right|^r dx\right)^{1/r}$$

is a norm in E. In the finite dimensional subspace E_1 the norms $|.|_r$ and $\|.\|$ are equivalent, so there exists a positive constant $C_3 = C_3(E_1)$ such that

$$\|u\| \leqslant C_3 \cdot |u|_r, \quad \forall u \in E_1.$$

The above remark and relation (23) imply

$$J_\lambda(u) \leqslant \|u\|^{p^0} + \frac{\lambda}{p} \cdot C_2 \cdot \|u\|^p - \frac{1}{r} \cdot C_3^{-1} \cdot \|u\|^r,$$

for all $u \in E_1$ with $\|u\| > 1$.

Hence

$$\|u\|^{p^0} + \frac{\lambda}{p} \cdot C_2 \cdot \|u\|^p - \frac{1}{r} \cdot C_3^{-1} \cdot \|u\|^r \geqslant 0, \tag{24}$$

for all $u \in S$ with $\|u\| > 1$. Since, by Remark 2 and the hypotheses of Theorem 1 we have $r > p^0 > p$, the above relation implies that S is bounded in E. □

Lemma 3. *Assume that $\{u_n\} \subset E$ is a sequence which satisfies the properties*

$$\left|J_\lambda(u_n)\right| < M, \tag{25}$$

$$J_\lambda'(u_n) \to 0 \quad \text{as } n \to \infty, \tag{26}$$

where M is a positive constant. Then $\{u_n\}$ possesses a convergent subsequence.

Proof. First, we show that $\{u_n\}$ is bounded in E. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\| > 1$ for any integer n.

By (26) we deduce that there exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\left\|J_\lambda'(u_n)\right\| \leqslant 1.$$

On the other hand, for any $n > N_1$ fixed, the application

$$E \ni v \to \left\langle J_\lambda'(u_n), v\right\rangle$$

is linear and continuous.

The above information yields

$$\left|\langle J_\lambda'(u_n), v\rangle\right| \leqslant \left\|J_\lambda'(u_n)\right\| \cdot \|v\| \leqslant \|v\|, \quad \forall v \in E,\ n > N_1.$$

Setting $v = u_n$ we have

$$\begin{aligned} -\|u_n\| &\leqslant \int_\Omega \log\left(1 + \left|\nabla u_n(u)\right|^q\right)\left|\nabla u_n(x)\right|^p dx + \lambda \int_\Omega \left|u_n(x)\right|^p dx - \int_\Omega \left|u_n(x)\right|^r dx \\ &\leqslant \|u_n\|, \end{aligned}$$

for all $n > N_1$. We obtain

$$-\|u_n\| - \int_\Omega \log\left(1 + \left|\nabla u_n(u)\right|^q\right)\left|\nabla u_n(x)\right|^p dx - \lambda \int_\Omega \left|u_n(x)\right|^p dx \leqslant -\int_\Omega \left|u_n(x)\right|^r dx, \tag{27}$$

for any $n > N_1$.

If $\|u_n\| > 1$, then relations (25) and (27) imply

$$\begin{aligned} M > J_\lambda(u_n) &= \int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx + \frac{\lambda}{p} \int_\Omega \left|u_n(x)\right|^p dx - \frac{1}{r} \int_\Omega \left|u_n(x)\right|^r dx \\ &\geqslant \int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx + \lambda \cdot \left(\frac{1}{p} - \frac{1}{r}\right) \cdot \int_\Omega \left|u_n(x)\right|^p dx \\ &\quad - \frac{1}{r} \cdot \int_\Omega \log\left(1 + \left|\nabla u_n(u)\right|^q\right)\left|\nabla u_n(x)\right|^p dx - \frac{1}{r} \cdot \|u_n\| \\ &= \int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx - \frac{1}{r} \cdot \int_\Omega \varphi\left(\left|\nabla u_n(x)\right|\right)\left|\nabla u_n(x)\right| dx \\ &\quad + \lambda \cdot \left(\frac{1}{p} - \frac{1}{r}\right) \cdot \int_\Omega \left|u_n(x)\right|^p dx - \frac{1}{r} \cdot \|u_n\|. \end{aligned}$$

Since

$$p^0 \geqslant \frac{t\varphi(t)}{\Phi(t)}, \quad \forall t > 0,$$

we find

$$\int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx - \frac{1}{r} \cdot \int_\Omega \varphi\left(\left|\nabla u_n(x)\right|\right)\left|\nabla u_n(x)\right| dx \geqslant \left(1 - \frac{p^0}{r}\right) \int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx.$$

Using the above relations we deduce that for any $n > N_1$ such that $\|u_n\| > 1$ we have

$$M > \left(1 - \frac{p^0}{r}\right) \cdot \int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx - \frac{1}{r} \cdot \|u_n\|. \tag{28}$$

Since $\Phi(t) \leqslant (t\varphi(t))/p$ for all $t \in \mathbb{R}$ we deduce by [8, Lemma C.9] that

$$\int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx \geqslant \|u_n\|^p, \tag{29}$$

for all $n > N_1$ with $\|u_n\| > 1$.

Relations (28) and (29) imply

$$M > \left(1 - \frac{p^0}{r}\right) \cdot \|u_n\|^p - \frac{1}{r} \cdot \|u_n\|,$$

for all $n > N_1$ with $\|u_n\| > 1$. Since $p^0 < r$, letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ is bounded in E we deduce that there exists a subsequence, still denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. Since E is compactly embedded in $L^p(\Omega)$ and $L^r(\Omega)$ it follows that $\{u_n\}$ converges strongly to u_0 in $L^p(\Omega)$ and $L^r(\Omega)$. Hence

$$\lim_{n\to\infty} J_{\lambda,2}(u_n) = J_{\lambda,2}(u_0) \quad \text{and} \quad \lim_{n\to\infty} J'_{\lambda,2}(u_n) = J'_{\lambda,2}(u_0). \tag{30}$$

Since

$$J_{\lambda,1}(u) = J_\lambda(u) - J_{\lambda,2}(u), \quad \forall u \in E,$$

relations (30) and (26) imply

$$\lim_{n\to\infty} J'_{\lambda,1}(u_n) = -J'_{\lambda,2}(u_0), \quad \text{in } E^\star. \tag{31}$$

Using the fact that Φ is convex and thus $J_{\lambda,1}$ is convex we have that

$$J_{\lambda,1}(u_n) \leqslant J_{\lambda,1}(u_0) + \left\langle J'_{\lambda,1}(u_n), u_n - u_0\right\rangle.$$

Passing to the limit as $n \to \infty$ and using (31) we deduce that

$$\limsup_{n\to\infty} J_{\lambda,1}(u_n) \leqslant J_{\lambda,1}(u_0). \tag{32}$$

Using again the fact that $J_{\lambda,1}$ is convex, it follows that $J_{\lambda,1}$ is weakly lower semicontinuous and hence

$$\liminf_{n\to\infty} J_{\lambda,1}(u_n) \geqslant J_{\lambda,1}(u_0). \tag{33}$$

By (32) and (33) we find

$$\lim_{n\to\infty} J_{\lambda,1}(u_n) = J_{\lambda,1}(u_0)$$

or

$$\lim_{n\to\infty} \int_\Omega \Phi\left(\left|\nabla u_n(x)\right|\right) dx = \int_\Omega \Phi\left(\left|\nabla u_0(x)\right|\right) dx. \tag{34}$$

Since Φ is increasing and convex, it follows that

$$\Phi\left(\frac{1}{2}\left|\nabla u_n(x) - \nabla u_0(x)\right|\right) \leqslant \Phi\left(\frac{1}{2}\left(\left|\nabla u_n(x)\right| + \left|\nabla u_0(x)\right|\right)\right)$$
$$\leqslant \frac{\Phi(|\nabla u_n(x)|) + \Phi(|\nabla u_0(x)|)}{2},$$

for all $x \in \Omega$ and all n. Integrating the above inequalities over Ω we find

$$0 \leqslant \int_\Omega \Phi\left(\frac{1}{2}\left|\nabla(u_n - u_0)(x)\right|\right) dx \leqslant \frac{\int_\Omega \Phi(|\nabla u_n(x)|)\, dx + \int_\Omega \Phi(|\nabla u_0(x)|)\, dx}{2},$$

for all n. We point out that Lemma C.9 in [8] implies

$$\int_\Omega \Phi\big(\big|\nabla u_n(x)\big|\big)\,dx \leqslant \|u_n\|^p < 1, \quad \text{provided that } \|u_n\| < 1,$$

while relation (20) yields

$$\int_\Omega \Phi\big(\big|\nabla u_n(x)\big|\big)\,dx \leqslant \|u_n\|^{p^0}, \quad \text{provided that } \|u_n\| > 1.$$

Since $\{u_n\}$ is bounded in E, the above inequalities prove the existence of a positive constant K_1 such that

$$\int_\Omega \Phi\big(\big|\nabla u_n(x)\big|\big)\,dx \leqslant K_1,$$

for all n. So, there exists a positive constant K_2 such that

$$0 \leqslant \int_\Omega \Phi\left(\frac{1}{2}\big|\nabla(u_n - u_0)(x)\big|\right)dx \leqslant K_2, \tag{35}$$

for all n.

On the other hand, since $\{u_n\}$ converges weakly to u_0 in E, Theorem 2.1 in [14] implies

$$\int_\Omega \frac{\partial u_n}{\partial x_i} v\,dx \to \int_\Omega \frac{\partial u_0}{\partial x_i} v\,dx, \quad \forall v \in L_{\Phi^\star}(\Omega),\ \forall i = 1, \ldots, N.$$

In particular, this holds for all $v \in L^\infty(\Omega)$. Hence $\{\frac{\partial u_n}{\partial x_i}\}$ converges weakly to $\frac{\partial u_0}{\partial x_i}$ in $L^1(\Omega)$ for all $i = 1, \ldots, N$. Thus we deduce that

$$\nabla u_n(x) \to \nabla u_0(x) \quad \text{a.e. } x \in \Omega. \tag{36}$$

Relations (35) and (36) and Lebesgue's dominated convergence theorem imply

$$\lim_{n\to\infty} \int_\Omega \Phi\left(\frac{1}{2}\big|\nabla(u_n - u_0)(x)\big|\right)dx = 0.$$

Taking into account that Φ satisfies the Δ_2-condition it follows by Lemma A.4 in [8] (see also [2, p. 236]) that

$$\lim_{n\to\infty} \left\|\frac{1}{2}(u_n - u_0)\right\| = 0$$

and thus

$$\lim_{n\to\infty} \big\|(u_n - u_0)\big\| = 0.$$

The proof of Lemma 3 is complete. □

Proof of Theorem 1. It is clear that the functional J_λ is even and verifies $J_\lambda(0) = 0$. Lemma 3 implies that J_λ satisfies the Palais–Smale condition. On the other hand, Lemmas 1 and 2 show that conditions (I1) and (I2) are satisfied. Thus the Mountain Pass lemma can be applied to the functional J_λ. We conclude that Eq. (3) has infinitely many weak solutions in E. The proof of Theorem 1 is complete. □

Remark 3. We point out the fact that the Orlicz–Sobolev space E cannot be replaced by a classical Sobolev space, since, in this case, condition (I1) in the Mountain Pass lemma cannot be satisfied. For a proof of that fact one can consult the proof of Remark 4 in [7, pp. 56–57].

4. Proof of Theorem 2

Let $\lambda > 0$ be arbitrary but fixed. Let $I_\lambda : E \to \mathbb{R}$ be defined by

$$I_\lambda(u) := \int_\Omega \Phi\left(\left|\nabla u(x)\right|\right) dx - \frac{\lambda}{p}\int_\Omega \left|u(x)\right|^p dx + \frac{1}{r}\int_\Omega \left|u(x)\right|^r dx.$$

The same arguments as those used in the case of functional J_λ show that I_λ is well defined on E and $I_\lambda \in C^1(E, \mathbb{R})$ with the Fréchet derivative given by

$$\left\langle I'_\lambda(u), v\right\rangle = \int_\Omega \log\left(1 + \left|\nabla u(x)\right|^q\right)\left|\nabla u(x)\right|^{p-2}\nabla u(x)\nabla v(x)\, dx$$
$$- \lambda\int_\Omega \left|u(x)\right|^{p-2}u(x)v(x)\, dx + \int_\Omega \left|u(x)\right|^{r-2}u(x)v(x)\, dx,$$

for all $u, v \in E$. This time our idea is to show that I_λ possesses a nontrivial global minimum point in E. We start with the following auxiliary result.

Lemma 4. *The functional I_λ is coercive on E.*

Proof. In order to prove Lemma 4, we first show that for any $b, d > 0$ and $0 < k < l$ the following inequality holds:

$$b \cdot t^k - d \cdot t^l \leqslant b \cdot \left(\frac{b}{d}\right)^{k/(l-k)}, \quad \forall t \geqslant 0. \tag{37}$$

Indeed, since the function

$$[0, \infty) \ni t \to t^\theta$$

is increasing for any $\theta > 0$ it follows that

$$b - d \cdot t^{l-k} < 0, \quad \forall t > \left(\frac{b}{d}\right)^{1/(l-k)},$$

and

$$t^k \cdot \left(b - d \cdot t^{l-k}\right) \leqslant b \cdot t^k < b \cdot \left(\frac{b}{d}\right)^{k/(l-k)}, \quad \forall t \in \left[0, \left(\frac{b}{d}\right)^{1/(l-k)}\right].$$

The above two inequalities show that (37) holds true.

Using (37) we deduce that for any $x \in \Omega$ and $u \in E$ we have

$$\frac{\lambda}{p} \cdot \left|u(x)\right|^p - \frac{1}{r} \cdot \left|u(x)\right|^r \leqslant \frac{\lambda}{p} \cdot \left[\frac{\lambda \cdot r}{p}\right]^{(p/(r-p))} = D_1,$$

where D_1 is a positive constant independent of u and x. Integrating the above inequality over Ω we find

$$\frac{\lambda}{p}\int_\Omega |u(x)|^p\,dx-\frac{1}{r}\int_\Omega |u(x)|^r\,dx\leqslant D_2,\quad \forall u\in E, \tag{38}$$

where D_2 is a positive constant independent of u.

Using inequalities (29) and (38) we obtain that for any $u\in E$ with $\|u\|>1$ we have

$$I_\lambda(u)\geqslant \|u\|^p-D_2.$$

Thus I_λ is coercive and the proof of Lemma 4 is complete. □

Proof of Theorem 2. First, we prove that I_λ is weakly lower semicontinuous on E. Indeed, using the definitions of $J_{\lambda,1}$ and $J_{\lambda,2}$ introduced in the above section we get

$$I_\lambda(u)=J_{\lambda,1}(u)-J_{\lambda,2}(u),\quad \forall u\in E.$$

Since Φ is convex it is clear that $J_{\lambda,1}$ is convex and thus weakly lower semicontinuous on E. By Remark 1 the functional $J_{\lambda,2}$ is also weakly lower semicontinuous on E. Thus, we obtain that I_λ is weakly lower semicontinuous on E.

By Lemma 4 we deduce that I_λ is coercive on E. Then Theorem 1.2 in [26] implies that there exists $u_\lambda\in E$ a global minimizer of I_λ and thus a weak solution of problem (4).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0>1$ be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1|>0$ we deduce that there exists $u_1\in C_0^\infty(\Omega)\subset E$ such that $u_1(x)=t_0$ for any $x\in\bar{\Omega}_1$ and $0\leqslant u_1(x)\leqslant t_0$ in $\Omega\setminus\Omega_1$. We have

$$\begin{aligned}I_\lambda(u_1)&=\int_\Omega \Phi\big(|\nabla u_1(x)|\big)\,dx-\frac{\lambda}{p}\int_\Omega |u_1(x)|^p\,dx+\frac{1}{r}\int_\Omega |u_1(x)|^r\,dx\\&\leqslant L-\frac{\lambda}{p}\int_{\Omega_1}|u_1(x)|^p\,dx\leqslant L-\frac{\lambda}{p}\cdot t_0^p\cdot|\Omega_1|\end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda_\star>0$ such that $I_\lambda(u_1)<0$ for any $\lambda\in[\lambda_\star,\infty)$. It follows that $I_\lambda(u_\lambda)<0$ for any $\lambda\geqslant\lambda_\star$ and thus u_λ is a nontrivial weak solution of problem (4) for λ large enough. The proof of Theorem 2 is complete. □

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