Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 305 (2006) 457–486
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Fixed conjugacy classes of normal subgroups and the $k(GV)$-problem

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Received 14 September 2005
Available online 22 December 2005
Communicated by Jan Saxl

Abstract

We establish several new bounds for the number of conjugacy classes of a finite group, all of which involve the maximal number c of conjugacy classes of a normal subgroup fixed by some element of a suitable subset of the group. To apply these formulas effectively, the parameter c, which in general is hard to control, is studied in some important situations.

These results are then used to provide a new, shorter proof of the most difficult case of the well-known $k(GV)$-problem, which occurs for $p = 5$ and V induced from the natural module of a 5-complement of $\mathrm{GL}(2,5)$. We also show how, for large p, the new results reduce the $k(GV)$-problem to the primitive case, thereby improving previous work on this. Furthermore, we discuss how they can be used in tackling the imprimitive case of the as of yet unsolved noncoprime $k(GV)$-problem.

1. Introduction and notation

Bounding the number of conjugacy classes of a finite group is a fundamental issue in finite group theory, as is evidenced by the large body of literature on the subject (for general results see e.g. [2,14,15], for asymptotic results on classical groups see [1], for the $k(GV)$-problem see e.g. [5,18]). This paper is another contribution to the subject, providing some

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.11.025

general bounds involving a parameter that, as far as we can tell, has hardly been used up to this point, but which will prove quite useful.

This new parameter is $|C_{\mathrm{cl}(N)}(g)|$. Here G is a group, $g \in G$, $N \trianglelefteq G$, $\mathrm{cl}(N)$ is the set of conjugacy classes of N, and $C_{\mathrm{cl}(N)}(g)$ is the set of classes of N which are fixed (as a set) by g under conjugation. Note that by Brauer's permutation lemma we have

$$\left|C_{\mathrm{cl}(N)}(g)\right| = \left|C_{\mathrm{Irr}(N)}(g)\right|,$$

where the latter is the number of irreducible complex characters of N fixed by g.

This parameter shows up in a number of bounds for the number $k(G)$ of conjugacy classes of G, such as the following:

A. Lemma. *Let G be a finite group and $N \trianglelefteq G$. Let $g_i \in G$ $(i = 1, \dots, k(G/N))$ such that the $g_i N$ are representatives of the conjugacy classes of G/N. Then*

$$k(G) \leqslant \sum_{i=1}^{k(G/N)} \left|C_{\mathrm{cl}(N)}(g_i)\right|.$$

(See Lemma 2.1 below.)

B. Lemma. *Let G be a finite group and $H \leqslant G$. Let N be the core of H in G. Then*

$$k(G) \leqslant k(H) + k(G/N) \max\left\{\left|C_{\mathrm{cl}(N)}(g)\right| \;\middle|\; g \in G - \bigcup_{x \in G} H^x\right\}.$$

(See Lemma 3.1 below.)

If one wants to use these and similar results effectively, one must somehow control $|C_{\mathrm{cl}(N)}(g)|$, which seems to be very difficult in general, and we are not aware of any result on this in the literature with the exception of our own first encounter with it in [13], where some very technical result on it was proved in [13, Lemma 4.7(b)]. In Sections 2, 4, and 5, therefore, we will prove some bounds on $|C_{\mathrm{cl}(N)}(g)|$ in some key special cases.

These techniques, while technical at times, turn out to be quite powerful. We will demonstrate this in Section 6, where we will give a short proof of the most difficult case of the $k(GV)$-problem that has only been solved recently by the combined efforts of Gluck, Magaard, Riese, and Schmid in [5]. Recall that the $k(GV)$-problem claims that $k(GV) \leqslant |V|$ whenever V is a finite faithful G-module of characteristic p not dividing $|G|$. This problem, which is equivalent to Brauer's well-known $k(B)$-problem for p-solvable groups, has kept mathematicians busy for the past 20 years, and the final step in its solution was a special case for $p = 5$ treated in [5] that had escaped all former attacks. So we will give a new proof of this case. More precisely, we will prove (see Theorem 6.2 below):

C. Theorem. *Let G be a finite $5'$-group and V be a faithful* GF(5)*-module such that V is induced from a G_1-module W, where G_1 is a suitable subgroup of G, $|W| = 25$ and $G_1/C_{G_1}(W) \neq 1$ is isomorphic to a subnormal subgroup of L, where L is a 5-complement* *in* $\mathrm{GL}(2,5)$. *Suppose that whenever* $U \leqslant G$ *and* $X \leqslant V$ *is a* U*-module with* $|UX| < |GV|$, *then* $k(UX) \leqslant |X|$. *Then*

$$k(GV) \leqslant |V|.$$

In Section 5 we will also use our techniques to directly reduce the $k(GV)$-problem to the case of V being primitive as G-module whenever $p > 2^{47}$ (see Theorem 5.3). This improves and shortens the corresponding reduction in our previous proof of the $k(GV)$-problem for large primes in [13, Theorem 4.1]. Finally, in Section 7 we turn to the more recent noncoprime $k(GV)$-problem (see Conjecture 7.1), research on which is still in its beginnings. While some work by Guralnick and Tiep [8] on primitive groups is underway, nothing is known on how to deal with the imprimitive case. We provide a few first steps in this direction that might be useful in an inductive argument (see Corollary 7.3 and Theorem 7.4 and the remarks following each of them). For instance, we will prove the following.

D. Theorem. *Let* G *be a finite group and* V *be a finite* G*-module. Suppose that* $N \trianglelefteq G$ *and* $V_N = V_1 \oplus \cdots \oplus V_n$ *for some* $n \geqslant 5$ *such that* G/N *primitively and faithfully permutes the* V_i. *Moreover, suppose that for some constant* $C > 0$ *we have*

$$k(NV) \leqslant C|V|\log_2|V|$$

and $k(UV_1) \leqslant C|V_1|\log_2|V_1|$ *for every* $U \leqslant N/C_N(V_1)$, *and*

$$\left|N/C_N(V_1)\right| \leqslant \frac{1}{50}C^{\frac{14}{3n}-\frac{8}{3}}|V_1|\left(\log_2|V|\right)^{\frac{14}{3n}-\frac{8}{3}}.$$

If $F^*(G/N)$ (*the generalized Fitting subgroup of* G/N) *is not a product of alternating groups, then*

$$k(GV) \leqslant C|V|\log_2|V|.$$

Our notation is as in [12,13]. In particular, if G acts on a set Ω, we denote by $n(G,\Omega)$ the number of orbits of G on Ω and by $C_\Omega(g)$ the set of fixed points of g on Ω. We will freely use the elementary formulas for $k(GV)$ as discussed in [13] as well as the well-known fact that if $N \trianglelefteq G$, then $k(G) \leqslant k(G/N)k(N)$. We will also use the latest improvement on upper bounds for the number of conjugacy classes of permutation groups. This is due to A. Maróti [16, Theorem 1.1] and states that for every $U \leqslant S_n$ with $n \neq 2$ we have $k(U) \leqslant 3^{(n-1)/2}$.

2. On conjugacy classes fixed by an automorphism

In this section we study the action of group elements on the conjugacy classes of some normal subgroup of the group.

Bits and pieces of what is to follow have already been foreshadowed in [12,13], but our treatment here is self-contained.

We start with a general lemma.

2.1. Lemma. *Let G be a finite group and $N \trianglelefteq G$. Let g_iN $(i = 1, \ldots, k(G/N))$ be representatives of the conjugacy classes of G/N, and write Ω_i for the set of N-orbits on g_iN. Then*

$$k(G) = \sum_{i=1}^{k(G/N)} n\bigl(C_{G/N}(g_i), \Omega_i\bigr) \leqslant \sum_{i=1}^{k(G/N)} \bigl|C_{\mathrm{cl}(N)}(g_iN)\bigr|.$$

Proof. Let $\Omega = \{g^N \mid g \in G\}$ be the set of N-orbits of G, and for $g \in G$ put $\Omega_{gN} = \{\omega \in \Omega \mid \omega \subseteq gN\}$. Hence $\Omega_i = \Omega_{g_iN}$ for $i = 1, \ldots, k(G/N)$. For each $\omega \in \Omega$ let $g_\omega \in G$, so that $g_\omega^N = \omega$. It is easy to check that $C_{G/N}(\omega) \leqslant C_{G/N}(g_\omega N)$ and that for $\omega \in \Omega_{gN}$ we have $g_\omega N = gN$. With this we conclude that

$$\begin{aligned} k(G) = n(G/N, \Omega) &= \frac{1}{|G/N|} \sum_{\omega \in \Omega} \bigl|C_{G/N}(\omega)\bigr| \\ &= \frac{1}{|G/N|} \sum_{\omega \in \Omega} \bigl|C_{C_{G/N}(\omega)}(g_\omega N)\bigr| \\ &= \frac{1}{|G/N|} \sum_{gN \in G/N} \sum_{\omega \in \Omega_{gN}} \bigl|C_{G/N}(\omega) \cap C_{G/N}(g_\omega N)\bigr| \\ &= \frac{1}{|G/N|} \sum_{gN \in G/N} \bigl|C_{G/N}(gN)\bigr| \frac{1}{|C_{G/N}(gN)|} \sum_{\omega \in \Omega_{gN}} \bigl|C_{C_{G/N}(gN)}(\omega)\bigr| \\ &= \frac{1}{|G/N|} \sum_{gN \in G/N} \bigl|C_{G/N}(gN)\bigr| n\bigl(C_{G/N}(gN), \Omega_{gN}\bigr) \\ &= \sum_{i=1}^{k(G/N)} n\bigl(C_{G/N}(g_iN), \Omega_i\bigr), \end{aligned} \tag{$*$}$$

which is the first part of the lemma. (Notice that $(*)$ was already proved in [13, Lemma 1.6] with a much longer proof.)

Next fix $g \in G$. We claim that

$$|\Omega_{gN}| = \bigl|C_{\Omega_N}(g)\bigr|, \tag{$**$}$$

that is, the number of N-orbits on gN is the same as the number of conjugacy classes of N fixed by g. To see this, let $x \in N$. Then $C_{gN}(x)$ is nonempty if and only if there is an $n \in N$ with $x^{-1}gnx = gn$ which means that $x^{n^{-1}} = x^g$. This is equivalent to $x^N \in C_{\Omega_N}(g)$. Moreover, then obviously $C_{gN}(x) = gnC_N(x)$. So altogether it follows that $|C_{gN}(x)| =$

$|C_N(x)|$ whenever $C_{gN}(x) \neq \emptyset$. Keeping all this in mind and putting $X = \{y \in N \mid y^N \in C_{\Omega_N}(g)\}$, we see that $y \in X$ if and only if $C_{gN}(y) \neq \emptyset$ and thus

$$\begin{aligned}\left|C_{\Omega_N}(g)\right| = n(N, X) &= \frac{1}{|N|}\sum_{y\in X}\left|C_N(y)\right| = \frac{1}{|N|}\sum_{y\in X}\left|C_{gN}(y)\right| \\ &= \frac{1}{|N|}\sum_{y\in N}\left|C_{gN}(y)\right| = |\Omega_{gN}|,\end{aligned}$$

where the last equation follows with the Cauchy–Frobenius orbit counting formula. So $(**)$ is proved.

Now as clearly $\Omega_N = \mathrm{cl}(N)$ and $C_{\Omega_N}(g) = C_{\Omega_N}(gN)$, by $(**)$ we conclude that $n(C_{G/N}(g_iN), \Omega_i) \leqslant |\Omega_i| = |C_{\mathrm{cl}(N)}(g_iN)|$, and so by $(*)$ the assertion of the lemma follows. □

2.2. Lemma. *Let G be a finite group. Suppose that $M \trianglelefteq G$ and that $M = \bigtimes_{i=1}^{l} M_i$, where the M_i $(i = 1, \ldots, l)$ are normal subgroups of G. We write elements $(a_1, \ldots, a_l) \in M$ simply as $a_1 \cdots a_l$ (for $a_i \in M_i$). Moreover, suppose that $G/M = \langle gM \rangle$ is cyclic of order m. Let $N \leqslant M$ with $N^g = N$, and put $L_i = M_i \times \cdots \times M_l$ for $i = 1, \ldots, l+1$ (so $L_{l+1} = 1$).*

- (a) *Let $x = x_1 \cdot \ldots \cdot x_l \in N$, where $x_i \in M_i$ $(i = 1, \ldots, l)$, and put $C_i = \bigcap_{j=1}^{i-1} C_N(x_j)$ for $i = 2, \ldots, l$, and put $C_1 = N$.*
  Then the following are equivalent:
  - (i) $x^g \in x^N$.
  - (ii) *For $i = 1, \ldots, l$ there exist $z_i \in C_i$ and $gz_1 \cdot \ldots \cdot z_i \in C_G(x_i)$.*
  - (iii) *Put $K_i = \{y \in M_i \mid x_1 \cdot \ldots \cdot x_{i-1} y L_{i+1} \in NL_{i+1}/L_{i+1}\}$ for $i = 1, \ldots, l$. Note that C_i acts on K_i (by conjugation). For $i = 1, \ldots, l-1$ there exist $z_i \in C_i$ such that $x_i^{gz_1\cdot\ldots\cdot z_{i-1}} \in K_i$ and $x_i^{gz_1\cdot\ldots\cdot z_{i-1}}$ and x_i lie in the same orbit of C_i on K_i. (The z_i here are actually the same as in* (ii).)
- (b) *Let $k_i = \max\{|C_{\mathrm{cl}(U)}(h)| \mid h \in G - M$ and $U \leqslant M_i$ with $U^h = U$ and $h^m \in M_1 \times \cdots \times M_{i-1} \times U \times M_{i+1} \times \cdots \times M_l\}$ for $i = 1, \ldots, l$. Then*

$$\left|C_{\mathrm{cl}(N)}(g)\right| \leqslant \prod_{i=1}^{l} k_i.$$

Proof. (a) (i) $\Rightarrow$ (ii): Put $x_0 = z_0 = 1$ and $C_0 = G$. We show by induction on i that there are $z_i \in C_i$ $(i = 0, \ldots, l)$ such that $gz_0z_1 \cdot \ldots \cdot z_i \in C_G(x_i)$. For $i = 0$ this is trivial. So let $1 \leqslant i \leqslant l$ and suppose that we already have $z_0, \ldots, z_{i-1}$. Then $x^{gz_0\cdot\ldots\cdot z_{i-1}} = \prod_{j=1}^{l} x_j^{gz_0\cdot\ldots\cdot z_{i-1}} = x_1 \cdot \ldots \cdot x_{i-1} \prod_{j=i}^{l} x_j^{gz_0\cdot\ldots\cdot z_{i-1}}$. As $x^g \in x^N$, we know that there is a $z_i \in N$ such that $x = x^{(gz_0\cdot\ldots\cdot z_{i-1})z_i} = x_1^{z_i} \cdot \ldots \cdot x_{i-1}^{z_i} \prod_{j=i}^{l} x_j^{gz_0\cdot\ldots\cdot z_i}$. This clearly forces $z_i \in C_i$ and $gz_0 \cdot \ldots \cdot z_i \in C_G(x_i)$, as wanted.

(ii) $\Rightarrow$ (iii): Let the $z_i \in C_i$ $(i = 1, \ldots, l)$ be as in (ii) and fix $i \in \{1, \ldots, l-1\}$. Then $x^{gz_1\cdots z_{i-1}} L_{i+1} = x_1 \cdot \ldots \cdot x_{i-1} x_i^{gz_1\cdots z_{i-1}} L_{i+1} \in NL_{i+1}/L_{i+1}$ which shows that $x_i^{gz_1\cdots z_{i-1}} \in$

K_i. Moreover, as $z_i \in C_i$ and $(x_i^{gz_1\cdots z_{i-1}})^{z_i} = x_i$ we see that $x_i^{gz_1\cdots z_{i-1}}$ lie in the same orbit of C_i on K_i.

(iii) $\Rightarrow$ (i): Suppose we have $z_i \in C_i$ $(i = 1, \dots, l-1)$ as in (iii). As $x_l^{gz_1\cdots z_{l-1}}$ and x_l lie in the same orbit of C_l on K_l, there is a $z_l \in C_l$ such that $x_l^{gz_1\cdots z_l} = x_l$. Thus clearly $x^{gz_1\cdots z_l} = x$, and so $x^g = x^{(z_1\cdots z_l)^{-1}} \in x^N$.

(b) We prove the statement by induction on l. If $l = 1$, the statement is easily seen to be true. Let $l \geqslant 1$. Observe that $M_1 \times \cdots \times M_{l-1} \cong M_0 := M/M_l \trianglelefteq G/M_l =: G_0$ and consider $N_0 = NM_l/M_l$. Then by induction we have

$$\left|C_{\mathrm{cl}(N_0)}(gM_l)\right| \leqslant \prod_{i=1}^{l-1} k_{0i},$$

where

$$\begin{aligned} k_{0i} = \max\bigl\{\left|C_{\mathrm{cl}(U)}(hM_l)\right| \mid\, & hM_l \in G_0 - M_0 \text{ and } U \leqslant M_iM_l/M_l \text{ with } U^{hM_l} = U \text{ and} \\ & (hM_l)^m \in M_1M_l/M_l \times \cdots \times M_{i-1}M_l/M_l \times U \times M_{i+1}M_l/M_l \times \cdots \\ & \times M_{l-1}M_l/M_l\bigr\}, \end{aligned}$$

and as $M_iM_l/M_l \cong M_i$ (as G-sets) and M_l centralizes M_i for $i < l_i$ we see that $k_{0i} = k_i$ for $i = 1, \dots, l-1$.

Now if $x_i \in M_i$ $(i = 1, \dots, l)$ such that for $x = x_1 \cdots x_l$ we have $x^g \in x^N$, then $xM_l = x_1 \cdots x_{l-1}M_l$ satisfies $(xM_l)^{gM_l} \in (xM_l)^{N_0}$, and so by the above xM_l is in one of at most $\prod_{i=1}^{l-1} k_i$ possible conjugacy classes of N_0.

Next suppose that $xM_l \in N_0$ with $(xM_l)^g \in (xM_l)^{N_0}$ has been chosen, i.e., $x_i \in M_i$ $(i = 1, \dots, l-1)$ are already fixed. Let K_l, C_l be as in (a). Now if $y_1,\ y_2 \in K_l$, then $x_1 \cdots x_{l-1}y_1$ and $x_1 \cdots x_{l-1}y_2$ obviously are in the same class of N if and only if y_1 and y_2 are in the same orbit of C_l on K_l, and thus there are exactly $n(C_l, K_l)$ different classes y^N of N such that $(yM_l)^{N_0} = (xM_l)^{N_0}$. If y^N is such a class and we choose the representative $y \in N$ such that $y = x_1 \cdots x_{l-1}x_l$ for a suitable $x_l \in K_l$, then by (a) y^N is fixed by g only if x_l and x_l^h lie in the same orbit of C_l on K_l for some element $h \in C_G(x_1 \cdots x_{l-1}) \cap gN$, i.e.,

$$y^N \in C_{\mathrm{cl}(N)}(g) \text{ only if } \bigl(x_l^{C_l}\bigr)^h = x_l^{C_l}$$

(note that h clearly normalizes $C_l = C_N(x_1 \cdots x_{l-1})$). Now put

$$L_l = \{z \in M_l \mid az \in C_l \text{ for some } a \in M_1 \times \cdots \times M_{l-1}\}.$$

Observe that $L_l \leqslant M_l$ and that $K_l \subseteq L_l$, in particular $x_l \in L_l$. Moreover, clearly $x_l^{C_l} = x_l^{L_l}$ and $L_l^h = L_l$ and $h^m \in M_1 \times \cdots \times M_{l-1} \times L_l$. Thus we see that $y^N \in C_{\mathrm{cl}(N)}(g)$ only if $x_l^{L_l} \in C_{\mathrm{cl}(L_l)}(h)$, in which case clearly $x_1 \cdots x_{l-1} x_l^{L_l} \subseteq y^N$. Thus, since obviously $|C_{\mathrm{cl}(L_l)}(h)| \leqslant k_l$, altogether we obtain

$$\left|C_{\mathrm{cl}(N)}(g)\right| \leqslant \left(\prod_{i=1}^{l-1} k_i\right) \cdot k_l,$$

and so we are done. □

2.3. Lemma. *Let G be a finite group. Suppose that $M \trianglelefteqslant G$ and that $M = \bigtimes_{i=1}^{p} M_i$, where p is a prime and the M_i are subgroups of G. Moreover, suppose that $G/M = \langle gM \rangle$ is cyclic of order p, and that g permutes the M_i transitively.*

Let $N \leqslant M$ with $N^g = N$. Put $L = \{y \in M_1 \mid ya \in N$ for some $a \in M_2 \times \cdots \times M_p\}$. (Clearly $L \leqslant M_1$.) Then

$$\left|C_{\mathrm{cl}(N)}(g)\right| \leqslant |L|.$$

Proof. Put $H = \langle N, g\rangle \leqslant G$. Then both N and H act on N by conjugation, and $k = |C_{\mathrm{cl}(N)}(g)|$ is exactly the number of common orbits of H and N on N. It is an elementary fact (see [9, Lemma 3.1]) that

$$k = \frac{1}{|N|} \sum_{n \in N} \left|C_N(gn)\right|.$$

As gn permutes the M_i transitively (for each $n \in N$), it is clear that $|C_N(gn)| \leqslant |L|$ for all $n \in N$. So the assertion follows. □

For $p = 2$, the above result turns out to be a little too weak for our purposes, and so later we will have to do some extra work to get around this. The bound provided by Lemma 2.3 is crude at times, and it is tempting to believe that

$$\left|C_{\mathrm{cl}(N)}(g)\right| \leqslant \max\left\{k(U) \mid U \leqslant M_1\right\}$$

holds. This is not true, however, as the following example shows:

2.4. Example. Let $G = S_3 \wr C_2$, where $C_2 = \langle g \rangle$, and let $N = (S_3' \times S_3') \cdot \langle ((12), (12)) \rangle \leqslant S_3 \times S_3$ (where S_3' is the commutator subgroup of S_3). So $|N| = 18$, and g normalizes N. It then can easily be checked that $k(N) = 6$ and $|C_{\mathrm{cl}(N)}(g)| = 4$, so in particular $|C_{\mathrm{cl}(N)}(g)| > \max\{k(U) \mid U \leqslant S_3\} = 3$.

(Note that Lemma 2.3 yields $|C_{\mathrm{cl}(N)}(g)| \leqslant 6$ here.)

In some sense, this seems to be a rare example, depending on the prime 2, as we see when we try to use the above example to create a more general one: Let q, p be primes with $q \mid p - 1$ and let F be the Frobenius group of order qp. Put $G = F \wr C_2$, $C_2 = \langle g \rangle$, and let, as above, N be of order $p^2 q$ such that g normalizes N. Then one can check (by hand) that if $q > 2$, then $|C_{\mathrm{cl}(N)}(g)| = 1 + \frac{p-1}{q} + q - 1 = k(F)$, so here $|C_{\mathrm{cl}(N)}(g)| \leqslant \max\{k(U) \mid U \leqslant F\}$, and if $q = 2$, then

$$\left|C_{\mathrm{cl}(N)}(g)\right| = 1 + 2\frac{p-1}{q} + q - 1 = p + 1,$$

so

$$\left|C_{\mathrm{cl}(N)}(g)\right| > \max\bigl\{k(U) \mid U \leqslant F\bigr\} = p.$$

3. The general tools

We now present our inductive arguments for proving results on $k(G)$.

3.1. Lemma. *Let G be a finite group and $H \leqslant G$. Put $N = \bigcap_{g\in G} H^g \trianglelefteqslant G$. Then*

$$k(G) \leqslant k(H) + k_0(G/N)\max\left\{\left|C_{\mathrm{cl}(N)}(g)\right| \,\middle|\, g \in G - \bigcup_{x\in G} H^x\right\},$$

where $k_0(G/N)$ is the number of conjugacy classes of G/N that are contained in $G/N - \bigcup_{x\in G}(H/N)^x$. (In particular, $k_0(G/N) \leqslant k(G/N)$.)

Proof. Note that by Lemma 2.1 we have

$$k(G) = \sum_{i=1}^{k(G/N)} n\bigl(C_{G/N}(g_iN), \Omega_i\bigr),$$

where the g_iN are representatives of the conjugacy classes of G/N, and Ω_i is the set of N-orbits on g_iN (where N acts by conjugation). Now let

$$T = \bigl\{(gN)^{G/N} \mid g \in H\bigr\}$$

be the set of conjugacy classes of G/N that intersects nontrivially with H/N and clearly we may assume that

$$T = \bigl\{(g_iN)^{G/N} \mid i = 1, \ldots, |T|\bigr\}$$

and that $g_i \in H$ for $i = 1, \ldots, |T|$, and that $g_1 = 1$. Then we have

$$k(G) = \sum_{i=1}^{|T|} n\bigl(C_{G/N}(g_iN), \Omega_i\bigr) + \sum_{i=|T|+1}^{k(G/N)} n\bigl(C_{G/N}(g_iN), \Omega_i\bigr).$$

The second sum is bounded above by

$$\begin{aligned}
&k_0(G/N)\max\bigl\{|\Omega_i| \bigm| i=|T|+1,\dots,k(G/N)\bigr\}\\
&\quad\leqslant k_0(G/N)\max\bigl\{\bigl|C_{\mathrm{cl}(N)}(g_iN)\bigr| \bigm| i=|T|+1,\dots,k(G/N)\bigr\}\\
&\quad\leqslant k_0(G/N)\max\Biggl\{\bigl|C_{\mathrm{cl}(N)}(g)\bigr| \Biggm| g\in G-\bigcup_{x\in G}H^x\Biggr\},
\end{aligned}$$

where the first inequality follows from the proof of Lemma 2.1. Thus it remains to show that

$$\sum_{i=1}^{|T|} n\bigl(C_{G/N}(g_iN),\Omega_i\bigr)\leqslant k(H).$$

Now let $h_i\in H$, $i=1,\dots,k(H/N)$, be such that the h_iN are representatives of the conjugacy classes of H/N and let Σ_i be that set of N-orbits on h_iN (with respect to conjugation). Clearly we may assume that $h_i=g_i$ for $i=1,\dots,|T|$. Then we conclude that

$$\sum_{i=1}^{|T|} n\bigl(C_{G/N}(g_iN),\Omega_i\bigr)=\sum_{i=1}^{|T|} n\bigl(C_{G/N}(h_iN),\Sigma_i\bigr)\leqslant\sum_{i=1}^{k(H/N)} n\bigl(C_{H/N}(h_iN),\Sigma_i\bigr)=k(H),$$

where the last equality again follows from Lemma 2.1. Hence the lemma is proved. □

Note that Lemma 3.1 always yields $k(G)>k(H)$ which, for typical applications where $k(H)$ is bounded by an inductive hypothesis, may give too weak a result unless additional information is known on $k(H)$. We therefore also present another lemma that is more specialized, but possibly more suitable for inductive arguments. This actually is a generalization of [12, Lemmas 2.2 and 2.3]. The proof is quite similar to the proofs of those lemmas, but for the convenience of the reader we outline the full argument.

3.2. Lemma. *Let G be a finite group and V be a finite G-module. Suppose that $N\trianglelefteq G$ and $V_N=V_1\oplus\cdots\oplus V_n$ for an $n\in\mathbb{N}$, where the V_i are N-modules, and assume that G/N transitively and faithfully permutes the V_i. Put $H=N_G(V_1)$ and let $f:\mathbb{N}\to\mathbb{R}$ be a function. Let $W\leqslant V$ be a G-submodule with $|W|\geqslant|V|^\delta$ for some $0\leqslant\delta\leqslant 1$. Assume that there is a G-module $W'\leqslant V$ such that $V=W\oplus W'$. Put*

$$m_0=\max\Biggl\{\bigl|C_{\mathrm{cl}(NV)}(g)\bigr| \Biggm| g\in G,\ g\text{ has at most }\frac{n}{2}\text{ fixed points in its permutation action on }\{V_1,\dots,V_n\}\Biggr\},$$

and suppose that the following hold:

(i) $k(HW)\leqslant f(|W|)$.
(ii) $k(UN/N)\leqslant\frac{1}{\sqrt{n+1}}\bigl(\frac{f(|W|)}{m_0}\bigr)^{1/2}$ *for all $U\leqslant G$.*

Then $k(GW)\leqslant f(|W|)$.

Proof. We may assume that $n \geqslant 2$. We consider the action of G/N on $\Omega := \mathrm{Irr}(NW)$. If $\omega \in \Omega$, we will write ω^G for the orbit of ω under G and $\omega\uparrow^G$ for the induced character. Let P be Gallagher's goodness property with respect to this action (see [13, Example 3.4(b)]). Then we have $k(HW) = \alpha_P(H/N, \Omega)$ and $k(GW) = \alpha_P(G/N, \Omega)$. Now let

$$R = \left\{ gN \in G/N \;\middle|\; gN \text{ normalizes at least } \frac{n}{2} \text{ of the } V_i \right\},$$

so R is a normal subset of G/N. Let

$$T = \left\{ \omega^{G/N} \mid \omega \in \Omega \text{ and } C_{G/N}(\omega) \nsubseteq R \right\},$$

so $\omega^{G/N}$ means that there is a $gN \in G/N - R$ such that gN fixes an element of $\omega^{G/N}$, i.e., $\omega^{G/N} \cap C_\Omega(gN) \neq \emptyset$. Hence $\omega^{G/N} \cap C_\Omega(g^h N) \neq 0$ for all $h \in G$. This shows that if $g_i N$, $i = 1, \ldots, t$ are representatives of the conjugacy classes of G/N which are not in R, then

$$T \subseteq \bigcup_{i=1}^{t} \left\{ \omega^{G/N} \mid \omega \in \Omega \text{ and } \omega^{G/N} \cap C_\Omega(g_i N) \neq \emptyset \right\}$$

and thus

$$|T| \leqslant t \cdot \max_{i=1,\ldots,t} \left| C_\Omega(g_i N) \right| \leqslant k(G/N) \max_{gN \in (G/N)-R} \left| C_\Omega(gN) \right|.$$

Now if $gN \in G/N - R$, then gN has at most $\frac{n}{2}$ fixed points on its permutation action on $\{V_1, \ldots, V_n\}$. Hence if we put $\Omega_1 = \mathrm{cl}(NV)$, then we have $|C_{\Omega_1}(g)| \leqslant m_0$. Let $\Omega_0 = \mathrm{Irr}(NV)$. Since by hypothesis there is a G-module W' such that $V = W \oplus W'$, clearly $NW \cong NV/W'$ and hence $\Omega \subseteq \Omega_0$. Now G/N acts on both Ω_1 and Ω_0 by conjugation, and so Brauer's permutation lemma (see e.g. [10, Theorem 18.5(b)]) yields

$$\left| C_\Omega(gN) \right| \leqslant \left| C_{\Omega_0}(gN) \right| = \left| C_{\Omega_1}(gN) \right| \leqslant m_0.$$

Hence we conclude that

$$|T| \leqslant k(G/N) m_0.$$

Now consider ω with $\omega^{G/N} \notin T$. Then $C_{G/N}(\omega) \subseteq R$, so all elements of $C_G(\omega)N/N$ have at least $\frac{n}{2}$ fixed points on $\{V_1, \ldots, V_n\}$. By [12, Lemma 2.1] there is an $i \in \{1, \ldots, n\}$ with $C_G(\omega) \leqslant N_G(V_i)$, and so we may assume that $C_G(\omega) \leqslant H$. As $H < G$, it follows that $|\omega^{G/N}| > |\omega^{H/N}|$, and so if $\omega_1, \ldots, \omega_k \in \omega^{G/N}$ are representatives of the orbits of H/N on $\omega^{G/N}$ with $\omega_1 = \omega$, then $k \geqslant 2$, and by the Theorem in [2] and [10, Exercise E17.2] we see that for $i = 2, \ldots, k$ we have

$$\begin{aligned} k_P\big(C_{H/N}(\omega)\big) &= k_P\big(C_{G/N}(\omega)\big) = k_P\big(C_{G/N}(\omega_i)\big) \\ &= \big|\big\{\psi \in \mathrm{Irr}\big(C_{GV}(\omega_i)\big) \mid \psi \text{ is a constituent of the induced} \\ &\quad\ \text{character } \omega_i{\uparrow}^{C_{GV}(\omega_i)}\big\}\big| \\ &\leqslant \big|C_{GV}(\omega_i) : C_{HV}(\omega_i)\big| \\ &\quad \cdot \big|\big\{\Theta \in \mathrm{Irr}\big(C_{HV}(\omega_i)\big) \mid \Theta \text{ is a constituent of } \omega_i{\uparrow}^{C_{HV}(\omega_i)}\big\}\big| \\ &\leqslant |G:H| \cdot k_P\big(C_{H/N}(\omega_i)\big) = n \cdot k_P\big(C_{H/N}(\omega_i)\big), \end{aligned}$$

and hence we obtain that

$$\begin{aligned} \sum_{j=1}^{k} k_P\big(C_{H/N}(\omega_j)\big) &\geqslant k_P\big(C_{H/N}(\omega_1)\big) + (k-1)\frac{1}{n}k_P\big(C_{H/N}(\omega)\big) \\ &\geqslant \frac{n+k-1}{n}k_P\big(C_{H/N}(\omega)\big) \geqslant \frac{n+1}{n}k_P\big(C_{H/N}(\omega)\big). \end{aligned}$$

Since these considerations hold for any $\omega^{G/N} \notin T$, we conclude that if $\omega_i \in \Omega$ $(i = 1, \dots, n(G/N, \Omega))$ are representatives of the orbits of G/N on Ω and the ω_{ij} $(j = 1, \dots, k_i)$ are representatives of the orbits of H/N on $\omega_i^{G/N}$, then we may assume that for all i with $\omega_i^{G/N} \notin T$ we have $C_{G/N}(\omega_i) \leqslant H/N$, and then the above yields

$$\begin{aligned} \sum_{i \text{ with } \omega_i^{G/N} \notin T} k_P\big(C_{G/N}(\omega_i)\big) &= \sum_{i \text{ with } \omega_i^{G/N} \notin T} k_P\big(C_{H/N}(\omega_i)\big) \\ &\leqslant \frac{n}{n+1} \sum_{i \text{ with } \omega_i^{G/N} \notin T} \sum_{j=1}^{k_i} k_P\big(C_{H/N}(\omega_{ij})\big) \\ &\leqslant \frac{n}{n+1}\alpha_P(H/N, \Omega) = \frac{n}{n+1}k(HW). \end{aligned}$$

Hence altogether we obtain

$$\begin{aligned} k(GW) &= \alpha_P(G/N, \Omega) \\ &= \sum_{i \text{ with } \omega_i^{G/N} \in T} k_P\big(C_{G/N}(\omega_i)\big) + \sum_{i \text{ with } \omega_i^{G/N} \notin T} k_P\big(C_{G/N}(\omega_i)\big) \\ &\leqslant |T| \cdot \max_{i=1,\dots,n(G/N,\Omega)} k_P\big(C_{G/N}(\omega_i)\big) + \frac{n}{n+1}k(HW) \\ &\leqslant k(G/N)m_0 \max_{U \leqslant G} k(UN/N) + \frac{n}{n+1}k(HW) \\ &\leqslant \Big(\max_{U \leqslant G} k(UN/N)\Big)^2 \cdot m_0 + \frac{n}{n+1}k(HW). \end{aligned}$$

Therefore by our Hypotheses (i) and (ii) we are done. □

The final lemma in this section will be useful in certain noncoprime situations.

3.3. Lemma. *Let G be a finite group and let $N \trianglelefteq G$. Then*

$$k(G) \leqslant \frac{k(N)}{|G/N|} + 2\big(k(G/N)-1\big)\max\big\{\big|C_{\mathrm{cl}(N)}(g)\big| \mid g \in G-N\big\}.$$

Proof. Let $g_i \in G$ ($i = 1, \ldots, k(G/N)$) such that $g_1 = 1$ and the $\overline{g_i} = g_i N$ are representatives of the conjugacy classes of G/N. Then by Lemma 2.1 we have $k(G) = \sum_{i=1}^{k(G/N)} n(C_{G/N}(\overline{g_i}), \Omega_i)$, where Ω_i is the set of N-orbits on $g_i N$. By the proof of Lemma 2.1 we have

$$n\big(C_{G/N}(\overline{g_i}), \Omega_i\big) \leqslant \big|C_{\mathrm{cl}(N)}(\overline{g_i})\big|,$$

so that we obtain

$$k(G) \leqslant n\big(G/N, \mathrm{cl}(N)\big) + \sum_{i=2}^{k(G/N)} \big|C_{\mathrm{cl}(N)}(\overline{g_i})\big|.$$

By the Cauchy–Frobenius orbit counting formula we have

$$\begin{aligned} n\big(G/N, \mathrm{cl}(N)\big) &= \frac{1}{|G/N|} \sum_{gN \in G/N} \big|C_{\mathrm{cl}(N)}(gN)\big| \\ &= \frac{1}{|G/N|} \sum_{i=1}^{k(G/N)} \big|G/N : C_{G/N}(\overline{g_i})\big|\big|C_{\mathrm{cl}(N)}(\overline{g_i})\big| \\ &= \frac{k(N)}{|G/N|} + \sum_{i=2}^{k(G/N)} \frac{1}{|C_{G/N}(\overline{g_i})|}\big|C_{\mathrm{cl}(N)}(\overline{g_i})\big| \\ &\leqslant \frac{k(N)}{|G/N|} + \sum_{i=2}^{k(G/N)} \big|C_{\mathrm{cl}(N)}(\overline{g_i})\big|. \end{aligned}$$

Thus altogether

$$k(G) \leqslant \frac{k(N)}{|G/N|} + 2\sum_{i=2}^{k(G/N)} \big|C_{\mathrm{cl}(N)}(\overline{g_i})\big|$$

which implies the assertion of the lemma. □

4. On the number of fixed conjugacy classes of normal subgroups in certain semidirect products

The aim of this section is to obtain strong bounds for $|C_{\mathrm{cl}(NV)}(g)|$, where $N \trianglelefteq G$, $g \in G$ and V is a faithful G-module.

We start with an easy lemma.

4.1. Lemma. *Let H be a finite group, $N \leqslant G \trianglelefteq H$ and $N \trianglelefteq H$. Let $g \in H$. Then*

$$\left|C_G(g)\right| \leqslant \left|C_{G/N}(gN)\right|\left|C_N(g)\right|.$$

Proof. Write $G/N = \{g_i N \mid i = 1, \ldots, |G/N|\}$ for suitable $g_i \in G$. If $h \in C_G(g)$, then $h = g_i x$ for a unique i and a unique $x \in N$. Now clearly $g_i N = hN \in C_{G/N}(gN)$, so there are $|C_{G/N}(gN)|$ possibilities for g_i. Once g_i is chosen, we see that $g_i x = h = h^g = (g_i x)^g = g_i x_0 x^g$ for some $x_0 \in N$ that depends on g and i. Hence x is a solution of the equation $[x^{-1}, g] = x_0$, and there are either 0 or exactly $|C_n(g)|$ solutions x for this equation, because if x_1 and x_2 are both solutions of the equation, then $x_1^{-1} x_2 \in C_N(g^{-1}) = C_N(g)$. Hence the assertion of the lemma follows. □

4.2. Lemma. *Let L be a finite group. Let $H \trianglelefteq L$, and suppose that $|L/H| = p$ is a prime and that $H = H_1 \times \cdots \times H_p$ for subgroups H_i of L that are permuted by L/H, i.e., $H_i^g = H_{i+1}$ for $i = 1, \ldots, p-1$ and $H_p^g = H_1$, where $L/H = \langle gH \rangle$. Let $N \leqslant H$ such that $N^g = N$ and $g^p \in N$, and put $G = \langle N, g \rangle$. Let $N_0 = H_1 \cap N$ and $N_1 = \prod_{i=0}^{p-1} N_0^{g_i}$. Then obviously $N_0 \trianglelefteq N$, $N_1 \trianglelefteq G$ and $N_1 = \bigtimes_{i=0}^{p-1} N_0^{g_i}$. Furthermore, if we put $J = N/N_1$, then*

$$\left|C_{\mathrm{cl}(N)}(g)\right| \leqslant k(J) \cdot k(N_0).$$

Proof. By [9, Lemma 3.1] we have the following elementary formula:

$$\left|C_{\mathrm{cl}(N)}(g)\right| = \frac{1}{|N|} \sum_{n \in N} \left|C_N(gn)\right|,$$

and so with Lemma 4.1 we get

$$\left|C_{\mathrm{cl}(N)}(g)\right| \leqslant \frac{1}{|N|} \sum_{n \in N} \left|C_J(gnN_1)\right|\left|C_{N_1}(gn)\right|.$$

Write $J = \{a_i N_1 \mid i = 1, \ldots, |J|\}$ for suitable $a_i \in N$, and also write $M_i = N_0^{g^{i-1}}$ for $i = 1, \ldots, p$, so that $N_1 = \bigtimes_{i=1}^{p} M_i$. Then we further have

$$\begin{aligned}\left|C_{\mathrm{cl}(N)}(g)\right| &\leqslant \frac{1}{|N|}\sum_{i=1}^{|J|}\sum_{x_1\in M_1}\cdots\sum_{x_p\in M_p}\left|C_J(ga_ix_1\cdots x_pN_1)\right|\left|C_{N_1}(ga_ix_1\cdots x_p)\right| \\ &= \left(\frac{1}{|J|}\sum_{i=1}^{|J|}\left|C_J(ga_iN_1)\right|\right)\cdot\left(\frac{1}{|N_1|}\sum_{x_1\in M_1}\cdots\sum_{x_p\in M_p}\left|C_{N_1}(ga_ix_1\cdots x_p)\right|\right).\end{aligned}$$

For convenience, call the first factor in the last product A and the second B. Then

$$A = \frac{1}{|J|}\sum_{a\in J}\left|C_J(ga)\right| = \frac{1}{|J|}\sum_{a\in J}\left|C_{gJ}(a)\right|,$$

and as it is easy to see that $C_{gJ}(a)$ either is empty or a coset of $C_J(a)$ (see e.g. [2, p. 176] for the argument), we have

$$A \leqslant \frac{1}{|J|}\sum_{a\in J}\left|C_J(a)\right| = k(J).$$

It thus remains to show that $B \leqslant k(N_0)$.

For the moment, fix $i \in \{1,\dots,|J|\}$ and $x_j \in M_j$ for $j = 2,\dots,p$, and put $g_0 = ga_ix_2\cdots x_p$. Then we clearly have

$$\left|C_{N_1}(ga_ix_1x_2\cdots x_p)\right| = \left|C_{N_1}(g_0x_1)\right|,$$

and if we define

$$U_1 = \left\{z_1 \in M_1 \mid z_1z \in C_{N_1}(g_0x_1) \text{ for some } z \in M_2\times\cdots\times M_p\right\}, \quad \text{then } U_1 \leqslant M_1,$$

and as g_0x_1 cyclically permutes the M_i, we see that for each $z_1 \in U_1$ there is a unique $z \in M_2\times\cdots\times M_p$ such that $z_1z \in C_{N_1}(g_0x_1)$, so that $|C_{N_1}(g_0x_1)| = |U_1|$. Moreover, $(g_0x_1)^p \in N$ and clearly $U_1 \leqslant C_{M_1}((g_0x_1)^p)$. Thus altogether

$$\begin{aligned}\left|C_{N_1}(ga_ix_1\cdots x_p)\right| \leqslant \left|C_{M_1}\left((g_0x_1)^p\right)\right| &= \left|C_{M_1}\left(g_0^px_1^{g_0^{p-1}}x_1^{g_0^{p-2}}\cdots x_1^{g_0}x_1\right)\right| \\ &= \left|C_{M_1}\left(g_0^px_1\right)\right|,\end{aligned}$$

where the last equality follows as $x_1^{g_0^j} \in M_{j+1}$ and M_{j+1} centralizes M_1 for $j = 1,\dots,p-1$. Moreover, we have

$$\sum_{x_1\in M_1}\left|C_{M_1}\left(g_0^px_1\right)\right| = \sum_{x_1\in M_1}\left|C_{g_0^pM_1}(x_1)\right| \leqslant \sum_{x_1\in M_1}\left|C_{M_1}(x_1)\right| = |M_1|k(M_1),$$

where the inequality again follows from the fact that $C_{g_0^pM_1}(x_1)$ is either empty or a coset of $C_{M_1}(x_1)$.

With this we finally have

$$
\begin{aligned}
B &\leqslant \frac{1}{|N_1|}\sum_{x_2\in M_2}\cdots\sum_{x_p\in M_p}\left(\sum_{x_1\in M_1}\bigl|C_{M_1}\bigl(g_0^p x_1\bigr)\bigr|\right)\\
&\leqslant \frac{1}{|M_1|^p}\sum_{x_2\in M_2}\cdots\sum_{x_p\in M_p}|M_1|k(M_1)=\frac{1}{|M_1|^p}|M_1|^{p-1}|M_1|k(M_1)\\
&= k(M_1)=k(N_0),
\end{aligned}
$$

so that the lemma is proved. □

We next recall an elementary result essentially obtained in [12, Lemma 3.3]. The version that follows, however, has been generalized to include noncoprime actions.

4.3. Lemma. *Suppose that G is a finite group and V is a finite G-module. Suppose that $V=V_1\oplus V_2$ for G-modules V_i $(i=1,2)$. Let $\lambda_i\in \operatorname{Irr}(V_1)$ $(i=1,\dots,n(G,V_1))$ be representatives of the orbits of G on $\operatorname{Irr}(V_1)$. Then*

$$
k(GV)=\sum_{i=1}^{n(G,V_1)} k\bigl(C_G(\lambda_i)V_2\bigr).
$$

In particular, $k(GV)\leqslant n(G,V_1)\cdot \max\{k(C_G(\lambda)V_2)\mid \lambda\in\operatorname{Irr}(V_1)\}$.

Moreover, if $(|G|,|V|)=1$ and $v_i\in V_1$ $(i=1,\dots,n(G,V_1))$ are representatives of the orbits of G on V_1, then

$$
k(GV)=\sum_{i=1}^{n(G,V_1)} k\bigl(C_G(v_i)V_2\bigr)\leqslant n(G,V_1)\max\bigl\{k\bigl(C_G(v)V_2\bigr)\mid v\in V_1\bigr\}.
$$

Proof. First observe that $n(G,V_1)=n(G,\operatorname{Irr}(V_1))$ and $n(G,V)=n(G,\operatorname{Irr}(V))$ by Brauer's permutation lemma. Note that any $\lambda\in\operatorname{Irr}(V)$ can be extended to its inertia group in GV, and therefore if μ_i $(i=1,\dots,n(G,V))$ are representatives of the orbits of G on $\operatorname{Irr}(V)$, then with Gallagher's result [11, Corollary (6.12)] we conclude that

$$
\begin{aligned}
k(GV)=\bigl|\operatorname{Irr}(GV)\bigr| &= \sum_{i=1}^{n(G,V)} k\bigl(C_G(\mu_i)\bigr)\\
&= \frac{1}{|G|}\sum_{\mu\in\operatorname{Irr}(V)}\bigl|C_G(\mu)\bigr|k\bigl(C_G(\mu)\bigr)\\
&= \frac{1}{|G|}\sum_{\mu_1\in\operatorname{Irr}(V_1),\ \mu_2\in\operatorname{Irr}(V_2)}\bigl|C_G(\mu_1\mu_2)\bigr|k\bigl(C_G(\mu_1\mu_2)\bigr)
\end{aligned}
$$

$$= \sum_{\mu_1 \in \mathrm{Irr}(V_1)} \frac{|C_G(\mu_1)|}{|G|} \left(\frac{1}{|C_G(\mu_1)|} \sum_{\mu_2 \in \mathrm{Irr}(V_2)} \left| C_{C_G(\mu_1)}(\mu_2) \right| k\big(C_{C_G(\mu_1)}(\mu_2)\big) \right)$$

$$= \sum_{\mu_1 \in \mathrm{Irr}(V_1)} \frac{|C_G(\mu_1)|}{|G|} k\big(C_G(\mu_1)V_2\big)$$

$$= \sum_{i=1}^{n(G,V_1)} k\big(C_G(\lambda_i)V_2\big)$$

as wanted. The remaining statements are immediate and well-known consequences of the first one. □

We also need a result on the number of orbits.

4.4. Lemma. *Let G be a finite group and let V be a finite G-module. Let $N \trianglelefteq G$. Then*

$$n(G, V) \leqslant \left(\frac{k(GV)|V|}{k(G)} \right)^{1/2}.$$

Proof. First note that $n(G, \mathrm{Irr}(V)) = n(G, V)$ by Brauer's permutation lemma. As every $\lambda \in \mathrm{Irr}(V)$ can be extended to its inertia group in GV, we have, if the λ_i are representatives of the orbits of G on $\mathrm{Irr}(V)$, that

$$\begin{aligned} k(GV) &= \sum_{i=1}^{n(G,V)} k\big(C_G(\lambda_i)\big) \\ &\geqslant \sum_{i=1}^{n(G,V)} \frac{k(G)}{|G : C_G(\lambda_i)|} = k(G)n(G, V) \frac{1}{n(G, V)} \sum_{i=1}^{n(G,V)} \frac{1}{|G : C_G(\lambda_i)|} \\ &\geqslant k(G)n(G, V)^2 \left(\sum_{i=1}^{n(G,V)} \big| G : C_G(\lambda_i) \big| \right)^{-1} = k(G)n(G, V)^2 |V|^{-1}, \end{aligned}$$

where the first inequality follows from Ernest's result (see [10, Problem E17.2]) and the second inequality follows from the arithmetic–harmonic–means inequality. Therefore $n(G, V) \leqslant (k(GV)|V|/k(G))^{1/2}$, and so we are done. □

We now can prove an important auxiliary result.

4.5. Lemma. *Let G be a finite group and V be a finite faithful G-module. Suppose that p is a prime and $V = V_1 \oplus \cdots \oplus V_p$ for subspaces V_i which are permuted nontrivially by G,* *and put* $N = \bigcap_{i=1}^{p} N_G(V_i) \trianglelefteq G$. *Moreover, assume that* $G/N = \langle gN \rangle$ *is cyclic of order* p. *Put*

$$K_i = C_N\left(\bigoplus_{j=1;\ j\neq i}^{p} V_j\right) \trianglelefteq N$$

for $i = 1, \ldots, p$, *so then* $N_1 = K_1 \cdots K_p = K_1 \times \cdots \times K_p \trianglelefteq G$.

Put $N_0 = K_1$, *and* $J = N/N_1$. *Then*

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant k(J) \cdot k(N_0 V_1).$$

Put $W_2 = V_2 \oplus \cdots \oplus V_p$, $U_1 = N_G(V_1)/C_G(V_1)$ *and*

$$M = \max\left\{k\left(C_N(\lambda_1)N_0/N_0\right)W_2 \mid \lambda_1 \in \mathrm{Irr}(V_1)\right\}$$

and $m = \max\{k(T) \mid T \leqslant N_0\}$. *Then for any* $S \leqslant U_1$ *with* $k(S) = \max\{k(U) \mid U \leqslant U_1\}$ *we have*

$$k(NV) \leqslant \left(\frac{k(SV_1)|V_1|}{k(N/N_0)^{1/(p-1)}}\right)^{1/2} \cdot M \cdot m \leqslant \left(\frac{k(SV_1)|V_1|}{k(J)^{1/(p-1)}}\right)^{1/2} \cdot M \cdot m.$$

Proof. Note that if we put $H_i = (N_G(V_i)/C_G(V_i))V_i$ for $i = 1, \ldots, p$ and $H = H_1 \times \cdots \times H_p$ (so that $NV \lesssim H$) and $L = \langle H, g\rangle$, then (after possibly relabeling the H_i) the hypotheses of Lemma 4.2 are fulfilled, so it follows easily that $|C_{\mathrm{cl}(NV)}(g)| \leqslant k(J) \cdot k(N_0 V_1)$ which proves the first inequality that we have to establish.

To prove the second one, put $W_1 = V_1$ and observe that W_1 and W_2 are N-modules. By Lemma 4.3, we have

$$k(NV) \leqslant n(N, W_1) \cdot M_1, \quad \text{where } M_1 = \max\left\{k\left(C_N(\lambda_1)W_2\right) \mid \lambda_1 \in \mathrm{Irr}(V_1)\right\}.$$

Now let $S \leqslant U_1$ such that $k(S) = \max\{k(U) \mid U \leqslant U_1\}$. Then by Lemma 4.4 we have

$$n(N, W_1) = n\left(N/C_N(W_1), V_1\right) \leqslant n(U_1, V_1) \leqslant n(S, V_1) \leqslant \left(\frac{k(SV_1)|V_1|}{k(S)}\right)^{1/2}.$$

Recall that $C_N(W_2) = N_0$. Furthermore it is easy to see that with $X_i = \bigoplus_{j=2}^{i} V_j$ $(i = 2, \ldots, p)$ we have

$$k(N/N_0) = k\left(N/C_N(W_2)\right) \leqslant \prod_{i=2}^{p} k\left(C_{N/N_0}(X_{i-1})/C_{N/N_0}(X_i)\right) \leqslant k(S)^{p-1},$$

where the second inequality follows by the choice of S. Thus $k(S) \geqslant (k(N/N_0))^{1/(p-1)}$ and hence

$$n(N, W_1) \leqslant \left(\frac{k(SV_1)|V_1|}{k(N/N_0)^{1/p}}\right)^{1/2} \leqslant \left(\frac{k(SV_1)|V_1|}{k(J)^{1/(p-1)}}\right)^{1/2}.$$

To complete the proof of the lemma, it remains to show that $M_1 \leqslant M \cdot m$. For any subgroup $T \leqslant N$ we have

$$k(TW_2) \leqslant k\big(\big(T/C_T(W_2)\big)W_2\big) \cdot k\big(C_T(W_2)\big) \leqslant k\big((TN_0/N_0)W_2\big) \cdot m,$$

and so the assertion follows and we are done. □

5. The coprime case

In this section we study what our results yield in the situation of the classical $k(GV)$-problem. For this, first recall the following result by Gambini and Gambini-Weigel [3], as stated in [4, Theorem 2.1].

5.1. Theorem. *Let G be a finite group and W a faithful primitive finite G-module with $(|G|, |W|) = 1$. Then*

$$|G| \leqslant |W| \log_2 |W|,$$

except when $|W| = 7^4$ and G is $\mathrm{Sp}(4,3)$ or $Z_3 \times \mathrm{Sp}(4,3)$.

5.2. Lemma. *Let G be a finite group and let V be a finite faithful G-module with $(|G|, |V|) = 1$, and assume that $k(UX) \leqslant |X|$ whenever U is a finite group, X is a faithful U-module with $(|U|, |X|) = 1$ and $|UX| < |GV|$. Suppose that $W < V$ and $H < G$ are such that $H = N_G(W)$, W is primitive as H-module, and $V = W^G$ is induced from W. So we can write $V = V_1 \oplus \cdots \oplus V_n$ for some $n > 1$ and subspaces V_i that are permuted faithfully by G/N, where $N = \bigcap_{g \in G} H^g$ and $V_1 = W$. Let p be a prime and $g \in G - N$ such that $g^p \in N$. Let f be the number of p-cycles in the permutation action of g on $\{V_1, \ldots, V_n\}$, so that g normalizes $n - pf$ of the V_i. Clearly $f \geqslant 1$. Put*

$$B = \begin{cases} 6 & \text{if } |V_1| = 7^4 \text{ and } N/C_N(V_1) \text{ is isomorphic to } \mathrm{Sp}(4,3) \text{ or } Z_3 \times \mathrm{Sp}(4,3), \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\big|C_{\mathrm{cl}(NV)}(g)\big| \leqslant A_i^f |V_1|^{n-pf}, \quad i = 1, 2,$$

for each of the following A_i:

$$A_1 = B|V_1|^2 \log_2 |V_1| \quad \text{and} \quad A_2 = |V_1|^{\frac{2p^2+1}{2p+1}} \big(B \log_2 |V_1|\big)^{\frac{2p-2}{2p+1}}.$$

Note that for $p = 2$, $A_2 = |V_1|^{9/5} (B \log_2 |V_1|)^{2/5}$.

Proof. Note that if we put $D_i = (N/C_N(V_i))V_i$, then $NV \lesssim M := \bigtimes_{i=1}^n D_i$ with G/N permuting the factors of this direct product. Now relabel the D_i such that $\mathcal{O}_i = \{D_{(i-1)p+1}, \ldots, D_{ip}\}$ $(i = 1, \ldots, f)$ are the orbits of $\langle g\rangle$ on $\{D_1, \ldots, D_n\}$ of size p, and $\mathcal{O}_i = \{D_{(p-1)f+i}\}$ $(i = f+1, \ldots, n-(p-1)f)$ are the remaining orbits. Put $M_i = \bigtimes_{j \text{ with } D_j \in \mathcal{O}_j} D_j$ for $i = 1, \ldots, n-(p-1)f$ and consider the group $G_0 = \langle g, M\rangle$.

This clearly satisfies the hypothesis of Lemma 2.2, and if we define the k_i as in Lemma 2.2(b), then by our hypothesis $k_i \leqslant |V_i| = |V_1|$ for $i = f+1, \ldots, n-(p-1)f$, and so by Lemma 2.2(b) we have

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant |V_1|^{n-pf} \prod_{i=1}^{f} k_i.$$

Thus it remains to show that $k_i \leqslant A_1$ and $k_i \leqslant A_2$ for $i = 1, \ldots, f$. For this we clearly may assume that $f = 1$, $n = p$ and G is embedded in $H = \langle g, H_1 \times \cdots \times H_p\rangle$, where $H_i = N_G(V_i)/C_G(V_i)$ for $i = 1, \ldots, p$ and $H_i^g = H_{i+1}$ for $i = 1, \ldots, p-1$ and $H_p^g = H_1$, and $N \leqslant H$, and we have to show that for $C := C_{\mathrm{cl}(NV)}(g)$ we have

$$|C| \leqslant A_i \quad \text{for } i = 1, 2.$$

Now by Theorem 5.1 we have $|D_1| \leqslant B|V_1|^2 \log_2 |V_1|$ and so by Lemma 2.3 we have

$$|C| \leqslant B|V_1|^2 \log_2 |V_1|.$$

This gives the first part of the lemma.

Next let N_0, N_1 and J be as in Lemma 4.5. With Lemma 4.5 and our hypothesis we obtain that

$$|C| \leqslant k(J)|V_1| \tag{1}$$

and

$$|C| \leqslant k(NV) \leqslant \frac{|V_1|}{k(J)^{1/(2(p-1))}} \cdot |V_1|^{p-1} \cdot \max\bigl\{k(T) \mid T \leqslant N_0\bigr\} \leqslant \frac{|V_1|^p}{k(J)^{1/(2(p-1))}} \cdot |N_0|.$$

Observe that as $C_N(V_2 \oplus \cdots \oplus V_p) = N_0$, by Theorem 5.1 we have

$$|J||N_0|^{p-1} = |N/N_0| = \bigl|N/C_N(V_2 \oplus \cdots \oplus V_p)\bigr| \leqslant |H_1|^{p-1} \leqslant \bigl(B|V_1| \log_2 |V_1|\bigr)^{p-1}$$

and thus

$$|N_0| \leqslant \frac{B|V_1| \log_2 |V_1|}{|J|^{1/(p-1)}},$$

so that we further get

$$|C| \leqslant \frac{B|V_1|^{p+1}\log_2(|V_1|)}{k(J)^{1/(2(p-1))}|J|^{1/(p-1)}} \leqslant \frac{B|V_1|^{p+1}\log_2(|V_1|)}{k(J)^{3/(2(p-1))}}. \tag{2}$$

Now the upper bounds in (1) and (2) are equal if and only if

$$k(J) = \left(B|V_1|^p \log_2|V_1|\right)^{\frac{2p-2}{2p+1}}. \tag{3}$$

Therefore either (1) or (2) will always yield a bound less than or equal to the one we obtain in case that $k(J)$ has the critical value in (3), therefore we always have

$$|C| \leqslant \left(B|V_1|^p \log_2|V_1|\right)^{\frac{2p-2}{2p+1}} \cdot |V_1| = |V_1|^{\frac{2p^2+1}{2p+1}}\left(B\log_2|V_1|\right)^{\frac{2p-2}{2p+1}}. \tag{4}$$

So the lemma is proved. □

Note that the A_1-bound in the previous lemma, which was relatively easy to establish, is always much stronger than the A_2-bound, with the only exception of $p = 2$, where the A_1-bound is trivial and useless, while the A_2-bound is nontrivial, albeit quite weak.

In view of the applications of Lemma 5.2, it would be highly desirable to improve the bound for $p = 2$; the current A_2-bound seems to be much too large, and in fact something like

$$|C| \leqslant |V_1|^{2/3}$$

instead of (4) should be possible. While our general bounds are larger than necessary, in specific situations, when more detailed information on the groups is available, such as good bounds for $k(J)$, then the formulas (1), (2) in the proof of Lemma 5.2 will yield much better results, as we shall see in Section 6. This is already so in case that $N/C_N(W_1)$ is isomorphic to Sp(4, 3) or $Z_3 \times$ Sp(4, 3), so that better bounds than the ones in the previous lemma can be obtained in that case, although we will not pursue this further here.

Next we look at the most general reduction of the imprimitive case of the $k(GV)$-problem that we can get here.

5.3. Theorem. *Let G be a finite group and V be a finite faithful G-module with $(|G|, |V|) = 1$. Assume that $k(UX) \leqslant |X|$ whenever U is a finite group, X is a faithful U-module with $(|U|, |X|) = 1$ and $|UX| < |GV|$. Suppose further that $W < V$ and $H < G$ are such that $H = N_G(W)$, W is primitive as H-module, and $V = W^G$ is induced from W. Put $\overline{H} = H/C_H(W)$. Then the following hold*:

(a) *If $|W| \geqslant 2^{47}$ then $k(GV) \leqslant |V|$.*
(b) *If $k(HV) \leqslant |V| - (3^{(n-1)/2} + 1)|V|^{9/10}(6\log_2|W|)^{n/5}$, then $k(GV) \leqslant |V|$. Moreover, if $k(\overline{H}W) \leqslant \frac{|W|}{2}$ and $|W| \geqslant 2^{19}$, then $k(GV) \leqslant |V|$.*

Proof. Let $N = \bigcap_{g\in G} H^g \trianglelefteq G$. Then we can write $V = V_1 \oplus \cdots \oplus V_n$ for $n = |G : H|$ and submodules $V_i \leqslant V$ such that $V_1 = W$, and G/N permutes the V_i transitively and faithfully.

(a) If $g \in G$ has at most $\frac{n}{2}$ fixed points in its permutation action on $\{V_1, \ldots, V_n\}$, then by Lemma 5.2 we know that with B as in Lemma 5.2 we have

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant \left(|W|^{9/5}\left(B\log_2|W|\right)^{2/5}\right)^{n/4}|W|^{n/2} = |W|^{(19/20)n}\left(B\log_2|W|\right)^{n/10} =: C$$

(as we clearly may assume that g is of prime order when checking this).

Now by Lemma 3.2 (with $\delta = 1$ and $f(x) = x$) we are done if

$$k(UN/N) \leqslant \frac{1}{\sqrt{n+1}}\left(\frac{|V|}{C}\right)^{1/2} \quad \text{for all } U \leqslant G,$$

and as for $n \neq 2$ we have $k(UN/N) \leqslant 3^{(n-1)/2}$ for all $U \leqslant G$, as UN/N is isomorphic to a subgroup of S_n, it suffices to have

$$3^{(n-1)/2} \leqslant \frac{1}{\sqrt{n+1}}\left(\frac{|V|}{C}\right)^{1/2} = \frac{1}{\sqrt{n+1}}\frac{|W|^{n/40}}{(B\log_2|W|)^{n/20}}$$

and also (in case that $n = 2$) that

$$2 \leqslant \frac{1}{\sqrt{3}}\frac{|W|^{1/20}}{(B\log_2|W|)^{1/10}}.$$

This is the case for $|W| \geqslant 2^{47}$ (as $B = 1$ in this case), as can easily be verified, so (a) is proved.

(b) If $g \in G$ permutes the V_i $(i = 1, \ldots, n)$ fixed point freely, then by Lemma 5.2 we know that

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant \left(|W|^{9/5}\left(B\log_2|W|\right)^{2/5}\right)^{n/2} =: D,$$

where B is as in Lemma 5.2.

By Lemma 3.1 we have

$$\begin{aligned} k(GV) &\leqslant k(HV) + k(G/N)\max\left\{\left|C_{\mathrm{cl}(NV)}(g)\right| \;\middle|\; g \in G - \bigcup_{x\in G} H^x\right\} \\ &\leqslant k(HV) \\ &\quad + k(G/N)\max\left\{\left|C_{\mathrm{cl}(NV)}(g)\right| \mid g \in G \text{ permutes the } V_i \text{ fixed point freely}\right\}. \end{aligned}$$

As $G/N \lesssim S_n$, again by [16] we have $k(G/N) \leqslant \lceil 3^{(n-1)/2}\rceil \leqslant 2^{n-1}$ (where $\lceil x\rceil$ denotes the upper integer part of x), and so we conclude that

$$k(GV) \leqslant k(HV) + \left\lceil 3^{(n-1)/2}\right\rceil D = k(HV) + \left\lceil 3^{(n-1)/2}\right\rceil |W|^{(9/10)n}\left(B\log_2|W|\right)^{n/5},$$

and so by our hypothesis the first assertion of (b) follows.

To prove the second one, first note that if $k(\overline{H}W) \leqslant \frac{|W|}{2}$, then

$$k(HV) \leqslant k(\overline{H}V_1) \cdot k\big(C_H(V_1)(V_2 \oplus \cdots \oplus V_n)\big) \leqslant \frac{|V_1|}{2} \cdot |V_2 \oplus \cdots \oplus V_n| = \frac{|V|}{2},$$

and as $|W| > 7^4$, clearly $B = 1$ here, so we obtain

$$k(GV) \leqslant \frac{|V|}{2} + 2^{n-1}|W|^{(9/10)n}\big(\log_2 |W|\big)^{n/5}.$$

Thus $k(GV) \leqslant |V|$ if

$$2^{n-1}|W|^{(9/10)n}\big(\log_2 |W|\big)^{n/5} \leqslant \frac{|W|^n}{2}$$

which is equivalent to

$$2^{10}\big(\log_2 |W|\big)^2 \leqslant |W|,$$

and this holds for $|W| \geqslant 2^{19}$. So the theorem is proved. □

So this is a general reduction of the imprimitive case of the $k(GV)$-problem to "small" cases. For large primes $p = \text{char}(V)$, this result even provides a complete reduction of the imprimitive case to the primitive case, saying that a minimal counterexample to the $k(GV)$-problem must be primitive. This is an improvement of the corresponding part in the proof of [12, Theorem 4.1]. (For ways to treat the primitive case for large primes, see [12].)

It would be nice if one could refine the methods here, in particular improve the bounds in Lemma 5.2, so as to further reduce the 2^{47} in Lemma 5.3 and reach a general reduction of the $k(GV)$-problem to primitive actions.

It would also be interesting to know whether with methods as the ones employed here it is possible (at least for large p) to reduce the problem further to tensorprimitive modules V.

6. The last case of the $k(GV)$-problem

While Lemma 5.3 seems to imply that our techniques only work for large primes $p = \text{char}(V)$, we will now see that they are also quite powerful in "small" situations. We demonstrate this by providing a new, short proof of the $k(GV)$-problem in the situation that turned out to be the most difficult in the original proof of the $k(GV)$-problem and that occupied all of [5]. Here $p = 5$ and V is induced from the irreducible module of order 5^2 of a 5-complement of $\text{GL}(2, 5)$.

6.1. Lemma. *Let G be a finite group and V be a finite faithful G-module. Let p be a prime, and suppose that $V = V_1 \oplus \cdots \oplus V_p$ for subspaces V_i which are permuted nontrivially* *by G. Assume that $G/N = \langle gN \rangle$ is cyclic of order p, where $N = \bigcap_{i=1}^{p} N_G(V_i) \trianglelefteq G$. Suppose further that $|V_1| = 5^2$ and let L be a 5-complement of* $\mathrm{GL}(2,5)$. *Assume that $U_1 := N_G(V_1)/C_G(V_1)$ is isomorphic to a subgroup of L in its natural action on V_1. Then*

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant |V|^{0.74}.$$

Proof. Put $C = C_{\mathrm{cl}(NV)}(g)$. By Lemma 2.3 we have

$$|C| \leqslant |LV_1| = 96 \cdot 25 = 2400.$$

Hence $|C| \leqslant |V|^{0.74}$ if $p \geqslant 5$, as can easily be checked.

Let $p \leqslant 3$. Let $N_0 \leqslant N$ and $N_1 \trianglelefteq G$ be as in Lemma 4.5, and write $\overline{N} = N/C_N(V_1)$ and observe that we may consider N_0 to be a subgroup of $\overline{N_1}$ as N_0 acts faithfully on V_1. Put $J = N/N_1$.

Let $p = 3$. If $N/C_N(V_1)$ is not isomorphic to L, then

$$\left|N/C_N(V_1)\right| \leqslant \frac{96}{2} = 48,$$

and then as for the primes $\geqslant 5$ by Lemma 2.3 we conclude that

$$|C| \leqslant 48 \cdot 25 \leqslant |V|^{0.74} = 25^{2.22}.$$

So we may assume that $\overline{N} \cong L$.

Now if $|N_0| \geqslant 8$, then N_0 contains $\overline{G_1}'' \cong Q_8$ (the quaternion group of order 8), and then it is easy to see that $k(J) \leqslant 50$, so by Lemma 4.5 we have

$$|C| \leqslant k(J)k(N_0V_1) \leqslant 50 \cdot 25 \leqslant |V|^{0.74},$$

so that we are done in this case. Hence $|N_0| \in \{1, 2, 4\}$.

If $|N_0| = 4$, then $\overline{N}/N_0 \cong S_4$ and thus again $k(J) \leqslant k(S_4)^2 = 5 \cdot 5 = 25$, so again by Lemma 4.5 we have

$$|C| \leqslant 25 \cdot 25 \leqslant |V|^{0.74}.$$

In the remaining cases we use Lemma 4.3. Note that as $\overline{N} \cong L$, we have $n(N, V_1) = 2$, so if $|N_0| = 1$, then N acts faithfully on $V_2 + V_3$, and by Lemma 4.3 for any $0 \neq v_1 \in V_1$ we have

$$\begin{aligned}|C| &\leqslant k(NV) = k\big(N(V_2 \oplus V_3)\big) + k\big(C_N(v_1)(V_2 \oplus V_3)\big)\\ &\leqslant k\big(N/C_N(V_2)V_2\big)k\big(C_N(V_2)V_3\big) + 25^2 \leqslant 20 \cdot 25 + 25^2 = 1125 \leqslant |V|^{0.74},\end{aligned}$$

as wanted. Thus let $|N_0| = 2$. Let $0 \neq v_1 \in V_1$. Then $C_N(v_1)$ acts faithfully on $V_2 \oplus V_3$, and $|C_N(V_2 \oplus V_3)| = 2$, and thus

$$J = N/N_1 = N/(N_1N_0) \cong (N/N_0)/(N_1/N_0),$$

and as

$$N/N_0 \lesssim N/C_N(V_2) \times N/C_N(V_3) \lesssim L \times L$$

and $|N_1/N_0| = |N_0|^2 = 4$, we see that

$$J \lesssim L/Z(L) \times L/Z(L)$$

and so $|J| \mid 48^2$. If $|J| \mid \frac{48^2}{2}$, then it is clear from the structure of $L/Z(L) \cong S_4 \times C_2$ that $k(J) \leqslant 50$, and then as $k(N_0 V_1) = 14$, by Lemma 4.5 we have $|C| \leqslant 14 \cdot 50 = 700 \leqslant |V|^{0.74}$. Hence we may assume that $J \cong L/Z(L) \times L/Z(L)$. Then

$$k\big(N(V_2 \oplus V_3)\big) \leqslant k\big(\big(N/C_N(V_2 \oplus V_3)\big)(V_2 \oplus V_3)\big) \cdot k\big(C_N(V_2 \oplus V_3)\big) \leqslant 20 \cdot 20 \cdot 2 = 800.$$

Moreover, $C_N(v_1)/C_N(V_1) \cong C_4$, and $C_N(V_1)/C_{N_1}(V_1) \cong S_4 \times C_2$, and $|C_N(V_1) \cap C_N(V_i)| = 2$ for $i = 2, 3$, and so $C_N(V_1)/C_{C_N(V_1)}(V_2) \cong L$. Hence $n(C_N(V_1), V_2) = 2$, and by Lemma 4.3 we conclude that

$$\begin{aligned} k\big(C_N(v_1)(V_2 \oplus V_3)\big) &\leqslant 4 \cdot k\big(C_N(V_1)(V_2 \oplus V_3)\big) \\ &\leqslant 4 \cdot 2 \cdot \max\big\{k\big(C_{C_N(V_1)}(v_2)V_3\big) \mid v_2 \in V_2\big\} \\ &\leqslant 4 \cdot 2 \cdot 25 \cdot 2 = 400. \end{aligned}$$

Thus altogether by Lemma 4.3 we have

$$|C| \leqslant k(NV) = k\big(N(V_2 \oplus V_3)\big) + k\big(C_N(v_1)(V_2 \oplus V_3)\big) \leqslant 800 + 400 = 1200 \leqslant |V|^{0.74}$$

which concludes the case $p = 3$.

It remains to consider the case $p = 2$. Here we have to show that $|C| \leqslant 117$.

Now if $k(J) \leqslant 4$, then again by Lemma 4.5 we have $|C| \leqslant 4 \cdot 25 = 100$ and we are done. Thus from now on let $k(J) \geqslant 5$.

If $3 \mid |N_0|$, then J is a 2-group and thus $|J| \geqslant 8$, so L has a section of order 24 with a normal Sylow 3-subgroup, which contradicts the structure of L. Thus $3 \nmid |N_0|$.

Next suppose that $3 \nmid |J|$, so $3 \nmid |U_1|$ and U_1 is a 2-group, more precisely a subgroup of $S := C_4 \wr C_2$ (which is a 2–Sylow subgroup of L). Observe that S' is of order 4 and acts fixed point freely on V_1.

Assume that $Z := Z(F(L)) \leqslant N_0$ (up to isomorphism). Then $|N_0| \in \{2, 4\}$ (as $k(J) \geqslant 5$). If $|N_0| = 4$, it is easy to check that then $k(N_0 V_1) \leqslant 16$, so by Lemma 4.5 we have $|C| \leqslant 16k(J)$ which forces $k(J) = 8$ (otherwise we are done), and so J is abelian of order 8. But then $N_0 = S'$ and thus $k(N_0 V_1) = 10$, so that Lemma 4.5 yields the wanted conclusion here. If $|N_0| = 2$, then $k(N_0 V_1) = 14$, so by Lemma 4.5 we may assume that $k(J) > 8$ forcing $|J| = 16$. Now we use Lemma 4.3 which in our coprime situation means that if $v_1 = 0, v_2, v_3$ are representatives of the three orbits of U_1 on V_1, then

$$k(NV) = k(NV_2) + k\big(C_N(v_2)V_2\big) + k\big(C_N(v_3)V_2\big),$$

and we may assume that $C_{U_1}(v_2)$ is cyclic of order 4 and $C_{U_1}(v_3)$ is of order 2. Hence $C_N(v_2) \cong C_4 \times C_2$ and $C_N(v_3)$ is elementary abelian of order 4, and both $C_N(v_2)$ and $C_N(v_3)$ act faithfully on V_2 (because if, say, $C_N(v_2)$ contained an element x acting trivially on V_2, then $x \in N_0$, contradicting the fact that N_0 acts fixed point freely on V_1), and both $C_N(v_2)$ and $C_N(v_3)$ contain an involution acting fixed point freely on V_2. Thus it is easy to check that $k(C_N(v_2)V_2) \leqslant 25$ and $k(C_N(v_3)V_2) \leqslant 25$, and as

$$k(NV_2) \leqslant 2 \cdot k\big(\big(N/C_N(V_2)\big)V_2\big) \leqslant 2 \cdot 25 = 50,$$

altogether we have $|C| \leqslant k(NV) \leqslant 25 + 25 + 50 = 100$, as wanted.

Therefore to complete the case $3 \nmid |J|$ we may assume that $Z \nleqslant N_0$. But from the structure of L it is then clear that then $|J| \leqslant 4$, a contradiction.

Hence for the rest of the proof we may assume that $3 \mid |J|$. From the structure of L and since $k(J) \geqslant 5$ it then follows that $|N_0| \leqslant 8$.

If $|N_0| = 8$, then necessarily $N_0 \cong Q_8$ acts fixed point freely on V_1 and thus $k(N_0V_1) = 8$, so by Lemma 4.5 we have $|C| \leqslant k(J)k(N_0V_1) \leqslant |J| \cdot 8 = 12 \cdot 8 = 96$, so we are done here.

If $|N_0| = 4$, then N_0 is cyclic of order 4 and acts fixed point freely on V_1, so that $k(N_0V_1) = 10$. Moreover, $k(J) \leqslant 10$ and so again by Lemma 4.5 we are done.

Next let $|N_0| = 2$. Then N_0 acts fixed point freely on V_1, so that $k(N_0V_1) = 14$. Thus by Lemma 4.5 we may assume that $k(J) \geqslant 9$, which implies that $U_1 \cong L$ is as large as possible. Then by Lemma 4.3 we have

$$k(NV) = k(NV_2) + k\big(C_N(v)V_2\big)$$

for any $0 \neq v \in V_1$, as U_1 has only one nontrivial orbit on V_1. As clearly $k(NV_2)$ and $k(C_N(v)V_2)$ are bounded above by $2 \cdot |V_2| = 50$, we are done here as well.

So finally let $|N_0| = 1$. Hence N acts faithfully on V_1 and on V_2, and as $k(UV_1) \leqslant |V_1| = 25$ for any $U \leqslant N$, by Lemma 4.3 we have $k(NV) \leqslant 25n(N, V_1)$, so that we are done whenever $n(N, V_1) \leqslant 4$. Thus let $n(N, V_1) \geqslant 5$. Then from the structure of H and its action on V_1 it is clear that N must be cyclic of order 3 or 6 and act fixed point freely on V_1. As $J \cong N$ here and $k(J) \geqslant 5$, only the case $|N| = 6$ remains, and then $|C| \leqslant k(NV) = 110$. This completes the proof of the lemma. □

Now we can prove the main result of this section. Observe that this includes the main result of [5], which constituted the last and in some sense most difficult case of the $k(GV)$-problem.

6.2. Theorem. *Let G be a finite $5'$-group and V be a faithful* GF(5)*-module such that V is induced from a G_1-module W, where G_1 is a suitable subgroup of G, $|W| = 25$ and $G_1/C_{G_1}(W) \neq 1$ is isomorphic to a subnormal subgroup of L, where L is a 5-complement in* GL(2, 5). *Suppose that whenever $U \leqslant G$ and $X \leqslant V$ is a U-module with $|UX| < |GV|$, then $k(UX) \leqslant |X|$. Then*

$$k(GV) \leqslant |V|.$$

Proof. Put $n = |G : G_1|$. Clearly we may assume that $n > 1$. Then $V = V_1 \oplus \cdots \oplus V_n$ for subspaces $V_i \cong W$ that are permuted transitively by G. Write $H = N_G(V_1)$ and $N = \bigcap_{x\in G} H^x \trianglelefteq G$. Now let $g \in G - \bigcup_{x\in G} H^x$ be of prime order, so that in particular g has no fixed point in its permutation action on $\{V_1, \ldots, V_n\}$. Then applying Lemmas 2.2(b) and 6.1 to the group $\langle g, NV\rangle$ yields

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant |V|^{0.74} \tag{1}$$

and hence (1) holds for all $g \in G - \bigcup_{x\in G} H^x$.

First suppose that $n = 2$. Then $|G/N| = 2$, and Lemma 2.1, together with (1), yields

$$k(GV) \leqslant k(NV) + |V|^{0.74}. \tag{2}$$

Put $N_1 = N/C_N(V_1)$, then

$$k(NV) \leqslant k(N_1V_1)\cdot k\big(C_N(V_1)V_2\big) \leqslant k(N_1V_1)\cdot 25,$$

the second inequality following by our hypothesis. So (2) yields $k(GV) \leqslant |V| = 625$ unless $k(N_1V_1) \geqslant 21$, but it can easily be checked that this happens only when V_1 is reducible as N_1-module and either $N_1 = 1$ or N_1 is cyclic of order 4. But as $N = G_1$ here, we have $N_1 \cong G_1/C_{G_1}(W) \neq 1$ by hypothesis, and if $|N_1| = 4$ and V_1 is reducible as N_1-module, then N_1 is not subnormal in (its copy in) L, again contradicting our hypothesis.

Thus for the rest of the proof we may assume that $n \geqslant 3$. Now by Lemma 3.1 and (1) we have

$$\begin{aligned} k(GV) &\leqslant k(HV) + k(G/N)\max\left\{\left|C_{\mathrm{cl}(NV)}(g)\right| \;\middle|\; g \in G - \bigcup_{x\in G} H^x\right\} \\ &\leqslant k(HV) + k(G/N)|V|^{0.74}. \end{aligned} \tag{3}$$

Recall that if $S \leqslant S_n$, then $k(S) \leqslant (\sqrt{3})^{n-1}$ for $n \geqslant 3$ (see [16]). With this, (3) becomes

$$k(GV) \leqslant k(HV) + 3^{\frac{1}{2}(n-1)}|V|^{0.74}. \tag{4}$$

Next we have to bound $k(HV)$. Note that V is reducible as H-module, and so with $H_1 := H/C_H(V_1)$ we have

$$k(HV) \leqslant k(H_1V_1)\cdot k\big(C_H(V_1)(V_2 \oplus \cdots \oplus V_n)\big) \leqslant k(H_1V_1)\cdot 25^{n-1}$$

(again by our hypothesis). Now in the case that $k(H_1V_1) \geqslant 21$ as for $n = 2$ we run into a contradiction, so we may assume that $k(H_1V_1) \leqslant 20$. Thus from (4) we get

$$k(GV) \leqslant \frac{4}{5}|V| + 3^{\frac{1}{2}(n-1)}|V|^{0.74}, \tag{5}$$

so that for $k(GV) \leqslant |V|$ it suffices to show that

$$\frac{4}{5}|V| + 3^{\frac{1}{2}(n-1)}|V|^{0.74} \leqslant |V|,$$

or, equivalently,

$$5 \cdot 3^{\frac{1}{2}(n-1)} \leqslant |V|^{0.26} = (25^n)^{0.26} = 5^{0.52n}. \tag{6}$$

For $n \geqslant 4$, this can indeed be checked to be true.

So finally let $n = 3$. Any subgroup of S_3 contains at most two conjugacy classes of fixed point free elements (namely, the two 3-cycles), and hence from Lemma 3.1 we see that

$$\begin{aligned} k(GV) &\leqslant k(HV) + \big(\text{number of conjugacy classes of elements of } G/N \text{ without fixed} \\ &\qquad \text{points on } \{V_1, V_2, V_3\}\big) \cdot \max\left\{ \left|C_{\mathrm{cl}(NV)}(g)\right| \;\middle|\; g \in G - \bigcup_{x \in G} H^x \right\} \\ &\leqslant \frac{4}{5}|V| + 2 \cdot |V|^{0.74} \leqslant 15038 < 15625 = |V|, \end{aligned}$$

and so the proof of the theorem is complete. □

Note that techniques as in the above result also will work in many other interesting situations, such as the ones that were left over by [17, Theorem A].

7. Reducing the noncoprime $k(GV)$-problem

In this section we present some ideas that might be helpful in dealing with the imprimitive case of the following conjecture that has sometimes been called the noncoprime $k(GV)$-problem.

7.1. Conjecture. *There is a universal constant C such that the following holds*:

Let G be a finite G-module and V be a finite faithful, completely reducible G-module. Then

$$k(GV) \leqslant C|V| \log_2 |V|.$$

Note that our reduction results used in Section 5 are not always useful here, because they require some knowledge (by induction) of $k(HV)$ where $H = N_G(V_1)$ for an imprimitivity decomposition $V = V_1 \oplus \cdots \oplus V_n$ of V, but in general there is no guarantee that V is completely reducible as an H-module (although this is the case for small dimensions, see [6]). Whenever $N \trianglelefteq G$, however, then by Clifford V is completely reducible as an N-module, and so we can use Lemma 3.3 in combination with the following result of Guralnick and Magaard see [7, Corollary 1]: If G is a primitive permutation group on a set Ω of size n and if $F^*(G)$ is not a product of alternating groups, then each nontrivial element of G fixes at least $\frac{4}{7}n$ elements of Ω.

With this we can prove the following result.

7.2. Theorem. *Let $f:\mathbb{N}\to\mathbb{R}$ be a function. Let G be a finite group and V be a finite G-module. Suppose that $N \trianglelefteq G$ and $V_N = V_1 \oplus \cdots \oplus V_n$ for an $n\in\mathbb{N}$, where the V_i are N-modules. Assume further that G/N primitively and faithfully permutes the V_i. Moreover, suppose that with $t_0 = \max\{k(UV_1) \mid U \leqslant N/C_N(V_1)\}$ where*

$$\left|N/C_N(V_1)\right| \leqslant \frac{(1-\frac{1}{|G/N|})^{14/(3n)} f(|V|)^{14/(3n)}}{2^{14/3}|V_1| t_0^{8/3}} \quad \text{and that } k(NV) \leqslant f\big(|V|\big).$$

Then one of the following holds:

(a) $k(GV) \leqslant f(|V|)$.
(b) *$F^*(G/N)$ is a product of alternating groups* (*where $F^*(G/N)$ is the generalized Fitting subgroup of G/N*).

Proof. Clearly we may assume that $n > 1$. Assume that $F^*(G/N)$ is not a product of alternating groups. We have to show that (a) holds. By [7, Corollary 1] we know that any $g \in G - N$ fixes at most $\frac{4}{7}n$ of the V_i, and hence with Lemmas 2.2(b) and 2.3 we see that with $n_1 := |N/C_N(V_1)|$ we have

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant t_0^{(4/7)n} \cdot \big(n_1|V_1|\big)^{(3/14)n}$$

for all $g \in G - N$.

Moreover, as $k(G/N) \leqslant 2^{n-1}$, with Lemma 3.3 we get

$$\begin{aligned} k(GV) &\leqslant \frac{k(NV)}{|G/N|} + 2\big(k(G/N)-1\big)\max\big\{\left|C_{\mathrm{cl}(NV)}(g)\right| \mid g \in G-N\big\} \\ &\leqslant \frac{f(|V|)}{|G/N|} + 2^n t_0^{(4/7)n}|V_1|^{(3/14)n} n_1^{(3/14)n} \leqslant f\big(|V|\big). \end{aligned}$$

Thus by our hypothesis on n_1 we are done. □

In view of Conjecture 7.1, the following special case provides a reduction to primitive groups in some situations.

7.3. Corollary. *Let G be a finite group and V be a finite G-module. Suppose that $N \trianglelefteq G$ and $V_N = V_1 \oplus \cdots \oplus V_n$ for an $n \in \mathbb{N}$, where the V_i are N-modules. Assume further that G/N primitively and faithfully permutes the V_i. Let $n \geqslant 5$ and t_0 be as in Theorem 7.2, and assume that*

$$\left|N/C_N(V_1)\right| \leqslant \frac{1}{50} C^{14/(3n)} \frac{|V_1|^{11/3}}{t_0^{8/3}} \big(\log_2|V|\big)^{14/(3n)}$$

for some constant C. If $F^*(G/N)$ is not a product of alternating groups and $k(NV) \leqslant C|V|\log_2|V|$, then

$$k(GV) \leqslant C|V|\log_2|V|.$$

Proof. Let $f(x) = Cx\log_2(x)$. As $n \geqslant 5$, we see that $|G/N| \geqslant 5$ and $\frac{14}{3n} \leqslant \frac{14}{15}$ and hence

$$\frac{1}{50} \leqslant \frac{(1-\frac{1}{5})^{14/15}}{2^{14/3}} \leqslant \frac{(1-\frac{1}{|G/N|})^{14/(3n)}}{2^{14/3}},$$

and thus our hypothesis on $|N/C_N(V_1)|$ implies the one in Theorem 7.2. Hence by Theorem 7.2 the assertion follows. □

Note that if G is a minimal counterexample to Conjecture 7.1 and if $\mathrm{char}(V) \geqslant \dim V_1 + 2$, then by the results in [6] we may assume that $t_0 \leqslant C|V_1|\log_2|V_1|$, so that by the hypothesis in Corollary 7.3 may be replaced by the stronger condition that

$$\left|N/C_N(V_1)\right| \leqslant \frac{1}{50}C^{14/(3n)-8/3}|V_1|\left(\log_2|V|\right)^{14/(3n)-8/3}.$$

This also yields Theorem D.

A similar result could be obtained with Lemma 3.2 in an obvious way, but we omit this here.

However, Lemma 3.1 gives a quite interesting result not involving t_0 (which in general can be hard to control).

7.4. Theorem. *Let G be a finite group and V be a finite G-module. Suppose that $N \trianglelefteq G$ and $V_N = V_1 \oplus \cdots \oplus V_n$ for an $n \in \mathbb{N}$, where the V_i are N-modules. Put $H = N_G(V_1)$. Suppose that*

$$k(HV) \leqslant C_1|V|\log_2|V|$$

for some constant C_1, and suppose that

$$\left|N/C_N(V_1)\right| \leqslant \frac{1}{4}(C_2 - C_1)^{2/n}|V_1|\left(\log_2|V|\right)^{2/n}$$

for some constant C_2. Then

$$k(GV) \leqslant C_2|V|\log_2|V|.$$

Proof. Put $n_1 = |N/C_N(V_1)|$. Then with Lemmas 2.2(b) and 2.3 for any $g \in G - \bigcup_{x\in G} H^x$ we have

$$\left|C_{\mathrm{cl}(NV)}(g)\right| \leqslant \left(n_1|V_1|\right)^{n/2},$$

and hence the assertion follows easily with Lemma 3.1. □

Note that the hypothesis on H in an inductive proof of Conjecture 7.1 is satisfied whenever we know that H acts completely reducibly on V, which, for instance, by results of Guralnick [6], is the case whenever $\mathrm{char}(V) \geqslant \dim V + 2$.

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