A new extension of Hermite matrix polynomials and its applications

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Abstract

In this paper, an extension of the Hermite matrix polynomials is introduced. Some relevant matrix functions appear in terms of the two-variable Hermite matrix polynomials. Furthermore, in order to give qualitative properties of this family of matrix polynomials, the Chebyshev matrix polynomials of the second kind are introduced.

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1. Introduction

An extension to the matrix framework of the classical families of Laguerre, Hermite and Gegenbauer polynomials have been introduced in [8,7,15]. In [11], the Laguerre and Hermite matrix polynomials are presented as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type.

Jódar and Company [7] introduced the class of Hermite matrix polynomials $H_n(x, A)$, which appear as finite series solutions of second order matrix differential equations $Y'' - xAY' + nAY = 0$, for a matrix $A$ in $\mathbb{C}^{N \times N}$ whose eigenvalues are all in the right open half-plane. An explicit expression for these matrix polynomials, the orthogonality property and a Rodrigues’s formula were given. Moreover, some properties of the Hermite matrix polynomials have been
given in [4,5,9,10] and a generalized form of the Hermite matrix polynomials has been introduced and studied in [14].

Our purpose here is to introduce and study a two variable extension of the Hermite matrix polynomials. This paper is organized as follows. Section 2 contains the definition of the two-variable Hermite matrix polynomials. In Section 3, we prove that these matrix polynomials have some recurrence relations and appear as finite series solution of matrix differential equations. Section 4 gives the two-variable Hermite matrix polynomials series expansion of \( \exp(Ax) \), \( \sin(Ax) \) and \( \cos(Ax) \) for a matrix \( A \) that satisfies an appropriate spectral property. Rodrigues’ formula of the two-variable Hermite matrix polynomials is given in Section 5. By exploiting this family of matrix polynomials, in Section 6, we give the definition of the Chebyshev matrix polynomials of the second kind.

Throughout this paper, for a matrix \( A \) in \( \mathbb{C}^{N \times N} \), its spectrum \( \sigma(A) \) denotes the set of all eigenvalues of \( A \). If \( f(z) \) and \( g(z) \) are holomorphic functions of the complex variable \( z \), which are defined in an open set \( \Omega \) of the complex plane and \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) with \( \sigma(A) \subset \Omega \), then from the properties of the matrix functional calculus [6, p. 558], it follows that:

\[
f(A)g(A) = g(A)f(A).
\]

If \( D_0 \) is the complex plane cut along the negative real axis and \( \log(z) \) denotes the principal logarithm of \( z \), then \( z^{1/2} \) represents \( \exp\left(\frac{1}{2} \log(z)\right) \). If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) with \( \sigma(A) \subset D_0 \), then \( A^{1/2} = \sqrt{A} \) denotes the image by \( z^{1/2} \) of the matrix functional calculus acting on the matrix \( A \).

Let \( A \) be a matrix in \( \mathbb{C}^{N \times N} \) such that

\[
\text{Re}(\lambda) > 0 \quad \text{for every eigenvalue } \lambda \in \sigma(A).
\] (1)

Then the \( n \)th Hermite matrix polynomials \( H_n(x, A) \) is defined by [7, p. 25]:

\[
H_n(x, A) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!} (x\sqrt{2A})^{n-2k}, \quad n \geq 0,
\] (2)

and satisfies the three terms recurrence relationship:

\[
H_n(x, A) = x\sqrt{2A}H_{n-1}(x, A) - 2(n-1)H_{n-2}(x, A); \quad n \geq 1;
\]

\[
H_{-1}(x, A) = 0, \quad H_0(x, A) = I,
\]

where \( I \) is the identity matrix in \( \mathbb{C}^{N \times N} \). By [7, p. 29], the following Rodrigues formula holds:

\[
H_n(x, A) = \exp\left(\frac{Ax^2}{2}\right) (-1)^n \left(\frac{A}{2}\right)^{-n/2} \left[ \frac{d^n}{dx^n} \exp\left(\frac{Ax^2}{2}\right) \right], \quad n \geq 0.
\] (3)

According to [7], we have

\[
\exp(xt\sqrt{2A - t^2 I}) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, A)t^n.
\]

We shall use the relations [7,4]:

\[
\sum_{n \geq 0} \sum_{k \geq 0} A(k, n) = \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/2 \rfloor} A(k, n-2k)
\] (4)
\[
\sum_{n \geq 0} \sum_{k \geq 0} A(k, n) = \sum_{n \geq 0} \sum_{k = 0}^{n} A(k, n - k), \tag{5}
\]
where \( A(k, n) \) is a matrix on \( \mathbb{C}^{N \times N} \).

2. The two-variable Hermite matrix polynomials

Let \( A \) be a matrix in \( \mathbb{C}^{N \times N} \) satisfying the condition (1). We define the two-variable Hermite matrix polynomials by

\[
F(x, y, t) = \exp(xt \sqrt{2A} - yt^2 I) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, y, A)t^n; \quad |t| < \infty. \tag{6}
\]

We can write

\[
\exp(xt \sqrt{2A} - yt^2 I) = \sum_{n=0}^{\infty} \sum_{k=0}^{n \infty} \frac{(-1)^k(\sqrt{2A})^n}{k!n!} x^n y^k t^{n+2k}.
\]

Using the relation (4) yields

\[
\exp(xt \sqrt{2A} - yt^2 I) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k(\sqrt{2A})^{n-2k}}{k!(n-2k)!} x^{n-2k} y^k t^n. \tag{7}
\]

Thus, from (6) and (7), we obtain an explicit representation for the two-variable Hermite matrix polynomials in the form:

\[
H_n(x, y, A) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k(\sqrt{2A})^{n-2k}}{k!(n-2k)!} x^{n-2k} y^k. \tag{8}
\]

It is clear that

\[
H_0(x, y, A) = I, \quad H_1(x, y, A) = x \sqrt{2A},
\]
\[
H_n(x, 0, A) = (x \sqrt{2A})^n \quad \text{and} \quad H_n(-x, y, A) = (-1)^n H_n(x, y, A).
\]

Also, we can write

\[
H_n(x, y, A) = y^{n/2} H_n(x/\sqrt{y}, A) \quad \text{and} \quad H_n(x, 1, A) = H_n(x, A),
\]
where \( H_n(z, A) \) is defined in (2).

Now, we study the uniform convergence of the matrix series (6) for a fixed value of \( t \). By taking norms in (8) we get

\[
\|H_n(x, y, A)\| \leq N_n(a\|\sqrt{2A}\|, b); \quad |x| < a, \quad |y| < b, \tag{9}
\]
where \( N_n(x, y) \) is the scalar Hermite Kampé de Fériet polynomials [1] in the form:

\[
n! \sum_{k=0}^{[n/2]} \frac{y^k x^{n-2k}}{k!(n-2k)!}.
\]
So that
\[
\exp(|a|\|\sqrt{2}A\| + b|t|^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(a\|\sqrt{2}A\|, b)|t|^n.
\]
(10)

In view of (9) and (10), the matrix series (6) is termwise differentiable with respect to \(x\) and \(y\).

3. Recurrence relations

This section deals with obtaining some recurrence relations for the two-variable Hermite matrix polynomials.

**Theorem 1.** The two-variable Hermite matrix polynomials \(H_n(x, y, A)\) satisfies the following relations:

\[
\frac{\partial^k}{\partial x^k} H_n(x, y, A) = (\sqrt{2}A)^k \frac{n!}{(n-k)!} H_{n-k}(x, y, A); \quad 0 \leq k \leq n
\]
(11)

and

\[
\frac{\partial^k}{\partial y^k} H_n(x, y, A) = (-1)^k \frac{n!}{(n-2k)!} H_{n-2k}(x, y, A); \quad 0 \leq k \leq \lfloor n/2 \rfloor.
\]
(12)

**Proof.** Taking termwise differentiation of (6) with respect to \(x\) yields

\[
t \sqrt{2}A \exp(xt\sqrt{2}A - yt^2 I) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial}{\partial x} H_n(x, y, A)t^n.
\]
(13)

From (6) and (13) we have

\[
\sqrt{2}A \sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_{n-1}(x, y, A)t^n = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial}{\partial x} H_n(x, y, A)t^n.
\]

Hence, it follows that

\[
\frac{\partial}{\partial x} H_n(x, y, A) = \sqrt{2}AnH_{n-1}(x, y, A).
\]
(14)

Iteration (14), for \(0 \leq k \leq n\), implies (11).

Again differentiating (6) with respect to \(y\) yields

\[
-t^2 \exp(xt\sqrt{2}A - yt^2 I) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial y} H_n(x, y, A)t^n.
\]

By (6) it follows that

\[
-\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, y, A)t^{n+2} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial y} H_n(x, y, A)t^n.
\]

Therefore, we have

\[
-\sum_{n=2}^{\infty} \frac{1}{(n-2)!} H_{n-2}(x, y, A)t^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial y} H_n(x, y, A)t^n.
\]
Note that \( \frac{\partial}{\partial y} H_0(x, y, A) = \frac{\partial}{\partial y} H_1(x, y, A) = 0. \) Thus
\[
\frac{\partial}{\partial y} H_n(x, y, A) = -n(n - 1)H_{n-2}(x, y, A); \quad n \geq 2. \tag{15}
\]
By iteration (15), for \( 0 \leq k \leq [n/2] \), one gets (12). \( \square \)

The following corollary is a consequence of Theorem 1.

**Corollary 2.** For the two-variable Hermite matrix polynomials the following holds:
\[
\frac{\partial^2}{\partial x^2} H_n(x, y, A) + 2A \frac{\partial}{\partial y} H_n(x, y, A) = 0. \tag{16}
\]

**Proof.** By (11) and (15), Eq. (16) follows directly. \( \square \)

According to Eq. (16), it is clear that \( H_n(x, y, A) \) are the solutions of the heat partial differential equation \([2, 12, 13]\).

Now, we can state and prove the following:

**Theorem 3.** Let \( A \) be a matrix in \( \mathbb{C}^{N \times N} \) satisfying (1). Then we have
\[
x^n(\sqrt{2}A)^n = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} y^k H_{n-2k}(x, y, A), \tag{17}
\]
and
\[
H_n(x, y, A) = x \sqrt{2} A H_{n-1}(x, y, A) - 2(n-1)y H_{n-2}(x, y, A), \quad n \geq 2. \tag{18}
\]

**Proof.** In view of (6), one gets
\[
\exp(xt \sqrt{2}A) = \sum_{n \geq 0} \sum_{k \geq 0} \frac{y^k}{k!n!} H_n(x, y, A)t^{n+k},
\]
which can be written, by applying (4), in the form:
\[
\exp(xt \sqrt{2}A) = \sum_{n \geq 0} \sum_{k=0}^{[n/2]} \frac{y^k}{k!(n-2k)!} H_{n-2k}(x, y, A)t^n. \tag{19}
\]

By expanding the left-hand side of (19) in powers of \( t \) and identification of the coefficients of \( t^n \) in both sides gives (17).

Differentiating (6) with respect to \( x \) and \( t \) we find respectively
\[
\frac{\partial F}{\partial x} = t \sqrt{2} A \exp(xt \sqrt{2}A - yt^2 I)
\]
and
\[
\frac{\partial F}{\partial t} = (x \sqrt{2} A - 2yt I) \exp(xt \sqrt{2}A - yt^2 I).
\]

Therefore, \( F(x, t) \) satisfies the partial matrix differential equation:
\[
(x \sqrt{2} A - 2yt I) \frac{\partial F}{\partial x} - t \sqrt{2} A \frac{\partial F}{\partial t} = 0,
\]
which, by using (6), becomes
\[
\sum_{n=1}^{\infty} nH_n(x, y, A)t^n = x \sum_{n=1}^{\infty} \frac{\partial}{\partial x} H_n(x, y, A)t^n - 2y(\sqrt{2A})^{-1} \sum_{n=2}^{\infty} n \frac{\partial}{\partial x} H_{n-1}(x, y, A)t^n.
\]
Since \(x \frac{\partial}{\partial x} H_1(x, y, A) = H_1(x, y, A)\), then for \(n \geq 2\) it follows:
\[
nH_n(x, y, A) = x \frac{\partial}{\partial x} H_n(x, y, A) - 2yn(\sqrt{2A})^{-1} \frac{\partial}{\partial x} H_{n-1}(x, y, A). \tag{20}
\]
Using (14) for \(\frac{\partial}{\partial x} H_n(x, y, A)\) and \(\frac{\partial}{\partial x} H_{n-1}(x, y, A)\) we get (18). \(\square\)

In the following result, the two-variable Hermite matrix polynomials appear as finite series solutions of a second order matrix differential equation.

**Corollary 4.** The two-variable Hermite matrix polynomials are a solution of the second order matrix differential equation in the form:
\[
\left[ y \frac{\partial^2}{\partial x^2} - x A \frac{\partial}{\partial x} + n A \right] H_n(x, y, A) = 0; \quad n \geq 0. \tag{21}
\]

**Proof.** From (14) we have
\[
(\sqrt{2A})^{-1} \frac{\partial^2}{\partial x^2} H_n(x, y, A) = n \frac{\partial}{\partial x} H_{n-1}(x, y, A),
\]
which can be used in (20) to obtain (21). \(\square\)

For \(y = 1\), the matrix differential equation (21) gives the Hermite matrix differential equation (3.15) of [7, p. 26].

4. **Two-variable Hermite matrix polynomials expansions**

We begin this section with the following lemma.

**Lemma 5.** For a real number \(K > 2\) we have
\[
\|H_n(x, y, A)\| \leq \sqrt{n!} \sqrt{(2y)^n} K^n \exp(x^2/y), \quad |x| < \frac{K}{\|\sqrt{2A}\|}, \quad y > 0. \tag{22}
\]

**Proof.** For \(n = 0\), the inequality (22) is true. By using the induction, let us suppose that (22) is true for \(j = 0, 1, \ldots, n\).

Taking norms in the formula (18) gives
\[
\|H_{n+1}(x, y, A)\| \leq |x| \|\sqrt{2A}\| \|H_n(x, y, A)\| + 2n|y|\|H_{n-2}(x, y, A)\|
\leq \sqrt{n!} \sqrt{(2y)^n} K^{n-1} \exp(x^2/y)(K^2 + \sqrt{2yn})
\leq \sqrt{(n + 1)!} \sqrt{(2y)^n} K^{n-1} \exp(x^2/y) \left( \frac{K^2 + \sqrt{2yn}}{\sqrt{n + 1}} \right).
\]
It is easy to show that
\[ K^2 + \sqrt{2y} \frac{n+1}{n+1} \leq \sqrt{2y} K^2 \]
and so that (22) holds true for \( n+1 \). Hence the inequality (22) is established. \[ \square \]

Let \( B \) be a matrix satisfies the spectral property
\[ |\text{Re}(\lambda)| > |\text{Im}(\lambda)| \quad \text{for all } \lambda \in \sigma(B). \tag{23} \]

The following result gives expansion of \( \exp(Bx) \), \( \cos(Bx) \) and \( \sin(Bx) \) in a series of the two-variable Hermite matrix polynomials.

**Theorem 6.** Let \( B \) be a matrix in \( \mathbb{C}^{N \times N} \) satisfying (23). For \( x, y \in \mathbb{R} \), it follows that
\[
\exp(xB) = \exp(y) \sum_{n \geq 0} \frac{1}{n!} H_n \left( x, y, \frac{1}{2} B^2 \right), \tag{24}
\]
\[
\sin(xB) = \exp(-y) \sum_{n \geq 0} \frac{1}{(2n+1)!} H_{2n+1} \left( x, y, \frac{1}{2} B^2 \right) \tag{25}
\]
and
\[
\cos(xB) = \exp(-y) \sum_{n \geq 0} \frac{1}{(2n)!} H_{2n} \left( x, y, \frac{1}{2} B^2 \right). \tag{26}
\]
Moreover, the convergence of the matrix series appearing in (25)–(26) to the respective matrix functions \( \exp(xB), \sin(xB) \) and \( \cos(xB) \) is uniform in any bounded interval of the real axis.

**Proof.** Define the matrix \( A = \frac{1}{2} B^2 \). In view of the spectral mapping theorem [6] it is easy to find that \( \sigma(A) = \{ \frac{1}{2} b^2 : b \in \sigma(B) \} \) and by (23) we have
\[ \text{Re} \left( \frac{1}{2} b^2 \right) = \frac{1}{2} \left[ (\text{Re}(b))^2 - (\text{Im}(b))^2 \right] > 0, \quad b \in \sigma(B). \]
That is, \( A \) satisfies the condition (1). In (6), putting \( t = 1 \) and \( B = \sqrt{2A} \) gives
\[ \exp(xB - yI) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n \left( x, y, \frac{1}{2} B^2 \right). \]
Therefore, (25) follows.

For \( A = \frac{1}{2} B^2 \), we can write (17) in the form:
\[ x^{2n+1} B^{2n+1} = \sum_{k=0}^{n} \frac{(2n+1)!}{k!(2(n-k)+1)!} y^k H_{2(n-k)+1} \left( x, y, \frac{1}{2} B^2 \right), \]
which can be used in the series expansion of \( \sin(xB) \) to obtain
\[
\sin(xB) = \sum_{n \geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} B^{2n+1} \]
\[ = \sum_{n \geq 0} \sum_{k=0}^{n} \frac{(-1)^n}{k!(2(n-k)+1)!} y^k H_{2(n-k)+1} \left( x, y, \frac{1}{2} B^2 \right). \]
From (5), it follows that
\[
\sin(xB) = \sum_{n \geq 0} \sum_{k \geq 0} \frac{(-1)^{n+k}}{k!(2n + 1)!} y^k H_{2n+1} \left( x, y, \frac{1}{2} B^2 \right)
\]
\[
= \sum_{n \geq 0} \left( \sum_{k \geq 0} \frac{(-y)^k}{k!} \right) \frac{(-1)^n}{(2n + 1)!} H_{2n+1} \left( x, y, \frac{1}{2} B^2 \right)
\]
and (25) follows.

In similar way to the previous steps one can obtain (26).

Now, we are going to study the uniform convergence of (25)–(26). To this aim and by Lemma 5 one gets
\[
\exp(y) \sum_{n \geq 0} \frac{1}{n!} \left\| H_n \left( x, y, \frac{1}{2} B^2 \right) \right\| \leq \exp \left( \frac{x^2 + y^2}{y} \right) \sum_{n \geq 0} \sqrt{\sqrt{2}y} \frac{K^n}{\sqrt{n!}}.
\]

Indeed, the series in the right-hand side is convergent, and thus the matrix series (24) is uniformly convergent in any bounded interval of the real axis.

In analogous way, one can show that the matrix series (25) and (26) are uniformly convergent in any bounded interval of the real axis. Therefore, the result is established.

Note that the important advantage of the series developments of the exponential, sine and cosine matrix functions as given in (25)–(26), respectively, with respect to the Taylor series of these matrix functions comes out from the fact that it is not requisite to compute powers $B^n$ of the matrix $B$, as well as the fact that using the three terms matrix relationship (18) and taking into account that $H_0(x, y, \frac{1}{2} B^2) = I$ and $H_1(x, y, \frac{1}{2} B^2) = xB$, the two-variable Hermite matrix polynomials can be computed recurrently.

5. Rodrigues’ formula

In this section, we provide a Rodrigues’ formula for the two-variable Hermite matrix polynomials. For this aim let us assume that $f(z)$ is an entire function of the complex variable $z$, and $g(z)$ is a holomorphic function in the neighborhood of $\sigma(A)$. If $F(A) = f(g(z))$, then by [6, p. 570] it follows that
\[
F(A) = f \left( g(A) \right).
\] (27)

The matrix generating function (6) with the aid of the Taylor’s theorem gives
\[
H_n(x, y, A) = \left[ \frac{d^n}{dt^n} \exp(xt\sqrt{2A} - yt^2I) \right]_{t=0}.
\] (28)

We can write
\[
xt\sqrt{2A} - yt^2I = \frac{Ax^2}{2y} - \left( \frac{x}{\sqrt{y}} \right) \sqrt{A/2} - \sqrt{yt}I \right)^2
\]
\[
= \frac{Ax^2}{2y} - (A/2) \left( \frac{x}{\sqrt{y}}I - (A/2)^{-1/2} \sqrt{y}I \right)^2.
\]
By (28) we see
\[ \exp \left( -\frac{Ax^2}{2y} \right) H_n(x, y, A) = \left[ \frac{d^n}{dt^n} \exp\left( -(A/2)[(x/\sqrt{y})I - (A/2)^{-1/2}\sqrt{yt}I]^2 \right) \right]_{t=0}. \]

By (27) and taking \( z = x/\sqrt{y} \) one gets
\[ \exp \left( -\frac{Az^2}{2} \right) H_n(x, y, A) = \left[ \frac{d^n}{dz^n} \exp\left( -(A/2)[zI - (A/2)^{-1/2}\sqrt{yt}I]^2 \right) \right]_{z=0}. \]

Therefore, the Rodrigues formula is accomplished in the form:
\[ H_n(x, y, A) = \exp \left( -\frac{Ax^2}{2y} \right) \left( -1 \right)^{n/2} \left( \frac{A}{2} \right)^{-n/2} \left[ \frac{d^n}{dz^n} \exp \left( -\frac{Az^2}{2} \right) \right]. \]

Note that setting \( y = 1 \) in (29) gives (3).

We conclude this section giving another representation for the two-variable Hermite matrix polynomials.

**Theorem 7.** Suppose that \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) satisfying (1). Then the two-variable Hermite matrix polynomials has the following representation:
\[ H_n(x, y, A) = \exp \left( -y(2A)^{-1} \frac{\partial^2}{\partial x^2} \right) (x\sqrt{2A})^n. \]

**Proof.** It is clear that
\[ \exp \left( -y(2A)^{-1} \frac{\partial^2}{\partial x^2} \right) \exp(xt\sqrt{2A}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} y^n \left[ \frac{\partial^2}{\partial x^2} \right]^{2n} \exp(xt\sqrt{2A}) \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} y^n t^{2n} \exp(xt\sqrt{2A}) \]
\[ = \exp(xt\sqrt{2A} - yt^2I). \]

Therefore, by (6), we have
\[ \exp \left( -y(2A)^{-1} \frac{\partial^2}{\partial x^2} \right) \exp(xt\sqrt{2A}) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, y, A)t^n, \]
which by expanding in powers of \( t \) becomes
\[ \exp \left( -y(2A)^{-1} \frac{\partial^2}{\partial x^2} \right) \sum_{n=0}^{\infty} \frac{(x\sqrt{2A})^n}{n!} t^n = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, y, A)t^n. \]

Thus by identification of the coefficients of \( t^n \) in both sides gives the representation (30).

It is worth noting that, for \( y = 1 \), the expression (30) gives another representation for the Hermite matrix polynomials in the form:
\[ H_n(x, A) = \exp \left( -(\sqrt{2A})^{-1} \frac{d^2}{dx^2} \right) (x\sqrt{2A})^n. \]
Furthermore, in view of (30), we can write

\[(x\sqrt{2A})^n = \exp \left( y(2A)^{-1} \frac{\partial^2}{\partial x^2} \right) H_n(x, y, A).\]

6. The Chebyshev matrix polynomials

The two-variable Hermite matrix polynomials will be exploited here to define a matrix version of Chebyshev polynomials. The Chebyshev polynomials of the second kind [3] are defined by

\[U_n(z) = \sum_{k=0}^{[n/2]} (-1)^k(n-k)!(2z)^{n-2k} k!(n-2k)!\]

Suppose that \(A\) is a matrix in \(\mathbb{C}^{N \times N}\) satisfying the condition (1). By (8) it follows that

\[\frac{1}{n!} \int_0^\infty e^{-t} t^n H_n\left( x, \frac{1}{t}, A \right) \, dt = \int_0^\infty e^{-t} t^n \sum_{k=0}^{[n/2]} (-1)^k(\sqrt{2A})^{n-2k} k!(n-2k)!x^{n-2k} t^{-k} \, dt.\]

Since the summation in the right-hand side of the above equality is finite, then the series and the integral can be permuted. Also, in view of

\[n! = \int_0^\infty e^{-t} t^n \, dt,\]

we can write

\[\frac{1}{n!} \int_0^\infty e^{-t} t^n H_n\left( x, \frac{1}{t}, A \right) \, dt = \sum_{k=0}^{[n/2]} (-1)^k(n-k)! (\sqrt{2A})^{n-2k} k!(n-2k)! x^{n-2k}.\]

Hence, the Chebyshev matrix polynomials of the second kind can be defined by

\[U_n(x, A) = \sum_{k=0}^{[n/2]} (-1)^k(n-k)! (\sqrt{2A})^{n-2k} k!(n-2k)! x^{n-2k} x^k,\]

and

\[U_n(x, A) = \frac{1}{n!} \int_0^\infty e^{-t} t^n H_n\left( x, \frac{y}{t}, A \right) \, dt.\]

In similar way, we define the generalized Chebyshev matrix polynomials of the second kind as follows:

\[U_n(x, y, A) = \sum_{k=0}^{[n/2]} (-1)^k(\sqrt{2A})^{n-2k} k!(n-2k)! x^{n-2k} y^k\]

and

\[U_n(x, y, A) = \frac{1}{n!} \int_0^\infty e^{-t} t^n H_n\left( x, \frac{y}{t}, A \right) \, dt.\]

It is evident that

\[U_n(x, y, A) = y^{n/2} U_n(x/\sqrt{y}, A).\]
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References