Exact enumeration of rooted 3-connected triangular maps on the projective plane

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Abstract

We study the relation between rooted 3-connected triangular maps and rooted 2-connected triangular maps on the projective plane. We then use this relation to derive a simple parametric expression for the generating function of rooted 3-connected triangular maps on the projective plane. We believe that this is the first simple exact result for 3-connected nonplanar maps.

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1. Introduction

A map on a surface is a 2-cell embedding of a graph on the surface. A map is rooted by selecting a vertex, an edge incident with the vertex, and a side of the edge in the map. The selected vertex and edge are called the root vertex and root edge of the map, respectively. The face corresponding to the selected side of the root edge is called the root face.

A rooted near-triangular map on a surface is a rooted map on the surface such that each face except possibly the root face has valence three. A rooted triangular map is a rooted near-triangular map whose root face valence is also three. Much work has
been done on enumerating various types of rooted planar maps since Tutte’s pioneer work in the 1960s [19–22]. (See also [5] for a survey.)

Major developments on enumeration of nonplanar maps occurred in the 1980s mainly because of the connections with modern physics [2,7,16,17,23] and other branches of mathematics [18]. Simple expressions for all rooted maps on the projective plane, the torus and the Klein Bottle, were first obtained by Bender et al. [6]. Arques [1] independently obtained the result on the torus. Bender and Canfield [4] derived recursions for all rooted maps, which in principle can be used to calculate the exact expressions of the generating functions for all rooted maps on any given surface. The generating functions for all rooted maps on the first few surfaces were calculated in [4,10]. Similar recursions for all rooted triangular maps were derived by Gao [13], and parametric expressions of the generating functions were also given for the projective plane and the torus. However, exact enumeration of nonplanar maps with connectivity higher than one is much more complicated. Brown [9] made the first attempt in enumerating rooted 2-connected maps in the projective plane, but his result appears quite complicated. A simple parametric expression was obtained in [14] for rooted 2-connected triangular maps on the projective plane. Edelman and Reiner [11] derived a simple expression for the generating function of rooted triangular maps on the Môbius band where all vertices lie on the boundary. In this paper, we obtain a simple expression for the generating function of rooted 3-connected triangular maps on the projective plane. Our approach is based on the study of the structures of 2-cycles in 2-connected triangular maps on the projective plane, and the well-known fact that a triangular map is 3-connected if and only if it contains no loops or multiple edges. We believe that this is the first simple exact result for 3-connected nonplanar maps.

Let \( P(x) \) be the generating function of rooted 3-connected triangular maps on the projective plane, where \( x \) marks the total number of vertices. In the rest of the paper, we use \([x^n]f(x)\) to denote the coefficient of \( x^n \) in the power series expansion of \( f(x) \).

Our main result is the following:

**Theorem 1.** Let \( t = t(x) = x + \cdots \) be the (unique) power series satisfying \( x = t(1 - t)^3 \). Then

\[
P(x) = \frac{1}{2} (1 - t)^3 (1 - 3t - (1 - t)\sqrt{1 - 4t})
- \frac{t^3 (1 - 5t + 12t^2 - 14t^3 + 4t^4 + 3t^5)}{(1 - 2t)(1 - t)^3}
= x^5 + 33x^7 + 615x^8 + 8948x^9 + 113820x^{10} + \cdots .
\]  

(1)

**Corollary 1.** As \( n \to \infty \), the number of rooted 3-connected triangular maps with \( n \) vertices on the projective plane has the asymptotic expression

\[
[x^n]P(x) \sim -\frac{81}{1024 \Gamma(-1/4)} 24^{1/4} n^{-5/4} \left( \frac{256}{27} \right)^n.
\]
Fig. 1. Rooted 3-connected triangular maps, with six or seven vertices, on the projective plane, where antipodal points on the circle are identified to form the cross-cap.

All rooted 3-connected triangular maps, with six or seven vertices, on the projective plane, are shown in Fig. 1.

2. Structures of 2-cycles and some known results

Let $T$ be a rooted 2-connected triangular map on the projective plane. A cycle in $T$ is called contractible if it encloses a disk, otherwise it is called noncontractible. The disk enclosed by a contractible cycle $C$ will be called the interior of $C$. It is not difficult to see that there are exactly five different types of 2-cycles, as described below, where $C$ refers to a 2-cycle in $T$.

1. $C$ is contractible and the root face is not in the interior of $C$.
2. $C$ is contractible and the root face is in the interior of $C$.
3. $C$ is the unique noncontractible 2-cycle of $T$.
4. $C$ is noncontractible (not unique), and all noncontractible 2-cycles share a common vertex.
5. $C$ is noncontractible (not unique), and not all noncontractible 2-cycles share a common vertex.
We shall start with 2-connected triangular maps on the projective plane, and remove all the 2-cycles to obtain our results for 3-connected triangular maps. Let $B(z)$ be the generating function for rooted 2-connected planar near-triangular maps with root face degree 2, and $P(z)$ be the generating function for rooted 2-connected triangular maps on the projective plane, where $z$ marks the total number of vertices. It is known [14] that

$$B(z) = u^2(1 - 3u)(1 - 2u)^2,$$

$$P(z) = \frac{(1 - 2u)^2(1 - 5u)}{2(1 - 3u)} - \frac{1 - 2u}{2} \sqrt{(1 - 2u)(1 - 6u)},$$

where $u = u(z) = z + \cdots$ is the (unique) power series satisfying

$$z = u(1 - 2u)^2.$$  \hspace{1cm} (4)

We will also deal with rooted planar 2-connected near-triangular maps with no multiple edges. Let $A_j(x)$ be the generating function for such maps with root face degree $j$, where $x$ marks the total number of vertices. It is known [8] that

$$A_3(x) = x^2 t(1 - 2t),$$

$$A_4(x) = x^2 t^2 (1 - t)(2 - 5t),$$

where $t = t(x)$ is the power series defined in Theorem 1.

3. Proof of Theorem 1

We first deal with all contractible 2-cycles of type 1. A 2-cycle of type 1 is called maximal if it is not contained in the interior of another 2-cycle of type 1. Let $G_1(x) = \sum_{n \geq 0} g_{1,n} x^n$ be the generating function for all rooted 2-connected triangular maps on the projective plane which have no contractible 2-cycles of type 1. Suppose $T$ is a 2-triangular map on the projective plane. Closing each maximal 2-cycle of type 1 in $T$ gives a triangular map $T_1$ counted by $G_1(z)$. Conversely, we can replace each edge in $T_1$ by a map counted by $B(z)$ to obtain a triangular map counted by $P(z)$. If $T_1$ has $n$ vertices, then it has $3n - 3$ edges, and hence

$$P(z) = \sum_{n \geq 0} g_{1,n} z^n (B(z)z^{-2})^{3n-3} = z^6 B^{-3}(z) G_1(z^{-5} B^3(z)).$$

Setting $x = z^{-5} B^3(z)$, $u = t/(1 + 2t)$, and using (2), we obtain

$$x = t(1 - t)^3,$$

$$G_1(x) = \frac{1}{2} (1 + 2t)(1 - t)^2(1 - 3t - (1 - t)\sqrt{1 - 4t}).$$  \hspace{1cm} (7)

Next we remove contractible 2-cycles of type 2. Among all contractible 2-cycles of type 2, there is one which contains the least number of vertices in its interior, which shall be denoted by $C_{\min}$ (See Fig. 2).
Let $T$ be a triangular map counted by $G_1(x)$. Cutting $T$ along $C_{min}$ gives two parts, the interior part of $C_{min}$ being planar, and the other being on the projective plane. By removing one of the two edges of $C_{min}$, the part on the projective plane becomes a rooted triangular map counted by $G_1(x)$ such that there is no edge parallel to the root edge (i.e., joining the end vertices of the root edge). Let $K(x)$ be the generating function for such triangular maps. The planar part has no multiple edges, and hence is a 3-connected triangular map. Since it has another distinguished edge corresponding to $C_{min}$, its generating function is

$$H(x) = \sum_{n \geq 0} ([x^n]A_3(x))(3n - 6)x^n = 3xA_3'(x) - 6A_3(x).$$

(8)

Let $G_2(x)$ be the generating function for rooted triangular maps on the projective plane with no contractible 2-cycles, the above argument gives

$$G_1(x) = G_2(x) + x^{-2}H(x)K(x).$$

(9)

Since $x^{-2}A_3(x)K(x)$ counts all triangular maps (counted by $G_1$) in which there is an edge parallel to the root edge, we have

$$G_1(x) = K(x) + x^{-2}A_3(x)K(x)$$

and hence

$$K(x) = \frac{x^2G_1(x)}{x^2 + A_3(x)}.$$

(10)

It follows from (7)–(10) that

$$G_2(x) = \frac{1}{2} (1 - t)^3(1 - 3t - (1 - t)\sqrt{1 - 4t}).$$

(11)

To deal with noncontractible 2-circles, we need to consider three special types of planar near-triangular maps counted by $A_4(x)$. Let $T$ be a rooted near-triangular map...
counted by $A_4(x)$, with root vertex $a$, root edge $ab$, and root face $abcd$. We say that $T$ is of type 1 if $b$ and $d$ are joined by an edge (See Fig. 3); $T$ is of type 2 if there is no interior path of length 1 or 2 between $a$ and $c$; $T$ is of type 3 if $T$ is of type 2 and there is no interior path of length 1 or 2 between $b$ and $d$. For $i = 1, 2, 3$, let $W_i(x)$ be the generating function for maps of type $i$, where $x$ marks the total number of vertices.

We first prove

**Lemma 1.**

$$W_1(x) = x^{-2}A_3^2(x), \quad (12)$$

$$W_2(x) = \frac{A_4(x) - W_1(x)}{1 + x^{-3}(A_4(x) - W_1(x))}, \quad (13)$$

$$W_3(x) = A_4(x) - 2W_1(x) - 2x^{-3}(A_4(x) - W_1(x))W_2(x) + x^{-3}W_1^2(x). \quad (14)$$

**Proof.** Eq. (12) is obvious. Let $\mathcal{F}$ be the family of maps counted by $A_4(x)$ such that $a$ and $c$ are not joined by an edge. Then $\mathcal{F}$ is counted by $A_4(x) - W_1(x)$. If we use $D(x)$ to denote the generating function for those maps in $\mathcal{F}$ such that $a$ and $c$ are joined by an interior path of length 2, then

$$W_2(x) = (A_4(x) - W_1(x)) - D(x). \quad (15)$$

On the other hand, each map counted by $D(x)$ can be decomposed into a map counted by $W_2(x)$ and a map in $\mathcal{F}$ by cutting through the “right-most” path $avc$ (See Fig. 4). Hence,

$$D(x) = x^{-3}(A_4(x) - W_1(x))W_2(x), \quad (16)$$

which, together with (15), gives (13).
We now derive (14) using the inclusion–exclusion principle. Let $F' \subseteq F$ be the family of maps such that $b$ and $d$ are not joined by an edge, $F'_1 \subseteq F'$ denote the family of maps such that $a$ and $c$ are joined by an interior path of length 2, and $F'_2 \subseteq F'$ denote the family of maps such that $b$ and $d$ are joined by an interior path of length 2. Then

$$W_3(x) = |F'| - |F'_1| - |F'_2| + |F'_1 \cap F'_2|.$$  \hspace{1cm} (17)

It is easy to see that

$$|F'| = A_4(x) - 2W_1(x).$$  \hspace{1cm} (18)

By Eq. (16), we have

$$|F'_1| = |F'_2| = x^{-3}(A_4(x) - W_1(x))W_2(x).$$  \hspace{1cm} (19)

Noting that each map in $F'_1 \cap F'_2$ can be decomposed into two maps counted by $W_1$, we have

$$|F'_1 \cap F'_2| = x^{-3}W_1^2(x).$$  \hspace{1cm} (20)

Now (14) follows from (17)–(20).

For $j = 3, 4, 5$, let $G_j(x)$ be the generating function for all rooted 2-c triangular maps on the projective plane which do not contain any contractible 2-cycle, and contain noncontractible 2-cycles of type $j$. Then

$$G_2(x) = G_3(x) + G_4(x) + G_5(x) + \bar{P}(x).$$  \hspace{1cm} (21)

Let $aba$ be the noncontractible 2-cycle of type 3 in a rooted triangular map $T$ counted by $G_2(x)$. If we cut through the 2-cycle, $T$ becomes a planar near-triangular map $T'$ with a quadrangular face $aba'b'$ (See Fig. 5).

We note that $T'$ has no multiple edges, there is no interior path of length 1 or 2 between $a$ and $a'$, or between $b$ and $b'$. If we choose a rooting on the quadrangular face $aba'b'$, then $T'$ becomes a near-triangular map counted by $W_3(x)$, which has a
secondary rooting on a triangular face (corresponding to the original rooting of $T$). It is important to note that there are $2 \times 4 = 8$ ways to choose a rooting on the quadrangular face of $T'$ and they all correspond to the same map $T$. Suppose $T'$ has $n$ vertices. Then, by Euler's formula, $T'$ has $2n - 6$ interior faces which give rise to $6(2n - 6)$ secondary rootings. Hence, we have

$$G_3(x) = (1/8)x^{-2} \sum_{n \geq 4} 6(2n - 6)([x^n]W_3(x))x^n$$

$$= (3/2)x^{-2}(xW_3'(x) - 3W_3(x)),$$  \hspace{1cm} (22)

where the factor $x^{-2}$ signifies the identification of $a$ with $a'$, and $b$ with $b'$.

Next we consider noncontractible 2-cycles of type 4 in a triangular map $T$ counted by $G_2(x)$. Note that any two noncontractible 2-cycles $aba$ and $aca$ divides $T$ into two separate parts (see Fig. 6).

Let $T_1$ be the part containing the root edge, and $T_2$ be the other part, both including the two 2-cycles. Choose $aba$ and $aca$ such that $T_1$ has minimum number of vertices. If we split the common vertex $a$ into two copies $a$ and $a'$, then $T_1$ becomes a rooted planar near-triangular map with a quadrangular face $aba'c$ such that there is no path of length 1 or 2 between $a$ and $a'$, and $T_2$ becomes a rooted near-triangular map with a quadrangular root face and with no edge between $a$ and $a'$. The argument leading to
(23) shows that $T_1$ is counted by $(\frac{3}{2})(xW_2'(x) - 3W_2(x))$. It is clear that $T_2$ is counted by $A_4(x) - W_1(x)$. We need to exclude the case when the edge $bc$ are present in both $T_1$ and $T_2$, since this creates a noncontractible 2-cycle $bcb$ in the original triangular map $T$ which does not contain the vertex $a$. When $T_1$ contains the edge $bc$, the argument leading to (23) shows that its generating function is given by $(\frac{3}{2})(xW_1'(x) - 3W_1(x))$. When $T_2$ contains the edge $bc$, its generating function is given by $W_1(x)$. Hence, we obtain

$$G_4(x) = (3/2)x^{-5}(xW_2'(x) - 3W_2(x))(A_4(x) - W_1(x))$$

$$- (3/2)x^{-5}(xW_1'(x) - 3W_1(x))W_1(x),$$

(24)

where the factor $x^{-5}$ signifies the fact that $T_1$ and $T_2$ are identified through the 4-cycle $aba'c$, and vertices $a$ and $a'$ are further identified to obtain $T$.

Finally we consider noncontractible 2-cycles of type 5. In this case, it is not too difficult to see that there are exactly three noncontractible 2-cycles $aba$, $aca$, and $bcb$ (See Fig. 7).

These three 2-cycles separate the original triangular map into four rooted planar triangular maps with no multiple edges. The part containing the original root face has another distinguished face, which is counted by

$$\sum_{n \geq 3} (2n - 5)([x^n]A_3(x)x^n = (2xA_3'(x) - 5A_3(x)).$$

Hence, we have

$$G_5(x) = (2xA_3'(x) - 5A_3(x))(A_3(x)/x^3)^3.$$  

(25)
Using (5), (6), (12)–(14) and (23)–(25), we obtain
\[ G_3(x) + G_4(x) + G_5(x) = \frac{t^3(1 - 5t + 12t^2 - 14t^3 + 4t^4 + 3t^5)}{(1 - 2t)(1 - t)^3}. \]  
(26)

Now Theorem 1 follows from (11), (21) and (26).

To prove Corollary 1, we note that \( t(x) \) is algebraic in \( x \) and has a unique singularity \( x = \frac{27}{256} \) on its circle of convergence. Using Lagrange inversion formula, we know \( t(x) \), as a power series in \( x \), has positive coefficients, hence \( t(x) < t(\frac{27}{256}) = \frac{1}{4} \) for \( |x| \leq \frac{27}{256} \) and \( x \neq \frac{27}{256} \). We also have, as \( x \to \frac{27}{256} \),
\[
t(x) = \frac{1}{4} - (\sqrt{6}/8)(1 - 256x/27)^{1/2} + (1/12)(1 - 256x/27) + O((1 - 256x/27)^{3/2}),
\]
and \( \tilde{P}(x) = 211/6912 - (81/1024)2^{1/4}(1 - 256x/27)^{1/4} + O((1 - 256x/27)^{1/2}) \).

Now Corollary 1 follows from Darboux’s theorem, or Flajolet–Odlyzko’s transfer theorem [12]. Note that this asymptotic formula first appeared in [15].

4. Uncited reference

[3]

References