Contractible Subgraphs in 3-Connected Graphs

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A subgraph \( H \) of a 3-connected finite graph \( G \) is called contractible if \( H \) is connected and \( G - V(H) \) is 2-connected. This work is concerned with a conjecture of McCuaig and Ota which states that for any given \( k \) there exists an \( f(k) \) such that any 3-connected graph on at least \( f(k) \) vertices possesses a contractible subgraph on \( k \) vertices. We prove this for \( k \leq 4 \) and consider restrictions to maximal planar graphs, Halin graphs, line graphs of 6-edge-connected graphs, 5-connected graphs of bounded degree, and \( AT \)-free graphs.

1. INTRODUCTION

The conjecture mentioned in the abstract is—if true—one among many possible extensions of Tutte’s result stating that every 3-connected graph on at least five vertices contains a contractible edge (\( k = 2 \)) [12]. For \( k = 1 \) it is trivial, and for \( k = 3 \) it has been settled by McCuaig and Ota [6, Theorem 1].

Here we shall give proofs for \( k \leq 4 \) based on some theorems on almost critical graphs of connectivity 2. Moreover, we will prove the conjecture for several graph classes.

An edge between distinct vertices \( x, y \) of a graph \( G = (V, E) \) will be denoted by \([x, y]\) and the neighborhood of \( x \) in \( G \) is defined as \( N_G(x) := \{ y \in V(G) : [x, y] \in E(G) \} \). More generally, \( N_G(X) := (\bigcup_{x \in X} N(x)) - X \) for an arbitrary \( X \subseteq V(G) \). We will omit any index \( G \) if it is clear from the context. As it is common in the context of vertex connectivity, we do not allow a graph to have multiple edges or loops. In addition, we consider only finite graphs. Furthermore, \( \bar{x} := V(G) - (X \cup N_G(X)) \).

By \( \kappa(G) \) we denote the vertex connectivity of a graph \( G \). Let \( \mathcal{S} \) be a set of subsets of \( V(G) \). Let \( \mathcal{F} := \{ T \subseteq V(G) : G - T \text{ is not connected} \} \). Let \( \mathcal{F}_G := \{ T \subseteq V(G) : G - T \text{ is not connected} \} \). Let \( \mathcal{F} \) be the set of all smallest separating sets of \( G \). Let \( \mathcal{T} \in \mathcal{F} \). The union of at least one but
not of all components of \( G - T \) is called a \( T - \mathcal{F} \)-fragment or simply an \( \mathcal{F} \)-fragment. Observe that \( \bar{F} \) is a \( T - \mathcal{F} \)-fragment if and only if \( F \) is a \( T - \mathcal{F} \)-fragment. A \( T - \mathcal{F} \)-fragment \( F \) is a \( T - \mathcal{F} \)-end or briefly an \( \mathcal{F} \)-end if no \( \mathcal{F} \)-fragment is properly contained in \( F \). If no \( \mathcal{F} \)-fragment has less than \( |F| \) vertices then \( F \) is called a \( T - \mathcal{F} \)-atom or an \( \mathcal{F} \)-atom. A graph is called \( \mathcal{F} \)-critical if \( \mathcal{F} \neq \emptyset \), for each \( S \in \mathcal{F} \) there exists a \( T \in \mathcal{F} \) containing \( S \), and for each \( \mathcal{F} \)-fragment \( B \) there exists an \( S \in \mathcal{F} \) and a \( T \in \mathcal{F} \) with \( S \subseteq T - \bar{B} \) and \( T \cap B \neq \emptyset \). This terminology is due to Mader, who also proved the following tool \([5]\).

**Lemma 1.** Let \( B \) be a \( T_{\mathcal{F}} - \mathcal{F} \)-end of a graph \( G \) and \( S \in \mathcal{F}, T \in \mathcal{F} \) satisfying \( S \subseteq T - \bar{B} \) and \( T \cap B \neq \emptyset \). Then one of the following is true.

1. \( B \subseteq T \) and \( |B| \leq \frac{\alpha(G)}{2} \), or
2. \( \bar{B} \subseteq T \), or
3. \( F \subseteq T_B \) for some \( T \)-fragment \( F \) with \( |F| < \frac{\alpha(G)}{2} \).

*If, in addition, \( B \) is an \( \mathcal{F} \)-atom then 1 holds.*

**2. ALMOST CRITICAL GRAPHS**

A graph is called *almost critical* if it is \( \{ \emptyset \} \)-critical, i.e., if it is noncomplete and each of its fragments is intersected by some smallest separating set. For a graph \( G \), let

\[ \mathcal{E}_G := \{ \{ x \} : \text{there exist an end } E \text{ of } G \text{ and a } T \in \mathcal{F} \text{ with } x \in E \cap T \}. \]

It is then easy to see that every almost critical graph is \( \mathcal{E}_G \)-critical.

**Lemma 2.** Let \( G \) be an almost critical graph of connectivity 2 nonisomorphic to \( C_4 \) or \( C_5 \). Then \( G \) contains two paths \( W_1, W_2 \) with the following three properties.

1. \( |W_1| \geq 2, |W_2| \geq 2 \).
2. All vertices in \( W_1 \cup W_2 \) have degree 2 in \( G \).
3. \( W_1 \leq W_2 \).

*Proof.* Without loss of generality, let \( G \) be different from a cycle. By Lemma 1 we obtain for each end \( B \) of \( G \) and each \( T \in \mathcal{F} \) intersecting \( B \) that

\[ B \leq T \wedge |B| = 1 \quad \text{or} \quad \bar{B} \leq T \wedge |\bar{B}| = 1. \quad (1) \]

This remains true if we consider an \( \mathcal{E}_G \)-end \( B \) and a \( T \) intersecting some end contained in \( B \).
By Lemma 1 there is a $T_A$-$A$-atom $A$ with $|A|=1$. $T_A$ is intersected by an end $B$ and either $|B|=1$ or $|B|=1$ by (1). Consequently, the vertices of $A \cup B$ or of $A \cap B$ induce a path consisting of two vertices of degree 2. Let $W$ be a path in $G$ whose vertices all have degree 2 in $G$ such that $W$ is not contained in a longer path whose vertices all have degree 2 in $G$. Since $G$ is not a cycle, $W$ must be a fragment. Therefore, $\bar{W}$ contains an end $B$.

There exists a $T \in \mathcal{F}$ with $T \cap B \neq \emptyset$. Since $|B| \geq |W| \geq 2$ it follows $|B|=1$ by (1). Moreover, we may choose $T$ in such a way that $T$ does not separate $W$; assume that this would be impossible, and consider a $T-\{B\}$-end $F$. By assumption, there exist $w \in W \cap T$ and $w' \in W \cap F \cap N(w)$. Since $N(W) \cap F \neq \emptyset$, we obtain that $F' := F - \{w\} \neq \emptyset$ and $N(F') = B \cup \{w\}$, contradicting the fact that $F$ is a $\{B\}$-end.

Thus there exists a $T-\{B\}$-fragment $F$ with $F \cap W = \emptyset$. $F$ contains an $\mathcal{E}_G$-end $B'$, which has cardinality 1 by (1), since $|B| \geq |F \cap W| + |F \cap N(W)| \geq 2$. $N(B')$ contains an end $B''$ of cardinality 1 also by (1). Neither $B'$ nor $B''$ is contained in $N(W)$ since all vertices of $N(W)$ have degree at least 3. Therefore, $B' \cup B'' \subseteq \bar{W}$ and thus $W_1 := W$ and $W_2 := B' \cup B''$ have properties 1, 2, and 3.

This implies a result of Nebesky on the distribution of vertices of degree 2 in almost critical graphs of connectivity 2 [8, Theorem 4A].

**Lemma 3.** Let $G$ be an almost critical graph of connectivity 2 nonisomorphic to a cycle. Then $G$ contains two paths $W_1, W_2$ with the following four properties.

1. $|W_1| \geq 2, |W_2| \geq 2$.
2. All vertices in $W_1 \cup W_2$ have degree 2 in $G$.
3. $W_1 \subseteq \bar{W}_2$.
4. Both $G-W_1$ and $G-W_2$ are 2-connected.

**Proof.** Let $W$ be a path of length at least 1 consisting of vertices of degree 2. If $G-W$ is not 2-connected then the contraction of an edge of $W$ yields an almost critical graph of connectivity 2 nonisomorphic to a cycle.

By contracting successively all edges of such paths one obtains an almost critical graph $G^*$ of connectivity 2 nonisomorphic to a cycle.

By Lemma 2, $G^*$ possesses two paths $W_1^*$ and $W_2^*$ satisfying the conditions 1, 2, and 3. By construction, 4 holds for them as well. Since we have contracted only edges whose end vertices have degree 2, the corresponding paths in $G$ satisfy 1, 2, 3, and 4, too.

A corollary of this is that every almost critical graph of connectivity 2 possesses four vertices of degree 2. This is sharp since any graph which
arises from a 3-connected graph by connecting the endvertices of two paths on at least two vertices each to distinct vertices of $G$ by four independent edges is almost critical and has connectivity 2.

3. EXTENDABILITY OF CONTRACTIBLE SUBGRAPHS

A contractible subgraph $H$ of a graph $G$ is called extendable if it is contained in a contractible subgraph of size $|H| + 1$. The following statement raises a connection between almost critical graphs and extendability.

**Lemma 4.** Let $G$ be 3-connected and $H$ be a contractible subgraph. Suppose that $G - H$ is not a triangle. Then $H$ is extendable unless $G - H$ is almost critical of connectivity 2.

**Proof.** Suppose that $H$ is not extendable and consider an arbitrary $x \in N(H)$. Since $H$ is contractible, $s(G - H) \geq 2$. Since $H$ is not extendable, $s((G - (H \cup \{x\})) < 2$. Thus $2 > s((G - H) - \{x\}) \geq s(G - H) - 1 \geq 1$ which implies $s(G - H) = 2$ and $x \in T$ for some smallest separating set $T$ of $G - H$.

It remains to show that $G - H$ is almost critical. Take a fragment $F$ of $G - H$. Since $G$ is 3-connected there must be an $x \in N(H) \cap F$. As we have seen above, $x$ is contained in a smallest separating set of $G - H$.

Assuming $|H| = 1$, Lemmas 3 and 4 imply that any vertex of a 3-connected graph nonisomorphic to $K_4$ is either on a contractible edge, i.e., an edge which induces a contractible subgraph, or possesses at least four neighbors of degree 3 which are on two contractible edges each.

This improves the bound in [1, Corollary 4] by one, implies Tutte’s result that every 3-connected graph nonisomorphic to $K_4$ contains a contractible edge ([12], for a short proof see [11]), and implies a result of Halin stating that every vertex of degree 3 of a graph nonisomorphic to $K_4$ is on a contractible edge [3]. Furthermore, it implies that if $C$ is a cycle of a noncomplete 3-connected graph $G$ such that $C$ contains the endvertices of every contractible edge of $G$ then every vertex in $G - C$ is adjacent to two disjoint pairs of consecutive vertices on $C$ of degree 3; in particular, $G$ must be Hamiltonian [9, Theorem 6], and it contains at least $2^{\ell - 6}$ distinct Hamiltonian cycles. Another consequence is that a vertex $x$ of a graph $G$ on at least seven vertices such that $N(x)$ does not contain three independent vertices must be on a contractible edge unless it has degree 4 and is the intersection of two contractible triangles.

Another easy corollary of this is that a contractible, not extendable subgraph $H$ of a graph $G$ has at least four neighbors unless $G - H$ forms a triangle.
For a 2-connected subgraph $H$ of a graph $G$ let $h_G(H)$ be the largest 2-connected supergraph of $H$ in $G$. In other words: $x \in G$ belongs to $h_G(H)$ if and only if there exists a path containing $x$ with both ends contained in $H$.

**Lemma 5.** Let $G$ be a 3-connected graph and $H$ be a contractible subgraph which is not extendable. Suppose that $G - H$ is not isomorphic to a cycle.

By Lemma 3, there exist two paths $W_1$ and $W_2$ satisfying conditions 1, 2, 3, and 4 as there.

Let $H_1 := h_{G - W_1}((G - H) - W_2) \cap H$, $H_2 := h_{G - W_2}((G - H) - W_1) \cap H$.

Then the following is true.

1. $N((G - H) - W_2) \cap H \subseteq H_1$ and $N((G - H) - W_1) \cap H \subseteq H_2$.

2. $H = H_1 \cup H_2$. Every component of $H^+_1, H^+_2$ has exactly one neighbor in $H_2, H_1$, respectively.

3. Suppose that $W$ is a subpath of $W_1$ with $N(H^+_1) \cap W \subseteq W$. Then $W \cup H^+_1$ is contractible.

4. If $|H^+_1| = 0$ then each subpath of $W_1$ is contractible.

5. Suppose that $G$ contains no contractible subgraph on $|H| + 1$ vertices. If $|H^+_1| = 1$ then $|W_1| \leq |H| - 1$ unless $|W_1| = |H| + 1$ and both endvertices of $W_1$ are adjacent to $H^+_1$ and each proper subpath of $W_1$ is contractible.

Since $W_1$ and $W_2$ do have neighbors in $H$, it follows $H_1 \neq \emptyset, H_2 \neq \emptyset$, and consequently $H^+_1 \neq H, H^+_2 \neq H$.

**Proof.**

1. Let $x \in N((G - H) - W_2) \cap H$. Then there exists a $v \in N(x) \cap ((G - H) - W_2)$. Since each vertex in $W_1$ has a neighbor in $H$, there must be a $w \in W_1 - \{v\}$ and a $y \in N(w) \cap H$. Thus there exists an $x, y$-path in $H$ whose endvertices are adjacent to distinct vertices of $(G - H) - W_2$. Since any such path is contained in $h_{G - W_1}((G - H) - W_2)$ we have $x \in H_1$. This proves 1.

2. Let $x \in H$ (we shall prove $x \in H_1 \cup H_2$). Choose an arbitrary $y \in (G - H) - W_2$. Then there are three openly disjoint $x, y$-paths $W_1', W_2', W_3'$ in $G$. We may assume by 1 that $x$ is neither in $N((G - H) - W_2')$ nor in $N((G - H) - W_2)$ and thus not in $N(G - H)$ at all. Thus, for $j \in \{1, 2, 3\}$, the longest subpath of $W_j'$ contained in $H$ has an endpoint $w_j'$ distinct from $x$. By 1, each $w_j'$ is in $H_1$ or in $H_2$. Without loss of generality, $w_1, w_2 \in H_1$ and thus $x \in H_1$. The second part of the assertion follows from the definition of $h_G$. This proves 2.

3. By 1, and 2, $N(H^+_1) \subseteq W \cup H_2$. By 2, every component of $H^+_1$ has one (even two) neighbors in $W$; consequently, $W \cup H^+_1$ is connected. Since
((G − H) − W_1) ∪ H_2 is 2-connected, it remains to show that each component C of W_1 − W is contained in h_{G−(W_1∪H_2)}((G − H) − W_1). This is true since C must be a c, d-path for some endvertex c of W_1, where d must have a neighbor in H − H_1^*, = H_2. This proves 3.

4. Follows straight from 3.

5. By 3, W_1 ∪ H_1^* is contractible; hence |W_1| ≠ |H|. We may assume that |W_1| ≥ |H| + 1. Let W be a minimum subpath of W_1 satisfying \(N(H_1^*) \cap W_1 \subseteq W\). Then both endvertices of W are adjacent to the vertex of H_1^*. There cannot exist a subpath \(W'\) of W_1 with \(W' \subseteq W\) and \(|W'| = |H|\), since \(W' ∪ H_1^*\) would be contractible by 3. In particular, \(|W| > |H| + 1\). Assume that \(|W_1| > |H| + 1\); then there exists a subpath of W_1 on |H| + 1 vertices which does not contain all neighbors of H_1^*; consequently, this subpath is contractible, which is absurd. This proves \(|W_1| = |H| + 1\) and thus \(W = W_1\). Hence 5 is proved.

We shall use Lemma 5 in the next two sections.

4. CONTRACTIBLE TRIPLES

Let us turn back to the conjecture of McCuaig and Ota. We first shall use Lemma 5 to give a new proof of their result on contractible triples, i.e., contractible subgraphs on three vertices.

**Theorem 1 [6].** Every 3-connected graph on at least 6 vertices which is isomorphic neither to K_3,3 nor to the skeleton Q of the 3-dimensional cube contains a contractible triple.

**Proof.** Let |G| ≥ 6. Let us assume that G contains no contractible triple.

**Part I.** Assume that there exists a contractible edge [x, y] such that G − {x, y} is a chordless cycle \(|C| ≥ 4\).

Consider a subpath \(a, b, c, d\) of C with \(a ∈ N(x)\). Assume that \(d ∈ N(y)\) for otherwise \(\{b, c, y\}\) would be contractible. Consequently, \(\{b, c\}\) must be contractible, and, since \(N(\{b, c\}) = \{a, d, x\}\), \(\{b, c\}\) must be extendable, too—a contradiction. Thus, \(d ∈ N(y) − N(x)\). This implies \(G ≅ K_{3,3}\) in case \(|C| = 4\).

Thus we may assume that \(|C| ≥ 5\). Consider a subpath \(a, b, c, d, e\) of C on five vertices and with \(a ∈ N(x)\). Then \(e ∈ N(x) − N(y)\), for \(e ∈ N(y)\) implies that \(\{b, c, d\}\) is contractible. By the same argument, \(N(y) − {x} \subseteq \{b, c, d\}\).

Let z be the neighbor of a in \(C − \{b\}\). Since \(z ∈ N(x)\) we may apply the latter paragraph to \(z, a, b, c, d\) instead of \(a, b, c, d, e\) (possibly \(z = e\)). It
follows $N(y) - \{x\} \subseteq \{a, b, c\}$ and, repeating this, $N(y) - \{x\} \subseteq \{z, a, b\}$. Thus, $N(y) - \{x\} \subseteq \{a\}$, which is absurd.

**Part II.** In the remaining proof we may assume that for each contractible $H$, $|H| = 2$, $G - H$ is not a chordless cycle.

We use the terminology of Lemma 5 and take $H = \{x, y\}$ such that $|W_1| + |W_2| \geq 5$. Furthermore, each edge of $W_1$ is contractible and has at most four neighbors in $G$, contradicting the choice of $H$.

It follows that $H_1^* = H_2^* = \emptyset$. Hence $H = H_1 = H_2$, and, by Lemma 5.4, $|W_1| = |W_2| = 2$. If $x$ has two neighbors in $(G - H) - W_1$ then $y$ has no neighbors in $W_1$ (for otherwise $W_1 \cup \{y\}$ would be contractible); hence $y$ has two neighbors in $(G - H) - W_1$. By symmetry, $x$ has no neighbors in $W_1$ and thus $W_1$ has no neighbors in $H$ at all, a contradiction.

Therefore, $W_1$ and $W_2$ are connected to $H$ by two independent edges each, and $N(H) = W_1 \cup W_2$.

Note that $W_1$, $W_2$ are contractible, too. In addition, $|N(W_1)| = |N(W_2)| = 4$. Thus we may apply our arguments to $W_1$ instead of $H$. It follows that $G$ is 3-regular and that $V(G)$ is partitionable into some contractible vertex sets $H_0, \ldots, H_{l-1}$ of size 2 such that there are two independent edges between $H_i$ and $H_{i+1}$, $i \in \{0, \ldots, l-1\}$, indices mod $l$.

Take a path $X$ on three vertices which intersects $H_0, H_1, H_2$. Then $G - X$ is 2-connected unless $G \cong Q$.

5. **CONTRACTIBLE QUADRUPLES**

We now investigate the case of contractible quadruples, i.e., contractible vertex sets of cardinality 4.

**Theorem 2.** Let $G$ be a 3-connected graph on at least seven vertices not isomorphic to $K_{3,4}$. Then $G$ contains a contractible quadruple.

**Proof.** Without loss of generality we may assume that $G$ is minimally 3-connected, i.e., that the removal of any edge yields a graph of connectivity 2. Halin has proved that any triangle of a minimally 3-connected graph has at least two vertices of degree 3 [3]. We shall use his result without any further reference.

If $G$ contains no contractible triple then $G \cong Q$ by Theorem 1; therein an arbitrary 4-cycle is contractible.

Thus we may assume that $G$ possesses at least one contractible triple $H$. We subdivide the remaining proof into two parts. In the first part we deal
with the case that $G - H$ is a chordless cycle. In the second part we suppose that for any choice of $H, G - H$ is not a chordless cycle. This allows numerous applications of Lemma 5.

**Part 1.** Assume that $G - H$ is a chordless cycle $C (|C| \geq 4)$.

First consider the case that $H$ is a triangle. Then $H$ contains two vertices $x, y$ of degree 3. Consider the two distinct unique neighbors $u, v$ in $C$ of $x, y$, respectively. $C - \{u, v\}$ consists of at most two paths whose vertices are linked to the vertex $z \in H - \{x, y\}$ each. If $C - \{u, v\}$ is connected, then $\{u, v, x, y\}$ is contractible, which is impossible. Consequently, $C - \{u, v\}$ consists of two paths. If there is a path $W$ in $C - \{u, v\}$ on more than one vertex, and if $w$ denotes the endvertex of $W$ adjacent to $u$, then $\{x, y, u, w\}$ is contractible, which is absurd. In the remaining case, $C$ itself is a contractible quadruple.

Second, consider the case that $H$ is a chordless path $x, y, z$. Consider a subpath $W$ of $C$ with $|W| = 4$. If each endvertex of $H$ has a neighbor in $C - W$, then $W$ is contractible. Thus, the neighbors in $C$ of at least one endvertex of $H$ are all contained in $W$.

Let $W$ be a subpath of $C$ such that $|W| = 4$ and $N(x) \subseteq W$. It is possible to choose $W$ in such a way that one of its endvertices $w$ is adjacent to $x$. If $|C| \geq 5$ then we may consider the subpath $W'$ of $C$ with $|W'| = 4$ and $W - W' = \{w\}$. Then $N(z) \subseteq W'$ since $N(x) \not\subseteq W'$.

If $|C| \geq 7$ then there exists a subpath $W''$ of $C$ with $|W''| = 4$ and $W \cap W'' = \emptyset$. This contradicts the fact that $x$ and $z$ both have neighbors in $W \cap W'' \subseteq C - W''$.

If $|C| = 6$ then let $W''$ be the subpath of $C$ on three vertices which is distinct from $W$ and has the same endvertices as $W$. Either $N(x) \subseteq W''$ or $N(z) \subseteq W''$. In the first case, $N(x) = \{y\} \cup (W \cap W'')$; then $W \cup \{x\}$ forms a 2-connected subgraph of $G$; since the vertex in $C - (W \cup W'')$ is adjacent to $y$, $G - (W \cup \{x\})$ is contractible. In the latter case, $N(z) = \{y\} \cup (W' \cap W'')$; then $\{x, w\} \cup (W - W'')$ is contractible.

Assume that $|C| = 5$. Let $u \neq v$ in $N(x) \cap C$. Suppose that they are non-adjacent. Then they have a common neighbor $c$ in $C$. Both vertices $s, t$ in $C - \{u, v, c\}$ are in $N(x) - N(\{y, z\})$, for otherwise $\{s, t, y, z\}$ would be contractible. Without loss of generality, $s \in N(u)$. Applying the same argument to $(s, v)$ instead of $(u, v)$ one obtains $\{c, u\} \subseteq N(x) - N(\{y, z\})$; consequently, $N(\{y, z\}) = \{x, v\}$, which is absurd. Thus, $x$ has exactly two neighbors $u, v$ in $C$ which are, moreover, adjacent. By symmetry, $z$ has exactly two (adjacent) neighbors in $C$. Without loss of generality, $u$ has a neighbor $c$ in $C$ which is not adjacent to $x$ or $z$ and thus is contained in $N(y)$ (otherwise, $v$ has such a neighbor). Hence the subgraph induced by $u, c, x, y$ is 2-connected. Since $z$ has two neighbors in $C - \{c\}$, $G - \{u, c, x, y\}$ is connected and, thus, contractible.
If \(|C| = 4\) then we consider the set \(S := \{x, y, z\} : x, y, z\) form a triangle in \(G\). If there would be a nonseparating triangle \(A\) then \(G - A\) is a contractible subgraph on four vertices—hence \(S \subseteq \mathcal{F}_0\). If \(S\) is empty then we are done since \(K_{3,4}\) is the only triangle-free 3-connected graph on seven vertices. Otherwise, we may take an \(T_B - \mathcal{F}\)-atom \(B\). Since \(|B| \leq 2\) there must be an \(S \in \mathcal{F}\) intersecting \(B\). Thus we find \(S\) and \(T\) as in Lemma 1. Then \(B \subseteq T\) which is impossible since \(T\) cannot separate the triangle \(T_B\).

**Part II.** Let us assume that \(G\) contains no contractible quadruple. We shall end up in several contradictions. Let \(H\) be an arbitrary contractible triple and the further terminology be as in Lemma 5. We prove the following series of claims.

1. \(|H_1^\uparrow| \neq 1, \ |H_2^\uparrow| \neq 1\).
2. \(|H_1^\uparrow| \neq 2, \ |H_2^\uparrow| \neq 2\).
3. \(H_1^\uparrow = H_2^\uparrow = \emptyset\). \(W_1\) and \(W_2\) are both contractible, \(|W_1|, |W_2| \in \{2, 3\}\).
4. If \(H\) is a chordless path and \(|W_1| = 2\) then each \(x \in H\) has at most one neighbor in \((G - H) - W_1\).
5. \(|W_1| = |W_2| = 2\).
6. If \(H\) is a chordless path then its endvertices have degree 3 and are connected to \(W_i\) by two independent edges, \(i \in \{1, 2\}\). The cutvertex of \(H\) has also degree 3 and its neighbor outside \(H\) is not in \(W_1 \cup W_2\). In particular, all vertices in \(W_1 \cup W_2 \cup H\) have degree 3 in \(G\).
7. \(H\) induces a triangle.

In the proof of 5, 6, 7, and in the remainder, we shall apply several of the claims to contractible subgraphs \(H'\) on three vertices distinct from \(H\).

1. Suppose that \(H_1^\uparrow = \{x\}\). (So \(H_2 = H - \{x\}\).) Assume that \(|W_1| = 2\). Since \(x\) has only one neighbor \(y\) in \(H\) by Lemma 5.2, \(H\) induces a path \(x, y, z\) and \(W_1 \subseteq N(x)\). \(H' := W_1 \cup \{x\}\) is contractible by 5.3. Since \(H'\) is not extendable, there exist paths \(W_1, W_2\) in \(G - H'\) as in Lemma 3. Since \(N(H') \subseteq N(W_1) - H\) \(\cup \{y, z\}\), we may assume \(W_1 = \{y, z\}\) without loss of generality. Since the vertices of \(W_1\) have degree 2 in \(G - H'\), they have at most one neighbor in \(W_2\) each. Since every vertex in \(W_2\) has a neighbor in \(H\) and therefore in \(\{y, z\} = W_1\), we obtain that \(|W_2| = 2\) and that \(W_2 \cup \{y, z\}\) is connected. Consequently, \(W_2 \cup \{y, z\}\) is contractible, a contradiction. This proves \(|W_1| > 2\).

By Lemma 5.5 it follows that \(|W_1| = 4\) and that the endvertices of \(W_1\) are both adjacent to \(x\). No endvertex \(w\) of \(W_1\) can be adjacent to a vertex in \(H_2\); for otherwise \((W_1 - \{w\}) \cup \{x\}\) would be contractible. None of the
inner vertices $b_1, b_2$ is adjacent to a vertex in $H_2$, for otherwise $\{b_1, b_2\} \cup H_2$ would be contractible. Thus $N(W_1) \cap H = N(x)$, and, consequently, any subpath of $W_1$ on three vertices is contractible and extendable, a contradiction. This proves 1.

2. $|H^*_1| = 2$ implies $|H^*_2| \leq 1$. By 1, $H^*_2 = \emptyset$. Lemma 5.4 implies $|W_2| \leq 3$. Recall that all neighbors of $H^*_1$ are in $W_1 \cup H_2$. Observe that $G - (W_2 \cup H_2)$ must be 2-connected. Hence $W_2 \cup H_2$ is contractible, which implies $|W_2| = 2$.

Suppose that $e$ is an endvertex of a spanning tree of $H$. If $e \in H^*_1$ then $e$ has at most one neighbor in $W_1$, for otherwise $(H - \{e\}) \cup W_2$ would be contractible. This implies that $H$ induces a triangle and both vertices $x, y$ of $H^*_1$ have exactly one neighbor $u, v$, respectively, in $W_1$. We have $u \neq v$, since $G$ is 3-connected, and we have $[u, v] \not\in E(G)$, otherwise $\{u, v, x, y\}$ would be contractible. There is no common neighbor $w \in W_1$ of $u$ and $v$ for otherwise $W_2 \cup H_2 \cup \{w\}$ would be contractible. Thus there exists a subpath $W$ of $W_1$ on three vertices which contains $u$ and does not contain $v$. $W \cup \{x\}$ is contractible—a contradiction. This proves 2.

3. Follows straightforward from 1, 2, and Lemma 5.4.

4. Suppose that $|W_1| = 2$ and that $H$ is a chordless path $x, y, z$. If $x$ has two neighbors in $(G - H) - W_1$ then $N(W_1) \cap H = \{x\}$, for otherwise $W_1 \cup \{y, z\}$ would be contractible. But then $z$ must have two neighbors in $(G - H) - W_1$ and thus $N(W_1) \cap H = \{z\}$, a contradiction. Thus, $x, z$ each have at most one neighbor in $(G - H) - W_1$ and therefore at least one neighbor in $W_1$. Also $y$ has at most one neighbor in $(G - H) - W_1$, for otherwise $\{x, z\} \cup W_1$ would be contractible. This proves 4.

5. We show the assertion of 5 by distinguishing the possible cardinals of $W_1, W_2$ determined by 3.

Assume that $|W_1| = 3$. Consider a spanning path $x, y, z$ of $H$. Note that $|N(W_1) \cap H| > 1, |N(W_2) \cap H| > 1$, since $W_1, W_2$ are not extendable (but contractible). Suppose for a while that $x \not\in N(W_1) \cap H$. Then $N(x) \cap W_2 = \emptyset$, for otherwise $W_2 \cup \{x\}$ would be contractible. Thus, $(G - H) - W_2 \cup \{x, y\}$ is 2-connected; consequently, $N(z) \cap W_2 = \emptyset$, which yields the contradiction $N(W_2) \cap H = \{y\}$. Thus we may assume that $x$ and, by symmetry, $z$, are contained in $N(W_1) \cap H$. Again, by symmetry, $x, z \in N(W_2) \cap H$. Without loss of generality, $y$ has a neighbor in $(G - H) - W_1$. We may choose it either distinct from some neighbor of $x$ in $W_2$ or distinct from some neighbor of $z$ in $W_2$. Thus, either $W_1 \cup \{z\}$ or $W_1 \cup \{x\}$ is contractible—a contradiction.

Assume that $|W_1| = 2, |W_2| = 3$, and $H$ is a chordless path $x, y, z$. Since 4 holds and since each vertex of $W_2$ has at least one neighbor in $H$, there are exactly three edges between $W_2$ and $H$, which are independent. Thus,
there exists an edge between an endvertex of \( H \), say \( x \), and an endvertex of \( W_2 \), say \( w \). Since \( x \) has a neighbor in \((G - H) - W_2\), \((W_2 \cup H) - \{x, w\}\) is contractible, a contradiction.

By the latter paragraphs it remains to show that the case \(|W_1| = 2\), \(|W_2| = 3\), \( H \) a triangle on \( x, y, z \), does not occur. Since \( W_2 =: H' \) is a contractible path, its neighborhood consists of two paths \( W'_1, W'_2 \) as in Lemma 5. We already know, by the preceding paragraphs, that \(|W'_1| = |W'_2| = 2\). \( H' \) has at most five neighbors in \( G \). Since \( H \) is a triangle, any two vertices in \( H \) induce neither \( W'_1 \) nor \( W'_2 \). Without loss of generality, \( x \in W'_1 \). Then \( x \) has a neighbor in \( W'_1 \); this neighbor is contained in \( N(W'_2) - H \). Thus \( x \) has degree larger than 2 in \( G - H' \)—which is impossible.

This proves 5.

6. Let \( H \) be an induced path \( x, y, z \). By 4, \( x \) has precisely one neighbor \( u_1 \) in \( W_1 \), precisely one neighbor \( u_2 \) in \( W_2 \), and no other neighbors except \( y \). By symmetry, \( N(z) = \{y, v_1, v_2\} \), where \( v_1 \in W_1 \), \( v_2 \in W_2 \). Let \( w \) be the vertex in \( W_1 - \{u_1\} \). Assume that \( u_1 = v_1 \) or that \( y \) is adjacent to \( u_1 \). If \( u_1 = v_1 \) then \( y \) is adjacent to \( w \) and \( z \) is adjacent to \( u_1 \), otherwise \( y \) is adjacent to \( u_1 \) and \( z \) is adjacent to \( w = v_1 \). In any case, \( H' := \{x\} \cup W_2 \) is a contractible path on three vertices. There exist corresponding paths \( W'_1, W'_2 \) as in Lemma 5. Again, by 4, the endvertex \( x \) of \( H' \) has one neighbor in \( W'_1 \) and another in \( W'_2 \). Consequently, \( u_1 \in W'_1 \cup W'_2 \) but \( u_1 \) has degree exceeding 2 in \( G - H' \), a contradiction. So far we have proved that \( u_1 \neq v_1 \) and that \( y \) is not adjacent to \( u_1 \). By symmetry, \( y \) is not adjacent to \( v_1 \). Again, by symmetry, \( u_2 \neq v_2 \) and \( y \) is neither adjacent to \( u_2 \) nor adjacent to \( v_2 \). (So \( W_1 = \{u_1, v_1\}, W_2 = \{u_2, v_2\}\).) It remains to show that \( y \) has degree 3. Assume that \( y \) has more than three neighbors; then two of them are contained in \((G - H) - W_1 \). So \( W'_1 \cup \{x, z\} \) is contractible, a contradiction.

This proves 6.

7. Assume that \( H \) induces a path. Let \( x, y, z, u_1, v_1, u_2, v_2 \) be as in the preceding section (proof of 6). Let \( w \) be the neighbor of \( y \) in \( G - H \). Let \( T_i \) be the neighborhood of \( W_i \) in \( G - H \), \( i \in \{1, 2\} \).

Let \( t \in T_1 \) be the neighbor of \( u_1 \) distinct \( x \) and \( v_1 \). It is easy to see that \( H' := \{u_1, x, y\} \) induces a contractible path. Take two paths \( W'_1, W'_2 \) of \( G - H' \) according to Lemma 5. By 6 applied to \( H' \) instead of \( H \) it follows that \( W'_1 = \{v_1, z\} \) and \( W'_2 = \{w, t\} \). In particular, \( t \) has degree 3 in \( G \) and is adjacent to \( w \), which has also degree 3 in \( G \).

By symmetry, all vertices in \( T_1 \cup T_2 \) have degree 3 in \( G \) and are adjacent to \( w \). Since \( w \) has degree 3, it follows \( T_1 = T_2 = N(w) - \{y\} \).
Now consider $X := W_1 \cup W_2 \cup T_1 \cup T_2 \cup H \cup \{w\}$. Then $X$ consists of ten vertices which have all degree 3 in $G$. Since they have all degree 3 in $G(X)$ as well, it follows $G(X) = G$. Hence $\{w, y\} \cup T_1 \cup T_2$ is a contractible quadruple.

This proves 7.

Now suppose that $H = \{x, y, z\}$ is a triangle. By 5, $|W_1| = |W_2| = 2$.

Assume that the two vertices $u, v$ of $W_1$ have a common neighbor in $H$, say $z$. Let $t$ be the neighbor of $u$ in $G - H$ distinct from $v$. Since $z$ has degree 4, it follows, by Halin’s result, that $u, v, x, y$ all have degree 3 in $G$. So $N_G(u) - \{t\}$ consists of two adjacent vertices $v, z$.

This implies that $H' := \{u, t\}$ is contractible. Assume that $H'$ is extendable. Then, by 7, either $\{u, t, v\}$ or $\{u, t, z\}$ induces a triangle. But neither $t, v$ are adjacent, nor, by Halin’s result, are $t, z$ adjacent (note that $t$ has degree at least 3 in $G - H$).

So $H'$ is not extendable. Since $G - H'$ contains the triangle $H$, $G - H'$ is not a cycle.

Therefore we may apply Lemma 5 to $H'$. Let $W'_1, W'_2, H'_1, H'_2, H'_1^*, H'_2^*$ be the corresponding vertex sets as in Lemma 5. Clearly, $|W'_1| \geq 3$, since $W'_1 \cup H'$ is contractible. By symmetry, $|W'_2| \geq 3$. By 7 and Lemma 5.4 it follows that $|H'_1^*| = |H'_2^*| = 1$. Without loss of generality, $H'_1^* = \{u\}$. By 5.1, $W'_1 \subseteq N(u)$, contradicting the fact that $u$ has degree 3 in $G$.

So $W'_1$ is connected to $H$ by a pair of independent edges. Without loss of generality, $x, y$ are their endpoints. By Halin’s result, $x$ or $y$ has degree 3 in $G$, say $x$. By symmetry, $W'_2$ is linked to $y, z$ by a pair of independent edges. But then $W'_1 \cup \{x\}$ must be a contractible subpath of $G$, contradicting 7.

6. CONTRACTIBLE SUBGRAPHS IN SELECTED GRAPH CLASSES

The following theorem implies that McCuaig’s and Ota’s conjecture is true if we consider only planar triangulations.

**Theorem 3.** Let $G$ be a maximal planar, 3-connected graph. Then every contractible subgraph on at most $|G| - 4$ vertices is extendable.

**Proof.** Let $H$ be a contractible subgraph on at most $|G| - 4$ vertices. If $H$ is not extendable, then $G - H$ is 2-connected and contains a vertex $x$ with $N(x) - H = \{y, z\}$ for nonadjacent vertices $y \neq z$ by Lemma 2. Suppose that the neighbors of $x$ in $G$ are cyclically ordered with respect to a certain embedding of $G$; i.e., any two consecutive neighbors belong to the same face. Let $y', z'$ be the successors of $y, z$, respectively. Then $y', z' \in H$. Thus the vertices of a $y', z'$-path in $H$ together with $x$ form a separating
cycle of $G$ that separates $y$ from $z$—and thus $x$ must be a cutvertex of $G - H$, which is impossible.

A different proof method allows us to verify the conjecture for a large class of line graphs.

**Theorem 4.** Let $G$ be the line graph of a 6-edge-connected graph. If $|G| \geq 3k - 2$ then $G$ contains a contractible subgraph on $k$ vertices.

**Proof.** Let $H$ be a 6-edge-connected graph and let $G = L(H)$. From a well known theorem of Tutte [13] and Nash-Williams [7] it follows that $H$ contains three edge-disjoint spanning trees $T_1$, $T_2$, $T_3$ (see [2, Corollary 3.5.2; 14]). Thus there exists a partition of $E(H)$ into three connected spanning subgraphs $T'_1$, $T'_2$, $T'_3$. Without loss of generality, $T'_1$ contains at least $k$ edges. Let $T'_1$ be a connected subgraph of $T'_1$ on $k$ edges. Since $T'_3 \cup T'_3$ is 2-edge-connected, $E(H) - T'_1$ must be, too. Thus, $T'_1 \subseteq E(H) = V(G)$ induces a contractible subgraph of $G$ on $k$ vertices.

The following theorem deals with a superclass of the Halin-graphs.

**Theorem 5.** Let $G$ be a 3-connected graph whose vertex set can be partitioned into a chordless cycle and a chordless tree. If $|G| \geq k^2$ then $G$ contains a contractible subgraph on $k$ vertices.

**Proof.** Let $C$ be a chordless cycle of $G$ and $T := G - C$ a chordless tree. Let $|G| \geq k^2$.

Suppose that $|T| \geq k$. There exists a minimum subtree $T'$ of $T$ such that $|T'| \geq k$ and $T'$ is contractible. If $|T'| = k$, then we are done. Otherwise we may take any endvertex $v$ of $T'$; since $|N(v) - T'| \geq 2$, $G - (T' - \{v\})$ is 2-connected, contradicting the choice of $T'$.

Thus we may assume that $|T| < k$. In particular, $T$ has at most $k - 1$ endvertices. For any subpath $W$ of $C$ on $k$ vertices we may assume that there is an endvertex $v$ of $T$ such that $N(v) - T \subseteq W$; for otherwise $W$ would be contractible. Thus we obtain $|C| \leq (k - 1) \cdot k$, hence $|G| < k^3$—a contradiction.

### 7. INDUCED PATHS IN 5-CONNECTED GRAPHS

Lovász conjectured that for each natural number $k$ there exists a smallest natural number $f^*(k)$ such that between any two vertices of an $f^*(k)$-connected graph there exists a chordless path such that the removal of its vertices leaves a $k$-connected graph (cf. [10, p. 262]).

Using Tutte's wheel theorem, it is easy to see that $f^*(1) = 3$. Moreover, it is possible to prove the following.
Theorem 6 [4]. Between any two vertices of a 5-connected graph there is a chordless path whose vertices can be removed such that the remaining graph is 2-connected.

This implies \( f^*(2) \leq 5 \), and the square of a cycle of even length shows \( f^*(2) = 5 \).

Theorem 6 implies McCuaig’s and Ota’s conjecture in case of 5-connected graphs of bounded degree.

Corollary 1. Let \( \ell \) be a natural number. Then for any natural number \( k \) there exists an \( f(k, \ell) \) such that any 5-connected graph of maximum degree \( \ell \) on at least \( f(k, \ell) \) vertices contains a contractible subpath on \( k \) vertices.

Proof. There exists an \( f(k, \ell) \) such that any connected graph of maximum degree \( \ell \) on at least \( f(k, \ell) \) vertices has a pair of vertices of distance \( k - 1 \). Now suppose that \( G \) is 5-connected of maximum degree \( \ell \) and \( |G| \geq f(k, \ell) \). Then there is a pair of vertices such that any chordless path \( P \) between them has at least \( k \) vertices. By Theorem 6, there exists a contractible subpath on at least \( k \) vertices. Since any subpath contained in this one is contractible as well, we are finished.

8. AT-FREE CRITICAL 2-CONNECTED GRAPHS

The goal of this section is to prove McCuaig’s and Ota’s conjecture for AT-free graphs.

Three independent vertices of a graph \( G \) are said to form an asteroidal triple, or briefly an AT of \( G \) if between any two of them there exists a path which avoids the neighborhood of the third. A graph is AT-free if it does not contain an AT.

AT-free graphs are suspected to have a “good” algorithmic behaviour in the sense that some problems which are \( NP \)-hard in general are solvable in polynomial time at least on some subclasses of AT-frees such as interval graphs or cocomparability graphs.

From the point of view of multiple connectivity, the fragments of an AT-free graph have an interesting intersection property (which is not the subject of this work): While, in general, it may happen that two fragments \( F, F' \) whose neighborhoods separate each other satisfy \( F \cap F' \neq \emptyset \neq F \cap F' \), this situation never occurs in AT-free graphs.

A noncomplete graph is called critical 2-connected if \( G - x \) is not 2-connected for every \( x \in V(G) \) (or, alternatively, if \( G \) is \( \{\{x\} : x \in V(G)\} \)-critical and of connectivity 2). We start with a structural property of critical 2-connected graphs with a minimum number of vertices of degree 2, namely with exactly four vertices of degree 2. It implies for example that
these graphs are Hamiltonian. For brevity, let us denote by $V_2(G)$ the set of vertices having degree 2 in $G$.

**Lemma 6.** Let $G$ be a critical 2-connected graph with precisely four vertices of degree 2. Then there exists a partition of $V(G)$ into two chordless paths whose endvertices are the vertices of degree 2 in $G$. Furthermore, each endvertex of one of these paths is adjacent to exactly one endvertex of the other path.

**Proof.** Let $G$ be a minimum counterexample. Since the assertion is trivially true if $G$ is a chordless cycle, we may assume that $G$ contains two paths $W_1, W_2$ as in Lemma 3, which form a $T_1$- and a $T_2$-fragment, respectively. The assertion is true for $|G|=6$, hence $|G|\geq 7$.

Consider a $T\in\mathcal{F}$ and suppose $T\cap(W_1\cup W_2)=\emptyset$; we call such a $T$ a good $T$.

Since $G-W_1$ and $G-W_2$ are 2-connected, any vertex in $G-(W_1\cup W_2)$ is contained in a good separator.

Let $T$ be a good separator and $i\neq j$ in $\{1,2\}$. If $W_i\subseteq F$ holds for some $T$-fragment $F$, then $W_j\subseteq F$, for otherwise $|F|\geq 4$ and thus $F$ would contain a vertex of degree 2 by Lemma 1, which contradicts $V_2(G)=W_1\cup W_2$.

Thus we have proved that for any good $T\in\mathcal{F}$ the graph $G-T$ consists of two components $C_1, C_2$ such that $W_1\subseteq C_1, W_2\subseteq C_2$. We refer to $C_1$ as a $W_1$-component throughout this proof. It is easy to see that the intersection (and union) of two $W_1$-components is again a $W_1$-component.

There exists a minimal $W_1$-component $F$ among all $W_1$-components distinct from $W_1$ (since $|G|\geq 7$).

Then $F\subseteq W_1\cup T_1$, for otherwise there would be a good separator $T'$ intersecting $F-(W_1\cup T_1)$; denoting its $W_1$-component by $F'$, it is then easy to see that $W_1\subseteq F\cap F'\subseteq F$, contradicting the choice of $F$.

If $|F|=4$, then, by choice of $F$, there have to be all four edges between $T:=N(F)$ and $T_1$. This implies $T''\cap F=\emptyset$ for any $T''\in\mathcal{F}$ with $T''\cap F\neq\emptyset$.

If $|F|=3$, then for any $T''\in\mathcal{F}$ with $T''\cap F\neq\emptyset$, either $T''\cap F=\emptyset$ or $T''\cap F=\emptyset$, in the latter case, $(T''\cup T)\setminus T_1$ is in $\mathcal{F}$ as well.

Thus, in any case, for every $x\in F\cup T$, there exists a $T''\in\mathcal{F}$ with $x\in T''\subseteq F\cup T$. Consequently, the graph $G^-$ which arises from $G$ by contracting an edge $\{t_1, t\}$ to a vertex $t^-$, where $t_1\in T_1-T$, $t\in T-T_1$, must be critical 2-connected and must contain precisely four vertices of degree 2.

By choice of $G$, there exists a partition of $G^-$ into two paths $P_1^-, P_2^-$ with the properties of the assertion. One of these paths, say $P_1^-$, is of the form $x, t^-, y, ..., z$, where $x\in W_1\cap N_0(t_1)$ and $y\in F\cap N_0(t)$, $z\in W_2$. Thus, the corresponding path $P_1=x, t_1, t, y, ..., z$ in $G$ is chordless. We conclude that $P_1$ and $P_2:=P_2^-$ form a partition of $G$ into two paths which has the properties of the assertion—a contradiction. □
We proceed by proving that almost all $AT$-free critical 2-connected graphs contain precisely four vertices of degree 2.

**Lemma 7.** Any $AT$-free critical 2-connected graph nonisomorphic to $C_5$ contains exactly four vertices of degree 2.

**Proof.** Let $G$ be an $AT$-free critical 2-connected graph. If $G$ is a cycle, then it is either of length 4 or of length 5 by $AT$-freeness. Thus we may assume that $G$ is not a cycle and take two paths $W_1, W_2$ as in Lemma 3. If $|W_1| \geq 3$, then the endvertices of $W_1$ together with an arbitrary vertex of $\overline{W_1}$ form an $AT$. Thus $|W_1| = |W_2| = 2$. Since $V_2(G) \cap (N(W_1) \cup N(W_2)) = \emptyset$, it suffices to prove that $V_2(G) \cap \overline{W_1} \cap \overline{W_2} = \emptyset$. Suppose that there would be a vertex $x \in V_2(G) \cap \overline{W_1} \cap \overline{W_2}$ and take arbitrary $w_1 \in W_1, w_2 \in W_2$. Since $G - W_i$ is 2-connected, there exists an $x, w_i$-path in $G$ which avoids the neighborhood of $w_i$ for all $i \neq j$ in $\{1, 2\}$. Since $G$ is 2-connected, there are two vertex disjoint $w_1, w_2$-paths. Since $G$ is $AT$-free, both paths intersect $N(x)$. This implies that the neighbors of $x$ are on a common cycle excluding $x$. Thus, $x$ cannot be contained in a smallest separating set of $G$, a contradiction.

Now we are prepared to prove the main theorem of this section.

**Theorem 7.** Let $G$ be a 3-connected, $AT$-free graph on at least $3k + 2$ vertices. Then $G$ contains a contractible subgraph on $k + 1$ vertices.

**Proof.** The assertion is true for $k = 1$. Let $H$ be a contractible subgraph of $G$ with $|H| = k$ and suppose that $H$ is not extendable. Then either $G - H$ is a cycle of length at least $2k + 2$ or $G - H$ is almost critical of connectivity 2. The first case cannot happen since every cycle on at least six vertices contains an $AT$.

Thus, $G - H$ contains two paths $W_1, W_2$ as in Lemma 3. Let $x \in (G - H) - (W_1 \cup W_2)$. Then there exist $w_1 \in W_1 - N(x), w_2 \in W_2 - N(x)$. Since $(G - H) - W_i$ is 2-connected, there must be an $x, w_j$-path which avoids the neighborhood of $w_i$ for all $i \neq j$ in $\{1, 2\}$. Since $w_1, w_2$ both have a neighbor in $H$, it follows by assumption that there exists a $w_1, w_2$-path whose inner vertices are contained in $H$. Since $G$ is $AT$-free, $x$ must have a neighbor in $H$.

Since every neighbor of $H$ is a critical vertex in $G - H$ (see proof of Lemma 4), we have proved that $G - H$ is critical 2-connected. By Lemmas 6 and 7, there exists a partition of $G - H$ into two chordless paths $P, P'$ with endvertices $w_1, w_2$ and $w'_1, w'_2$, respectively, where $W_i = \{w, w'_i\}$ for $i \in \{1, 2\}$. Without loss of generality, $|P'| \geq k + 1$. Consider an arbitrary subpath $P''$ of $P'$ on $k + 1$ vertices.

\[\text{CONTRACTIBLE SUBGRAPHS}\]
Let us assume for a while that $G - P'$ contains a cutvertex. Since any vertex of $G - P'$ has a neighbor in $H$, any cutvertex of $G - P'$ must be contained in $H$. Since $(G - H) - P'$ is connected, there exists an end $B$ of $G - P'$ satisfying $B \subseteq H$ and $N(B) - H \subseteq P'$. Take an arbitrary vertex $b \in B$. Since $|N(B) - H| \geq 2$, there exists a neighbor of $B$ in $P'$ which is nonadjacent to $w_i$; since the edges of $W_1, W_2, P, P'$ form a cycle, there exists a $b, w_i$-path in $G$ which avoids the neighborhood of $w_i$ for all $i \neq j$ in $\{1, 2\}$. Since $N(B) - H \subseteq P' \subseteq P'$, $P$ is a $w_1, w_2$-path which avoids the neighborhood of $b$. Thus, $w_1, w_2, b$ form an $AT$—a contradiction.

Thus, $G - P'$ contains no cutvertex, and therefore $P'$ is a contractible subgraph on $k + 1$ vertices.

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