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Certain Classes of Meromorphic Functions with Positive and Missing Coefficients

S. B. JOSHI

Department of Mathematics, Walchand College of Engineering, Sangli 416415, Maharashtra, India

S. R. KULKARNI

Department of Mathematics, Willingdon College, Sangli 416415, Maharashtra, India

AND

H. M. SRIVASTAVA

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada

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The main object of the present paper is to introduce certain novel subclasses of meromorphic functions with positive and missing coefficients and investigate their basic properties including, for example, coefficient estimates, distortion theorem, and the radii of starlikeness and convexity. It is also proved that one of the classes (considered here) is closed under convex linear combinations. The various results obtained in this paper are shown to be sharp. © 1995 Academic Press, Inc.

1. INTRODUCTION AND DEFINITIONS

Let Ω denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

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which are *analytic* in the *punctured* unit disk

$$\mathcal{D} := \{z: z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\}$$

and which have a simple pole at the origin ($z = 0$) with residue 1 there. Also let Ω_S denote the class of all functions in Ω which are *univalent* in $\mathcal{D}$. We denote by $\Omega^*(\alpha)$ and $\Omega_K(\alpha)$ the subclasses of Ω_S consisting of all functions which are, respectively, *meromorphically starlike* and *meromorphically convex of order* α in $\mathcal{D}$ ($0 \leqq \alpha < 1$), that is,

$$\Omega^*(\alpha) := \left\{ f : f \in \Omega_S \text{ and } \operatorname{Re}\left(-\frac{zf'(z)}{f(z)} \right) > \alpha \quad (0 \leqq \alpha < 1; z \in \mathcal{D}) \right\} \tag{1.2}$$

and

$$\Omega_K(\alpha) := \left\{ f : f \in \Omega_S \text{ and } \operatorname{Re}\left(-1 - \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (0 \leqq \alpha < 1; z \in \mathcal{D}) \right\}. \tag{1.3}$$

The class $\Omega^*(\alpha)$ and various other subclasses of Ω have been studied rather extensively by Nehari and Netanyahu [9], Clunie [4], Pommerenke [10], Miller [7], Royster [11], and others (cf., e.g., Bajpai [2], Goel and Sohi [6], Mogra *et al.* [8], Uralegaddi and Ganigi [13], Cho *et al.* [3], Aouf [1], and Uralegaddi and Somanatha [14, 15]; see also Duren [5, pp. 29 and 137], and Srivastava and Owa [12, pp. 86 and 429]). Motivated especially by the aforementioned works of Aouf [1] and Cho *et al.* [3], we aim at presenting here a systematic study of the basic properties (coefficient estimates, distortion theorem, radii of starlikeness and convexity, and closure under convex linear combinations) of some novel subclasses of the class $\Omega(p)$, where $\Omega(p)$ denotes the class of functions $f(z)$ defined by (1.1) *with*

$$a_j = 0 \qquad (j = 1, \ldots, p - 1; p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.4}$$

that is, by

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N}), \tag{1.5}$$

which are analytic in $\mathcal{D}$.

DEFINITION 1. A function $f \in \Omega(p)$ is said to be in the class $\Omega_S(p)$ if it is also univalent in $\mathcal{D}$.

DEFINITION 2. A function $f(z)$, analytic *and* univalent in $\mathfrak{D}$, is said to be in the class $\Omega_S^+(p)$ if it is given by (1.1) *with*

$$a_j = 0\,(j = 1, \ldots, p-1; p \in \mathbb{N}) \qquad \textit{and} \qquad a_{p+j} \geqq 0\,(j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.6}$$

that is, by

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geqq 0; p \in \mathbb{N}). \tag{1.7}$$

Clearly, we have the relationships:

$$\Omega_S^+(p) \subset \Omega_S(p) \subseteq \Omega_S \subset \Omega \text{ and } \Omega(p) \subseteq \Omega \qquad (p \in \mathbb{N}). \tag{1.8}$$

With a view to defining the classes

$$\Omega(p; \alpha, \beta, \gamma, A, B; \lambda) \qquad \text{and} \qquad \Omega_S^+(p; \alpha, \beta, \gamma, A, B; \lambda),$$

we begin by setting

$$F(z) := (1-\lambda) f(z) + \lambda z f'(z) \qquad (f \in \Omega(p); 0 \geqq \lambda < \tfrac{1}{2}), \tag{1.9}$$

so that, obviously,

$$F(z) = \frac{1-2\lambda}{z} + \sum_{n=0}^{\infty} \{1 + \lambda(p+n-1)\} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N}; 0 \leqq \lambda < \tfrac{1}{2}), \tag{1.10}$$

since $f \in \Omega(p)$ is given by (1.5).

DEFINITION 3. A function $f \in \Omega(p)$ is said to be in the class $\Omega(p; \alpha, \beta, \gamma, A, B; \lambda)$ if the function $F(z)$ defined by (1.9) satisfies the inequality:

$$\left| \frac{z^2 F'(z) + 1 - 2\lambda}{\{(B-A)\gamma - B\} z^2 F'(z) + (1-2\lambda)\{(B-A)\gamma\alpha - B\}} \right| < \beta \qquad (z \in \mathfrak{D}), \tag{1.11}$$

where (and *throughout this paper*) the parameters α, β, γ, A, and B are constrained as follows:

$$0 \leqq \alpha < 1; 0 < \beta \leqq 1; -1 \leqq A < B \leqq 1; 0 < B \leqq 1;$$
$$\frac{B}{B-A} < \gamma \leqq \begin{cases} B/\{(B-A)\alpha\} & (\alpha \neq 0) \\ 1 & (\alpha = 0). \end{cases} \tag{1.12}$$

We shall denote by $\Omega^+(p;\ \alpha,\ \beta,\ \gamma,\ A,\ B;\ \lambda)$ the subclass of $\Omega(p;\alpha,\beta,\gamma,A,B;\lambda)$, given by Definition 3, consisting of functions $f(z)$ of the form (1.7), that is,

$$\Omega^+(p;\ \alpha,\ \beta,\ \gamma,\ A,\ B;\ \lambda) = \Omega^+(p) \cap \Omega(p;\ \alpha,\ \beta,\ \gamma,\ A,\ B;\ \lambda), \quad (1.13)$$

where $\Omega^+(p)$ denotes the class of functions $f(z)$ defined by (1.7), which are analytic in $\mathfrak{D}$. For convenience, we shall also write

$$\Omega_S^+(p;\alpha,\beta,\gamma,A,B;\lambda) = \Omega_S^+(p) \cap \Omega(p;\alpha,\beta,\gamma,A,B;\lambda) \quad (1.14)$$

or, equivalently,

$$\Omega_S^+(p;\alpha,\beta,\gamma,A,B;\lambda) = \Omega_S(p) \cap \Omega^+(p;\alpha,\beta,\gamma,A,B;\lambda), \quad (1.15)$$

where the classes $\Omega_S(p)$ and $\Omega_S^+(p)$ are given by Definitions 1 and 2, respectively.

A slight variant of the special case of Definition 3 when $\lambda = 0$ and $p = 1$ was considered earlier by Aouf [1]. Furthermore, the classes $\Sigma(\alpha,\ \beta,\ \gamma)$ and $\Sigma_p(\alpha,\ \beta,\ \gamma)$, studied systematically by Cho *et al.* [3], correspond essentially to the special case of the classes

$$\Omega(p;\alpha,\beta,\gamma,A,B;\lambda) \qquad \text{and} \qquad \Omega_S^+(p;\alpha,\beta,\gamma,A,B;\lambda),$$

respectively, when

$$\lambda = 0,\ p = 1,\ A = -1,\ \text{and}\ B = 1.$$

2. Coefficient Estimates and Distortion Theorem

Our first result (Theorem 1 below) provides a sufficient condition for a function, analytic in $\mathfrak{D}$, to be in the class $\Omega(p;\ \alpha,\ \beta,\ \gamma,\ A,\ B;\ \lambda)$.

THEOREM 1. *Let the function $f(z)$ defined by* (1.5) *be analytic in $\mathfrak{D}$, that is, let $f \in \Omega(p)$. If*

$$\sum_{n=0}^{\infty} \{1 + \beta[(B-A)\gamma - B]\}(p+n)\{1 + \lambda(p+n-1)\}|a_{p+n}| \\ \leqq (B-A)\beta\gamma(1-2\lambda)(1-\alpha) \qquad (p \in \mathbb{N};\ 0 \leqq \lambda < \tfrac{1}{2}), \quad (2.1)$$

where the parameters α, β, γ, A, and B are constrained as in (1.12), *then*

$$f \in \Omega(p;\ \alpha,\ \beta,\ \gamma,\ A,\ B;\ \lambda). \quad (2.2)$$

Proof. Suppose that (2.1) holds true for all admissible values of the parameters α, β, γ, A, and B. In view of Definition 3, we consider the expression

$$\begin{aligned}\Lambda(f) := |z^2F'(z) + 1 - 2\lambda)| - \beta|\{(B-A)\gamma - B\}z^2F'(z) \\ + (1-2\lambda)\{(B-A)\gamma\alpha - B\}| \qquad (z \in \mathfrak{D}),\end{aligned} \tag{2.3}$$

where the function $F(z)$ is defined, in terms of $f \in \Omega(p)$, by (1.9).

In (2.3) we now replace $F'(z)$ by its series expansion given by (1.10), and let

$$0 < |z| = r < 1.$$

We thus find from (2.3) that

$$\begin{aligned}\Lambda(f) = &\left|\sum_{n=0}^{\infty} (p+n)\{1+\lambda(p+n-1)\}a_{p+n}z^{p+n+1}\right| \\ &-\beta\Bigg|(B-A)\gamma(1-2\lambda)(1-\alpha) - [(B-A)\gamma - B]\sum_{n=0}^{\infty}(p+n) \\ &\cdot\{1+\lambda(p+n-1)\}a_{p+n}z^{p+n+1}\Bigg| \\ \leqq &\sum_{n=0}^{\infty}(p+n)\{1+\lambda(p+n-1)\}|a_{p+n}|r^{p+n+1} \\ &-(B-A)\beta\gamma(1-2\lambda)(1-\alpha) \\ &+\beta[(B-A)\gamma - B]\sum_{n=0}^{\infty}(p+n)\{1+\lambda(p+n-1)\}|a_{p+n}|r^{p+n+1} \\ = &\sum_{n=0}^{\infty}\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}|a_{p+n}|r^{p+n+1} \\ &-(B-A)\beta\gamma(1-2\lambda)(1-\alpha).\end{aligned} \tag{2.4}$$

The inequality in (2.4) holds true for all r $(0 < r < 1)$. Therefore, letting $r \to 1-$ in (2.4), we obtain

$$\begin{aligned}\Lambda(f) \leqq &\sum_{n=0}^{\infty}\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}|a_{p+n}| \\ &-(B-A)\beta\gamma(1-2\lambda)(1-\alpha) \\ \leqq &\,0 \qquad (p \in \mathbb{N}; 0 \leqq \lambda < \tfrac{1}{2}),\end{aligned} \tag{2.5}$$

by the hypothesis (2.1), and the assertion (2.2) of Theorem 1 follows immediately from Definition 3.

Next we give a necessary *and* sufficient condition for a function $f \in \Omega^+(p)$ to be in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$.

THEOREM 2. *Let the function $f(z)$ defined by* (1.7) *be analytic in $\mathcal{D}$, that is, let $f \in \Omega^+(p)$.*
Then

$$f \in \Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda) \tag{2.6}$$

if and only if

$$\sum_{n=0}^{\infty} \{1 + \beta[(B - A)\gamma - B]\}(p + n)\{1 + \lambda(p + n - 1)\}a_{p+n} \\ \leqq (B - A)\beta\gamma(1 - 2\lambda)(1 - \alpha) \qquad (p \in \mathbb{N}; 0 \leqq \lambda < \tfrac{1}{2}), \tag{2.7}$$

provided that the parameters α, β, γ, A, and B are constrained as in (1.12). *The result is sharp.*

Proof. Since $f \in \Omega^+(p)$, we have

$$|a_{p+n}| = a_{p+n} \qquad (a_{p+n} \geqq 0; p \in \mathbb{N}; n \in \mathbb{N}_0), \tag{2.8}$$

and it is obvious from Theorem 1 that the condition (2.7) is sufficient for the assertion (2.6) of Theorem 2.

In order to prove that the condition (2.7) is also necessary, let us assume that the assertion (2.6) holds true. Then, in view of (1.13), we find from Definition 3, and from the series expansion (1.10) with $f \in \Omega^+(p)$, that

$$\left|\frac{z^2F'(z) + 1 - 2\lambda}{\{(B - A)\gamma - B\}z^2F'(z) + (1 - 2\lambda)\{(B - A)\gamma\alpha - B\}}\right| \\ = \left|\frac{\sum_{n=0}^{\infty} a^*_{p+n} z^{p+n+1}}{(B - A)\gamma(1 - 2\lambda)(1 - \alpha) - [(B - A)\gamma - B]\sum_{n=0}^{\infty} a^*_{p+n} z^{p+n+1}}\right| \\ < \beta \qquad (z \in \mathcal{D}; a^*_{p+n} := (p + n)\{1 + \lambda(p + n - 1)\}a_{p+n}). \tag{2.9}$$

Since $\mathrm{Re}(z) \leqq |z|$ for all z, it follows from (2.9) that

$$\mathrm{Re}\left(\frac{\sum_{n=0}^{\infty} a^*_{p+n} z^{p+n+1}}{(B - A)\gamma(1 - 2\lambda)(1 - \alpha) - [(B - A)\gamma - B]\sum_{n=0}^{\infty} a^*_{p+n} z^{p+n+1}}\right) \\ < \beta \qquad (z \in \mathcal{D}; a^*_{p+n} := (p + n)\{1 + \lambda(p + n - 1)\}a_{p+n}). \tag{2.10}$$

We now choose the values of z on the real axis so that $z^2F'(z)$ is real $(0 < z < 1)$. Upon clearing the denominator in (2.10) and letting $z \to 1-$ through real values, we obtain

$$\sum_{n=0}^{\infty}(p+n)\{1+\lambda(p+n-1)\}a_{p+n} \leqq (B-A)\beta\gamma(1-2\lambda)(1-\alpha) - \beta[(B-A)\gamma - B]\sum_{n=0}^{\infty}(p+n)\{1+\lambda(p+n-1)\}a_{p+n}, \tag{2.11}$$

which immediately yields the required condition (2.7).

Our assertion in Theorem 2 is sharp for functions $f_n \in \Omega^+(p)$ of the form:

$$f_n(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{\{1+\beta[(B-A)\gamma-B]\}(p+n)\{1+\lambda(p+n-1)\}}z^{p+n} \quad (n \in \mathbb{N}_0). \tag{2.12}$$

A distortion property for functions in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$ is contained in

THEOREM 3. *If the function $f(z)$ defined by* (1.7) *is in the class* $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$, *then*

$$\frac{1}{|z|} - \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{p\{1+\beta[(B-A)\gamma-B]\}\{1+\lambda(p-1)\}}|z|^p \leqq |f(z)| \leqq \frac{1}{|z|} + \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{p\{1+\beta[(B-A)\gamma-B]\}\{1+\lambda(p-1)\}}|z|^p \quad (z \in \mathfrak{D}; p \in \mathbb{N}). \tag{2.13}$$

The result (2.13) *is sharp with the extremal function* $f_0(z)$ *given by* (2.12) *for* $n = 0$.

Proof. Since $f \in \Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$, Theorem 2 readily yields the inequality:

$$\sum_{n=0}^{\infty} a_{p+n} \leqq \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{p\{1+\beta[(B-A)\gamma-B]\}\{1+\lambda(p-1)\}} \quad (p \in \mathbb{N}). \tag{2.14}$$

Making use of (2.14), we find from the definition (1.7) that

$$|f(z)| \leqq \frac{1}{|z|} + \sum_{n=0}^{\infty} a_{p+n}|z|^{p+n} \leqq \frac{1}{|z|} + |z|^p \sum_{n=0}^{\infty} a_{p+n} \leqq \frac{1}{|z|} + \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{p\{1+\beta[(B-A)\gamma-B]\}\{1+\lambda(p-1)\}}|z|^p \quad (z \in \mathfrak{D}; p \in \mathbb{N})$$

and

$$|f(z)| \geqq \frac{1}{|z|} - \sum_{n=0}^{\infty} a_{p+n} |z|^{p+n} \geqq \frac{1}{|z|} - |z|^p \sum_{n=0}^{\infty} a_{p+n}$$

$$\geqq \frac{1}{|z|} - \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{p\{1+\beta[(B-A)\gamma - B]\}\{1+\lambda(p-1)\}} |z|^p \qquad (z \in \mathfrak{D}; p \in \mathbb{N}),$$

which evidently prove the assertion (2.13) of Theorem 3.

It is not difficult to verify that the equality in (2.13) holds true for the function $f_0(z)$ given by (2.12) for $n = 0$ at the points

$$z = r, \pm ir \qquad (0 < r < 1). \tag{2.15}$$

3. Further Properties

The radii of starlikeness and convexity for the class

$$\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$$

is given by

THEOREM 4. *If the function $f(z)$ defined by (1.7) is in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$, then $f(z)$ is meromorphically starlike of order δ $(0 \leqq \delta < 1)$ in*

$$|z| < r_p = r_p(\alpha, \beta, \gamma, \delta, A, B; \lambda),$$

where

$$r_p(\alpha, \beta, \gamma, \delta, A, B; \lambda) := \inf_{n \in \mathbb{N}} \left\{ \frac{(p+n-1)(1-\delta)\{1+\beta[(B-A)\gamma - B]\}\{1+\lambda(p+n-2)\}}{(p+n+1-\delta)(B-A)\beta\gamma(1-2\lambda)(1-\alpha)} \right\}^{1/(p+n)} \qquad (p \in \mathbb{N}). \tag{3.1}$$

Furthermore, under the same hypothesis, $f(z)$ is meromorphically convex of order δ $(0 \leqq \delta < 1)$ in

$$|z| < R_p = R_p(\alpha, \beta, \gamma, \delta, A, B; \lambda),$$

where

$$R_p(\alpha, \beta, \gamma, \delta, A, B; \lambda) := \inf_{n \in \mathbb{N}} \left\{ \frac{(1-\delta)\{1+\beta[(B-A)\gamma - B]\}\{1+\lambda(p+n-2)\}}{(p+n+1-\delta)(B-A)\beta\gamma(1-2\lambda)(1-\alpha)} \right\}^{1/(p+n)} \quad (p \in \mathbb{N}). \tag{3.2}$$

Each of the above results is sharp for the functions $f_n(z)$ given by (2.12).

Proof. Let $f(z)$ be in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$. Then, by Theorem 2, we have

$$\sum_{n=0}^{\infty} \frac{\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}}{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)} a_{p+n} \leqq 1 \tag{3.3}$$
$$(p \in \mathbb{N}; 0 \leqq \lambda < \tfrac{1}{2}).$$

In view of Definition (1.2), the first assertion of Theorem 4 would follow if we can show that

$$\left| 1 + \frac{zf'(z)}{f(z)} \right| \leqq 1 - \delta \qquad (|z| < r_p(\alpha, \beta, \gamma, \delta, A, B; \lambda)), \tag{3.4}$$

where $r_p(\alpha, \beta, \gamma, \delta, A, B; \lambda)$ is given by (3.1). Indeed, we find from (1.7) and (3.3) that

$$\left| 1 + \frac{zf'(z)}{f(z)} \right| = \left| \frac{\sum_{n=0}^{\infty} (p+n+1)\, a_{p+n}\, z^{p+n+1}}{1 + \sum_{n=0}^{\infty} a_{p+n}\, z^{p+n+1}} \right|$$
$$\leqq \frac{\sum_{n=0}^{\infty} (p+n+1) a_{p+n} |z|^{p+n+1}}{1 - \sum_{n=0}^{\infty} a_{p+n} |z|^{p+n+1}}$$
$$\leqq 1 - \delta \qquad (0 \leqq \delta < 1),$$

provided that

$$\sum_{n=0}^{\infty} \left(\frac{p+n+2-\delta}{1-\delta} \right) a_{p+n} |z|^{p+n+1} \leqq 1 \qquad (0 \leqq \delta < 1). \tag{3.5}$$

Now, by virtue of (3.3), the inequality (3.5) does hold true if

$$|z|^{p+n+1} \leqq \frac{(p+n)(1-\delta)\{1+\beta[(B-A)\gamma-B]\}\{1+\lambda(p+n-1)\}}{(p+n+2-\delta)(B-A)\beta\gamma(1-2\lambda)(1-\alpha)} \qquad (n \in \mathbb{N}_0),$$

that is, if

$$|z| \leqq \left\{\frac{(p+n-1)(1-\delta)\{1+\beta[(B-A)\gamma-B]\}\{1+\lambda(p+n-2)\}}{(p+n+1-\delta)(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}\right\}^{1/(p+n)} \qquad (n \in \mathbb{N}). \tag{3.6}$$

Writing $r_p(\alpha, \beta, \gamma, \delta, A, \beta; \lambda)$ for $|z|$ in (3.6), we immediately obtain the radius of starlikeness asserted by (3.1).

In precisely the same manner, we can find the radius of convexity asserted by (3.2), by requiring that

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| \leqq 1 - \delta \qquad (0 \leqq \delta < 1; |z| < R_p(\alpha, \beta, \gamma, \delta, A, B; \lambda)),$$

in view of Definition 3.

The proof of Theorem 4 is completed upon observing that each of its assertions is sharp for the functions $f_n(z)$ given by (2.12).

Our next result (Theorem 5 below) involves a linear combination of several functions of the type (2.12).

THEOREM 5. *Let*

$$\phi_0(z) = \frac{1}{z} \qquad (z \in \mathcal{D}) \tag{3.7}$$

and

$$\phi_{p+n}(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{\{1+\beta[(B-A)\gamma-B]\}(p+n)\{1+\lambda(p+n-1)\}} z^{p+n} \tag{3.8}$$

$$(p \in \mathbb{N}; n \in \mathbb{N}_0; z \in \mathcal{D}).$$

Then the function $\phi(z)$ is in the class

$$\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$$

if and only if it can be expressed in the form:

$$\phi(z) = \sum_{n=0}^{\infty} \lambda_n \phi_n(z) \tag{3.9}$$

$$(\lambda_n \geqq 0; \lambda_j = 0\,(j = 1, \ldots, p-1, \textit{if}\, p \geqq 2); \sum_{n=0}^{\infty} \lambda_n = 1).$$

Proof. From (3.7), (3.8), and (3.9), it is easily seen that

$$\begin{aligned}\phi(z) &= \sum_{n=0}^{\infty} \lambda_n \phi_n(z) \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}} \lambda_{p+n} z^{p+n}.\end{aligned} \tag{3.10}$$

Since

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}}{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)} \\ &\quad\cdot \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}} \lambda_{p+n} \\ &= \sum_{n=0}^{\infty} \lambda_{p+n} = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leqq 1,\end{aligned}$$

it follows from Theorem 2 that the function $\phi(z)$ given by (3.10) is in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$.

Conversely, let us suppose that

$$\phi(z) \in \Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda).$$

Then, by Theorem 2, $\phi(z)$ possesses a representation of the form:

$$\phi(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_{p+n} z^{p+n} \qquad (b_{p+n} \geqq 0; p \in \mathbb{N}; z \in \mathfrak{D}), \tag{3.11}$$

where [cf. Eq. (2.7)]

$$0 \leqq b_{p+n} \leqq \frac{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)}{\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}} \tag{3.12}$$

$$(p \in \mathbb{N}; n \in \mathbb{N}_0; 0 \leqq \lambda < \tfrac{1}{2}).$$

In view of (3.11), we may set

$$\lambda_{p+n} = \frac{\{1+\beta[(B-A)\gamma - B]\}(p+n)\{1+\lambda(p+n-1)\}}{(B-A)\beta\gamma(1-2\lambda)(1-\alpha)} b_{p+n}$$

$$(p \in \mathbb{N};\, n \in \mathbb{N}_0),$$

$$\lambda_j = 0 \qquad (j = 1, \ldots, p-1, \text{if } p \geqq 2),$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n \geqq 0,$$

and the representation for $\phi(z)$, given by (3.9), would follow readily from (3.11) and (3.10).

Finally, we prove

THEOREM 6. *The class* $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$ *is closed under convex linear combinations.*

Proof. Suppose that the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n,j} z^{p+n} \qquad (j = 1, 2; p \in \mathbb{N}; z \in \mathfrak{D}) \tag{3.13}$$

are in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$.

Setting

$$f(z) = (1-\mu) f_1(z) + \mu f_2(z) \qquad (0 \leqq \mu \leqq 1), \tag{3.14}$$

we find from (3.13) that

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \{(1-\mu) a_{p+n,1} + \mu a_{p+n,2}\} z^{p+n} \tag{3.15}$$

$$(0 \leqq \mu \leqq 1; p \in \mathbb{N}; z \in \mathfrak{D}).$$

Since, by hypothesis, Theorem 2 applies to each of the functions $f_1(z)$ and $f_2(z)$ defined by (3.13), we have

$$\sum_{n=0}^{\infty} \{1 + \beta[(B - A)\gamma - B]\}(p + n)\{1 + \lambda(p + n - 1)\}$$

$$\cdot \{(1 - \mu)a_{p+n,1} + \mu\, a_{p+n,2}\}$$

$$= (1 - \mu) \sum_{n=0}^{\infty} \{1 + \beta[(B - A)\gamma - B]\}(p + n)\{1 + \lambda(p + n - 1)\}a_{p+n,1}$$

$$+ \mu \sum_{n=0}^{\infty} \{1 + \beta[(B - A)\gamma - B]\}(p + n)\{1 + \lambda(p + n - 1)\}a_{p+n,2}$$

$$\leqq (1 - \mu)\{(B - A)\beta\gamma(1 - 2\lambda)(1 - \alpha)\} + \mu\{(B - A)\beta\gamma(1 - 2\lambda)(1 - \alpha)\}$$

$$= (B - A)\beta\gamma(1 - 2\lambda)(1 - \alpha),$$

which shows, again by Theorem 2, that $f(z)$ defined by (3.14) is also in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B; \lambda)$.

This evidently completes the proof of Theorem 6.

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