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Boundary Harnack principle for subordinate Brownian motions

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Abstract

We establish a boundary Harnack principle for a large class of subordinate Brownian motions, including mixtures of symmetric stable processes, in κ-fat open sets (disconnected analogue of John domains). As an application of the boundary Harnack principle, we identify the Martin boundary and the minimal Martin boundary of bounded κ-fat open sets with respect to these processes with their Euclidean boundaries.

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1. Introduction

The boundary Harnack principle for nonnegative classical harmonic functions is a very deep result in potential theory and has very important applications in probability and potential theory.

In [4] Bogdan showed that the boundary Harnack principle is valid in bounded Lipschitz domains for nonnegative harmonic functions of rotationally invariant stable processes and then in [27] Song and Wu extended the boundary Harnack principle for rotationally invariant stable processes to bounded κ-fat open sets. Subsequently Bogdan–Stos–Sztonyk [7] and Sztonyk [29] extended the boundary Harnack principle to symmetric (not necessarily rotationally invariant)

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stable processes. In a recent paper [6], Bogdan, Kulczycki and Kwasnicki proved a version of the boundary Harnack inequality for nonnegative harmonic functions of rotationally invariant stable processes in arbitrary open sets.

By using some perturbation methods, the boundary Harnack principle has been generalized to some classes of rotationally invariant Lévy processes including relativistic stable processes and truncated stable processes. These processes can be regarded as perturbations of rotationally invariant stable processes and their Green functions on bounded smooth domains are comparable to their counterparts for rotationally invariant stable processes (see [9,12,15–17,22]). This comparison of Green functions played a crucial role in the arguments of [12,16,17].

In this paper, we will show that, under minimal conditions, the boundary Harnack principle is valid for subordinate Brownian motions with characteristic exponents of the form $\Phi(\xi) = |\xi|^\alpha \ell(|\xi|^2)$ for some $\alpha \in (0, 2)$ and some positive function ℓ which is slowly varying at ∞. Examples of this class of subordinate Brownian motions include, among others, relativistic stable processes and mixtures of rotationally invariant stable processes. The Green functions of subordinate Brownian motions considered here behave like $c|x|^{-d+\alpha}(\ell(|x|^{-2}))^{-1}$ near the origin. So these subordinate Brownian motions cannot be regarded as perturbations of rotationally invariant stable processes in general and their Green functions in bounded smooth domains are not comparable to their counterparts for rotationally invariant stable processes.

Our proof of the boundary Harnack principle will be similar to the arguments in [4,27] for rotationally invariant stable processes. One of the key ingredients is a sharp upper bound for the expected exit time from a ball which, in the case of stable processes, follows easily from the explicit formula for the Green function of a ball. However, the known methods seem to fail to get the desired upper bound here and a substantially new idea needs to be introduced. We rely on the fluctuation theory for real-valued Lévy processes and borrow some results from [26] to accomplish the desired upper bound.

The organization of this paper is as follows. In Section 2 we use the fluctuation theory for real-valued Lévy processes to establish a nice upper bound on the expected exit time from an interval for a one-dimensional subordinate Brownian motion. In Section 3, we use the results of Section 2 to establish the desired upper bound on the expected exit time from a ball for a multi-dimensional subordinate Brownian motion and an upper bound on the Poisson kernel of a ball. The proof of the boundary Harnack principle is given in Section 4 and in the last section we apply our boundary Harnack principle to study the Martin boundary with respect to subordinate Brownian motions.

In this paper we will use the following convention: the values of the constants $r_1, r_2, \ldots$ will remain the same throughout this paper, while the values of the constants $c_1, c_2, \ldots$ or $C, C_1, C_2, \ldots$ might change from one appearance to another. The dependence of the constants on the dimension, the index α and the slowly varying function will not be mentioned explicitly, while the dependence of the constants on other quantities will be expressed using $c(\cdot)$ with the arguments representing the quantities the constant depends on. In this paper, we use ":=" to denote a definition, which is read as "is defined to be". $f(t) \sim g(t)$, $t \to 0$ ($f(t) \sim g(t)$, $t \to \infty$, respectively) means $\lim_{t\to 0} f(t)/g(t) = 1$ ($\lim_{t\to\infty} f(t)/g(t) = 1$, respectively).

2. Some results on one-dimensional subordinate Brownian motion

Suppose that $W = (W_t : t \geq 0)$ is a one-dimensional Brownian motion with

$$\mathbb{E}\left[e^{i\xi(W_t - W_0)}\right] = e^{-t\xi^2}, \quad \forall \xi \in \mathbb{R}, t > 0,$$

and $S = (S_t : t \geq 0)$ is a subordinator (a nonnegative increasing Lévy process) independent of W and with Laplace exponent ϕ, that is

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \forall t, \lambda > 0.$$

A C^∞ function $g : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n D^n g \leq 0$ for every positive integer n. Any Bernstein function g can be written in the following form

$$g(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt)$$

where $a, b \geq 0$ and μ is a measure on $(0, \infty)$ with $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$. μ is called the Lévy measure of g. It is well known that a function g is the Laplace exponent of a subordinator if and only if g is a Bernstein function with $\lim_{\lambda \to 0} g(\lambda) = 0$. A Bernstein function g is called a complete Bernstein function if its Lévy measure μ has a completely monotone density with respect to the Lebesgue measure. For details on examples and properties of complete Bernstein functions, one can see [13,23] or [26]. One of the important properties of complete Bernstein functions is that f is complete Bernstein if and only if $\lambda \mapsto \lambda/f(\lambda)$ is complete Bernstein. We will use this property in the paper.

Throughout this paper we will assume that ϕ is a complete Bernstein function such that

$$\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda) \tag{2.1}$$

for some $\alpha \in (0, 2)$ and some positive function ℓ which is slowly varying at ∞, that is, $\ell(\lambda t)/\ell(t) \to 1$ as $t \to \infty$ for every $\lambda > 0$. For concepts and results related to the slowly varying functions, we refer our readers to [3].

Using Corollary 2.3 of [25] or Theorem 2.3 of [21] we know that the potential measure U of S defined by

$$U(A) := \mathbb{E}\int_0^\infty 1_{(S_t \in A)} dt = \int_0^\infty \mathbb{P}(S_t \in A) dt$$

has a decreasing density u.

By using the Tauberian theorem (Theorem 1.7.1 in [3]) and the monotone density theorem (Theorem 1.7.2 in [3]), one can easily check that

$$u(t) \sim \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)} \frac{1}{\ell(t^{-1})}, \quad t \to 0. \tag{2.2}$$

Let $\mu(t)$ be the density of the Lévy measure of ϕ. It follows from Proposition 2.23 of [26] that

$$\mu(t) \sim \frac{\alpha}{2\Gamma(1-\alpha/2)} \frac{\ell(t^{-1})}{t^{1+\alpha/2}}, \quad t \to 0. \tag{2.3}$$

The subordinate Brownian motion $X = (X_t : t \geq 0)$ defined by $X_t = W_{S_t}$ is a symmetric Lévy process with the characteristic exponent

$$\Phi(\theta) = \phi(\theta^2) = |\theta|^\alpha \ell(\theta^2), \quad \forall \theta \in \mathbb{R}.$$

Let $\overline{X}_t := \sup\{0 \vee X_s : 0 \leq s \leq t\}$ and let L_t be a local time of $\overline{X} - X$ at 0. L is also called a local time of the process X reflected at the supremum. Then the right continuous inverse L_t^{-1} of L is a possibly killed subordinator and is called the ladder time process of X. The process

$\overline{X}_{L_t^{-1}}$ is also a possibly killed subordinator and is called the ladder height process of X. (For the basic properties of the ladder time and ladder height processes, we refer the readers to Chapter 6 of [1].)

It follows from Corollary 9.7 of [10] that the Laplace exponent χ of the ladder height process of X is given by

$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\Phi(\lambda\theta))}{1+\theta^2}\mathrm{d}\theta\right)$$
$$= \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(|\theta|^\alpha\lambda^\alpha\ell(\theta^2\lambda^2))}{1+\theta^2}\mathrm{d}\theta\right), \quad \forall\lambda > 0. \tag{2.4}$$

Under our assumptions, we have the following result.

Proposition 2.1. *The Laplace exponent χ of the ladder height process of X is a special Bernstein function. i.e., $\lambda/\chi(\lambda)$ is also a Bernstein function.*

Proof. Define $\psi(\lambda) = \lambda/\phi(\lambda)$. Let T be a subordinator independent of W and with Laplace exponent ψ and let $Y = (Y_t : t \geq 0)$ be the subordinate Brownian motion defined by $Y_t = W_{T_t}$. Let Ψ be the characteristic exponent of Y. Then

$$\Phi(\theta)\,\Psi(\theta) = \phi(\theta^2)\psi(\theta^2) = \theta^2, \quad \forall\theta \in \mathbb{R}.$$

Let ρ be the Laplace exponent of the ladder height process of Y. Then by (2.4) we have

$$\chi(\lambda)\rho(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\Phi(\theta\lambda)) + \log(\Psi(\theta\lambda))}{1+\theta^2}\,\mathrm{d}\theta\right)$$
$$= \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\Phi(\theta\lambda)\,\Psi(\theta\lambda))}{1+\theta^2}\,\mathrm{d}\theta\right)$$
$$= \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\theta^2\lambda^2)}{1+\theta^2}\,\mathrm{d}\theta\right) = \lambda.$$

Thus χ is a special Bernstein function. □

Proposition 2.2. *If there are $M > 1$, $\delta \in (0, 1)$ and a nonnegative integrable function f on $(0, \delta)$ such that*

$$\left|\log\left(\frac{\ell(\lambda^2\theta^2)}{\ell(\lambda^2)}\right)\right| \leq f(\theta), \quad \forall(\theta, \lambda) \in (0, \delta) \times (M, \infty), \tag{2.5}$$

then

$$\lim_{\lambda\to\infty} \frac{\chi(\lambda)}{\lambda^{\alpha/2}(\ell(\lambda^2))^{1/2}} = 1. \tag{2.6}$$

Proof. Using the identity

$$\lambda^{\beta/2} = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\theta^\beta\lambda^\beta)}{1+\theta^2}\,\mathrm{d}\theta\right), \quad \forall\lambda, \beta > 0,$$

we get easily from (2.4) that

$$\chi(\lambda) = \lambda^{\alpha/2} \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\ell(\lambda^2\theta^2))}{1+\theta^2} d\theta\right)$$
$$= \lambda^{\alpha/2}(\ell(\lambda^2))^{1/2} \exp\left(\frac{1}{\pi}\int_0^\infty \log\left(\frac{\ell(\lambda^2\theta^2)}{\ell(\lambda^2)}\right)\frac{1}{1+\theta^2} d\theta\right).$$

By Potter's Theorem (Theorem 1.5.6 (1) in [3]), there exists $\lambda_0 > 1$ such that

$$\left|\log\left(\frac{\ell(\lambda^2\theta^2)}{\ell(\lambda^2)}\right)\right| \frac{1}{1+\theta^2} \le 2\frac{\log\theta}{1+\theta^2}, \quad \forall(\theta,\lambda) \in [1,\infty)\times[\lambda_0,\infty).$$

Thus by using the dominated convergence theorem in the first integral below, the uniform convergence theorem (Theorem 1.2.1 in [3]) in the second integral, and the assumption (2.5) in the third integral, we have

$$\lim_{\lambda\to\infty}\int_0^\infty \log\left(\frac{\ell(\lambda^2\theta^2)}{\ell(\lambda^2)}\right)\frac{1}{1+\theta^2}\, d\theta$$
$$= \lim_{\lambda\to\infty}\left(\int_1^\infty + \int_\delta^1 + \int_0^\delta\right)\log\left(\frac{\ell(\lambda^2\theta^2)}{\ell(\lambda^2)}\right)\frac{1}{1+\theta^2}\, d\theta = 0. \quad \square$$

In the case $\phi(\lambda) = \lambda^{\alpha/2}$ for some $\alpha \in (0, 2)$, the assumption of the proposition above is trivially satisfied. Now we give some other examples.

Example 2.3. Suppose that $\alpha \in (0, 2)$ and define

$$\phi(\lambda) = (\lambda+1)^{\alpha/2} - 1.$$

Then ϕ is a complete Bernstein function which can be written as $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ with

$$\ell(\lambda) = \frac{(\lambda+1)^{\alpha/2} - 1}{\lambda^{\alpha/2}}.$$

Using elementary analysis one can easily check that there is a nonnegative integrable function f on (0, 1) such that (2.5) is satisfied.

Example 2.4. Suppose $0 < \beta < \alpha < 2$ and define

$$\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}.$$

Then ϕ is a complete Bernstein function which can be written as $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ with

$$\ell(\lambda) = 1 + \lambda^{(\beta-\alpha)/2}.$$

Using elementary analysis one can easily check that there is a nonnegative integrable function f on (0, 1) such that (2.5) is satisfied.

Example 2.5. Suppose that $\alpha \in (0, 2)$ and $\beta \in (0, 2-\alpha)$. Define

$$\phi(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{\beta/2}.$$

By using the facts that λ and $\log(1+\lambda)$ are complete Bernstein functions and properties of complete Bernstein functions (see [26]), one can easily check that ϕ is a complete Bernstein function. ϕ can be written as $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ with

$$\ell(\lambda) = (\log(1+\lambda))^{\beta/2}.$$

To check that there is a nonnegative integrable function f on $(0, 1)$ such that (2.5) is satisfied, we only need to bound the function

$$\left|\log\left(\frac{\log(1+\lambda^2\theta^2)}{\log(1+\lambda^2)}\right)\right|$$

for large λ and small θ. We will consider two cases separately. Fix an $M > 1$ and a $\theta < 1$.

(1) $\lambda \geq M, \theta < 1$ and $\lambda > 1/\theta$. In this case, by using the fact that for any $a > 0$ the function $x \mapsto \frac{x}{x-a}$ is decreasing on (a, ∞), we get that

$$\begin{aligned}\left|\log\left(\frac{\log(1+\lambda^2\theta^2)}{\log(1+\lambda^2)}\right)\right| &= \log\left(\frac{\log(1+\lambda^2)}{\log(1+\lambda^2\theta^2)}\right)\\ &\leq \log\left(\frac{\log(1+\lambda^2)}{\log(\theta^2)+\log(1+\lambda^2)}\right)\\ &\leq \log\left(\frac{\log(1+\theta^{-2})}{\log(\theta^2)+\log(1+\theta^{-2})}\right)\\ &= \log\left(\frac{\log(1+\theta^2)-\log(\theta^2)}{\log(1+\theta^2)}\right).\end{aligned}$$

(2) $\lambda \geq M, \theta < 1$ and $\lambda \leq 1/\theta$. In this case we have

$$\begin{aligned}\left|\log\left(\frac{\log(1+\lambda^2\theta^2)}{\log(1+\lambda^2)}\right)\right| &= \log\left(\frac{\log(1+\lambda^2)}{\log(1+\lambda^2\theta^2)}\right)\\ &\leq \left(\frac{\log(1+\lambda^2)}{\log(1+M^2\theta^2)}\right)\\ &\leq \left(\frac{\log(1+\theta^{-2})}{\log(1+M^2\theta^2)}\right).\end{aligned}$$

Combining the results above one can easily check that there is a nonnegative integrable function f on $(0, 1)$ such that (2.5) is satisfied.

Example 2.6. Suppose that $\alpha \in (0, 2)$ and $\beta \in (0, \alpha)$. Define

$$\phi(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{-\beta/2}.$$

By using the facts that λ and $\log(1 + \lambda)$ are complete Bernstein functions and properties of complete Bernstein functions (see [26]), one can easily check that ϕ is a complete Bernstein function. ϕ can be written as $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ with

$$\ell(\lambda) = (\log(1+\lambda))^{-\beta/2}.$$

Similarly to the example above, one can use elementary analysis to check that there is a nonnegative integrable function f on $(0, 1)$ such that (2.5) is satisfied.

The method of Example 2.5 can be used to construct a whole class of complete Bernstein functions satisfying the assumptions of this paper. For instance, by using arguments similar to the one used in Example 2.5, one can check for $\alpha \in (0, 2)$, $\beta \in (0, 2 - \alpha)$, functions like $\lambda^{\alpha/2}(\log(1+\log(1+\lambda)))^{\beta/2}$, $\lambda^{\alpha/2}(\log(1+\log(1+\log(1+\lambda))))^{\beta/2}, \ldots$ are complete Bernstein functions satisfying the assumptions of this paper. Similar to Example 2.6, for any $\alpha \in (0, 2)$, $\beta \in (0, \alpha)$, functions like $\lambda^{\alpha/2}(\log(1+\log(1+\lambda)))^{-\beta/2}$, $\lambda^{\alpha/2}(\log(1+\log(1+\log(1+\lambda))))^{-\beta/2}, \ldots$ are complete Bernstein functions satisfying the assumptions of this paper.

In the remainder of this section we will always assume that the assumption of Proposition 2.2 is satisfied. It follows from Propositions 2.1 and 2.2 and Corollary 2.3 of [25] that the potential measure V of the ladder height process of X has a decreasing density v. Since X is symmetric, we know that the potential measure $\hat{V}$ of the dual ladder height process is equal to V.

In light of Proposition 2.2, one can easily apply the Tauberian theorem (Theorem 1.7.1 in [3]) and the monotone density theorem (Theorem 1.7.2 in [3]) to get the following result.

Proposition 2.7. *As $x \to 0$, we have*

$$V((0,x)) \sim \frac{x^{\alpha/2}}{\Gamma(1+\alpha/2)(\ell(x^{-2}))^{1/2}},$$
$$v(x) \sim \frac{x^{\alpha/2-1}}{\Gamma(\alpha/2)(\ell(x^{-2}))^{1/2}}.$$

It follows from Proposition 2.2 and Lemma 7.10 of [19] that the process X does not creep upwards. Since X is symmetric, we know that X also does not creep downwards. Thus if, for any $a \in \mathbb{R}$, we define

$$\tau_a = \inf\{t > 0 : X_t < a\}, \qquad \sigma_a = \inf\{t > 0 : X_t \le a\},$$

then we have

$$\mathbb{P}_x(\tau_a = \sigma_a) = 1, \quad x > a. \tag{2.7}$$

Let $G^{(0,\infty)}(x, y)$ be the Green function of $X^{(0,\infty)}$, the process obtained by killing X upon exiting from $(0, \infty)$. Then we have the following result.

Proposition 2.8. *For any $x, y > 0$ we have*

$$G^{(0,\infty)}(x,y) = \begin{cases} \int_0^x v(z)v(y+z-x)\mathrm{d}z, & x \le y, \\ \int_{x-y}^x v(z)v(y+z-x)\mathrm{d}z, & x > y. \end{cases}$$

Proof. By using (2.7) and Theorem 20 on page 176 of [1] we get that for any nonnegative function f on $(0, \infty)$,

$$\mathbb{E}_x\left[\int_0^\infty f(X_t^{(0,\infty)})\mathrm{d}t\right] = k\int_0^\infty \int_0^x v(z)f(x+z-y)v(y)\mathrm{d}z\mathrm{d}y, \tag{2.8}$$

where k is a constant depending on the normalization of the local time of the process X reflected at its supremum. We choose $k = 1$. Then

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^\infty f(X_t^{(0,\infty)})\mathrm{d}t\right] &= \int_0^\infty v(y)\int_0^x v(z)f(x+y-z)\mathrm{d}z\mathrm{d}y \\
&= \int_0^x v(z)\int_0^\infty v(y)f(x+y-z)\mathrm{d}y\mathrm{d}z = \int_0^x v(z)\int_{x-z}^\infty v(w+z-x)f(w)\mathrm{d}w\mathrm{d}z \\
&= \int_0^x f(w)\int_{x-w}^x v(z)v(w+z-x)\mathrm{d}z\mathrm{d}w \\
&\quad + \int_x^\infty f(w)\int_0^x v(z)v(w+z-x)\mathrm{d}z\mathrm{d}w.
\end{aligned} \tag{2.9}$$

On the other hand,

$$\mathbb{E}_x\left[\int_0^\infty f(X_t^{(0,\infty)})\mathrm{d}t\right] = \int_0^\infty G^{(0,\infty)}(x,w)f(w)\mathrm{d}w$$
$$= \int_0^x G^{(0,\infty)}(x,w)f(w)\mathrm{d}w + \int_x^\infty G^{(0,\infty)}(x,w)f(w)\mathrm{d}w. \tag{2.10}$$

By comparing (2.9) and (2.10) we arrive at our desired conclusion. □

For any $r > 0$, let $G^{(0,r)}$ be the Green function of $X^{(0,r)}$, the process obtained by killing X upon exiting from $(0, r)$. Then we have the following result.

Proposition 2.9. *For any $R > 0$, there exists $C = C(R) > 0$ such that*

$$\int_0^r G^{(0,r)}(x,y)\mathrm{d}y \le C\frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}}\frac{x^{\alpha/2}}{(\ell(x^{-2}))^{1/2}}, \quad x\in(0,r),\ r\in(0,R).$$

Proof. For any $x \in (0, r)$, we have

$$\int_0^r G^{(0,r)}(x,y)\mathrm{d}y \le \int_0^r G^{(0,\infty)}(x,y)\mathrm{d}y$$
$$= \int_0^x\int_{x-y}^x v(z)v(y+z-x)\mathrm{d}z\mathrm{d}y + \int_x^r\int_0^x v(z)v(y+z-x)\mathrm{d}z\mathrm{d}y$$
$$= \int_0^x v(z)\int_{x-z}^x v(y+z-x)\mathrm{d}y\mathrm{d}z$$
$$+ \int_0^x v(z)\int_x^r v(y+z-x)\mathrm{d}y\mathrm{d}z \le 2V((0,r))V((0,x)).$$

Now the desired conclusion follows easily from Proposition 2.7 and the continuity of $V((0, x))$ and $x^{\alpha/2}/(\ell(x^{-2}))^{1/2}$. □

As a consequence of the result above, we immediately get the following.

Proposition 2.10. *For any $R > 0$, there exists $C = C(R) > 0$ such that*

$$\int_0^r G^{(0,r)}(x,y)\mathrm{d}y \le C\frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}}\left(\frac{x^{\alpha/2}}{(\ell(x^{-2}))^{1/2}} \wedge \frac{(r-x)^{\alpha/2}}{(\ell((r-x)^{-2}))^{1/2}}\right),$$
$$x\in(0,r),\ r\in(0,R).$$

3. Key estimates on multi-dimensional subordinate Brownian motions

In the remainder of this paper we will always assume that $d \ge 2$ and that $\alpha \in (0, 2)$. From now on we will assume that $B = (B_t : t \ge 0)$ is a Brownian motion on $\mathbb{R}^d$ with

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi\cdot(B_t-B_0)}\right] = \mathrm{e}^{-t|\xi|^2}, \quad \forall\xi\in\mathbb{R}^d, t > 0.$$

Suppose that $S = (S_t : t \ge 0)$ is a subordinator independent of B and that its Laplace exponent ϕ is a complete Bernstein function satisfying all the assumptions of the previous section. More precisely we assume that there is a positive function ℓ on $(0, \infty)$ which is slowly varying at ∞ such that $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ for all $\lambda > 0$ and that there is a nonnegative integrable function f on

$(0, \delta)$ for some $\delta > 0$ such that (2.5) holds. As in the previous section, we will use $u(t)$ and $\mu(t)$ to denote the potential density and the Lévy density of S respectively.

In what follows, we will use $X = (X_t : t \geq 0)$ to denote the subordinate Brownian motion defined by $X_t = B_{S_t}$. Then it is easy to check that when $d \geq 3$ the process X is transient. In the case of $d = 2$, we will always assume the following:

A1. *The potential density u of S satisfies the following assumption*:

$$u(t) \sim ct^{\gamma-1}, \quad t \to \infty \tag{3.1}$$

for some constants $c > 0$ and $\gamma < 1$.

Under this assumption, one can easily see that the integral

$$\int_0^\infty (4\pi t)^{-1} \exp\left(-\frac{r^2}{4t}\right) u(t)\mathrm{d}t$$

is finite for all $r > 0$ and thus the process X is also transient for $d = 2$.

We will use $G(x, y) = G(x - y)$ to denote the Green function of X. The Green function G of X is given by the following formula

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} \mathrm{e}^{-|x|^2/(4t)} u(t)\mathrm{d}t, \quad x \in \mathbb{R}^d.$$

Using this formula, we can easily see that G is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$.

In order to get the asymptotic behavior of G near the origin, we need some additional assumption on the slowly varying function ℓ. For any $y, t, \xi > 0$, define

$$\Lambda_{\ell,\xi}(y, t) := \begin{cases} \dfrac{\ell(1/y)}{\ell(4t/y)}, & y < \dfrac{t}{\xi}, \\ 0, & y \geq \dfrac{t}{\xi}. \end{cases}$$

We will always assume that

A2. *There is a $\xi > 0$ such that*

$$\Lambda_{\ell,\xi}(y, t) \leq g(t), \quad \forall y, t > 0,$$

for some positive function g on $(0, \infty)$ with

$$\int_0^\infty t^{(d-\alpha)/2-1} \mathrm{e}^{-t} g(t)\mathrm{d}t < \infty.$$

It is easy to check (see the proofs of Theorem 3.6 and Theorem 3.11 in [26]) that for the subordinators corresponding to Examples 2.3–2.6, **A1** and **A2** are satisfied.

Under these assumptions we have the following.

Theorem 3.1. *The Green function G of X satisfies the following*

$$G(x) \sim \frac{\alpha\Gamma((d-\alpha)/2)}{2^{\alpha+1}\pi^{d/2}\Gamma(1+\alpha/2)} \frac{1}{|x|^{d-\alpha}\ell(|x|^{-2})}, \quad |x| \to 0.$$

Proof. This follows easily from **A1**, **A2**, (2.2) and Lemma 3.3 of [26]. We omit the details. □

Let J be the jumping function of X, then

$$J(x) = \int_0^\infty (4\pi t)^{-d/2} \mathrm{e}^{-|x|^2/(4t)} \mu(t) \mathrm{d}t, \quad x \in \mathbb{R}^d.$$

Thus $J(x) = j(|x|)$ with

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} \mathrm{e}^{-r^2/(4t)} \mu(t) \mathrm{d}t, \quad r > 0.$$

It is easy to see that j is continuous and decreasing on $(0, \infty)$. In order to get the asymptotic behavior of j near the origin, we need some additional assumption on the slowly varying function ℓ. For any $y, t, \xi > 0$, define

$$\Upsilon_{\ell,\xi}(y,t) := \begin{cases} \dfrac{\ell(4t/y)}{\ell(1/y)}, & y < \dfrac{t}{\xi}, \\ 0, & y \geq \dfrac{t}{\xi}. \end{cases}$$

We will always assume that

A3. *There is a* $\xi > 0$ *such that*

$$\Upsilon_{\ell,\xi}(y,t) \leq h(t), \quad \forall y, t > 0$$

for some positive function h *on* $(0, \infty)$ *with*

$$\int_0^\infty t^{(\mathrm{d}+\alpha)/2-1} \mathrm{e}^{-t} h(t) \mathrm{d}t < \infty.$$

It is easy to check (see the proofs of Theorem 3.6 and Theorem 3.11 in [26]) that for the subordinators corresponding to Examples 2.3–2.6, **A3** is satisfied.

Theorem 3.2. *The function* j *satisfies the following*

$$j(r) \sim \frac{\alpha \Gamma((d+\alpha)/2)}{2^{1-\alpha} \pi^{d/2} \Gamma(1-\alpha/2)} \frac{\ell(r^{-2})}{r^{d+\alpha}}, \quad r \to 0.$$

Proof. This follows easily from **A1**, **A3**, (2.3) and Lemma 3.3 of [26]. We omit the details. □

For any open set D, we use τ_D to denote the first exit time from D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Given an open set $D \subset \mathbb{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a cemetery state. We now recall the definition of harmonic functions with respect to X.

Definition 3.3. Let D be an open subset of $\mathbb{R}^d$. A function u defined on $\mathbb{R}^d$ is said to be

(1) harmonic in D with respect to X if

$$\mathbb{E}_x \left[|u(X_{\tau_B})| \right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x \left[u(X_{\tau_B}) \right], \quad x \in B,$$

for every open set B whose closure is a compact subset of D;

(2) regular harmonic in D with respect to X if it is harmonic in D with respect to X and for each $x \in D$,

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) \right];$$

(3) harmonic for X^D if it is harmonic for X in D and vanishes outside D.

In order for a scale invariant Harnack inequality to hold, we need to assume some additional conditions on the Lévy density μ of S. We will always assume that

A4. *The Lévy density μ of S satisfies the following conditions: there exists $C_1 > 0$ such that*

$$\mu(t) \le C_1 \mu(t+1), \quad \forall t > 1.$$

It follows from (2.3) that for any $M > 0$ there exists $C_2 > 0$ such that

$$\mu(t) \le C_2 \mu(2t), \quad \forall t \in (0, M).$$

Using **A4** and repeating the proof of Lemma 4.2 of [21] we get that

(1) For any $M > 0$, there exists $C_3 > 0$ such that

$$j(r) \le C_3 j(2r), \quad \forall r \in (0, M). \tag{3.2}$$

(2) There exists $C_4 > 0$ such that

$$j(r) \le C_4 j(r+1), \quad \forall r > 1. \tag{3.3}$$

It is easy to check (see [26]) that for the subordinators corresponding to Examples 2.3–2.6, **A4** is satisfied. Therefore by Theorem 4.14 of [26] (see also [21]) we have the following Harnack inequality.

Theorem 3.4 (*Harnack Inequality*). *There exist $r_1 \in (0, 1)$ and $C > 0$ such that for every $r \in (0, r_1)$, every $x_0 \in \mathbb{R}^d$, and every nonnegative function u on $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ with respect to X, we have*

$$\sup_{y \in B(x_0, r/2)} u(y) \le C \inf_{y \in B(x_0, r/2)} u(y).$$

For any bounded open set D in $\mathbb{R}^d$, we will use $G_D(x, y)$ to denote the Green function of X^D. Using the continuity and the radial decreasing property of G, we can easily check that G_D is continuous in $(D \times D) \setminus \{(x, x) : x \in D\}$.

Proposition 3.5. *For any $R > 0$, there exists $C = C(R) > 0$ such that for every open subset D with diam $(D) \le R$,*

$$G_D(x, y) \le G(x, y) \le C \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}}, \quad \forall (x, y) \in D \times D. \tag{3.4}$$

Proof. The results of this proposition are immediate consequences of Theorem 3.1 and the continuity and positivity of $\ell(r^{-2})r^{d-\alpha}$ on $(0, \infty)$. $\square$

The idea of the proof of the next lemma comes from [29].

Lemma 3.6. *For any $R > 0$, there exists $C = C(R) > 0$ such that for every $r \in (0, R)$ and $x_0 \in \mathbb{R}^d$,*

$$\mathbb{E}_x[\tau_{B(x_0,r)}] \le C \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \frac{(r - |x - x_0|)^{\alpha/2}}{(\ell((r - |x - x_0|)^{-2}))^{1/2}}, \quad x \in B(x_0, r).$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. For $x \neq 0$, put $Z_t = \frac{X_t \cdot x}{|x|}$. Then Z_t is a Lévy process on $\mathbb{R}$ with

$$\mathbb{E}(e^{i\theta Z_t}) = \mathbb{E}(e^{i\theta \frac{x}{|x|} \cdot X_t}) = e^{-t|\theta|^\alpha \ell(\theta^2)}, \quad \theta \in \mathbb{R}.$$

Thus Z_t is of the type of one-dimensional subordinate Brownian motion we studied in the previous section. It is easy to see that, if $X_t \in B(0,r)$, then $|Z_t| < r$, hence

$$\mathbb{E}_x[\tau_{B(0,r)}] \le \mathbb{E}_{|x|}[\tilde{\tau}],$$

where $\tilde{\tau} = \inf\{t > 0 : |Z_t| \ge r\}$. Now the desired conclusion follows easily from Proposition 2.10. □

Lemma 3.7. *There exist $r_2 \in (0, r_1]$ and $C > 0$ such that for every positive $r \le r_2$ and $x_0 \in \mathbb{R}^d$,*

$$\mathbb{E}_{x_0}[\tau_{B(x_0,r)}] \ge C\frac{r^\alpha}{\ell(r^{-2})}.$$

Proof. The conclusion of this lemma follows easily from Theorem 3.2 and Lemma 3.2 of [24]. □

Using the Lévy system for X, we know that for every bounded open subset D and every $f \ge 0$ and $x \in D$,

$$\mathbb{E}_x\left[f(X_{\tau_D}); X_{\tau_D-} \neq X_{\tau_D}\right] = \int_{\overline{D}^c}\int_D G_D(x,z)J(z-y)\mathrm{d}z f(y)\mathrm{d}y. \tag{3.5}$$

(See, for example, Appendix A.3 of [11].) For notational convenience, we define

$$K_D(x,y) := \int_D G_D(x,z)J(z-y)\mathrm{d}z, \quad (x,y) \in D \times \overline{D}^c. \tag{3.6}$$

Thus (3.5) can be simply written as

$$\mathbb{E}_x\left[f(X_{\tau_D}); X_{\tau_D-} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} K_D(x,y)f(y)\mathrm{d}y.$$

Using the continuity of G_D and J, one can easily check that K_D is continuous on $D \times \overline{D}^c$.

As a consequence of Lemmas 3.6, 3.7 and (3.6), we get the following proposition.

Proposition 3.8. *There exist $C_5, C_6 > 0$ such that for every $r \in (0, r_2)$ and $x_0 \in \mathbb{R}^d$,*

$$K_{B(x_0,r)}(x,y) \le C_5\, j(|y-x_0|-r)\frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}}\frac{(r-|x-x_0|)^{\alpha/2}}{(\ell((r-|x-x_0|)^{-2}))^{1/2}} \tag{3.7}$$

for all $(x,y) \in B(x_0,r) \times \overline{B(x_0,r)}^c$ and

$$K_{B(x_0,r)}(x_0,y) \ge C_6 J(y-x_0)\frac{r^\alpha}{\ell(r^{-2})}, \quad \forall y \in \overline{B(x_0,r)}^c. \tag{3.8}$$

Proof. Without loss of generality, we assume $x_0 = 0$. For $z \in B(0,r)$ and $r < |y| < 2$,

$$|y| - r \le |y| - |z| \le |z-y| \le |z| + |y| \le r + |y| \le 2|y|,$$

and for $z \in B(0,r)$ and $y \in B(0,2)^c$,

$$|y| - r \le |y| - |z| \le |z-y| \le |z| + |y| \le r + |y| \le |y| + 1.$$

Thus by the monotonicity of J, (3.2) and (3.3), there exists a constant $c > 0$ such that

$$cJ(y) \le J(z-y) \le j(|y|-r), \quad (z,y) \in B(0,r) \times \overline{B(0,r)}^c.$$

Applying the above inequality and Lemmas 3.6 and 3.7 to (3.6), we have proved the proposition. □

Proposition 3.9. *For every $a \in (0,1)$, there exists $C = C(a) > 0$ such that for every $r \in (0, r_2)$, $x_0 \in \mathbb{R}^d$ and $x_1, x_2 \in B(x_0, ar)$,*

$$K_{B(x_0,r)}(x_1, y) \le C K_{B(x_0,r)}(x_2, y), \quad y \in \overline{B(x_0, r)}^c.$$

Proof. This follows easily from the Harnack inequality (Theorem 3.4) and the continuity of $K_{B(x_0,r)}$. For details, see the proof of Lemma 4.2 in [29]. □

As an immediate consequence of Theorem 3.2, we have

Lemma 3.10. *There exists $r_3 \in (0, r_2]$ such that for every $y \in \mathbb{R}^d$ with $|y| \le r_3$,*

$$\frac{\alpha\Gamma((d+\alpha)/2)}{2^{2-\alpha}\pi^{d/2}\Gamma(1-\alpha/2)} \frac{\ell(|y|^{-2})}{|y|^{d+\alpha}} \le J(y) \le \frac{2^\alpha \alpha\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)} \frac{\ell(|y|^{-2})}{|y|^{d+\alpha}}.$$

The inequalities below will be used several times in the remainder of this paper.

Lemma 3.11. *There exist $r_4 \in (0, r_3]$ and $C > 0$ such that*

$$\frac{s^{\alpha/2}}{\left(\ell(s^{-2})\right)^{1/2}} \le C \frac{r^{\alpha/2}}{\left(\ell(r^{-2})\right)^{1/2}}, \quad \forall 0 < s < r \le 4r_4, \tag{3.9}$$

$$\frac{s^{1-\alpha/2}}{\left(\ell(s^{-2})\right)^{1/2}} \le C \frac{r^{1-\alpha/2}}{\left(\ell(r^{-2})\right)^{1/2}}, \quad \forall 0 < s < r \le 4r_4, \tag{3.10}$$

$$s^{1-\alpha/2}\left(\ell(s^{-2})\right)^{1/2} \le C r^{1-\alpha/2}\left(\ell(r^{-2})\right)^{1/2}, \quad \forall 0 < s < r \le 4r_4, \tag{3.11}$$

$$\int_r^\infty \frac{\left(\ell(s^{-2})\right)^{1/2}}{s^{1+\alpha/2}} \mathrm{d}s \le C \frac{\left(\ell(r^{-2})\right)^{1/2}}{r^{\alpha/2}}, \quad \forall 0 < r \le 4r_4, \tag{3.12}$$

$$\int_0^r \frac{\left(\ell(s^{-2})\right)^{1/2}}{s^{\alpha/2}} \mathrm{d}s \le C \frac{\left(\ell(r^{-2})\right)^{1/2}}{r^{\alpha/2-1}}, \quad \forall 0 < r \le 4r_4, \tag{3.13}$$

$$\int_r^\infty \frac{\ell(s^{-2})}{s^{1+\alpha}} \mathrm{d}s \le C \frac{\ell(r^{-2})}{r^\alpha}, \quad \forall 0 < r \le 4r_4, \tag{3.14}$$

$$\int_0^r \frac{\ell(s^{-2})}{s^{\alpha-1}} \mathrm{d}s \le C \frac{\ell(r^{-2})}{r^{\alpha-2}}, \quad \forall 0 < r \le 4r_4 \tag{3.15}$$

and

$$\int_0^r \frac{s^{\alpha-1}}{\ell(s^{-2})} \mathrm{d}s \le C \frac{r^\alpha}{\ell(r^{-2})}, \quad \forall 0 < r \le 4r_4. \tag{3.16}$$

Proof. The first three inequalities follow easily from Theorem 1.5.3 of [3], while the last five from the 0-version of Theorem 1.5.11 of [3]. □

Proposition 3.12. *For every $a \in (0,1)$, there exists $C = C(a) > 0$ such that for every $r \in (0, r_4]$ and $x_0 \in \mathbb{R}^d$,*

$$K_{B(x_0,r)}(x,y) \le C \frac{r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \frac{(\ell((|y-x_0|-r)^{-2}))^{1/2}}{(|y-x_0|-r)^{\alpha/2}}, \quad \forall x \in B(x_0, ar),\ y \in \{r < |x_0 - y| \le 2r\}.$$

Proof. By Proposition 3.9

$$K_{B(x_0,r)}(x,y) \le \frac{c_1}{r^d} \int_{B(x_0,ar)} K_{B(x_0,r)}(w,y) \mathrm{d}w$$

for some constant $c_1 = c_1(a) > 0$. Thus from Lemma 3.6 and (3.7), we have that

$$\begin{aligned} K_{B(x_0,r)}(x,y) &\le \frac{c_2}{r^d} \int_{B(x_0,r)} \int_{B(x_0,r)} G_{B(x_0,r)}(w,z) J(z-y) \mathrm{d}z \mathrm{d}w \\ &= \frac{c_2}{r^d} \int_{B(x_0,r)} \mathbb{E}_z[\tau_{B(x_0,r)}] J(z-y) \mathrm{d}z \\ &\le \frac{c_3}{r^d} \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \int_{B(x_0,r)} \frac{(r-|z-x_0|)^{\alpha/2}}{(\ell((r-|z-x_0|)^{-2}))^{1/2}} J(z-y) \mathrm{d}z \end{aligned}$$

for some constants $c_2 = c_2(a) > 0$ and $c_3 = c_3(a) > 0$. Now applying Lemma 3.10, we get

$$K_{B(x_0,r)}(x,y) \le \frac{c_4 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{B(x_0,r)} \frac{(r-|z-x_0|)^{\alpha/2}}{(\ell((r-|z-x_0|)^{-2}))^{1/2}} \frac{\ell(|z-y|^{-2})}{|z-y|^{d+\alpha}} \mathrm{d}z$$

for some constant $c_4 = c_4(a) > 0$. Since $r - |z-x_0| \le |y-z| \le 3r \le 3r_4$, from (3.9) we see that

$$\frac{(r-|z-x_0|)^{\alpha/2}}{(\ell((r-|z-x_0|)^{-2}))^{1/2}} \le c_5 \frac{(|y-z|)^{\alpha/2}}{(\ell(|y-z|^{-2}))^{1/2}}$$

for some constant $c_5 > 0$. Thus we have

$$\begin{aligned} K_{B(x_0,r)}(x,y) &\le \frac{c_6 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{B(x_0,r)} \frac{(\ell(|z-y|^{-2}))^{1/2}}{|z-y|^{d+\alpha/2}} \mathrm{d}z \\ &\le \frac{c_6 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{B(y,|y-x_0|-r)^c} \frac{(\ell(|z-y|^{-2}))^{1/2}}{|z-y|^{d+\alpha/2}} \mathrm{d}z \\ &\le \frac{c_7 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{|y-x_0|-r}^{\infty} \frac{\left(\ell(s^{-2})\right)^{1/2}}{s^{1+\alpha/2}} \mathrm{d}s \end{aligned}$$

for some constants $c_6 = c_6(a) > 0$ and $c_7 = c_7(a) > 0$. Using (3.12) in the above equation, we conclude that

$$K_{B(x_0,r)}(x,y) \le \frac{c_8 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \frac{(\ell((|y-x_0|-r)^{-2}))^{1/2}}{(|y-x_0|-r)^{\alpha/2}}$$

for some constant $c_8 = c_8(a) > 0$. $\square$

4. Boundary Harnack principle

In this section, we give the proof of the boundary Harnack principle for X.

Using an argument similar to the first part of the proof of Lemma 3.3 in [27] and using Lemma 3.10 and (3.14)–(3.15) we can easily get the following lemma. We skip the details.

Lemma 4.1. *There exists $C > 0$ such that for any $r \in (0, r_4)$ and any open set D with $D \subset B(0, r)$ we have*

$$\mathbb{P}_x \left(X_{\tau_D} \in B(0, r)^c \right) \leq C r^{-\alpha} \ell(r^{-2}) \int_D G_D(x, y) \mathrm{d}y, \quad x \in D \cap B(0, r/2).$$

Lemma 4.2. *There exists $C > 0$ such that for any open set D with $B(A, \kappa r) \subset D \subset B(0, r)$ for some $r \in (0, r_4)$ and $\kappa \in (0, 1)$, we have that for every $x \in D \setminus B(A, \kappa r)$,*

$$\int_D G_D(x, y) \mathrm{d}y \leq C r^{\alpha} \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})} \right) \mathbb{P}_x \left(X_{\tau_{D \setminus B(A, \kappa r)}} \in B(A, \kappa r) \right).$$

Proof. Fix a point $x \in D \setminus B(A, \kappa r)$ and let $B := B(A, \frac{\kappa r}{2})$. Note that, by the harmonicity of $G_D(x, \cdot)$ in $D \setminus \{x\}$ with respect to X, we have

$$G_D(x, A) \geq \int_{D \cap \overline{B}^c} K_B(A, y) G_D(x, y) \mathrm{d}y \geq \int_{D \cap B(A, \frac{3\kappa r}{4})^c} K_B(A, y) G_D(x, y) \mathrm{d}y.$$

Since $\frac{3\kappa r}{4} \leq |y - A| \leq 2r$ for $y \in B(A, \frac{3\kappa r}{4})^c \cap D$ and j is a decreasing function, it follows from (3.8) in Proposition 3.8 and Lemma 3.10 that

$$\begin{aligned} G_D(x, A) &\geq c_1 \frac{(\frac{\kappa r}{2})^{\alpha}}{\ell\left((\frac{\kappa r}{2})^{-2}\right)} \int_{D \cap B(A, \frac{3\kappa r}{4})^c} G_D(x, y) J(y - A) \mathrm{d}y \\ &\geq c_1 j(2r) \frac{(\frac{\kappa r}{2})^{\alpha}}{\ell\left((\frac{\kappa r}{2})^{-2}\right)} \int_{D \cap B(A, \frac{3\kappa r}{4})^c} G_D(x, y) \mathrm{d}y \\ &\geq c_2 \kappa^{\alpha} r^{-d} \frac{\ell((2r)^{-2})}{\ell((\frac{\kappa r}{2})^{-2})} \int_{D \cap B(A, \frac{3\kappa r}{4})^c} G_D(x, y) \mathrm{d}y \end{aligned}$$

for some positive constants c_1 and c_2. On the other hand, applying Theorem 3.4 we get

$$\int_{B(A, \frac{3\kappa r}{4})} G_D(x, y) \mathrm{d}y \leq c_3 \int_{B(A, \frac{3\kappa r}{4})} G_D(x, A) \mathrm{d}y \leq c_4 r^d \kappa^d G_D(x, A)$$

for some positive constants c_3 and c_4. Combining these two estimates we get that

$$\int_D G_D(x, y) \mathrm{d}y \leq c_5 \left(r^d \kappa^d + r^d \kappa^{-\alpha} \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})} \right) G_D(x, A) \tag{4.1}$$

for some constant $c_5 > 0$.

Let $\Omega = D \setminus B(A, \frac{\kappa r}{2})$. Note that for any $z \in B(A, \frac{\kappa r}{4})$ and $y \in \Omega$, $2^{-1}|y - z| \leq |y - A| \leq 2|y - z|$. Thus we get from (3.6) that for $z \in B(A, \frac{\kappa r}{4})$,

$$c_6^{-1} K_{\Omega}(x, A) \leq K_{\Omega}(x, z) \leq c_6 K_{\Omega}(x, A) \tag{4.2}$$

for some $c_6 > 1$. Using the harmonicity of $G_D(\cdot, A)$ in $D \setminus \{A\}$ with respect to X, we can split $G_D(\cdot, A)$ into two parts:

$$\begin{aligned} G_D(x, A) =& \ \mathbb{E}_x\left[G_D(X_{\tau_\Omega}, A)\right] \\ =& \ \mathbb{E}_x\left[G_D(X_{\tau_\Omega}, A) :\ X_{\tau_\Omega} \in B\left(A, \frac{\kappa r}{4}\right)\right] \\ &+ \mathbb{E}_x\left[G_D(X_{\tau_\Omega}, A) :\ X_{\tau_\Omega} \in \left\{\frac{\kappa r}{4} \le |y - A| \le \frac{\kappa r}{2}\right\}\right] \\ :=& \ I_1 + I_2. \end{aligned}$$

Using (3.4) and (4.2), we have

$$\begin{aligned} I_1 &\le c_6\, K_\Omega(x, A) \int_{B(A, \frac{\kappa r}{4})} G_D(y, A)\mathrm{d}y \\ &\le c_7\, K_\Omega(x, A) \int_{B(A, \frac{\kappa r}{4})} \frac{1}{|y - A|^{d-\alpha}} \frac{\mathrm{d}y}{\ell(|y - A|^{-2})} \end{aligned}$$

for some constant $c_7 > 0$. Since $|y - A| \le 4r \le 4r_4$, by (3.9),

$$\frac{|y - A|^{\alpha/2}}{\ell(|y - A|^{-2})} \le c_8 \frac{(4r)^{\alpha/2}}{\ell((4r)^{-2})} \tag{4.3}$$

for some constant $c_8 > 0$. Thus

$$I_1 \le c_7 c_8 K_\Omega(x, A) \int_{B(A, \frac{\kappa r}{4})} \frac{1}{|y - A|^{d-\alpha/2}} \frac{(4r)^{\alpha/2}}{\ell((4r)^{-2})}\mathrm{d}y \le c_9 \kappa^{\alpha/2} r^\alpha \frac{1}{\ell((4r)^{-2})} K_\Omega(x, A)$$

for some constant $c_9 > 0$. Now using (4.2) again, we get

$$I_1 \le c_{10} \kappa^{\alpha/2-d} r^{\alpha-d} \frac{1}{\ell((4r)^{-2})} \int_{B(A, \frac{\kappa r}{4})} K_\Omega(x, z)\mathrm{d}z$$

for some constant $c_{10} > 0$. On the other hand, by (3.4),

$$\begin{aligned} I_2 &= \int_{\{\frac{\kappa r}{4} \le |y-A| \le \frac{\kappa r}{2}\}} G_D(y, A)\mathbb{P}_x(X_{\tau_\Omega} \in \mathrm{d}y) \\ &\le c_{11} \int_{\{\frac{\kappa r}{4} \le |y-A| \le \frac{\kappa r}{2}\}} \frac{1}{|y - A|^{d-\alpha}} \frac{1}{\ell(|y - A|^{-2})} \mathbb{P}_x(X_{\tau_\Omega} \in \mathrm{d}y) \end{aligned}$$

for some constant $c_{11} > 0$. Using (4.3), the above is less than or equal to

$$c_{12} \kappa^{\alpha/2-d}\, r^{\alpha-d} \frac{1}{\ell((4r)^{-2})} \mathbb{P}_x\left(X_{\tau_\Omega} \in \left\{\frac{\kappa r}{4} \le |y - A| \le \frac{\kappa r}{2}\right\}\right)$$

for some constant $c_{12} > 0$. Therefore

$$G_D(x, A) \le c_{13}\, \kappa^{\alpha/2-d}\, r^{\alpha-d} \frac{1}{\ell((4r)^{-2})}\, \mathbb{P}_x\left(X_{\tau_\Omega} \in B\left(A, \frac{\kappa r}{2}\right)\right)$$

for some constant $c_{13} > 0$. Combining the above with (4.1), we get

$$\begin{aligned} \int_D G_D(x, y)\mathrm{d}y \le&\ c_{14}\, r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})}\right) \\ &\times \mathbb{P}_x\left(X_{\tau_{D \setminus B(A, \frac{\kappa r}{2})}} \in B\left(A, \frac{\kappa r}{2}\right)\right) \end{aligned}$$

for some constant $c_{14} > 0$. It follows immediately that

$$\int_D G_D(x,y)\mathrm{d}y \le c_{14} r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})}\left(1+\frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})}\right) \times \mathbb{P}_x\left(X_{\tau_{D\setminus B(A,\kappa r)}} \in B(A,\kappa r)\right). \quad \square$$

Combining Lemmas 4.1 and 4.2 and using the translation invariant property, we have the following:

Lemma 4.3. *There exists $c_1 > 0$ such that for any open set D with $B(A,\kappa r) \subset D \subset B(Q,r)$ for some $r \in (0, r_4)$ and $\kappa \in (0,1)$, we have that for every $x \in D \cap B(Q, \frac{r}{2})$,*

$$\mathbb{P}_x\left(X_{\tau_D} \in B(Q,r)^c\right) \le c_1 \kappa^{-d-\alpha/2} \frac{\ell(r^{-2})}{\ell((4r)^{-2})}\left(1+\frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})}\right) \times \mathbb{P}_x\left(X_{\tau_{D\setminus B(A,\kappa r)}} \in B(A,\kappa r)\right).$$

Let $A(x,a,b) := \{y \in \mathbb{R}^d : a \le |y-x| < b\}$.

Lemma 4.4. *Let D be an open set and $0 < 2r < r_4$. For every $Q \in \mathbb{R}^d$ and any positive function u vanishing on $D^c \cap B(Q, \frac{11}{6}r)$, there is a $\sigma \in (\frac{10}{6}r, \frac{11}{6}r)$ such that for any $x \in D \cap B(Q, \frac{3}{2}r)$,*

$$\mathbb{E}_x\left[u(X_{\tau_{D\cap B(Q,\sigma)}}); X_{\tau_{D\cap B(Q,\sigma)}} \in B(Q,\sigma)^c\right] \le C \frac{r^\alpha}{\ell((2r)^{-2})} \int_{B(Q,\frac{10r}{6})^c} J(y-Q)u(y)\mathrm{d}y$$

for some constant $C > 0$ independent of Q and u.

Proof. Without loss of generality, we may assume that $Q = 0$. Note that by (3.13)

$$\begin{aligned}
&\int_{\frac{10}{6}r}^{\frac{11}{6}r} \int_{A(0,\sigma,2r)} \ell((|y|-\sigma)^{-2})^{1/2}(|y|-\sigma)^{-\alpha/2} u(y)\mathrm{d}y\mathrm{d}\sigma \\
&\quad = \int_{A(0,\frac{10}{6}r,2r)} \int_{\frac{10}{6}r}^{|y|\wedge\frac{11}{6}r} \ell((|y|-\sigma)^{-2})^{1/2}(|y|-\sigma)^{-\alpha/2}\mathrm{d}\sigma\, u(y)\mathrm{d}y \\
&\quad \le c_1 \int_{A(0,\frac{10}{6}r,2r)} \left(\int_0^{|y|-\frac{10}{6}r} \ell(s^{-2})^{1/2} s^{-\alpha/2}\mathrm{d}s\right) u(y)\mathrm{d}y \\
&\quad \le c_2 \int_{A(0,\frac{10r}{6},2r)} \ell\left(\left(|y|-\frac{10r}{6}\right)^{-2}\right)^{1/2} \left(|y|-\frac{10r}{6}\right)^{1-\alpha/2} u(y)\mathrm{d}y
\end{aligned}$$

for some positive constants c_1 and c_2. Using (3.11), we get that there is a constant $c_3 > 0$ such that

$$\begin{aligned}
&\int_{A(0,\frac{10r}{6},2r)} \ell\left(\left(|y|-\frac{10r}{6}\right)^{-2}\right)^{1/2} \left(|y|-\frac{10r}{6}\right)^{1-\alpha/2} u(y)\mathrm{d}y \\
&\quad \le c_3 \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2})^{1/2}|y|^{1-\alpha/2}u(y)\mathrm{d}y,
\end{aligned}$$

which is less than or equal to

$$c_4 \frac{r^{1-\alpha/2}}{\ell((2r)^{-2})^{1/2}} \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2}) u(y) \mathrm{d}y$$

for some constant $c_4 > 0$ by (3.10). Thus, by taking $c_5 > 6c_2c_4$, we can conclude that there is a $\sigma \in (\frac{10}{6}r, \frac{11}{6}r)$ such that

$$\int_{A(0,\sigma,2r)} \ell((|y|-\sigma)^{-2})^{1/2} (|y|-\sigma)^{-\alpha/2} u(y) \mathrm{d}y \leq c_5 \frac{r^{-\alpha/2}}{\ell((2r)^{-2})^{1/2}} \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2}) u(y) \mathrm{d}y. \tag{4.4}$$

Let $x \in D \cap B(0, \frac{3}{2}r)$. Note that, since X satisfies the hypothesis **H** in [28], by Theorem 1 in [28] we have

$$\begin{aligned}
&\mathbb{E}_x \left[u(X_{\tau_{D\cap B(0,\sigma)}}); X_{\tau_{D\cap B(0,\sigma)}} \in B(0,\sigma)^c \right] \\
&\quad = \mathbb{E}_x \left[u(X_{\tau_{D\cap B(0,\sigma)}}); X_{\tau_{D\cap B(0,\sigma)}} \in B(0,\sigma)^c,\ \tau_{D\cap B(0,\sigma)} = \tau_{B(0,\sigma)} \right] \\
&\quad = \mathbb{E}_x \left[u(X_{\tau_{B(0,\sigma)}}); X_{\tau_{B(0,\sigma)}} \in B(0,\sigma)^c,\ \tau_{D\cap B(0,\sigma)} = \tau_{B(0,\sigma)} \right] \\
&\quad \leq \mathbb{E}_x \left[u(X_{\tau_{B(0,\sigma)}}); X_{\tau_{B(0,\sigma)}} \in B(0,\sigma)^c \right] = \int_{B(0,\sigma)^c} K_{B(0,\sigma)}(x,y) u(y) \mathrm{d}y.
\end{aligned}$$

In the first equality above we have used the fact that u vanishes on $D^c \cap B(0,\sigma)$. Since $\sigma < 2r < r_4$, from (3.7) in Propositions 3.8, 3.12 and Lemma 3.10 we have

$$\begin{aligned}
&\mathbb{E}_x \left[u(X_{\tau_{D\cap B(0,\sigma)}}); X_{\tau_{D\cap B(0,\sigma)}} \in B(0,\sigma)^c \right] \leq \int_{B(0,\sigma)^c} K_{B(0,\sigma)}(x,y) u(y) \mathrm{d}y \\
&\quad \leq c_6 \int_{A(0,\sigma,2r)} \frac{\sigma^{\alpha/2-d}}{\left(\ell(\sigma^{-2})\right)^{1/2}} \frac{(\ell((|y|-\sigma)^{-2}))^{1/2}}{(|y|-\sigma)^{\alpha/2}} u(y) \mathrm{d}y \\
&\qquad + c_6 \int_{B(0,2r)^c} j(|y|-\sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma-|x|)^{\alpha/2}}{(\ell((\sigma-|x|)^{-2}))^{1/2}} u(y) \mathrm{d}y
\end{aligned}$$

for some constant $c_6 > 0$. When $y \in A(0, 2r, 4)$ we have $\frac{1}{12}|y| \leq |y| - \sigma$, while when $|y| \geq 4$ we have $|y| - \sigma \geq |y| - 1$. Since $\sigma - |x| \leq \sigma \leq 2r$, we have by (3.9) and the monotonicity of j,

$$j(|y|-\sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma-|x|)^{\alpha/2}}{(\ell((\sigma-|x|)^{-2}))^{1/2}} \leq c_7 j\left(\frac{|y|}{12}\right) \frac{r^\alpha}{\ell((2r)^{-2})}, \quad y \in A(0,2r,4)$$

and

$$j(|y|-\sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma-|x|)^{\alpha/2}}{(\ell((\sigma-|x|)^{-2}))^{1/2}} \leq c_7 j(|y|-1) \frac{r^\alpha}{\ell((2r)^{-2})}, \quad |y| \geq 4$$

for some constant $c_7 > 0$. Thus by applying (3.2) and (3.3), we get

$$j(|y|-\sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma-|x|)^{\alpha/2}}{(\ell((\sigma-|x|)^{-2}))^{1/2}} \leq c_8 j(|y|) \frac{r^\alpha}{\ell((2r)^{-2})}$$

for some constant $c_8 > 0$. On the other hand, by (3.9) and (4.4), there exist positive constants c_9 and c_{10} such that

$$\begin{aligned}
&\int_{A(0,\sigma,2r)} \frac{\sigma^{\alpha/2-d}}{\left(\ell(\sigma^{-2})\right)^{1/2}} \frac{(\ell((|y|-\sigma)^{-2}))^{1/2}}{(|y|-\sigma)^{\alpha/2}} u(y)\mathrm{d}y \\
&\quad \le \left(\frac{10r}{6}\right)^{-d} \frac{\sigma^{\alpha/2}}{\left(\ell(\sigma^{-2})\right)^{1/2}} \int_{A(0,\sigma,2r)} \frac{(\ell((|y|-\sigma)^{-2}))^{1/2}}{(|y|-\sigma)^{\alpha/2}} u(y)\mathrm{d}y \\
&\quad \le c_9 r^{-d} \frac{(2r)^{\alpha/2}}{\left(\ell((2r)^{-2})\right)^{1/2}} \frac{r^{-\alpha/2}}{\left(\ell((2r)^{-2})\right)^{1/2}} \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2}) u(y)\mathrm{d}y \\
&\quad \le c_{10} \frac{r^{\alpha}}{\ell((2r)^{-2})} \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2})|y|^{-d-\alpha} u(y)\mathrm{d}y,
\end{aligned}$$

which is less than or equal to

$$c_{11} \frac{r^{\alpha}}{\ell((2r)^{-2})} \int_{A(0,\frac{10r}{6},2r)} J(y)u(y)\mathrm{d}y,$$

for some constants $c_{11} > 0$ by Lemma 3.10. Hence

$$\begin{aligned}
&\mathbb{E}_x\left[u(X_{\tau_{D\cap B(0,\sigma)}}); X_{\tau_{D\cap B(0,\sigma)}} \in B(0,\sigma)^c\right] \\
&\quad \le c_{12} \frac{r^{\alpha}}{\ell((2r)^{-2})} \int_{B(0,\frac{10r}{6})^c} J(y)u(y)\mathrm{d}y
\end{aligned}$$

for some constant $c_{12} > 0$. □

Lemma 4.5. *Let D be an open set. Assume that $B(A,\kappa r) \subset D \cap B(Q,r)$ for some $0 < r < 2r_4$ and $\kappa \in (0, \frac{1}{2}]$. Suppose that $u \ge 0$ is regular harmonic in $D \cap B(Q,2r)$ with respect to X and $u = 0$ in $D^c \cap B(Q,2r)$. If w is a regular harmonic function with respect to X in $D \cap B(Q,r)$ such that*

$$w(x) = \begin{cases} u(x), & x \in B\left(Q, \frac{3r}{2}\right)^c \cup (D^c \cap B(Q,r)), \\ 0, & x \in A\left(Q, r, \frac{3r}{2}\right), \end{cases}$$

then

$$u(A) \ge w(A) \ge C\kappa^{\alpha} \frac{\ell((2r)^{-2})}{\ell((\kappa r)^{-2})} u(x), \quad \forall x \in D \cap B\left(Q, \frac{3}{2}r\right)$$

for some constant $C > 0$.

Proof. Without loss of generality, we may assume $Q = 0$ and $x \in D \cap B(0, \frac{3}{2}r)$. The left-hand side inequality in the conclusion of the lemma is obvious, so we only need to prove the right-hand side inequality. Since u is regular harmonic in $D \cap B(0,2r)$ with respect to X, we know from Lemma 4.4 that there exists $\sigma \in (\frac{10r}{6}, \frac{11r}{6})$ such that

$$\begin{aligned}
u(x) &= \mathbb{E}_x\left[u(X_{\tau_{D\cap B(0,\sigma)}}); X_{\tau_{D\cap B(0,\sigma)}} \in B(0,\sigma)^c\right] \\
&\le c_1 \frac{r^{\alpha}}{\ell((2r)^{-2})} \int_{B(0,\frac{10r}{6})^c} J(y)u(y)\mathrm{d}y
\end{aligned}$$

for some constant $c_1 > 0$. On the other hand, by (3.8) in Proposition 3.8, we have that

$$\begin{aligned} w(A) &= \int_{B(0,\frac{3r}{2})^c} K_{D\cap B(0,r)}(A,y)u(y)\mathrm{d}y \\ &\geq \int_{B(0,\frac{3r}{2})^c} K_{B(A,\kappa r)}(A,y)u(y)\mathrm{d}y \\ &\geq c_2 \int_{B(0,\frac{3r}{2})^c} J(A-y)\frac{(\kappa r)^\alpha}{\ell((\kappa r)^{-2})}u(y)\mathrm{d}y \end{aligned}$$

for some constant $c_2 > 0$. Note that $|y - A| \leq 2|y|$ in $A(0, \frac{3r}{2}, 4)$ and that $|y - A| \leq |y| + 1$ for $|y| \geq 4$. Hence by the monotonicity of j, (3.2) and (3.3),

$$w(A) \geq c_3 \frac{(\kappa r)^\alpha}{\ell((\kappa r)^{-2})} \int_{B(0,\frac{3r}{2})^c} J(y)u(y)\mathrm{d}y$$

for some constant $c_3 > 0$. Therefore

$$w(A) \geq c_4 \kappa^\alpha \frac{\ell((2r)^{-2})}{\ell((\kappa r)^{-2})} u(x)$$

for some constant $c_4 > 0$. □

We recall the definition of κ-fat set from [27].

Definition 4.6. Let $\kappa \in (0, 1/2]$. We say that an open set D in $\mathbb{R}^d$ is κ-fat if there exists $R > 0$ such that for each $Q \in \partial D$ and $r \in (0, R)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (R, κ) is called the characteristics of the κ-fat open set D.

Note that all Lipschitz domains and all non-tangentially accessible domains (see [14] for the definition) are κ-fat. Moreover, every *John domain* is κ-fat (see Lemma 6.3 in [20]). The boundary of a κ-fat open set can be highly nonrectifiable and, in general, no regularity of its boundary can be inferred. κ-fat open set may be disconnected.

Since ℓ is slowly varying at ∞, we get the Carleson's estimate from Lemma 4.5.

Corollary 4.7. *Suppose that D is a κ-fat open set with the characteristics (R, κ). There exists a constant R_1 such that if $r \leq R_1$, $Q \in \partial D$, $u \geq 0$ is regular harmonic in $D \cap B(Q, 2r)$ with respect to X and $u = 0$ in $D^c \cap B(Q, 2r)$, then*

$$u(A_r(Q)) \geq Cu(x), \quad \forall x \in D \cap B\left(Q, \frac{3}{2}r\right)$$

for some constant $C > 0$.

The next theorem is a boundary Harnack principle for bounded κ-fat open set and it is the main result of this section.

Theorem 4.8. *Suppose that D is a κ-fat open set with the characteristics (R, κ). There exists a constant $r_5 := r_5(D, \alpha, \ell) \leq r_4 \wedge R$ such that if $2r \leq r_5$ and $Q \in \partial D$, then for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have*

$$C^{-1}\frac{u(A_r(Q))}{v(A_r(Q))} \leq \frac{u(x)}{v(x)} \leq C\frac{u(A_r(Q))}{v(A_r(Q))}, \quad \forall x \in D \cap B\left(Q, \frac{r}{2}\right)$$

for some constant $C = C(D) > 1$.

Proof. Since ℓ is slowly varying at ∞, there exists a constant $r_5 := r_5(D, \alpha, \ell) \le r_4 \wedge R$ such that for every $2r \le r_5$,

$$\max\left(\frac{\ell(r^{-2})}{\ell((\kappa r)^{-2})}, \frac{\ell((2r)^{-2})}{\ell((4r)^{-2})}, \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}, \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} \right) \le 2. \tag{4.5}$$

Fix $2r \le r_5$ throughout this proof. Without loss of generality we may assume that $Q = 0$ and $u(A_r(0)) = v(A_r(0))$. For simplicity, we will write $A_r(0)$ as A in the remainder of this proof. Define u_1 and u_2 to be regular harmonic functions in $D \cap B(0, r)$ with respect to X such that

$$u_1(x) = \begin{cases} u(x), & r \le |x| < \frac{3r}{2}, \\ 0, & x \in B\left(0, \frac{3r}{2}\right)^c \cup (D^c \cap B(0, r)) \end{cases}$$

and

$$u_2(x) = \begin{cases} u(x), & x \in B\left(0, \frac{3r}{2}\right)^c \cup (D^c \cap B(0, r)), \\ 0, & r \le |x| < \frac{3r}{2}, \end{cases}$$

and note that $u = u_1 + u_2$. If $D \cap \{r \le |y| < \frac{3r}{2}\}$ is empty, then $u_1 = 0$ and the inequality (4.8) holds trivially. So we assume $D \cap \{r \le |y| < \frac{3r}{2}\}$ is not empty. Then by Lemma 4.5,

$$u(y) \le c_1 \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} u(A), \quad \forall y \in D \cap B\left(0, \frac{3r}{2}\right)$$

for some constant $c_1 > 0$. For $x \in D \cap B(0, \frac{r}{2})$, we have

$$\begin{aligned} u_1(x) &= \mathbb{E}_x \left[u(X_{\tau_{D\cap B(0,r)}}) : X_{\tau_{D\cap B(0,r)}} \in D \cap \left\{ r \le |y| < \frac{3r}{2} \right\} \right] \\ &\le \left(\sup_{D\cap\left\{r\le|y|<\frac{3r}{2}\right\}} u(y) \right) \mathbb{P}_x \left(X_{\tau_{D\cap B(0,r)}} \in D \cap \left\{ r \le |y| < \frac{3r}{2} \right\} \right) \\ &\le \left(\sup_{D\cap\{r\le|y|<\frac{3r}{2}\}} u(y) \right) \mathbb{P}_x \left(X_{\tau_{D\cap B(0,r)}} \in B(0, r)^c \right) \\ &\le c_1 \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} u(A) \mathbb{P}_x \left(X_{\tau_{D\cap B(0,r)}} \in B(0, r)^c \right). \end{aligned}$$

Now using Lemma 4.3 and (4.5) we have that for $x \in D \cap B(0, \frac{r}{2})$,

$$\begin{aligned} u_1(x) &\le c_2 \kappa^{-d-\frac{3}{2}\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} \frac{\ell(r^{-2})}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})} \right) \\ &\quad \times u(A)\, \mathbb{P}_x \left(X_{\tau_{(D\cap B(0,r))\setminus B(A,\frac{\kappa r}{2})}} \in B\left(A, \frac{\kappa r}{2}\right) \right) \\ &\le c_3\, u(A)\, \mathbb{P}_x \left(X_{\tau_{(D\cap B(0,r))\setminus B(A,\frac{\kappa r}{2})}} \in B\left(A, \frac{\kappa r}{2}\right) \right) \end{aligned} \tag{4.6}$$

for some positive constants c_2 and $c_3 = c_3(\kappa)$. Since $2r < r_4$, Theorem 3.4 implies that

$$u(y) \geq c_4\, u(A), \quad y \in B\left(A, \frac{\kappa r}{2}\right)$$

for some constant $c_4 > 0$. Therefore for $x \in D \cap B(0, \frac{r}{2})$

$$u(x) = \mathbb{E}_x\left[u(X_{\tau_{(D\cap B(0,r))\setminus B(A,\frac{\kappa r}{2})}})\right] \geq c_4\, u(A)\, \mathbb{P}_x\left(X_{\tau_{(D\cap B(0,r))\setminus B(A,\frac{\kappa r}{2})}} \in B\left(A, \frac{\kappa r}{2}\right)\right). \quad (4.7)$$

Using (4.6), the analogue of (4.7) for v and the assumption that $u(A) = v(A)$, we get that for $x \in D \cap B(0, \frac{r}{2})$,

$$u_1(x) \leq c_3\, v(A)\, \mathbb{P}_x\left(X_{\tau_{(D\cap B(0,r))\setminus B\left(A,\frac{\kappa r}{2}\right)}} \in B\left(A, \frac{\kappa r}{2}\right)\right) \leq c_5\, v(x) \quad (4.8)$$

for some constant $c_5 = c_5(\kappa) > 0$. For $x \in D \cap B(0, r)$, we have

$$\begin{aligned} u_2(x) &= \int_{B(0,\frac{3r}{2})^c} K_{D\cap B(0,r)}(x, z) u(z)\mathrm{d}z \\ &= \int_{B(0,\frac{3r}{2})^c} \int_{D\cap B(0,r)} G_{D\cap B(0,r)}(x, y) J(y - z)\mathrm{d}y u(z)\mathrm{d}z. \end{aligned}$$

Let

$$s(x) := \int_{D\cap B(0,r)} G_{D\cap B(0,r)}(x, y)\mathrm{d}y.$$

Note that for every $y \in B(0, r)$ and $z \in B(0, \frac{3r}{2})^c$,

$$\frac{1}{3}|z| \leq |z| - r \leq |z| - |y| \leq |y - z| \leq |y| + |z| \leq r + |z| \leq 2|z|$$

and that for every $y \in B(0, r)$ and $z \in B(0, 12)^c$,

$$|z| - 1 \leq |y - z| \leq |z| + 1.$$

So by the monotonicity of j, for every $y \in B(0, r)$ and $z \in A(0, \frac{3r}{2}, 12)$,

$$j(12|z|) \leq j(2|z|) \leq J(y - z) \leq j\left(\frac{1}{3}|z|\right) \leq j\left(\frac{1}{12}|z|\right)$$

and for every $y \in B(0, r)$ and every $z \in B(0, 12)^c$,

$$j(|z| - 1) \leq J(y - z) \leq j(|z| + 1).$$

Using (3.2) and (3.3), we have that, for every $y \in B(0, r)$ and $z \in B(0, \frac{3r}{2})^c$,

$$c_6^{-1} j(|z|) \leq J(y - z) \leq c_6 j(|z|)$$

for some constant $c_6 > 0$. Thus we have

$$c_7^{-1} \leq \frac{u_2(x)}{u_2(A)} \Big/ \frac{s(x)}{s(A)} \leq c_7, \quad (4.9)$$

for some constant $c_7 > 1$. Applying (4.9) to u and v and Lemma 4.5 to v and v_2, we obtain for $x \in D \cap B(0, \frac{r}{2})$,

$$\begin{aligned} u_2(x) &\le c_7\, u_2(A) \frac{s(x)}{s(A)} \le c_7^2 \frac{u_2(A)}{v_2(A)}\, v_2(x) \\ &\le c_8\, \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} \frac{u(A)}{v(A)}\, v_2(x) = c_8\, \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})}\, v_2(x) \end{aligned} \tag{4.10}$$

for some constant $c_8 > 0$. Combining (4.8) and (4.10) and applying (4.5), we have

$$u(x) \le c_9\, v(x), \quad \forall x \in D \cap B\left(0, \frac{r}{2}\right)$$

for some constant $c_9 = c_9(\kappa) > 0$. □

5. Martin boundary and Martin representation

In this section we will always assume that D is a bounded κ-fat open set in $\mathbb{R}^d$ with the characteristics (R, κ). We are going to apply Theorem 4.8 to study the Martin boundary of D with respect to X.

We recall from Definition 4.6 that for each $Q \in \partial D$ and $r \in (0, R)$, $A_r(Q)$ is a point in $D \cap B(Q, r)$ satisfying $B(A_r(Q), \kappa r) \subset D \cap B(Q, r)$. From Theorem 4.8, we get the following boundary Harnack principle for the Green function of X which will play an important role in this section. Recall that $r_5 \le R$ is the constant defined in Theorem 4.8.

Theorem 5.1. *There exists a constant $c = c(D, \alpha, \ell) > 1$ such that for any $Q \in \partial D$, $r \in (0, r_5)$ and $z, w \in D \setminus B(Q, 2r)$, we have*

$$c^{-1} \frac{G_D(z, A_r(Q))}{G_D(w, A_r(Q))} \le \frac{G_D(z, x)}{G_D(w, x)} \le c \frac{G_D(z, A_r(Q))}{G_D(w, A_r(Q))}, \quad x \in D \cap B\left(Q, \frac{r}{2}\right).$$

Since ℓ is slowly varying at ∞, there exists a positive constant $r_6 := r_6(\kappa, \ell) \le r_5$ such that for every $2r \le r_6$,

$$\begin{aligned} \frac{1}{2} &\le \min\left(\frac{\ell(\frac{\kappa^2}{64} r^{-2})}{\ell(r^{-2})}, \frac{\ell(\frac{4}{\kappa^2} r^{-2})}{\ell(r^{-2})} \right) \\ &\le \max\left(\frac{\ell(\frac{\kappa^2}{64} r^{-2})}{\ell(r^{-2})}, \frac{\ell(\frac{4}{\kappa^2} r^{-2})}{\ell(r^{-2})} \right) \le 2. \end{aligned} \tag{5.1}$$

Lemma 5.2. *There exist positive constants $c = c(D, \alpha)$ and $\gamma = \gamma(D, \alpha) < \alpha$ such that for any $Q \in \partial D$ and $r \in (0, r_6)$, and nonnegative function u which is harmonic with respect to X in $D \cap B(Q, r)$ we have*

$$u(A_r(Q)) \le c \left(\frac{2}{\kappa}\right)^{\gamma k} \frac{\ell\left((\kappa/2)^{-2k} r^{-2}\right)}{\ell\left(r^{-2}\right)} u(A_{(\kappa/2)^k r}(Q)), \quad k = 0, 1, \ldots. \tag{5.2}$$

Proof. Without loss of generality, we may assume $Q = 0$. Fix $r < r_6$ and let

$$\eta_k := \left(\frac{\kappa}{2}\right)^k r, \qquad A_k := A_{\eta_k}(0) \quad \text{and} \quad B_k := B(A_k, \eta_{k+1}), \quad k = 0, 1, \ldots.$$

Note that the B_k's are disjoint. So by the harmonicity of u, we have

$$u(A_k) \geq \sum_{l=0}^{k-1} \mathbb{E}_{A_k}\left[u(Y_{\tau_{B_k}}) : Y_{\tau_{B_k}} \in B_l\right] = \sum_{l=0}^{k-1} \int_{B_l} K_{B_k}(A_k, z)u(z)\mathrm{d}z.$$

Theorem 3.4 implies that

$$\int_{B_l} K_{B_k}(A_k, z)u(z)\mathrm{d}z \geq c_0\, u(A_l) \int_{B_l} K_{B_k}(A_k, z)\mathrm{d}z$$

for some constant $c_0 = c_0(d, \alpha) > 0$. Since $\mathrm{dist}(A_k, B_l) \leq 2\eta_l$, by (3.8) in Proposition 3.8 and the monotonicity of j we have

$$K_{B_k}(A_k, z) \geq\ c_1 J(2(A_k - z)) \frac{(\eta_{k+1})^\alpha}{\ell((\eta_{k+1})^{-2})} \geq\ c_1 J(4\eta_l) \frac{(\eta_{k+1})^\alpha}{\ell((\eta_{k+1})^{-2})}, \quad z \in B_l.$$

Applying Lemma 3.10 and (5.1), we get

$$\begin{aligned} K_{B_k}(A_k, z) &\geq c_2 \frac{(\eta_{k+1})^\alpha}{(4\eta_l)^{d+\alpha}} \frac{\ell((4\eta_l)^{-2})}{\ell((\eta_{l+1})^{-2})} \frac{\ell((\eta_{l+1})^{-2})}{\ell((\eta_{k+1})^{-2})} \\ &\geq 2\, c_2 \left(\frac{\kappa}{8}\right)^{d+\alpha} \frac{(\eta_{k+1})^\alpha}{(\eta_{l+1})^{d+\alpha}} \frac{\ell((\eta_{l+1})^{-2})}{\ell((\eta_{k+1})^{-2})}, \quad z \in B_l \end{aligned}$$

for some constant $c_2 = c_2(d, \alpha, \ell) > 0$. Thus we have

$$\int_{B_l} K_{B_k}(A_k, z)\mathrm{d}z \geq\ c_3 \frac{(\eta_{k+1})^\alpha}{(\eta_{l+1})^\alpha} \frac{\ell((\eta_{l+1})^{-2})}{\ell((\eta_{k+1})^{-2})}, \quad z \in B_l$$

for some constant $c_3 = c_3(d, \alpha, \ell) > 0$. Therefore,

$$(\eta_k)^{-\alpha}\, u(A_k)\ell((\eta_{k+1})^{-2}) \geq c_4 \sum_{l=0}^{k-1} (\eta_l)^{-\alpha}\, u(A_l)\ell((\eta_{l+1})^{-2})$$

for some constant $c_4 = c_4(d, \alpha, \kappa, \ell) > 0$. Let $a_k := (\eta_k)^{-\alpha} u(A_k)\ell(\frac{1}{(\eta_{k+1})^2})$ so that $a_k \geq c_4 \sum_{l=0}^{k-1} a_l$. By induction, one can easily check that $a_k \geq c_5(1 + c_4/2)^k a_0$ for some constant $c_5 = c_5(d, \alpha) > 0$. Thus, with $\gamma = \alpha - \ln(1 + \frac{c_4}{2})(\ln(2/\kappa))^{-1}$, we get

$$u(A_r(Q)) \leq c \left(\frac{2}{\kappa}\right)^{\gamma k} \frac{\ell\left((\kappa/2)^{-2(k+1)} r^{-2}\right)}{\ell\left((\kappa/2)^{-2} r^{-2}\right)} u(A_{(\kappa/2)^k r}(Q)).$$

Applying (5.1), we conclude that (5.2) is true. □

Lemma 5.3. *Suppose $Q \in \partial D$ and $r \in (0, r_5)$. If $w \in D \setminus B(Q, r)$, then*

$$G_D(A_r(Q), w) \geq c \frac{\kappa^\alpha r^\alpha}{\ell((\kappa r/2)^{-2})} \int_{B(Q,r)^c} J\left(\frac{1}{2}(z - Q)\right) G_D(z, w)\mathrm{d}z$$

for some constant $c = c(D, \alpha, \ell) > 0$.

Proof. Without loss of generality, we may assume $Q = 0$. Fix $w \in D \setminus B(0, r)$ and let $A := A_r(0)$ and $u(\cdot) := G_D(\cdot, w)$. Since u is regular harmonic in $D \cap B(0, (1 - \kappa/2)r)$ with respect to X, we have

$$
\begin{aligned}
u(A) &\geq \mathbb{E}_A\left[u\left(X_{\tau_{D\cap B(0,(1-\kappa/2)r)}}\right); X_{\tau_{D\cap B(0,(1-\kappa/2)r)}} \in B(0,r)^c\right] \\
&= \int_{B(0,r)^c} K_{D\cap B(0,(1-\kappa/2)r)}(A,z)u(z)\mathrm{d}z \\
&= \int_{B(0,r)^c}\int_{D\cap B(0,(1-\kappa/2)r)} G_{D\cap B(0,(1-\kappa/2)r)}(A,y)\, J(y-z)\mathrm{d}y u(z)\mathrm{d}z.
\end{aligned}
$$

Since $B(A,\kappa r/2) \subset D \cap B(0,(1-\kappa/2)r)$, by the monotonicity of the Green functions,

$$
G_{D\cap B(0,(1-\kappa/2)r)}(A,y) \geq G_{B(A,\kappa r/2)}(A,y), \quad y \in B(A,\kappa r/2).
$$

Thus

$$
\begin{aligned}
u(A) &\geq \int_{B(0,r)^c}\int_{B(A,\kappa r/2)} G_{B(A,\kappa r/2)}(A,y)J(y-z)\mathrm{d}y u(z)\mathrm{d}z \\
&= \int_{B(0,r)^c} K_{B(A,\kappa r/2)}(A,z)u(z)\mathrm{d}z,
\end{aligned}
$$

which is greater than or equal to

$$
c_1\int_{B(0,r)^c} J(z-A)\frac{(\kappa r/2)^\alpha}{\ell((\kappa r/2)^{-2})}\, u(z)\mathrm{d}z
$$

for some positive constant $c_1 = c_1(d,\alpha,\ell)$ by (3.8) in Proposition 3.8. Note that $|z-A| \leq 2|z|$ for $z \in B(0,r)^c$. Let $M :=\mathrm{diam}(D)$. Hence

$$
\begin{aligned}
u(A) &\geq c_2\frac{\kappa^\alpha r^\alpha}{\ell((\kappa r/2)^{-2})}\int_{A(0,r,M)} u(z)J(2z)\mathrm{d}z \\
&\geq c_3\frac{\kappa^\alpha r^\alpha}{\ell((\kappa r/2)^{-2})}\int_{A(0,r,M)} u(z)J\left(\frac{1}{2}z\right)\mathrm{d}z \qquad (5.3)
\end{aligned}
$$

for some constant $c_3 = c_3(d,\alpha,\ell,M) > 0$. We have used (3.2) in the last inequality above. □

Lemma 5.4. *There exist positive constants $c_1 = c_1(D,\alpha,\ell)$ and $c_2 = c_2(D,\alpha,\ell) < 1$ such that for any $Q \in \partial D$, $r \in (0,r_6)$ and $w \in D\setminus B(Q,2r/\kappa)$, we have*

$$
\mathbb{E}_x\left[G_D(X_{\tau_{D\cap B_k}},w):\ X_{\tau_{D\cap B_k}} \in B(Q,r)^c\right] \leq c_1c_2^k\, G_D(x,w), \quad x \in D\cap B_k,
$$

where $B_k := B(Q,(\kappa/2)^k r)$, $k = 0,1,\ldots$.

Proof. Without loss of generality, we may assume $Q = 0$. Fix $r < r_6$ and $w \in D\setminus B(0,4r)$. Let $\eta_k := (\kappa/2)^k r$, $B_k := B(0,\eta_k)$ and

$$
u_k(x) := \mathbb{E}_x\left[G_D(X_{\tau_{D\cap B_k}},w); X_{\tau_{D\cap B_k}} \in B(0,r)^c\right], \quad x \in D\cap B_k.
$$

Note that for $x \in D\cap B_{k+1}$

$$
\begin{aligned}
u_{k+1}(x) &= \mathbb{E}_x\left[G_D(X_{\tau_{D\cap B_{k+1}}},w); X_{\tau_{D\cap B_{k+1}}} \in B(0,r)^c\right] \\
&= \mathbb{E}_x\left[G_D(X_{\tau_{D\cap B_{k+1}}},w); \tau_{D\cap B_{k+1}} = \tau_{D\cap B_k},\ X_{\tau_{D\cap B_{k+1}}} \in B(0,r)^c\right] \\
&= \mathbb{E}_x\left[G_D(X_{\tau_{D\cap B_k}},w); \tau_{D\cap B_{k+1}} = \tau_{D\cap B_k},\ X_{\tau_{D\cap B_k}} \in B(0,r)^c\right] \\
&\leq \mathbb{E}_x\left[G_D(X_{\tau_{D\cap B_k}},w); X_{\tau_{D\cap B_k}} \in B(0,r)^c\right].
\end{aligned}
$$

Thus

$$u_{k+1}(x) \le u_k(x), \quad x \in D \cap B_{k+1}. \tag{5.4}$$

Let $A_k := A_{\eta_k}(0)$ and $M :=$diam(D). Since $G_D(\,\cdot\,, w)$ is zero on D^c, we have

$$\begin{aligned} u_k(A_k) &= \mathbb{E}_{A_k}\left[G_D(X_{\tau_{D\cap B_k}}, w); X_{\tau_{D\cap B_k}} \in A(0,r,M)\right] \\ &\le \mathbb{E}_{A_k}\left[G_D(X_{\tau_{B_k}}, w); X_{\tau_{B_k}} \in A(0,r,M)\right] \\ &\le \int_{A(0,r,M)} K_{B_k}(A_k, z) G_D(z,w)\mathrm{d}z. \end{aligned}$$

Since $r < r_4$, by (3.7) in Proposition 3.8, we get that for $z \in A(0,r,M)$,

$$K_{B_k}(A_k, z) \le c_1 j(|z| - \eta_k) \frac{\eta_k^{\alpha/2}}{(\ell(\eta_k^{-2}))^{1/2}} \frac{(\eta_k - |A_k|)^{\alpha/2}}{(\ell((\eta_k - |A_k|)^{-2}))^{1/2}}$$

for some constant $c_1 = c_1(D,\alpha) > 0$ and $k = 1, 2, \ldots$. Since $\eta_k - |A_k| \le \eta_k \le r_6$, from (3.9) we see that

$$\frac{(\eta_k - |A_k|)^{\alpha/2}}{(\ell((\eta_k - |A_k|)^{-2}))^{1/2}} \le c \frac{\eta_k^{\alpha/2}}{(\ell(\eta_k^{-2}))^{1/2}}.$$

Thus

$$K_{B_k}(A_k, z) \le c_2 j(|z| - \eta_k) \frac{\eta_k^{\alpha}}{\ell(\eta_k^{-2})}$$

for some constant $c_2 = c_2(D, \alpha, \ell) > 0$ and $k = 1, 2, \ldots$. Therefore by the monotonicity of j

$$u_k(A_k) \le c_2 \frac{\eta_k^{\alpha}}{\ell(\eta_k^{-2})} \int_{A(0,r,M)} J\left(\frac{1}{2}z\right) G_D(z,w)\mathrm{d}z, \quad k = 1, 2, \ldots. \tag{5.5}$$

From Lemma 5.3, we have

$$G_D(A_0, w) \ge c_3 \frac{\kappa^{\alpha} r^{\alpha}}{\ell((\kappa r/2)^{-2})} \int_{A(0,r,M)} J\left(\frac{1}{2}z\right) G_D(z,w)\mathrm{d}z \tag{5.6}$$

for some constant $c_3 = c_3(D, \alpha, \ell) > 0$. Therefore (5.5) and (5.6) imply that

$$u_k(A_k) \le c_4 \left(\frac{\kappa}{2}\right)^{k\alpha} \frac{\ell\left((\kappa/2)^{-2} r^{-2}\right)}{\ell\left((\kappa/2)^{-2k} r^{-2}\right)} G_D(A_0, w)$$

for some constant $c_4 = c_4(D, \alpha, \ell) > 0$. On the other hand, using Lemma 5.2, we get

$$G_D(A_0, w) \le c_5 \left(\frac{2}{\kappa}\right)^{\gamma k} \frac{\ell\left((\kappa/2)^{-2k} r^{-2}\right)}{\ell\left(r^{-2}\right)} G_D(A_k, w)$$

for some constant $c_5 = c_5(D, \alpha) > 0$. Thus by (5.1)

$$u_k(A_k) \le c_6 \left(\frac{2}{\kappa}\right)^{-k(\alpha-\gamma)} G_D(A_k, w)$$

for some constant $c_6 = c_6(D, \alpha) > 0$ and $k = 1, 2, \ldots$. By Theorem 5.1, we have

$$\frac{u_k(x)}{G_D(x, w)} \le \frac{u_{k-1}(x)}{G_D(x, w)} \le c_6 \frac{u_{k-1}(A_{k-1})}{G_D(A_{k-1}, w)} \le c_4 c_5 c_6 \left(\frac{2}{\kappa}\right)^{-(k-1)(\alpha-\gamma)}$$

for $k = 1, 2, \ldots$. □

Let $x_0 \in D$ be fixed and set

$$M_D(x, y) := \frac{G_D(x, y)}{G_D(x_0, y)}, \quad x, y \in D, y \neq x_0.$$

M_D is called the Martin kernel of D with respect to X.

Now the next theorem follows from Theorem 5.1 and Lemma 5.4 (instead of Lemmas 13 and 14 in [4] respectively) in very much the same way as in the case of symmetric stable processes in Lemma 16 of [4] (with Green functions instead of harmonic functions). We omit the details.

Theorem 5.5. *There exist positive constants R_1, M_1, c and β depending on D, α and l such that for any $Q \in \partial D$, $r < R_1$ and $z \in D \setminus B(Q, M_1 r)$, we have*

$$|M_D(z, x) - M_D(z, y)| \le c \left(\frac{|x-y|}{r}\right)^{\beta}, \quad x, y \in D \cap B(Q, r).$$

In particular, the limit $\lim_{D \ni y \to w} M_D(x, y)$ exists for every $w \in \partial D$.

There is a compactification D^M of D, unique up to a homeomorphism, such that $M_D(x, y)$ has a continuous extension to $D \times (D^M \setminus \{x_0\})$ and $M_D(\cdot, z_1) = M_D(\cdot, z_2)$ if and only if $z_1 = z_2$. (See, for instance, [18].) The set $\partial^M D = D^M \setminus D$ is called the Martin boundary of D. For $z \in \partial^M D$, set $M_D(\cdot, z)$ to be zero in D^c.

A positive harmonic function u for X^D is minimal if, whenever v is a positive harmonic function for X^D with $v \le u$ on D, one must have $u = cv$ for some constant c. The set of points $z \in \partial^M D$ such that $M_D(\cdot, z)$ is minimal harmonic for X^D is called the minimal Martin boundary of D.

For each fixed $z \in \partial D$ and $x \in D$, let

$$M_D(x, z) := \lim_{D \ni y \to z} M_D(x, y),$$

which exists by Theorem 5.5. For each $z \in \partial D$, set $M_D(x, z)$ to be zero for $x \in D^c$.

Lemma 5.6. *For every $z \in \partial D$ and $B \subset \overline{B} \subset D$, $M_D(X_{\tau_B}, z)$ is $\mathbb{P}_x$-integrable.*

Proof. Take a sequence $\{z_m\}_{m \ge 1} \subset D \setminus \overline{B}$ converging to z. Since $M_D(\cdot, z_m)$ is regular harmonic for X in B, by Fatou's lemma and Theorem 5.5,

$$\begin{aligned} \mathbb{E}_x\left[M_D\left(X_{\tau_B}, z\right)\right] &= \mathbb{E}_x\left[\lim_{m \to \infty} M_D\left(X_{\tau_B}, z_m\right)\right] \\ &\le \liminf_{m \to \infty} M_D(x, z_m) = M_D(x, z) < \infty. \quad \square \end{aligned}$$

Lemma 5.7. *For every $z \in \partial D$ and $x \in D$,*

$$M_D(x, z) = \mathbb{E}_x\left[M_D\left(X^D_{\tau_{B(x,r)}}, z\right)\right], \quad \text{for every } 0 < r < r_6 \wedge \frac{1}{2}\rho_D(x). \tag{5.7}$$

Proof. Fix $z \in \partial D$, $x \in D$ and $r < r_6 \wedge \frac{1}{2}\rho_D(x) < R$. let

$$\eta_m := \left(\frac{\kappa}{2}\right)^m r \quad \text{and} \quad z_m := A_{\eta_m}(0), \quad m = 0, 1, \dots.$$

Note that

$$B(z_m, \eta_{m+1}) \subset B\left(z, \frac{1}{2}\eta_m\right) \cap D \subset B(z, \eta_m) \cap D \subset B(z, r) \cap D \subset D \setminus B(x, r)$$

for all $m \geq 0$. Thus by the harmonicity of $M_D(\cdot, z_m)$, we have

$$M_D(x, z_m) = \mathbb{E}_x\left[M_D\left(X_{\tau_{B(x,r)}}, z_m\right)\right].$$

On the other hand, by Theorem 5.1, there exist constants $m_0 \geq 0$ and $c_1 > 0$ such that for every $w \in D \setminus B(z, \eta_m)$ and $y \in D \cap B(z, \eta_{m+1})$,

$$M_D(w, z_m) = \frac{G_D(w, z_m)}{G_D(x_0, z_m)} \leq c_1 \frac{G_D(w, y)}{G_D(x_0, y)} = c_1 M_D(w, y), \quad m \geq m_0.$$

Letting $y \to z \in \partial D$ we get

$$M_D(w, z_m) \leq c_1 M_D(w, z), \quad m \geq m_0, \tag{5.8}$$

for every $w \in D \setminus B(z, \eta_m)$.

To prove (5.7), it suffices to show that $\{M_D(X_{\tau_{B(x,r)}}, z_m) : m \geq m_0\}$ is $\mathbb{P}_x$-uniformly integrable. Since $M_D(X_{\tau_{B(x,r)}}, z)$ is $\mathbb{P}_x$-integrable by Lemma 5.6, for any $\varepsilon > 0$, there is an $N_0 > 1$ such that

$$\mathbb{E}_x\left[M_D\left(X_{\tau_{B(x,r)}}, z\right); M_D\left(X_{\tau_{B(x,r)}}, z\right) > N_0/c_1\right] < \frac{\varepsilon}{4c_1}. \tag{5.9}$$

Note that by (5.8) and (5.9)

$$\begin{aligned}
&\mathbb{E}_x\left[M_D\left(X_{\tau_{B(x,r)}}, z_m\right); M_D\left(X_{\tau_{B(x,r)}}, z_m\right) > N_0 \text{ and } X_{\tau_{B(x,r)}} \in D \setminus B(z, \eta_m)\right] \\
&\quad \leq c_1 \mathbb{E}_x\left[M_D\left(X_{\tau_{B(x,r)}}, z\right); c_1 M_D\left(X_{\tau_{B(x,r)}}, z\right) > N_0\right] < c_1 \frac{\varepsilon}{4c_1} = \frac{\varepsilon}{4}.
\end{aligned}$$

By (3.7) in Proposition 3.8, we have for $m \geq m_0$,

$$\begin{aligned}
&\mathbb{E}_x\left[M_D\left(X^D_{\tau_{B(x,r)}}, z_m\right);\ X_{\tau_{B(x,r)}} \in D \cap B(z, \eta_m)\right] \\
&\quad = \int_{D \cap B(z,\eta_m)} M_D(w, z_m) K_{B(x,r)}(x, w) \mathrm{d}w \\
&\quad \leq c_2 \int_{D \cap B(z,\eta_m)} M_D(w, z_m) j(|w - x| - r) \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \frac{(r - |w|)^{\alpha/2}}{(\ell((r - |w|)^{-2}))^{1/2}} \mathrm{d}w
\end{aligned}$$

for some $c_2 = c_2(d, \alpha, \ell) > 0$. Since $|w - x| \geq |x - z| - |z - w| \geq \rho_D(x) - \eta_m \geq 2r - r = r$, using the monotonicity of J and (3.9) to the above equation, we see that

$$\begin{aligned}
&\mathbb{E}_x\left[M_D\left(X^D_{\tau_{B(x,r)}}, z_m\right); X_{\tau_{B(x,r)}} \in D \cap B(z, \eta_m)\right] \\
&\quad \leq c_3\, j(r) \frac{r^\alpha}{\ell(r^{-2})} \int_{D \cap B(z,\eta_m)} M_D(w, z_m) \mathrm{d}w \\
&\quad \leq c_4 \int_{B(z,\eta_m)} M_D(w, z_m) \mathrm{d}w = c_4 G_D(x_0, z_m)^{-1} \int_{B(z,\eta_m)} G_D(w, z_m) \mathrm{d}w
\end{aligned} \tag{5.10}$$

for some $c_3 = c_3(D, \alpha, \ell) > 0$ and $c_4 = c_4(D, \alpha, \ell, r) > 0$. Note that, by Lemma 5.2, there exist $c_5 = c_5(D, \alpha, \ell, m_0) > 0$, $c_6 = c_6(D, \alpha, \ell, m_0, r) > 0$ and $\gamma < \alpha$ such that

$$G_D(x_0, z_m)^{-1} \le c_5 \left(\frac{\kappa}{2}\right)^{-\gamma m} \frac{\ell\left((\kappa/2)^{-2(m+1)}(\kappa/2)^{-2m_0}r^{-2}\right)}{\ell\left((\kappa/2)^{-2}(\kappa/2)^{-2m_0}r^{-2}\right)} G_D(x_0, z_{m_0})^{-1}$$
$$\le c_6 \left(\frac{\kappa}{2}\right)^{-\gamma m} \ell\left((\kappa/2)^{-2m}(\kappa/2)^{-2(m_0+1)}r^{-2}\right). \tag{5.11}$$

On the other hand, by (3.4)

$$\int_{B(z,\eta_m)} G_D(w, z_m)\mathrm{d}w \le c_7 \int_{B(z_m,2\eta_m)} \frac{\mathrm{d}w}{\ell(|w - z_m|^{-2})|w - z_m|^{d-\alpha}}$$
$$\le c_8 \int_0^{2\eta_m} \frac{s^{\alpha-1}}{\ell(s^{-2})}\mathrm{d}s \le c_9 \frac{(\eta_m)^\alpha}{\ell((2\eta_m)^{-2})}. \tag{5.12}$$

In the last inequality above, we have used (3.16). It follows from (5.10)–(5.12) that there exists $c_{10} = c_{10}(D, \alpha, \ell, m_0, r) > 0$ such that

$$\mathbb{E}_x \left[M_D(X^D_{\tau_{B(x,r)}}, z_m); X_{\tau_{B(x,r)}} \in D \cap B(z, 2r/m)\right]$$
$$\le c_{10} \left(\frac{\kappa}{2}\right)^{(\alpha-\gamma)m} \frac{\ell\left((\kappa/2)^{-2m}(\kappa/2)^{-2(m_0+1)}r^{-2}\right)}{\ell\left((\kappa/2)^{-2m}(2r)^{-2}\right)}.$$

Since ℓ is slowly varying at ∞, we can take $N = N(\varepsilon, D, m_0, r)$ large enough so that for $m \ge N$,

$$\mathbb{E}_x \left[M_D\left(X_{\tau_{B(x,r)}}, z_m\right); M_D\left(X_{\tau_{B(x,r)}}, z_m\right) > N\right]$$
$$\le \mathbb{E}_x \left[M_D\left(X_{\tau_{B(x,r)}}, z_m\right); X_{\tau_{B(x,r)}} \in D \cap B(z, 2r/m)\right]$$
$$+ \mathbb{E}_x \left[M_D\left(X_{\tau_{B(x,r)}}, z_m\right); M_D\left(X_{\tau_{B(x,r)}}, z_m\right) > N \text{ and } X_{\tau_{B(x,r)}} \in D \setminus B(z, 2r/m)\right]$$
$$< c_{10} \left(\frac{\kappa}{2}\right)^{(\alpha-\gamma)m} \frac{\ell\left((\kappa/2)^{-2m}(\kappa/2)^{-2(m_0+1)}r^{-2}\right)}{\ell\left((\kappa/2)^{-2m}(2r)^{-2}\right)} + \frac{\varepsilon}{4} < \varepsilon.$$

As each $M_D(X_{\tau_{B(x,r)}}, z_m)$ is $\mathbb{P}_x$-integrable, we conclude that $\{M_D(X_{\tau_{B(x,r)}}, z_m) : m \ge m_0\}$ is uniformly integrable under $\mathbb{P}_x$. □

Using the fact that $\mathbb{P}_x(X_{\tau_U} \in \partial U) = 0$ for every smooth open set U (Theorem 1 in [28]), one can follow the proof of Theorem 2.2 of [8] or the proof of Theorem 4.8 of [17] and show that the two lemmas above imply that $M_D(\cdot, z)$ is harmonic for X. We skip the details.

Theorem 5.8. *For every $z \in \partial D$, the function $x \mapsto M_D(\cdot, z)$ is harmonic in D with respect to X.*

Recall that a point $z \in \partial D$ is said to be a regular boundary point for X if $\mathbb{P}_z(\tau_D = 0) = 1$ and an irregular boundary point if $\mathbb{P}_z(\tau_D = 0) = 0$. It is well known that if $z \in \partial D$ is regular for X, then for any $x \in D$, $G_D(x, y) \to 0$ as $y \to z$.

Lemma 5.9. (1) *If $z, w \in \partial D$, $z \ne w$ and w is a regular boundary point for Y, then $M_D(x, z) \to 0$ as $x \to w$.*
(2) *The mapping $(x, z) \mapsto M_D(x, z)$ is continuous on $D \times \partial D$.*

Proof. Both of the assertions can be proved easily using our Theorems 5.1 and 5.5. We skip the proof since the argument is almost identical to the one on page 235 of [5]. □

Lemma 5.10. *Suppose that h is a bounded singular α-harmonic function in a bounded open set D. If there is a set N of zero capacity such that for any $z \in \partial D \setminus N$,*

$$\lim_{D \ni x \to z} h(x) = 0,$$

then h is identically zero.

Proof. Take an increasing sequence of open sets $\{D_m\}_{m\geq 1}$ satisfying $\overline{D_m} \subset D_{m+1}$ and $\bigcup_{m=1}^{\infty} D_m = D$. Set $\tau_m = \tau_{D_m}$. Then $\tau_m \uparrow \tau_D$ and $\lim_{m\to\infty} X_{\tau_m} = X_{\tau_D}$ by the quasi-left continuity of X. Since N has zero capacity, we have

$$\mathbb{P}_x(X_{\tau_D} \in N) = 0, \quad x \in D.$$

Therefore by the bounded convergence theorem we have for any $x \in D$,

$$h(x) = \lim_{m\to\infty} \mathbb{E}_x(h(X_{\tau_m}), \tau_m < \tau_D)$$
$$= \lim_{m\to\infty} \mathbb{E}_x(h(X_{\tau_m}) 1_{\partial D \setminus N}(X_{\tau_D}); \tau_m < \tau_D) = 0. \quad \square$$

So far we have shown that the Martin boundary of D can be identified with a subset of the Euclidean boundary ∂D.

If I is the set of irregular boundary points of D for X, then I is semi-polar by Proposition II.3.3 in [2], which is polar in our case (Theorem 4.1.2 in [11]). Thus $\mathrm{Cap}(I) = 0$. Using this observation and the above lemma, now we can follow the proof of Theorem 4.1 in [27] and show the following theorem, which is the main result of this section.

Theorem 5.11. *The Martin boundary and the minimal Martin boundary of D with respect to X can be identified with the Euclidean boundary of D.*

As a consequence of Theorem 5.11, we conclude that for every nonnegative harmonic function h for X^D, there exists a unique finite measure μ on ∂D such that

$$h(x) = \int_{\partial D} M_D(x, z)\mu(dz), \quad x \in D.$$

μ is called the Martin measure of h.

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