Differential Geometry techniques in the Black-Scholes option pricing; theoretical results and approximations

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Abstract

Black-Scholes model for the basket options is used to valuate S & P 500, DAX and other Stock market index options. We explain the lack of closed formulas for the multi-asset European call options, using a differential geometric approach initiated by Labordere and V. Linetski. Our theoretical results can be tested on some extensions of the SABR model, Heston's and Bergomi stochastic volatility models, usually approached using Monte Carlo or numerical partial differential equation simulations.

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1. Introduction

Here introduce the paper, and put a nomenclature if necessary, in a box with the same font size as the rest of the paper. The paragraphs continue from here and are only separated by headings, subheadings, images and formulae. The section headings are arranged by numbers, bold and 10 pt. Here follows further instructions for authors.

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The evaluation of the European call options is based on generalized Black-Scholes models of local and stochastic volatility. The stochastic differential equations which drive the assets imply that the option prices are solutions of second order partial differential equations. The economic reality and statistical techniques select a finite number of mathematical models, but the academic databases are poor in closed, specific formulas for the option pricing. We identify the cause of this lack of solvable models: it is not a result of a small number of local and stochastic volatility models. The coefficients of the Black-Scholes partial differential equations used in option pricing have to satisfy a specific system of differential equations, as a necessary condition to be solvable from the point of view of path integral, differential geometry or stochastic calculus techniques. And of course it is unrealistic to expect that the assets are driven by a partial differential equation connected with basic concepts of Differential Geometry: Killing vector fields and the Ricci flow. In any case we can approximate the solutions of BSM equation using our approach. We also compute differential geometry parameters for the SABR, Heston and Bergomi volatility models, successfully applied on Wall Street and considered as representing four generations of option pricing models.

1.1. Differential Geometry and Option pricing

F Henry-Labordere uses a differential geometry approach: a heat kernel approximation in his study of Heston and SABR stochastic volatility models.

A path integral approach to option pricing was studied by Linetzky (1998) and Taddei (1999). They associate to any stochastic differential equation a Lagrangian functional and a Van Vleck determinant required to compute a path integral, which is computable in very few cases: Gaussian models and models that can be reduced to Gaussians by changes of variables, reparametrizations of time and projections (Ex: the Black–Scholes model, Ornstein–Uhlenbeck, Cox–Ingersoll–Ross model, Bessel process.)

Given an n-dimensional stochastic differential equation which describes the evolution of a basket of n asset prices, when does the option pricing Black-Scholes equation can be transformed to the heat equation in \( \mathbb{R}^n \)? \( \{ W_i \} \) are n independent standard Brownian motions. The risk neutral dynamics of n stocks is given by:

\[
\frac{dX_i}{\tau} = r X_i dt + \sum \sigma_i^j (X(t), t) dW_j.
\]

\( r \) is a constant interest rate. Define \( G^{\alpha\beta}(x,t) = \sum \sigma_i^\alpha \sigma_j^\beta \)

The multidimensional Black-Scholes option pricing equation is:

\[
\partial_t f(x,t) + 1/2 \sum_{i=1}^{n} \sigma_i^j (x,t) \sigma_j^i f(x,t) = r f - x \cdot \partial_x f(x,t)
\]

Then \( h(x_1, x_2, \ldots, x_n, t) = e^{-r\tau} f(x_1 e^{\theta_1}, x_2 e^{\theta_2}, \ldots, x_n e^{\theta_n}, t) \) satisfies

\[
h + \frac{1}{2} g^{\alpha\beta} h_{\alpha\beta} = 0 \quad (2)
\]

where \( g^{\alpha\beta}(x,t) = G^{\alpha\beta}(x e^{\theta}, t e^{-\theta}) \cdot h_{\alpha\beta} \) are partial derivatives of \( h \) with respect to 2 variables; summation understood

Definition: We say that the eq. 2 is equivalent to the heat equation if there are n functions

\[
H_1(x_1,t), \ldots, H_n(x_n,t), \text{ t time and x in } \mathbb{R}^n \text{ such that for any } W, \text{ solution of the equation (3) below,}
\]

\[
W \frac{1}{\tau} + \frac{1}{2} \sum \frac{\partial^2}{\partial x_i^2} = 0 \quad (3)
\]

\( W (H_1(x_1,t), \ldots, H_n(x_n,t), t) = h(x,t) \) is a solution of (2). And for any \( h \) solution of (2), there is a \( W \) as above.

There is a path integral approach in computing certain financial and quantum mechanics observables. A path integral is a limit of a sequence of finite dimensional integrals. Also, Feynman–Kac formula connects the solution of the B.S. equation with probability theory. The transition probability function, or the Green kernel is computed as an integral over the space of all paths, where we assign a probability to each path.

\[
f(t,x) = \mathcal{E}(A) \left[ e^{-(\tau-t)A} h(X(T)) \right] = \int dX(T) e^{-(\tau-t)A} h(X(T)) p(X(T), T | x(T), t)
\]

\( \mathcal{O}(S(t), t) = \mathcal{E}(\mathcal{A}) \left[ e^{-(\tau-t)A} f(X(T)) \right] \quad (4) \]

Taddei and Linetzki() proposed a Lagrangian functional which can be used to approximate the Green kernel. Instantons are solutions of the Euler-Lagrange equation,
which are paths with extreme probabilities and which can be used to approximate \( p(a,b|x,y) \). Batard and Henry-Labordere wrote down a covariant formulation of the Black-Scholes equation, involving a generalized Laplacian associated with a connection in a vector bundle.

**The Christoffel symbols** are defined using the non-degenerate metric- inverse of the matrix \( g^{\alpha\beta} \):

\[
\Gamma^a_{bc} = \frac{1}{2} g^{a\gamma} \left( \frac{\partial g_{\gamma\beta}}{\partial X^b} + \frac{\partial g_{\gamma\alpha}}{\partial X^c} - \frac{\partial g_{\alpha\beta}}{\partial X^c} \right)
\]

**The Ricci-DeTurck vector field** is defined as \( v=\{\nu_k\} \), where \( \nu_k = \sum_k \Gamma^k_{ij} g^{ij} \). It is a globally well-defined vector field on the time-prices manifold \( M \) only if \( M \) is a chart. For local-chart-dependent or \( n \)-computations it is still a well-defined vector field. It appears in the theory of Ricci and mean curvature flows as a technical tool.

The main theorem of our article is the following local necessary condition for the local volatilities to generate a solvable model. The rest of the paper will show the necessary concepts and the main applications.

**Theorem:** A necessary condition for the equation \( h + \frac{1}{2} g^{\alpha\beta} h_{\alpha\beta} = 0 \) to be equivalent with the equation:

\[
\frac{\partial f}{\partial t} + \frac{1}{2} g^{\alpha\beta} h_{\alpha\beta} = 0
\]

- the heat equation for a time-independent metric defined on a local chart- or even on the entire price-domain is the following system of PDE to be satisfied for any \( \alpha \) and \( \beta \) from 1 to \( n \):

\[
\frac{\partial g_{\alpha\beta}}{\partial t} = \sum_p \nu_p \frac{\partial X^p}{\partial X^\alpha} \frac{\partial g_{\beta\beta}}{\partial X^\alpha} + \frac{\partial \nu_p}{\partial X^\alpha} g_{\beta\beta} + \frac{\partial \nu_p}{\partial X^\beta} g_{\alpha\alpha}
\]

- \( V \) is the Ricci-DeTurck vector.

Why this solvability criterion is important? Any Black-Scholes option pricing equation can be transformed using diffeomorphisms \( (t, H_1(x,t),\ldots,H_n(x,t)) \) into the equation: \( \Delta_x + \frac{\partial f}{\partial t} = 0 \), the heat equation of a (time-dependent) metric. If this heat equation is given by a time-dependent metric, we cannot change it into a heat equation of a time-independent metric. Even if for both equations there is a heat kernel which generate all solutions and there are approximation (geodesic) techniques applied in Finance by: Henry-Labordere, Linetsky, Vassilevich, Gatheral, J, exact solutions for time-dependent heat equations or even for time-independent Laplacians are given in very few cases: Calin, Kuznetsova, A., Craddock, M. K.A. Lennox C.F. Lo, C.H. Hui, J. Derezinski, M. Wrochna.

### 1.2. Approximations

If we begin with time-dependent volatilities \( \sigma_t^\alpha(x,t) = \sigma_t^\alpha(1+f^\beta(t)) \) the PDE’s from the theorem above will be a system of ordinary differential equations for \( f \)’s, so knowing volatilities- from reverse option pricing valuations or statistical data at a specific time and knowing the tendency- the Cauchy data given by the derivatives of \( f \)’s or by the constant volatilities at two different moments, it is possible to build a stochastic model with solvable time-dependent Black-Scholes equation.

An example of a solvable model is given by the matrix of the metric:

\[
g_{\alpha\beta} = \begin{pmatrix} 0 & f(x,y,t) \\ 0 & 1 \end{pmatrix}
\]

where \( f=p(t) \) or \( f=c' \cdot d'' \), for certain constants \( c \) and \( d \), \( P(t) \)-any function of time.

### 2. Differential Geometry background

If the equation were already written in harmonic coordinates- which means that for any fixed...
time \( t \), the Laplacian of the metric \( g_{ab} (x,t) \) applied to coordinate axes =0, then the equation is exactly the equation
\[
\left( \Delta_x + \frac{\partial}{\partial t} \right) f = 0.
\]
If the initial coordinates are not harmonic for every \( t \), there is a time-dependent diffeomorphism \( f(x,t) \) such that the BS equation is equivalent to
\[
\left( \Delta_x + \frac{\partial}{\partial t} \right) w = 0,
\]
which is the heat equation for the Laplacian of the metric \( g_{ab} \) = the pull-back of the metric \( g_{ab} \) via the diffeomorphism \( f(x,t) \). If the resulting metric \( g_{ab} \) is time-dependent, there are few cases when we have closed-form solutions for the heat kernel.

If instead, \( g_{ab} = g(0) \) does not depend on time, then the technical PDE condition stated in the theorem applies. If \( g(x,t) = [\text{the inverse with respect to } x \text{ of } f(x,t)]^* (g(0)) \), then \( \frac{\partial}{\partial t} g_{ij} = -\text{The Lie derivative of the metric } g(t) \text{ with respect to the infinitesimal generator of the inverse with respect to } x \text{ of } f(x,t) = \text{the Lie derivative of the metric } g(t) \text{ with respect to the infinitesimal generator of } f(x,t) \). \( f \) is the flow of \( v \).

Solutions of
\[
K(t, \text{ the inverse of } f(x,t)), \text{ where } K \text{ is any solution of } \left( \Delta g(0) + \frac{\partial}{\partial t} \right) K = 0.
\]
We can assume that the \( (x) \)-coordinates are harmonic with respect to \( g(0) \)-Laplacian - otherwise we compose the inverse of \( f \) by a time-independent-diffeomorphism to work in harmonic coordinates. In this case, \( \frac{\partial}{\partial t} J = 0 \) for any coordinate axes, we obtain that the \( n \) components of \( J=\text{the inverse of the diffeo-} f(x,t) \text{satisfies} \)
\[
\frac{\partial}{\partial t} J = 0,
\]
so the infinitesimal generator of \( J \) is the DeTurck-Ricci vector field. This implies that the infinitesimal generator of \( f \) is \( v \):
\[
\nu(f(x,t),t)= d/dt f(x,t).
\]
A much stronger condition which implies the PDE’s from the Theorem is given by: \( (\frac{\partial}{\partial t}, \nu) \) is a Killing vector field for the metric \( g(x,t) \). Given \( g_{ab} \) at \( t=0 \) and an isometry \( f(x,0) \) for \( g(0) \), we build \( n \) solutions \( f(x,t) \) of the heat equation for \( g(0) \) with boundary condition for \( t=0 \) to be \( f(x,0) \). The metrics \( g(x,t) \) are the pull-backs of the metric \( g(0) \) using \( f(x,t) \). There are the following technical problems: we need a diffeomorphism \( f(x,t) \) for every \( t \), well-defined on the entire range of \( x \). For small \( t \), \( f(x,t) \) is indeed a diffeomorphism. Also, many models have boundary conditions.


The theorem above implies that the coefficients \( G(x,t) \) verify a system of PDE’s.

The Heston, CEV, SABR and Bergomi models show that they don’t satisfy these equations. Particular cases of these models (SABR for \( \beta = 1 \)) with closed-form analytical solutions show that there is a time-independent Laplacian such that if \( f \) satisfies
\[
\left( \Delta g(0) + \frac{\partial}{\partial t} \right) f = 0,
\]
then
exp ( Aff((t, H_1 (x,t)...H_n (x,t)f((tJ(x)+K(x)), H_1 (x,t)...H_n (x ,t)) is a solution for the option pricing PDE. Aff means a linear combination, and usually J(x) is a linear combination, including constants, of the vector x.

So, the class of admissible diffeomorphisms (t, H_1 (x,t)...H_n (x,t)) was slightly extended. Even in this case, it is possible to write down a system of PDE’s satisfied by the coefficients of the pricing equation. Closed form solutions can be found in Stohny, Desmettre, Avramidi and Dell’Era.

The system of ODE’s required by Lo and Hui to solve a multi-asset financial derivative is a particular case of the system of PDE’s from our theorem, where the involved functions depend only on time.

Zhao considers a multi-asset Black-Scholes equations with time dependent coefficients, a generalization of the CEV model, where each component of the metric depends on time and on one coordinate axes. Derezinski and Labordere consider solvable models of type \( \Delta g_{\alpha\beta} + \mathcal{Q} f = 0 \), where \( \mathcal{Q} \) is a function. The way an European option pricing equation is equivalent to this one is also controlled by a system of PDE’s.

In a given chart, or if M is the n-dimensional real space, the Ricci-DeTurck vector field can depend on time or not. Its flow \( f(x,t) \) exists anyway if this special vector field is not identically zero. If the metric \( g \) does not depends on time, this vector field will be a Killing vector field.

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