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The Landau problem. III

Motions on special curves and time-optimal control problems

by I.J. Schoenberg

Math. Research Center, 610 Walnut street,
Madison, Wisconsin 53706, U.S.A.

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1. INTRODUCTION

This is the third and last of my papers on the Landau problem (L.P.) (See [2] and [3]) and is a direct continuation of L.P.II. In Part I of this paper we apply the general approach of L.P.II to the Landau motions on a few simple curves Γ. In Part II we collect as consequences of Theorem 1 of L.P.II several theorems on time-optimal control problems. Part III deals with skew motions, a conjecture , and a question from Murray Klamkin.

I. MOTIONS ON SPECIAL CURVES

2. Γ IS A CIRCLE C_R, OR AN ARC OF A CIRCLE

Circular uniform motions have been around since the dawn of Astronomy. The following theorem seems to describe a new extremum property of these simple motions.

THEOREM 2. *If $\Gamma = C_R$ is a circle of radius R, then the Landau motion for the control constant A is the uniform circular rotation*

(2.1) $\qquad \mathfrak{f}(t) = Re^{it\sqrt{A/R}}.$

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PROOF: The statement that the uniform rotation (2.1) is the Landau motion (or L-motion) on C_R for the control constant A, means the following: *That the motion* (2.1) *is the fastest motion on* C_R, *at every point of* C_R, *within the class of motions*

(2.2) $\quad (C_R)_A = \{f(t); f(t) \in C_R \text{ and } |\ddot{f}(t)| \leq A \text{ for all } t\}.$

From (2.1) we obtain $\dot{f}(t) = \sqrt{AR}\; i \exp(it\sqrt{A/R})$ and therefore that $|\dot{f}(t)| = \sqrt{AR}$. To establish Theorem 2 we are therefore to show that

(2.3) $\quad f(t) \in (C_R)_A$ *implies that* $|\dot{f}(t)| \leq \sqrt{AR}$ *for all* t.

Following our general approach of L.P.II we prove (2.3) by passing from $v = ds/dt$ to the new variable $u = v^4$. Then the inequality (2.3) amounts to $u^{\frac{1}{4}} \leq \sqrt{AR}$, or

(2.4) $\quad u \leq A^2R^2.$

That (2.4) holds for all motions of the class (2.2) follows from Lemma 1 of L.P.II § 3, in particular from its differential inequality (3.15). For the L-motion we have $ds/dt = \sqrt{AR}$, whence

$$t = \int_0^s \frac{ds}{\sqrt{AR}} = \frac{s}{\sqrt{AR}} \text{ for all } t.$$

Therefore $s = t\sqrt{AR}$, and for the angular variable $\theta = s/R = t\sqrt{A/R}$. This establishes (2.1).

From $\dot{f}(t) = \sqrt{AR}\; i \exp(it\sqrt{A/R})$ we find that $\|\dot{f}\| = \sqrt{AR}$, and the Corollary 2 of L.P.II § 2 shows that

(2.5) $\quad L(C_R) = \sqrt{AR}/\sqrt{A} = \sqrt{R},$

in agreement with the equation (6) of L.P.I § 2.

We pass now to the case when

(2.6) $\quad \Gamma = C_{R,l}$ *is an arc of the circle of length* l.

This problem gives rise to the simplest possible fig. 2 of L.P.II § 6. Because $R = R(s)$ is constant, the upper boundary γ of the domain Ω is the horizontal line $u = A^2R^2$, except that we have the discontinuities

$$R(s) = 0, \text{ if } s = 0, \pm l, \pm 2l, \ldots.$$

Being in Case 4 we may even consider only one half of the period, which is the interval $[0, l]$. Thus we deal with fig. 2 where

(2.7) $$\gamma\colon u = \begin{cases} A^2R^2 & \text{if } 0 < s < l, \\ 0 & \text{if } s = 0, \text{ or } s = l. \end{cases}$$

Within this rectangular domain Ω we are to solve our basic D.E.

$$(2.8) \qquad \left(\frac{du}{ds}\right)^2 = 16u\left(A^2 - \frac{u}{R^2}\right).$$

Writing $U(u,s) = +4\sqrt{u(A^2 - uR^{-2})}$, this splits into the two D. equations

$$(2.9) \qquad \frac{du}{ds} = U(u,s) \text{ and } \frac{du}{ds} = -U(u,s).$$

Since variables separate, we easily find that the solutions of the first D.E. (2.9) are given by the translates of the *increasing branch* of the function

$$(2.10) \qquad \begin{cases} u = g(s) = \frac{1}{2}A^2R^2\left(1 - \cos\frac{4s}{R}\right) = A^2R^2 \sin^2\frac{2s}{R}, \\ (-R\pi/4 \leqq s \leqq R\pi/4), \end{cases}$$

while the solutions of the second equation (2.9) are the translates of the *decreasing branch* of (2.10). (The reader is asked to draw a rough diagram).

In order to carry out the construction of the function $\tilde{u}(s)$ for the L-motion as given by equation (6.2) of L.P.II, we must realize that we have in $[0, l]$ precisely *three* critical motions:

1) $\tilde{u}_0(s) = g(s)$ for $0 \leqq s \leqq R\pi/4$,
2) $\tilde{u}_{l/2}(s) = A^2R^2$ for $0 \leqq s \leqq l$,
3) $\tilde{u}_l(s) = g(s-l)$ for $l - R\pi/4 \leqq s \leqq l$.

The minimum (6.2) of L.P.II evidently depends on the size of l. There are two cases: If

$$(2.11) \qquad l \geqq R\pi/2,$$

i.e. the arc $C_{R,l}$ is a quarter circle or larger, then in the half-period $[0, l]$ of $\tilde{u}(s)$ we have

$$(2.12) \qquad \tilde{u}(s) = \begin{cases} g(s) & \text{if } 0 \leqq s \leqq R\pi/4, \\ A^2R^2 & \text{if } R\pi/4 \leqq s \leqq l - R\pi/4, \\ g(s-l) & \text{if } l - R\pi/4 \leqq s \leqq l. \end{cases}$$

However, if

$$(2.13) \qquad l \leqq R\pi/2,$$

then

$$(2.14) \qquad \tilde{u}(s) = \begin{cases} g(s) & \text{if } 0 \leqq s \leqq l/2, \\ g(s-l) & \text{if } l/2 \leqq s \leqq l. \end{cases}$$

If (2.11) holds, then $\|\dot{f}\| = (\max \tilde{u}(s))^{\frac{1}{4}} = \sqrt{AR}$, while if (2.13) holds, then $\|\dot{f}\|^4 = g(l/2) = A^2R^2 \sin^2(l/R)$, whence $\|\dot{f}\| = \sqrt{AR \sin(l/R)}$. Now Corollary 2 of L.P.II § 2 establishes our

THEOREM 3. *The Landau constant of the arc $C_{R,l}$ is given by*

$$(2.15)\qquad L(C_{R,l}) = \begin{cases} \sqrt{R}\sin(l/R) & \text{if } l \leq R\pi/2, \\ \sqrt{R} & \text{if } l \geq R\pi/2. \end{cases}$$

If we let $R \to \infty$ then $C_{R,l}$ becomes a straight segment $I_{l/2}$ of length l; letting $R \to \infty$ in (2.15) we obtain $L(I_{l/2}) = \sqrt{l}$, which is Landau's original result (1) of L.P.I § 2.

As an application of (2.15) let OD be a segment of length d; we divide OD into n equal parts, and on each part as diameter we construct a half-circle alternatingly above and below OD. These n semi-circles of radii $R = d/(2n)$ form a (corrugated) smooth arc Γ_n of diameter $= d$. By Theorem 3 we find that $L(\Gamma_n) = \sqrt{R} = \sqrt{d/(2n)}$. For large n this establishes our statement (1.7) of L.P.II.

3. THE ARC Γ IS A PARABOLA

The parabolic motion of a material particle in a constant field of forces is also an old motion due to Galileo. We state an apparently new extremum property of this motion as

THEOREM 4. *On the parabola*

$$(3.1)\qquad \Pi: y = \frac{1}{2p} x^2$$

the Landau motion $\tilde{f}(t)$ for the control constant A, is identical with the Galilean motion

$$(3.2)\qquad f_G(t) = t\sqrt{Ap} + i \cdot \tfrac{1}{2} A t^2$$

having the constant acceleration $\ddot{f}_G(t) = i \cdot A$.

PROOF: We select the vertex V of Π as origin of its arc-length s. The radius of curvature $R(s)$ of Π is an elementary but complicated function whose explicit expression we do not need. We only use the readily verified fact that in the (s, u)-plane the graph of the curve

$$(3.3)\qquad \gamma: u = A^2 R^2(s), \quad (-\infty < s < \infty),$$

is a convex curve symmetric in the u-axis (fig. 1). From $R(0) = 2p$, the curve γ has a unique minimum at the point $M = (0, 4p^2A^2)$. There being therefore only a single critical motion $\tilde{u}_M(s)$, we conclude that the L-motion $\tilde{f}(t)$ is defined by

$$(3.4)\qquad \tilde{u}(s) = \tilde{u}_M(s) \text{ for all real } s.$$

According to our general discussion of L.P.II § 5, the graph $L'ML$ of $\tilde{u}_M(s)$ passes through the point M of γ and is composed of two arcs

$$(3.5)\qquad L'M \in V_{-}^{II}, \text{ and } ML \in V_{+}^{II},$$

which are decreasing and increasing solutions, respectively, of the D.E.

$$(3.6)\qquad \left(\frac{du}{ds}\right)^2 = 16u(A^2 - uR^{-2}(s)).$$

The function $\tilde{u}(s)$, defined by (3.4) has the following properties:

(3.7) $\tilde{u}(s) > 0$ *for all* s,

(3.8) $u'(s)$ *is everywhere continuous,*

(3.9) $\tilde{u}(s)$ *satisfies the D.E.* (3.6) *for all real* s.

Also the Galilean function

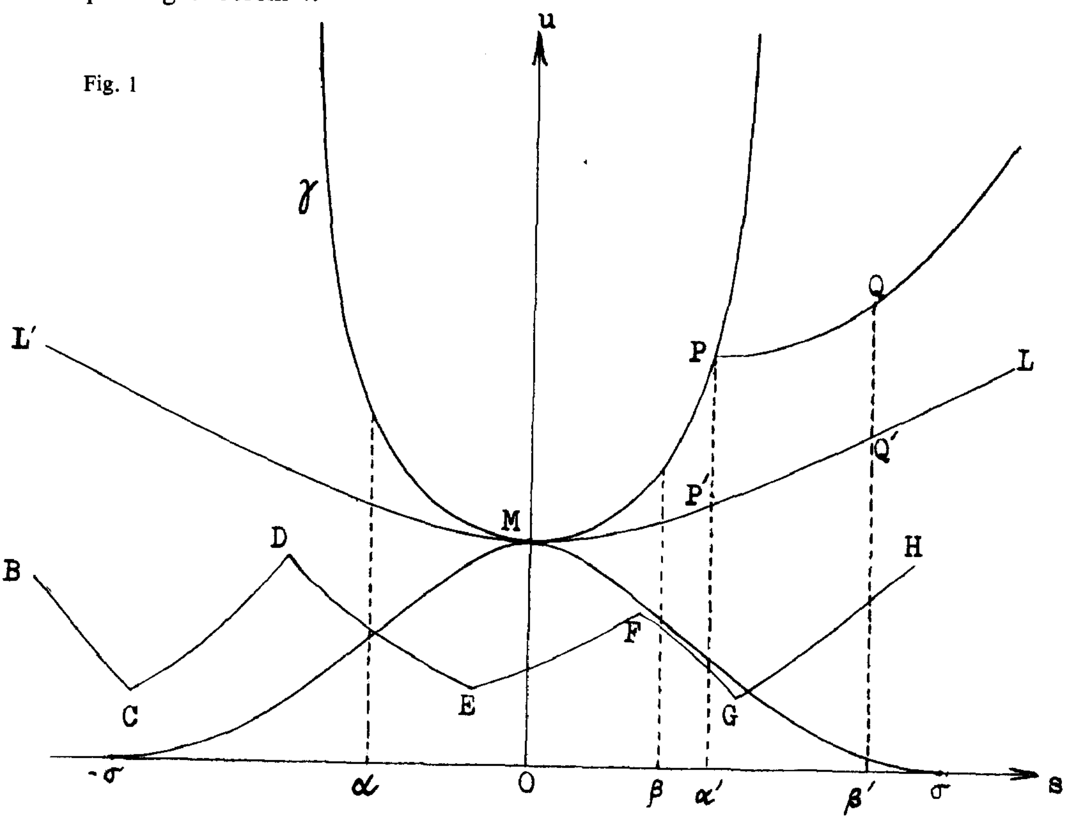
$$(3.10)\qquad u_G(s) = |\dot{f}_G(t)|^4,$$

arising from (3.2), *enjoys the three properties* (3.7), (3.8), (3.9). Let us point out that there are many functions $u(s)$, made up of a succession of arcs from V_+ and V_-, which satisfy the two conditions (3.7) and (3.9). An example is the function $u(s)$ defined by the curve *BCDEFGH* of fig. 1. However, *the function $\tilde{u}(s)$ is among them the only one whoses graph is free* of corners. The reason is that the point *M* is the only smooth junction point of an arc from V_+ (V_-) with an arc from V_- (V_+). It follows that we must have that

$$u_G(s) = \tilde{u}(s) \text{ for all } s,$$

proving Theorem 4.

Fig. 1



4. THE CURVE Γ IS AN ELLIPSE

Let it be

$$E = E_{a,b}\colon \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (0 < b < a). \tag{4.1}$$

This is an example of Case 3. A period $[0, 2l]$ of the curve

$$\gamma\colon u = A^2 R^2(s), \quad (-\infty < s < \infty), \tag{4.2}$$

is sketched in fig. 2. Taking on E its vertex V as origin for the arc-length, we have $u(0) = A^2R^2(0) = A^2b^4/a^2$ and $u(l/2) = A^2R^2(l/2) = A^2a^4/b^2$. Fig. 2 also shows a sketch of the periodic graph $M_0C_1M_1C_2M_2$ of the Landau motion according to § 6 of L.P.II.

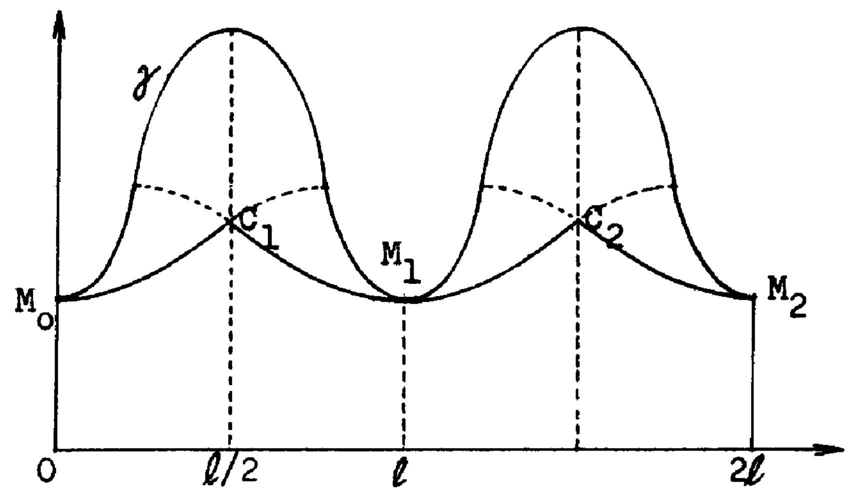


Fig. 2

Because $R(s)$ is an elementary but complicated function of s, we use the representation

$$E\colon x = a \cos\theta, \ y = b \sin\theta \tag{4.3}$$

in terms of which we find

$$R = (a^2 \sin^2\theta + b^2 \cos^2\theta)^3 a^{-2} b^{-2}. \tag{4.4}$$

Passing to the independent variable θ we find the D.E. for the increasing solutions of the class V_+ to be

$$\frac{d\tilde{u}}{d\theta} = 4\sqrt{\tilde{u}\left(A^2(a^2 \sin^2\theta + b^2\cos^2\theta) - \frac{a^2b^2\tilde{u}}{(a^2\sin^2\theta + b^2\cos^2\theta)^2}\right)} \tag{4.5}$$

with the initial condition for the arc M_0C_1 being

$$\tilde{u}\,|_{\theta=0} = A^2b^4/a^2. \tag{4.6}$$

By Corollary 2 of L.P.II § 2 we find that

$$L(E_{a,b}) = (\tilde{u}(l/2))^{\frac{1}{4}} A^{-\frac{1}{2}}, \tag{4.7}$$

and it suffices to determine the value of $\tilde{u}(l/2)$.

Here is a simplification: Observe that by similitude

$$f(t) \in (E_{a,b}) \text{ iff } f_1(t) = \frac{1}{a} f(t) \in (E_{1,b/a}).$$

In terms of the functional $F(f) = \|\dot{f}\| / \sqrt{\|\ddot{f}\|}$ we have that

$$F(f) = \sqrt{a} F(f_1).$$

By taking suprema on both sides we find that

$$(4.8) \qquad L(E_{a,b}) = \sqrt{a} L(E_{1,r}), \text{ where } r = b/a.$$

To determine the value of $u(l/2)$ we have chosen in (4.5)

$$A = 1, \ a = 1, \text{ and } b = r = .1, .2, \ldots, .9$$

and integrated numerically the D.E. (4.5) from $\theta = 0$ to $\theta = \pi/2$. This was beautifully done by C. Vargas, of the MRC Computing Staff, by a Runge-Kutta method. A source of difficulty is that (4.5) does *not* satisfy a Lipschitz condition in a neighborhood of the starting point $\theta = 0$, $\tilde{u} = A^2 b^4 / a^2$. For this reason the solution was started by an appropriate power series expansion. The results are shown in table 1 for the values of the Landau constant.

Table 1

r	$L(E_{1,r})$	r	$L(E_{1,r})$
.0	$1.41421 = \sqrt{2}$	.5	1.30011
.1	1.40978	.6	1.24811
.2	1.39643	.7	1.18609
.3	1.37394	.8	1.11782
.4	1.34199	.9	1.05409
		1.0	1.00000

A glance at fig. 2 shows the Landau motion $\bar{f}(t)$ to be accelerated on the arcs VW and $V'W'$ of fig. 3, and descellerated on WV' and $W'V$, the maximal speeds occuring at W and W'. At these points are the only discontinuities of $\ddot{\bar{f}}$, as indicated in fig. 3.

We mention the following three limiting cases in fig. 3:

1. If b decreases and tends to zero, then the angle $\alpha = \angle(\ddot{\bar{f}}_+, \ddot{\bar{f}}_-)$ increases to 180° and $\bar{f}$ becomes the to and fro Landau motion on the segment VV'.

2. If b increases and tends to the value a, then the angle α decreases to zero, while $\bar{f}$ becomes the uniform rotation on the circle $E_{a,a} = C_a$.

3. If we keep fixed the focus F of E, and also its vertex V, while we let its second focus F' tend to $-\infty$, then E approaches a parabola Π. Also the acceleration pattern $\ddot{\bar{f}}$ on the arc $W'VW$ is fanning out and approaches in the limit the horizontal acceleration vectors of the Galilean motion on the parabola Π.

The last example of a closed curve Γ which we mention here, leaving out the simple details, is the following. Let $RT_{a,b}$ denote a *racetrack* composed of a rectangle $(2a) \times (2b)$, whose sides of length $2b$ have been replaced by two semi-

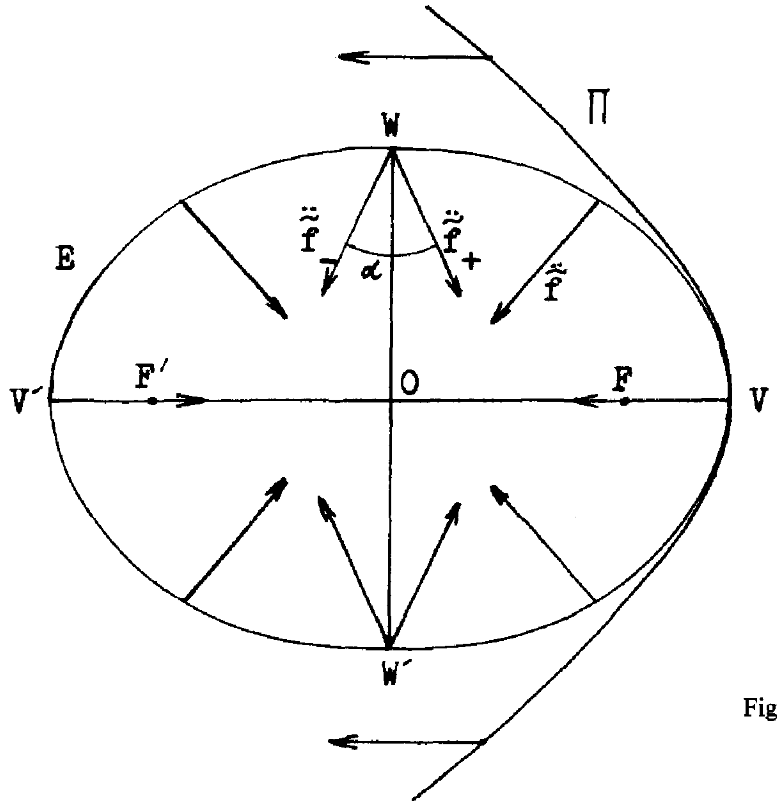


Fig. 3

circles of radius b. Using our approach and the results of § 2, it is easy to show that

$$L(RT_{a,b}) = \sqrt{2a+b}.$$

For details see § 9_4 of Reference [1] of L.P.II.

II. TIME - OPTIMAL CONTROL PROBLEMS ON CURVES

5. THE CASE WHEN Γ IS A SEGMENT AND ITS GENERALIZATIONS

Let $\Gamma = \{0 \leqq s \leqq l\}$ be a straight segment of length l. Let

(5.1) $\quad s = f(t), \quad (0 \leqq t \leqq T),$

be a motion on Γ restricted by the requirement that

(5.2) $\quad |\ddot{f}(t)| \leqq A$, where A is prescribed,

and satisfying the boundary conditions

(5.3) $\quad f(0) = 0, \dot{f}(0) = 0$, and $f(T) = l, \dot{f}(T) = 0,$

the problem being *to determine the shortest time T in which such a motion is possible.*

This is a well-known problem (See e.g. [6, pp. 233–236]). Its connection with the Landau problem is shown by the general

THEOREM 5. *Let $\Gamma = 0L$ be an arc of length l* (Case 4), *and let $\bar{f}(t)$ be its Landau motion for the control constant A. We know $\bar{f}(t)$ to be an even function* *of period* $2\tilde{T}$, say. *In particular, its restriction*

(5.4) $\quad \tilde{f}(t), \quad (0 \leqq t \leqq \tilde{T}).$

is known to satisfy the boundary conditions

(5.5) $\quad \tilde{f}(0) =$ *the endpoint* 0 *of* Γ, *with* $\dot{\tilde{f}}(0) = 0$,

and

(5.6) $\quad \tilde{f}(\tilde{T}) =$ *the endpoint* L *of* Γ, *with* $\dot{\tilde{f}}(\tilde{T}) = 0$.

Furthermore, let

(5.7) $\quad f(t), \quad (0 \leqq t \leqq T),$

be any motion along Γ *such that* $f(t) \in (\Gamma) \cap \mathcal{M}$ *and*

(5.8) $\quad |\ddot{f}(t)| \leqq A.$

We also assume that

(5.9) $\quad f(0) =$ *the endpoint* 0 *of* Γ, *with* $\dot{f}(0) = 0$,

and

(5.10) $\quad f(T) =$ *the endpoint* L *of* Γ, *with* $\dot{f}(T) = 0$.

Also that $f(t)$ *moves throughout* $0 \leqq t \leqq T$ *in the direction of increasing arc-length* s, *so that*

(5.11) $$v = \frac{ds}{dt} \geqq 0.$$

Then

(5.12) $\quad \tilde{T} \leqq T,$

with equality only if the motions $\tilde{f}$ *and* f *are identical.*

This shows that the Landau motion is the solution of this time-optimal control problem.

PROOF: Following L.P.II § 3 we introduce the variable $u = v^4$, where $v = ds/dt$. Let $\tilde{u}(s)$ and $u(s)$ be the functions corresponding to $\tilde{f}$ and f, respectively. Moreover, let the two motions be at the same point of arc-length s at the times $\tilde{t}$ and t, respectively, so that $\tilde{f}(\tilde{t}) = f(t)$. This implies the relations

(5.13) $$\tilde{v} = \frac{ds}{d\tilde{t}} = (\tilde{u}(s))^{\frac{1}{4}} = |\dot{\tilde{f}}(\tilde{t})|, \quad v = \frac{ds}{dt} = +(u(s))^{\frac{1}{4}} = |\dot{f}(t)|.$$

Notice that in the second relation we have used the assumption (5.11). From (5.13) we derive for the corresponding times the values

(5.14) $$\tilde{T} = \int_0^l \frac{ds}{(\tilde{u}(s))^{\frac{1}{4}}}, \quad T = \int_0^l \frac{ds}{(u(s))^{\frac{1}{4}}}.$$

From Theorem 1 of L.P.II § 2 we know that $|\dot{\tilde{f}}(\tilde{t})| \geqq |\dot{f}(t)|$, and therefore that $\tilde{u}(s) \geqq u(s)$, for all s. Now (5.14) evidently implies (5.12), including the condition for the equality sign to hold.

6. THE CURVE Γ IS A CLOSED CURVE, OR AN ARC

Let Γ be a closed curve (Case 3). We propose the problem of determining the shortest time $2T$ in which a motion along Γ restricted by

(6.1) $|\ddot{f}(t)| \leqq A$ for all t,

can perform one complete revolution around Γ. The solution is given by

THEOREM 6. *Let $\tilde{f}(t)$ be the Landau motion of period $2\tilde{T}$ along Γ for the constant A. Then*

(6.2) $\quad 2\tilde{T} \leqq 2T,$

with equality only if the motions $\tilde{f}(t)$ and $f(t)$ coincide, except for a shift in the time-origin.

A proof which is much like the proof of Theorem 5, may be omitted.

THE CASE WHEN Γ IS AN ARC. Somewhat different from the two problems so far discussed is the following: Let

(6.3) $\quad \Gamma = XY$ be an arc of length l, $\alpha \leqq s \leqq \beta = \alpha + l$, say.

What is the shortest time T for a motion $f(t)$, $(0 \leqq t \leqq T)$ to move from $s = \alpha$ to $s = \beta$, satisfying (6.1) *and the boundary conditions $f(0) = X$, $f(T) = Y$?*

Notice that our previous boundary conditions $\dot{f}(0) = 0$, $\dot{f}(T) = 0$, of § 5, have been omitted.

The solution is given by a slight modification of the construction of the L-motion as given in L.P.II §§ 5 and 6. Let

(6.4) $\quad \gamma\colon u = A^2R^2(s), \quad (\alpha \leqq s \leqq \beta),$

be the curve (5.2) of L.P.II § 5, but restricted to our arc $\Gamma = XY$. The modification is this: *If the point $s = \alpha$ happens to be a relative minimum point of γ in the interval $[\alpha, \beta]$, we include in the construction* (6.2) (L.P.II) *of $\tilde{u}(s)$, the critical motion attached to the minimum at $s = \alpha$. Similarly for the endpoint $s = \beta$.*

We discuss this problem when $\Gamma = XY$ is an arc of the parabola Π defined by (3.1). Referring to its (s, u)-diagram of fig. 1, there are two cases:

1. If the arc XY contains the vertex V of the parabola, hence $\alpha \leqq 0 \leqq \beta$, then the least time T is furnished by the Landau motion. By Theorem 4 this is identical with the Galilean motion (3.2).
2. If the vertex V is outside the arc XY, $0 < \alpha' < \beta'$ say, then we must construct the arc $PQ \in V^{II}_{+}$, defined as the graph of

(6.5) $\quad$ *the solution $u = u(s)$ of the D.E. $du/ds = +U(u, s)$*

such that $u(\alpha')=A^2R^2(\alpha')$. This graph defines the fastest motion, and the shortest time is given by

$$(6.6) \qquad T=\int_{\alpha'}^{\beta'} \frac{ds}{(u(s))^{\frac{1}{4}}}.$$

Observe that the time T, given by (6.6), is *below the corresponding traversal time for the Galilean motion given by the arc $P'Q'$ of* fig. 1. The reason is that now the motion need not worry about negotiating safely the perilous vertex V of the parabola.

III. MOTIONS ON SKEW CURVES, A CONJECTURE, AND AN ANSWER TO A QUESTION FROM MURRAY KLAMKIN

Here we collect three disconnected subjects.

7. MOTIONS ON SKEW CURVES

Let Γ be a curve in the space R^3 belonging to any of the four cases of L.P.II § 2. If we apply our analysis used for plane curves, and use for this purpose the Frenet formulae (See [5, Chap. III, § 7]) we find that the acceleration $\ddot{f}(t)$ depends on the curvature $\varrho=1/R(s)$ of Γ, but disregards entirely its torsion τ. It follows that the Landau motion on Γ depends on the arc-length s in the same way as the L-motion on a plane curve Γ^* having, for all relevant values of s, the same curvature as Γ. For example, the L-motion on a circular helix of curvature $1/R$, represented by $s=\tilde{s}(t)$, is the same as for a circle C_R of same curvature.

Let us determine the Landau constant $L(S_a)$ of the spherical shell

$$(7.1) \qquad S_a\colon x^2+y^2+z^2=a^2.$$

If Γ is any smooth curve on S_a, it is known [5, Chap. IV, § 12] that for its radius of curvature we have $R(s)\leqq a$, for all s. It follows that the curve γ defined by (5.2) of L.P.II, will never rise above the horizontal line $u=A^2a^2$. But then for the L-motion on Γ we must have $\tilde{u}(s)\leqq A^2a^2$, for all s. By Corollary 2 of L.P.II § 2 we conclude that

$$L(S_a)=\sqrt{a}.$$

This also follows from Theorem 1 of L.P.I concerning the L-constant of a solid spherical shell on letting $b=a$.

8. A CONJECTURED GENERAL EXTREMUM PROPERTY OF UNIFORM ROTATION

The content of Theorem 2, for $R=1$, is evidently the following: For the uniform rotation $\tilde{f}(t)=e^{it}$ we have $\|\tilde{f}\|=1$, $\|\ddot{\tilde{f}}\|=1$. *If $f(t)$ is any smooth motion on the circle C_1 such that*

$$\|\ddot{f}\|\leqq\|\ddot{\tilde{f}}\|=1, \textit{ then } \|\dot{f}\|\leqq\|\dot{\tilde{f}}\|=1.$$

Let me state the rather bold.

CONJECTURE 1. *Let n be an integer* $\geqq 2$. *If* $f(t)$ *is any smooth motion on the circle* C_1, such that

$$(8.1)\qquad \|f^{(n)}\| \leqq \|\bar{f}^{(n)}\| = 1,$$

then

$$(8.2)\qquad \|f^{(k)}\| \leqq \|\bar{f}^{(k)}\| = 1, \textit{ for } k = 1, 2, \ldots, n-1.$$

We may rephrase this in connection with the famous result of Kolmogorov: *If* $f\colon R \to \mathbb{C}$ *is a smooth map, and we write* $M_k = \|f^{(k)}\|$, *then*

$$(8.3)\qquad M_k \leqq C_{n,k} M_0^{1-(k/n)} M_n^{k/n}, \textit{ for } k = 1, \ldots, n-1.$$

Here we have equality for the Euler spline $\mathscr{E}_n(t)$ (For references see [4]). The Conjecture 1 is equivalent with the statement that for *circular motions* $f(t)$ we may sharpen (8.3) to

$$M_k \leqq M_0^{1-(k/n)} M_n^{k/n}, \textit{ for } k = 1, \ldots, n-1.$$

9. AN AFFIRMATIVE ANSWER TO A QUESTION FROM MURRAY KLAMKIN

On the occasion of a lecture of mine at Edmonton, Alberta, Murray Klamkin raised the following interesting question:

The acceleration $\ddot{f}(t)$ of the Landau motion on a circle is *continuous*, while it is *discontinuous* for a non-circular ellipse. *Are there closed convex curves* Γ, *besides circles, having a continuous* $\ddot{f}$?

We give here an affirmative answer by constructing a family of closed convex Γ having this property. Let α be a parameter such that

$$(9.1)\qquad 0 \leqq \alpha \leqq \tfrac{1}{2},$$

and let us prescribe $\ddot{f}(t)$ by

$$(9.2)\qquad \ddot{f}(t) = e^{i(\pi + t - \alpha \sin 2t)}.$$

A function $f(t)$ is readily recovered by using Bessel functions. From the identity (See [1, page 361])

$$(9.3)\qquad \exp\{\tfrac{1}{2}\alpha(z - z^{-1})\} = \sum_{k=-\infty}^{\infty} J_k(\alpha) z^k, \quad (z \neq 0),$$

we find that

$$(9.4)\qquad \ddot{f}(t) = \sum_{-\infty}^{\infty} (-1)^{k+1} J_k(\alpha) e^{(2k+1)ti},$$

whence by two integrations

$$(9.5)\qquad f(t) = \sum_{-\infty}^{\infty} (-1)^k \frac{J_k(\alpha)}{(2k+1)^2} e^{(2k+1)ti}.$$

If $\alpha = .4$, say, this series converges very fast as seen from the table

k	$J_k(.4)$
0	.96040
1	.19603
2	.01973
3	.00132
4	.00007

The curve

(9.6) $\quad \Gamma\colon z = f(t), \quad (0 \leqq t \leqq 2\pi), \quad (\alpha = .4)$

is an oval symmetric in both coordinate axes, because the function $\operatorname{Re} f(t)$ is even and $\operatorname{Im} f(t)$ is odd. Moreover, the function $\psi(t) = \pi + t - \alpha \sin 2t$ is *strictly increasing*, and this implies by (9.2), that the curves described by the points $\dot{f}(t)$, and $f(t)$, are both *strictly convex*. We have plotted accurately the transcendental curve (9.6) and found it to differ little from an ellipse, which it can not be. Why? The reason is that we will show in (9.7) that the motion $f(t)$ is its own Landau motion, and then the *continuity* of $\ddot{f}(t)$ shows that Γ must be different from an ellipse, whose L-motion has a *discontinuous* acceleration. Concerning Γ we found

$$f(0) = -f(\pi) = 1.13330, \quad f\left(\frac{\pi}{2}\right) = -f\left(\frac{3\pi}{2}\right) = (.78470)i.$$

I claim: *The motion*

(9.7) $\quad f(t) = \bar{f}(t)$

is also the Landau motion of the curve (9.6).

PROOF: Here $f(t)$ is the motion on Γ defined by (9.5), while $\bar{f}(t)$ denotes the L-motion on Γ for $A = 1$.

As mentioned before, the vector $\ddot{f}(t)$, of (9.2), turns steadily by 360° as t varies from 0 to 2π. This being the hodograph of the motion of the point $\dot{f}(t)$, it follows that also $z = \dot{f}(t)$ describes a closed and convex curve with center of symmetry at 0. The symmetry follows from $\dot{f}(t+\pi) = -\dot{f}(t)$ and implies that

(9.8) $\quad \dot{f}(t) \neq 0, \text{ if } 0 \leqq t \leqq 2\pi.$

This already implies the identity (9.7) for the following reasons: From $|\ddot{f}(t)| = 1$ for all t, we infer that the graph $u = u(s)$ for the motion $f(t)$ is composed of a succession of arcs from the classes V_+ and V_-. A junction point M_1 of two such arcs C_1M_1 and M_1C_2 say, must be a stationary point of the curve

(9.9) $\quad \gamma\colon u = A^2R^2(s), \quad (A = 1),$

because there is no corner at M_1.

Suppose this junction point M_1 to be a minimum point of γ. It is then impossible that $C_1M_1 \in V_+$ and $M_1C_2 \in V_-$, because this would forcibly lead to a point where $u = 0$, in contradiction to (9.8). We must therefore have the situation shown by fig. 4, where M_0, M_1, M_2 are the minimum points of γ. This already shows that $u(s) = \tilde{u}(s)$ gives rise to the L-motion, and the claim (9.7) is established.

Notice that we have placed the junction points C_1 and C_2 at the maximum points of γ. This follows from the continuity of $u'(s)$ implied by the continuity of $\ddot{f}(t)$. Indeed $M_0C_1 \in V_+$ and $C_1M_1 \in V_-$ can only join smoothly at a maximum point of γ, because of the web-like structure of the solutions of the D.E.

$$\left(\frac{du}{ds}\right)^2 = 16u(A^2 - uR^{-2}(s)).$$

CONCLUDING REMARKS. Let us say that a Landau motion f is *a Landau-Klamkin motion*, or *L-K* motion, *provided that the acceleration* $\ddot{f}$ *is continuous for all* t. Now that Murray Klamkin mentioned them, we can see many *L-K* motions, not necessarily on closed curves. A nice example is a symmetric parabolic arc Γ which is part of the parabola Π of (3.1). We refer to fig. 1 and let the interval $(-\sigma, \sigma)$ be the support of the bell-shaped curve formed by the arc of V_+ with endpoint at M, and by the arc of V_- starting at M. I remind the reader that these arcs are, respectively, the solutions of the differential equations (2.9) for the parabola Π, both passing through M. Let

(9.10) $$u = \tilde{u}(s), \quad (-\sigma \leqq s \leqq \sigma),$$

denote the function having the bell-shaped graph of fig. 1. Writing

(9.11) $$t = t(s) = \int_0^s \frac{ds}{(\tilde{u}(s))^{\frac{1}{4}}}, \quad (-\sigma \leqq s \leqq \sigma),$$

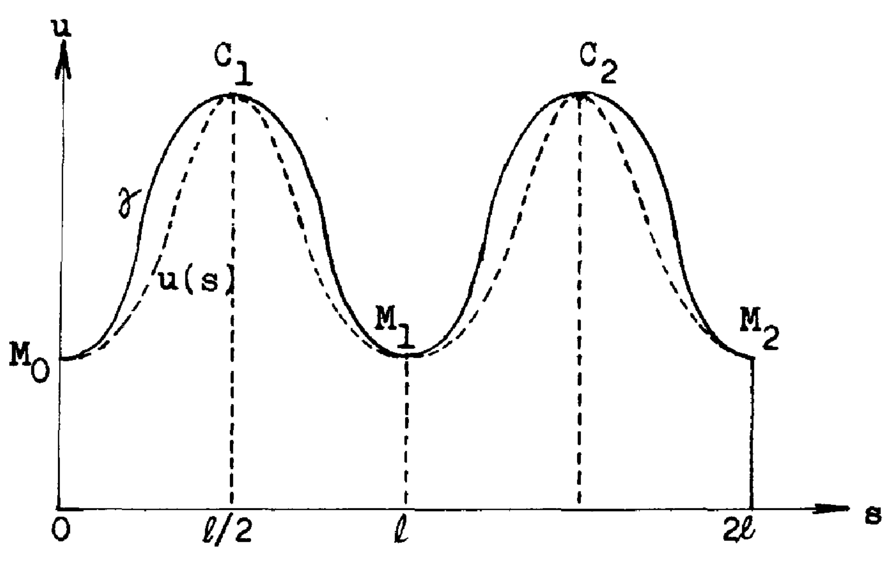


Fig. 4

and solving for s to get

(9.12) $s = s(t), \quad (-T/2 \leqq t \leqq T/2),$

we obtain precisely a half-period of the Landau motion $\bar{f}(t)$ on the arc $\Gamma = \Pi[-\sigma, \sigma] = XY$ of Π (Case 4). This is a to and fro motion $\bar{f}$ on Γ with

(9.13) $\dot{\bar{f}}(-T/2) = \dot{\bar{f}}(T/2) = 0.$

Of course, we do know that $|\ddot{\bar{f}}(t)| = A$ for all t. However

(9.14) *$\ddot{\bar{f}}(t)$ is continuous for all times t.*

PROOF: This is evident for all t except if $t = (T/2) + kT$. However, Taylor's series shows that we must have

(9.15) $$\ddot{\bar{f}}\left(\frac{T}{2}+0\right) = \ddot{\bar{f}}\left(\frac{T}{2}-0\right),$$

the common value being a vector of length A which is tangent to Π at the point Y and pointing into the arc. Here we have used the equations (9.13).

It is interesting to compare the L-motion for $\Gamma = \Pi[-\sigma, \sigma]$ with Landau's original motion on a straight segment I_σ of the same length 2σ. *In passing from I_σ to Γ the discontinuity of $\ddot{\bar{f}}$ has disappeared.* As a matter of fact we achieved the same effect in § 2, without pointing it out, that the discontinuity for I_σ disappears if we bend the segment into a circular arc of radius $R \leqq 4\sigma/\pi$.

Let me close with the following problem: *Can the above L-K motion on the parabolic arc $\Pi[-\sigma, \sigma]$ be explicitly determined in terms of known functions?*

REFERENCES

1. Abramowitz, M. and I.A. Stegun – Editors, Handbook of Mathematical functions, National Bureau of Standards, Washington, 1965.
2. Schoenberg, I.J. – The Landau problem I. The case of motions on sets, Indagationes Math., 276–286 (1978).
3. Schoenberg, I.J. – The Landau problem II. The case of motions on curves, Indagationes Mathematicae, **43**, 325–335 (1981).
4. Schoenberg, I.J. – The elementary cases of Landau's problem of inequalities between derivatives, American Math. Monthly, **80**, 121–158 (1973).
5. Stoker, J.J. – Differential Geometry, Wiley-Interscience, New York, 1969.
6. Young, L.C. – Lectures on the Calculus of Variations and Optimal Control Theory, W.B. Saunders Co., Philadelphia, 1969.