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Traveling wavefronts for time-delayed reaction–diffusion equation: (I) Local nonlinearity

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ABSTRACT

In this paper, we study a class of time-delayed reaction–diffusion equation with local nonlinearity for the birth rate. For all wavefronts with the speed $c > c_*$, where $c_* > 0$ is the critical wave speed, we prove that these wavefronts are asymptotically stable, when the initial perturbation around the traveling waves decays exponentially as $x \to -\infty$, but the initial perturbation can be arbitrarily large in other locations. This essentially improves the stability results obtained by Mei, So, Li and Shen [M. Mei, J.W.-H. So, M.Y. Li, S.S.P. Shen, Asymptotic stability of traveling waves for the Nicholson's blowflies equation with diffusion, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 579–594] for the speed $c > 2\sqrt{D_m(\varepsilon p - d_m)}$ with small initial perturbation and by Lin and Mei [C.-K. Lin, M. Mei, On travelling wavefronts of the Nicholson's blowflies equations with diffusion, submitted for publication] for $c > c_*$ with sufficiently small delay time $r \approx 0$. The approach adopted in this paper is the technical weighted energy method used in [M. Mei, J.W.-H. So, M.Y. Li, S.S.P. Shen, Asymptotic stability of traveling waves for the Nicholson's blowflies equation with diffusion, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 579–594], but inspired by Gourley [S.A. Gourley, Linear stability of travelling fronts in an age-structured reaction–diffusion population model, Quart. J. Mech. Appl. Math. 58 (2005) 257–268] and based on the property of the critical wavefronts, the weight function is carefully

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selected and it plays a key role in proving the stability for any $c > c_*$ and for an arbitrary time-delay $r > 0$.

1. Introduction

In this series of study, we consider a class of time-delayed reaction–diffusion equation arising from the population dynamics (see, for example [27])

$$\frac{\partial v}{\partial t} - D_m \frac{\partial^2 v}{\partial x^2} + d_m v = \varepsilon \int_{-\infty}^{\infty} b\big(v(t-r, x-y)\big) f_\alpha(y)\, dy, \quad t > 0, \ x \in R, \tag{1.1}$$

with the initial condition

$$v(s,x) = v_0(s,x), \quad s \in [-r, 0], \ x \in R, \tag{1.2}$$

which describes the population distribution of single species with age-structure and spatial diffusion. Here, $v(t,x)$ denotes the total mature population at time t and location x. $D_m > 0$ is the diffusion rate of the mature population in space, $d_m > 0$ is the death rate of the mature population, and $r > 0$ denotes the maturation age. Let $d_i(z)$ and $D_i(z)$ be the death rate and the spatial diffusion rate for the immature population for age $z \in [0, r]$, respectively. Define $\varepsilon > 0$ and $\alpha \geqslant 0$ as follows:

$$\varepsilon = e^{-\int_0^r d_i(z)\, dz} \quad \text{and} \quad \alpha = \int_0^r D_i(z)\, dz. \tag{1.3}$$

They represent the impact of the death rate of the immature and the effect of the dispersal rate of the immature, respectively. If the mature population move more effectively than the immature, i.e. $D_i(z) \leqslant D_m$, then, according to (1.3) for α, we have

$$\alpha \leqslant r D_m. \tag{1.4}$$

The function $b(v(t,x))$ is the birth function. As we know, the reaction term in a reaction–diffusion equation, which is some what like the birth function in our equation (1.1), plays an important role to depict the character of the equation. One interesting type of such a function is the Fisher–KPP's reaction function $b(v) = v(1-v)$ with bi-stable nodes $v = 0$ and $v = 1$ as the equilibria (see [4, 10,24,25]). The related nonlinear wave stability for the time-delayed reaction–diffusion equation was studied by Smith and Zhao [25] by the upper–lower solution method. The other interesting type is the reaction term with two nodes as the equilibria, but one is stable and the other one is unstable, for example, the Nicholson's blowflies birth rate $b(v) = pve^{-av}$ (see [6,8,11,14,15,20,21,26–29,33,34,37], etc.). The related wave stability was then investigated in [15,20,21] by the technical weighted energy method, because, as mentioned therein, the upper–lower solution method is defective in this case. For such kind birth functions with one stable node and one unstable node, in this paper, as considered in [14], we take the birth function as

$$b_1(v) = pve^{-av^q} \quad \text{or} \quad b_2(v) = \frac{pv}{1 + av^q}, \tag{1.5}$$

where $p > 0$, $q > 0$ and $a > 0$ are constants. Here, $b_1(v)$, as an exponential type, is a generalization of the Nicholson's blowflies birth rate as considered before, and $b_2(v)$ is the kind of important rational-type function. Note that, for $b_1(v)$ with $q = 1$, it is the well-known Nicholson's blowflies birth function, and the corresponding equation (1.1) is the so-called Nicholson's blowflies equation with diffusion. Lastly, $f_\alpha(y)$ is the heat kernel

$$f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{2\alpha}} \quad \text{with} \int_{-\infty}^{\infty} f_\alpha(y)\,dy = 1 \text{ for all } \alpha > 0. \tag{1.6}$$

When $\alpha \to 0$, namely $D_i(z) \approx 0$, which means that the immature population is almost non-mobile, by the property of the heat kernel we have

$$\lim_{\alpha\to 0} \int_{-\infty}^{\infty} b\big(v(t-r, x-y)\big) f_\alpha(y)\,dy = b\big(v(t-r, x)\big).$$

In this case, the nonlocal equation (1.1) is reduced to a local time-delayed reaction–diffusion equation for $\alpha = 0$:

$$\frac{\partial v}{\partial t} - D_m \frac{\partial^2 v}{\partial x^2} + d_m v = \varepsilon b\big(v(t-r, x)\big). \tag{1.7}$$

The models (1.1) and (1.7) as well as related equations have been extensively studied recently, see [1–31] and [33–37] and the references therein. The existence of traveling wavefronts for the local equation (1.7) and the nonlocal equation (1.1) were obtained by So and Zou [29], So, Wu and Zou [26] and Liang and Wu [14] using the method of upper–lower solutions. The boundedness and the asymptotic behavior for the critical wave speed with respect to the delay time r as well as the diffusion rate D_m were analyzed by Wu, Wei and Mei in [33,34]. Furthermore, the stability of the traveling wavefronts was showed by Mei, So, Li and Shen [21], Mei and So [20] and recently in Lin and Mei [15]. For the local equation (1.7) with $b(v) = b_1(v)$ and $q = 1$, i.e. the Nicholson's blowflies equation, by using the weighted energy method, Mei, So, Li and Shen [21] proved that the wavefronts are stable for the wave speed $c > 2\sqrt{D_m(\varepsilon p - d_m)}$ with a sufficiently small initial perturbation, but the case when $c_* < c \leqslant 2\sqrt{D_m(\varepsilon p - d_m)}$ was left open, where $c_* > 0$ is the critical wave speed. Later by using the comparison principle together with the weighted energy method, Lin and Mei [15] further improved it to $c > c_*$ for the delay time $r \ll 1$, because $c_* \approx 2\sqrt{D_m(\varepsilon p - d_m)}$ as $r \ll 1$. The initial perturbation is also improved to be arbitrarily large in a weighted Sobolev space by the comparison principle and the squeeze technique. For the nonlocal equation (1.1) with $b(v) = b_1(v)$ and $q = 1$, the stability was obtained in [20] with a much strong restriction on the wave speed $c > 2\sqrt{D_m(3\varepsilon p - 2d_m)}$. The main goal in the present paper and our second paper [18] is to study the stability of all traveling wavefronts with $c > c_*$ and arbitrary delay time $r > 0$. In this paper, we focus on the local equation (1.7) and prove the stability of all wavefronts with arbitrary delay time r. Our approach is still the technical weighted energy method together with the comparison principle. But, inspired by Gourley [5] (see also an extension to the nonlinear stability by Li, Mei and Wong [11]) and based on the property of the critical wavefront speed, by selecting an ideal weight function and carefully taking the energy estimates, we obtain the stability for all waves including those slower waves. No restriction is needed for the delay time r and the speed c. For the nonlocal equation (1.1), the improved stability for all wavefronts with $c > c_*$ is the content of our second paper [18], where some new techniques are introduced to overcome the difficulty of the energy estimates caused by the nonlocal integral term.

For research related to other models of population dynamics, we refer to [1–3,5,6,11–13,16] and the references therein. See also the review articles [7,8] and the textbooks [32,35,36].

Notations. Throughout the paper, $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0, 1, 2, \ldots)$ represents a specific constant. Let I be an interval, typically $I = R$. $L^2(I)$ is the space of the square integrable functions defined on I, and $H^k(I)$ $(k \geqslant 0)$ is the Sobolev space of the L^2-functions $f(x)$

defined on the interval I whose derivatives $\frac{d^i}{dx^i}f$ $(i=1,\dots,k)$ also belong to $L^2(I)$. $L^2_w(I)$ denotes the weighted L^2-space with a weight function $w(x)>0$ and its norm is defined by

$$\|f\|_{L^2_w}=\biggl(\int_I w(x)\bigl|f(x)\bigr|^2\,dx\biggr)^{1/2}.$$

$H^k_w(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{H^k_w}=\left(\sum_{i=0}^k\int_I w(x)\left|\frac{d^i}{dx^i}f(x)\right|^2dx\right)^{1/2}.$$

Let $T>0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C^0([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$. $L^2([0,T];\mathcal{B})$ as the space of the $\mathcal{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

The rest of the paper is organized as follows. In Section 2, we introduce the travelling wavefronts, their properties and state the nonlinear stability theorem. In Section 3, after establishing the comparison principle and some key energy estimates in the weighted Sobolev spaces, we prove the nonlinear stability result.

2. Traveling wavefronts and stability theorem

It is easily seen that Eq. (1.7) has two constant equilibria $v_\pm$, where

$$\text{for } b_1(v):\quad v_-=0\quad\text{and}\quad v_+=\left(\frac{1}{a}\ln\frac{\varepsilon p}{d_m}\right)^{1/q},\tag{2.1}$$

$$\text{for } b_2(v):\quad v_-=0\quad\text{and}\quad v_+=\left(\frac{\varepsilon p-d_m}{ad_m}\right)^{1/q}.\tag{2.2}$$

A traveling wavefront of (1.7) connecting with $v_\pm$ is a monotone solution $v(t,x)=\phi(x+ct)$ satisfying

$$\begin{cases} c\phi'(\xi)-D_m\phi''(\xi)+d_m\phi(\xi)=\varepsilon b\bigl(\phi(\xi-cr)\bigr),\\ \phi(\pm\infty)=v_\pm,\end{cases}\tag{2.3}$$

where $\xi=x+ct$ and $'=\frac{d}{d\xi}$.

By using the upper–lower solutions method as in [14,26,29], we can, similarly, prove the existence of the traveling wavefronts.

Proposition 2.1 *(Existence of traveling wavefronts). Assume that*

$$\textit{for } b(v)=b_1(v):\quad 1<\frac{\varepsilon p}{d_m}\leqslant e^{1/q},\tag{2.4}$$

$$\begin{aligned}\textit{for } b(v)=b_2(v):\quad &\textit{either }\ 1<\frac{\varepsilon p}{d_m}\leqslant\frac{q}{q-1}\quad \textit{if } q>1,\\ &\textit{or } 1<\frac{\varepsilon p}{d_m}<\infty\quad \textit{if } 0<q\leqslant 1.\end{aligned}\tag{2.5}$$

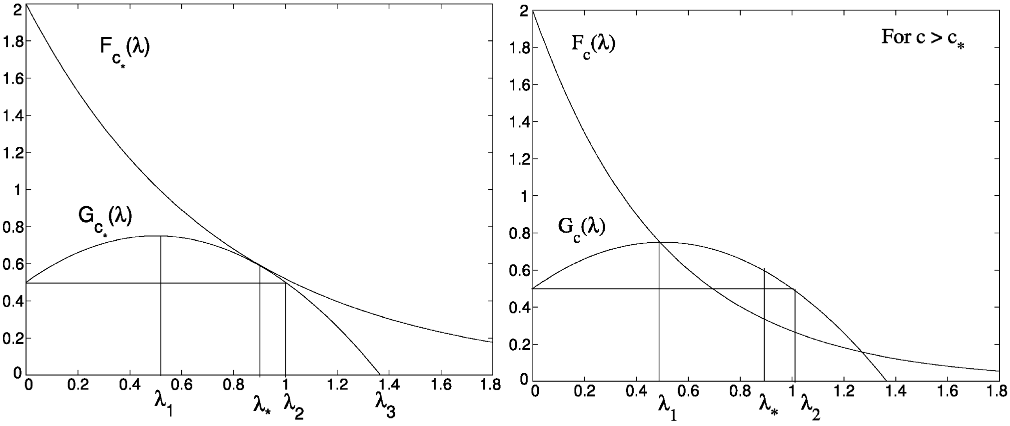

Fig. 2.1. The graphs of $F_c(\lambda)$ and $G_c(\lambda)$ for $c = c_*$ and $c > c_*$, respectively.

Then there exist a minimum speed $c_* = c_*(r, \varepsilon, D_m, d_m, p) > 0$ and a corresponding number $\lambda_* = \lambda(c_*) > 0$ satisfying

$$F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*), \qquad F'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*), \tag{2.6}$$

where

$$F_c(\lambda) = \varepsilon p e^{-\lambda c r}, \qquad G_c(\lambda) = c\lambda - D_m\lambda^2 + d_m, \tag{2.7}$$

and (c_*, λ_*) is the tangent point of $F_c(\lambda)$ and $G_c(\lambda)$, such that for all $c > c_*$, the travelling wavefront $\phi(x+ct)$ of Eq. (1.7) *connecting $v_\pm$ exists uniquely* (*up to a shift*)*.*

The graphs of $F_c(\lambda)$ and $G_c(\lambda)$ are shown in Fig. 2.1 for $c = c_*$ and $c > c_*$, respectively.

From the graphs, we can easily see that $F_c(\lambda_*) = G_c(\lambda_*)$ for $c = c_*$ and $F_c(\lambda_*) < G_c(\lambda_*)$ for $c > c_*$, that is,

$$\varepsilon p e^{-\lambda_* c_* r} = c_*\lambda_* - D_m\lambda_*^2 + d_m, \quad \text{for } c = c_*, \tag{2.8}$$

$$\varepsilon p e^{-\lambda_* c r} < c\lambda_* - D_m\lambda_*^2 + d_m, \quad \text{for } c > c_*. \tag{2.9}$$

By a straightforward calculation, it can be verified that the wavefront $\phi(\xi)$ decays to the constant state v_- as follows

$$\left|\phi(\xi) - v_-\right| = O(1)e^{-\lambda_1|\xi|}, \quad \text{as } \xi \to -\infty, \tag{2.10}$$

where $\lambda_1 = \lambda_1(c)$ is a c-depending function as showed in Fig. 2.1. It is also noted that

$$\varepsilon p e^{-\lambda c r} < c\lambda - D_m\lambda^2 + d_m, \quad \text{for } c > c_*,\ \lambda \in (\lambda_1, \lambda_*]. \tag{2.11}$$

Thus, we define a weight function as

$$w(x) = \begin{cases} e^{-2\lambda(x - x_0 - cr)}, & \text{for } x \leqslant x_0 + cr, \\ 1, & \text{for } x > x_0 + cr, \end{cases} \tag{2.12}$$

where λ is a positive number between $\lambda_1 = \lambda_1(c)$ and $\lambda_* = \lambda_*(c_*)$, i.e., $\lambda_1 < \lambda \leqslant \lambda_*$ (cf. Fig. 2.1), and the number x_0 is chosen to be sufficiently large such that (3.31) holds, i.e.,

$$b'\big(\phi(x_0 - cr)\big) < \frac{\min\{\lambda c + d_m - D_m\lambda^2, d_m\}}{\varepsilon \cosh(\lambda cr)}.$$

We can now state our main theorem.

Theorem 2.2 *(Nonlinear stability). Let $\frac{\varepsilon p}{d_m}$ satisfy* (2.4) *for $b(v) = b_1(v)$, or* (2.5) *for $b(v) = b_2(v)$, and let $b'(v_+)$ be sufficiently small. For any given wavefront $\phi(x+ct)$ with the speed $c > c_*$, if the initial data satisfy*

$$v_- \leqslant v_0(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R, \tag{2.13}$$

and the initial perturbation $v_0(s,x) - \phi(x+cs)$ is in $C([-r,0]; H^1_w(R))$, then the solution of (1.7) *and* (1.2) *satisfies*

$$v_- \leqslant v(t,x) \leqslant v_+, \quad \text{for } (t,x) \in R_+ \times R, \tag{2.14}$$

and

$$v(t,x) - \phi(x+ct) \in C\big([0,\infty); H^1_w(R)\big). \tag{2.15}$$

In particular, the solution $v(t,x)$ converges to the wavefront $\phi(x+ct)$ exponentially in time, that is,

$$\sup_{x\in R}\big|v(t,x) - \phi(x+ct)\big| \leqslant Ce^{-\mu t}, \quad t \geqslant 0, \tag{2.16}$$

for some positive number μ.

Remark 1.

1. As showed in Theorem 2.2, the wavefront stability holds for all wavefronts $\phi(x+ct)$ with $c > c_*$, and for all delay time $r > 0$, which essentially improves the previous stability result for the wavefronts with speed $c > 2\sqrt{D_m - \varepsilon p}$ in [21]. In another word, when the initial perturbation around the wavefront at $x \to -\infty$ is sufficiently small, the solution of (1.7) and (1.2) can be sufficiently approximated by this specified traveling wavefront, which is very important and significant in the physical application, as well as the issue of numerical simulation.
2. As the asymptotical profile, the wavefront $\phi(x+ct)$ satisfying $v_0(s,x) - \phi(x+cs) \in H^1_w(R)$ is uniquely selected. In fact, from the definition of the weight function (2.12) and the weighted Sobolev's space $H^1_w(R)$, we have

$$f \in H^1_w \quad \Leftrightarrow \quad \sqrt{w(x)}f \in H^1 \hookrightarrow C(R),$$

see also a proof in [19]. Thus, the condition $v_0(s,x) - \phi(x+cs) \in H^1_w(R)$ implies that

$$\big|v_0(s,x) - \phi(x+cs)\big| = O(1)e^{-\lambda|x|}, \quad \text{as } x \to -\infty,\ s \in [-r,0], \tag{2.17}$$

for $\lambda \in (\lambda_1, \lambda_*]$, while, from (2.11), we know that, for any two wavefronts $\phi(x+ct)$ and $\phi(x+ct+x_1)$, where x_1 is a shift, the difference is

$$\big|\phi(x+cs) - \phi(x+cs+x_1)\big| = O(1)e^{-\lambda_1|x|}, \quad \text{as } x \to -\infty,\ s \in [-r,0]. \tag{2.18}$$

Since $\lambda > \lambda_1$, i.e., the decay of $|v_0(s,x) - \phi(x+cs)|$ is much faster than $|\phi(x+cs) - \phi(x+cs+x_1)|$, the condition (2.17) determines a unique wavefront, which is the asymptotical profile of the original solution $v(t,x)$. In another word, once the initial perturbation around the wavefront $\phi(x+cs)$ at $x \to -\infty$ decays faster than $O(e^{-\lambda_1|x|})$, then the original solution $v(t,x)$ to Eqs. (1.7) and (1.2) behaves exactly as this traveling wavefront $\phi(x+ct)$.

3. Regarding the condition $b'(v_+) \ll 1$, actually it is natural. In fact, we can easily check that, for $b(v) = b_1(v)$,

$$b_1'(v_+) = \frac{d_m}{\varepsilon}\left[1 - \ln\left(\frac{\varepsilon p}{d_m}\right)^q\right] \to 0, \quad \text{as } \frac{\varepsilon p}{d_m} \to e^{1/q}.$$

Thus,

$$b_1'(v_+) = 0, \quad \text{for } \frac{\varepsilon p}{d_m} = e^{1/q}.$$

In this case, the wavefront strength $|v_+ - v_-|$ is the largest, and the wavefront is called the strongest wave. So we prove the stability for the strong wavefronts.
Similarly, for $b(v) = b_2(v_+)$, we have

$$b_2'(v_+) = \frac{d_m^2}{\varepsilon^2 p}\left[1 - (q-1)\frac{\varepsilon p - d_m}{d_m}\right] \to 0, \quad \text{as } \frac{\varepsilon p}{d_m} \to \frac{q}{q-1}, \text{ for } q > 1,$$

and thus

$$b_2'(v_+) = 0, \quad \text{for } \frac{\varepsilon p}{d_m} = \frac{q}{q-1}, \ q > 1.$$

3. Proof of main theorem

First of all, we state the following two lemmas which are given in [15].

Lemma 3.1 *(Boundedness). Let*

$$v_- = 0 \leqslant v_0(s,x) \leqslant v_+, \quad \textit{for } (s,x) \in [-r,0] \times R. \tag{3.1}$$

Then the solution $v(t,x)$ of the Cauchy problem (1.7) *and* (1.2) *satisfies*

$$v_- \leqslant v(t,x) \leqslant v_+, \quad \textit{for } (t,x) \in [0,\infty) \times R. \tag{3.2}$$

Lemma 3.2 *(Comparison principle). Let $\overline{v}(t,x)$ and $\underline{v}(t,x)$ be the solutions of* (1.7) *and* (1.2) *with the initial data $\overline{v}_0(s,x)$ and $\underline{v}_0(s,x)$, respectively. If*

$$v_- \leqslant \underline{v}_0(s,x) \leqslant \overline{v}_0(s,x) \leqslant v_+, \quad \textit{for } (s,x) \in [-r,0] \times R, \tag{3.3}$$

then

$$v_- \leqslant \underline{v}(t,x) \leqslant \overline{v}(t,x) \leqslant v_+, \quad \textit{for } (t,x) \in R_+ \times R. \tag{3.4}$$

As in [11,15], we will use the comparison principle and the weighed energy method to prove Theorem 2.2. Let the initial data $v_0(s,x)$ be such that $v_- \leqslant v_0(s,x) \leqslant v_+$ for $(s,x) \in [-r,0] \times R$, and define

$$\begin{cases} V_0^+(s,x) = \max\{v_0(s,x), \phi(x+cs)\}, \\ V_0^-(s,x) = \min\{v_0(s,x), \phi(x+cs)\}, \end{cases} \quad \text{for } (s,x) \in [-r,0] \times R. \tag{3.5}$$

Obviously,

$$v_- \leqslant V_0^-(s,x) \leqslant v_0(s,x) \leqslant V_0^+(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R, \tag{3.6}$$

$$v_- \leqslant V_0^-(s,x) \leqslant \phi(x+cs) \leqslant V_0^+(s,x) \leqslant v_+, \quad \text{for } (s,x) \in [-r,0] \times R. \tag{3.7}$$

Let $V^+(t,x)$ and $V^-(t,x)$ be the corresponding solutions of (1.7) and (1.2) with the initial data $V_0^+(s,x)$ and $V_0^-(s,x)$ respectively, i.e.,

$$\begin{cases} \dfrac{\partial V^\pm}{\partial t} - D_m \dfrac{\partial^2 V^\pm}{\partial x^2} + d_m V^\pm = \varepsilon b\big(V^\pm(t-r,x)\big), & (t,x) \in R_+ \times R, \\ V^\pm(s,x) = V_0^\pm(s,x), & x \in R,\ s \in [-r,0]. \end{cases} \tag{3.8}$$

By the comparison principle (Lemma 3.2), we have

$$v_- \leqslant V^-(t,x) \leqslant v(t,x) \leqslant V^+(t,x) \leqslant v_+, \quad \text{for } (t,x) \in R_+ \times R, \tag{3.9}$$

$$v_- \leqslant V^-(t,x) \leqslant \phi(x+ct) \leqslant V^+(t,x) \leqslant v_+, \quad \text{for } (t,x) \in R_+ \times R. \tag{3.10}$$

We will now prove the stability result (Theorem 2.2) in three steps.

Step 1. *The convergence of $V^+(t,x)$ to $\phi(x+ct)$.*
Let $\xi := x + ct$ and

$$u(t,\xi) := V^+(t,x) - \phi(x+ct), \qquad u_0(s,\xi) := V_0^+(s,x) - \phi(x+cs). \tag{3.11}$$

Then by (3.10) and (3.7), we have

$$u(t,\xi) \geqslant 0, \qquad u_0(s,\xi) \geqslant 0. \tag{3.12}$$

From Eq. (1.7), $u(t,\xi)$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial \xi} - D_m \dfrac{\partial^2 u}{\partial \xi^2} + d_m u - \varepsilon b'\big(\phi(\xi - cr)\big) u(t-r, \xi - cr) \\ \quad = \varepsilon Q(t-r, \xi - cr), \quad (t,\xi) \in R_+ \times R, \\ u(s,\xi) = u_0(s,\xi), \quad (s,\xi) \in [-r,0] \times R, \end{cases} \tag{3.13}$$

where

$$Q(t-r, \xi - cr) = b(\phi + u) - b(\phi) - b'(\phi) u \tag{3.14}$$

with $\phi = \phi(\xi - cr)$ and $u = u(t-r, \xi - cr)$.

Let $w(\xi) > 0$ be the weight function defined in (2.12). Multiplying (3.13) by $e^{2\mu t} w(\xi) u(t,\xi)$, where $\mu > 0$ will be specified later in Lemma 3.4, we obtain

$$\begin{aligned}
&\left\{\frac{1}{2}e^{2\mu t}wu^2\right\}_t + \left\{\frac{1}{2}e^{2\mu t}cwu^2 - D_m e^{2\mu t}wuu_\xi\right\}_\xi + D_m e^{2\mu t}wu_\xi^2 + D_m e^{2\mu t}w'uu_\xi \\
&\quad + \left\{-\frac{c}{2}\frac{w'}{w} + d_m - \mu\right\}e^{2\mu t}wu^2 - \varepsilon e^{2\mu t}wub'\big(\phi(\xi - cr)\big)u(t-r, \xi - cr) \\
&= \varepsilon e^{2\mu t}wuQ(t-r, \xi - cr). \qquad (3.15)
\end{aligned}$$

By the Cauchy–Schwarz inequality $|xy| \leqslant \frac{\delta}{2}x^2 + \frac{1}{2\delta}y^2$ for any $\delta > 0$ and then taking $\delta = 2$, we have

$$\left|D_m e^{2\mu t}w'uu_\xi\right| = D_m e^{2\mu t}w\left|u_\xi \cdot \frac{w'}{w}v\right| \leqslant D_m e^{2\mu t}wu_\xi^2 + \frac{D_m}{4}e^{2\mu t}\left(\frac{w'}{w}\right)^2 wu^2.$$

Applying the above inequality to (3.15), we obtain

$$\begin{aligned}
&\left\{\frac{1}{2}e^{2\mu t}wu^2\right\}_t + \left\{\frac{1}{2}e^{2\mu t}cwu^2 - D_m e^{2\mu t}wuu_\xi\right\}_\xi + \left\{-\frac{c}{2}\frac{w'}{w} + d_m - \mu - \frac{D_m}{4}\left(\frac{w'}{w}\right)^2\right\}e^{2\mu t}wu^2 \\
&\quad - \varepsilon e^{2\mu t}wub'\big(\phi(\xi - cr)\big)u(t-r, \xi - cr) \\
&\leqslant \varepsilon e^{2\mu t}wuQ(t-r, \xi - cr). \qquad (3.16)
\end{aligned}$$

Integrating (3.16) over $R \times [0, t]$ with respect to ξ and t yields

$$\begin{aligned}
&e^{2\mu t}\|u(t)\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu\tau}\left\{-c\frac{w'(\xi)}{w(\xi)} + 2d_m - 2\mu - \frac{D_m}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2\right\}w(\xi)u^2(\tau, \xi)\,d\xi\,d\tau \\
&\quad - 2\varepsilon \int_0^t \int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi - cr)\big)u(\tau, \xi)u(\tau - r, \xi - cr)\,d\xi\,d\tau \\
&\leqslant \|u_0(0)\|_{L_w^2}^2 + 2\varepsilon \int_0^t \int_R e^{2\mu\tau}w(\xi)u(\tau, \xi)Q(\tau - r, \xi - cr)\,d\xi\,d\tau. \qquad (3.17)
\end{aligned}$$

Using (2.1) and (2.4) for $b(v) = b_1(v)$, or, (2.2) and (2.5) for $b(v) = b_2(v)$, a straightforward computation gives

$$b_1'(\phi) = p\big(1 - aq\phi^q\big)e^{-a\phi^q} \geqslant 0, \quad \text{or} \quad b_2'(\phi) = \frac{p[1 - a(q-1)\phi^q]}{(1 + a\phi^q)^2} \geqslant 0$$

for $v_- \leqslant \phi \leqslant v_+$ with the corresponding constant equilibria $v_\pm$ given in (2.1) or (2.2), which gives

$$0 \leqslant b'(\phi) \leqslant p, \quad \text{for } v_- \leqslant \phi \leqslant v_+. \qquad (3.18)$$

Furthermore, using the Cauchy–Schwarz inequality again, we obtain

$$\begin{aligned}
&\left|2\varepsilon e^{2\mu\tau}w(\xi)b'\big(\phi(\xi - cr)\big)u(\tau, \xi)u(\tau - r, \xi - cr)\right| \\
&\quad \leqslant \varepsilon e^{2\mu\tau}w(\xi)b'\big(\phi(\xi - cr)\big)\left[\eta u^2(\tau, \xi) + \frac{1}{\eta}u^2(\tau - r, \xi - cr)\right]
\end{aligned}$$

for any positive constant η, which will be specified later in Lemma 3.3. Thus, by the change of variables (see [15,21] for details), the third term on the left-hand side of (3.17) can be bounded as follows:

$$\begin{aligned}
&\left|2\varepsilon\int_0^t\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-cr)\big)u(\tau,\xi)u(\tau-r,\xi-cr)\,d\xi\,d\tau\right|\\
&\quad\leqslant \varepsilon\eta\int_0^t\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-cr)\big)u^2(\tau,\xi)\,d\xi\,d\tau\\
&\qquad+\frac{\varepsilon}{\eta}\int_0^t\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-cr)\big)u^2(\tau-r,\xi-cr)\,d\xi\,d\tau\\
&\quad= \varepsilon\eta\int_0^t\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-cr)\big)u^2(\tau,\xi)\,d\xi\,d\tau\\
&\qquad+\frac{\varepsilon}{\eta}e^{2\mu r}\int_{-r}^{t-r}\int_R e^{2\mu\tau}w(\xi+cr)b'\big(\phi(\xi)\big)u^2(\tau,\xi)\,d\xi\,d\tau\\
&\quad\leqslant \varepsilon\eta\int_0^t\int_R e^{2\mu\tau}w(\xi)b'\big(\phi(\xi-cr)\big)u^2(\tau,\xi)\,d\xi\,d\tau\\
&\qquad+\frac{\varepsilon}{\eta}e^{2\mu r}\int_0^t\int_R e^{2\mu\tau}w(\xi+cr)b'\big(\phi(\xi)\big)u^2(\tau,\xi)\,d\xi\,d\tau\\
&\qquad+\frac{\varepsilon}{\eta}e^{2\mu r}\int_{-r}^0\int_R e^{2\mu\tau}w(\xi+cr)b'\big(\phi(\xi)\big)u_0^2(\tau,\xi)\,d\xi\,d\tau.
\end{aligned}\tag{3.19}$$

Substituting (3.19) into (3.17), we then have

$$\begin{aligned}
&e^{2\mu t}\|u(t)\|^2_{L^2_w}+\int_0^t\int_R e^{2\mu\tau}B_{\eta,\mu,w}(\xi)w(\xi)u^2(\tau,\xi)\,d\xi\,d\tau\\
&\quad\leqslant \|u_0(0)\|^2_{L^2_w}+2\varepsilon\int_0^t\int_R e^{2\mu\tau}w(\xi)u(\tau,\xi)Q(\tau-r,\xi-cr)\,d\xi\,d\tau\\
&\qquad+\frac{\varepsilon}{\eta}e^{2\mu r}\int_{-r}^0\int_R e^{2\mu\tau}w(\xi+cr)b'\big(\phi(\xi)\big)u_0^2(s,\xi)\,d\xi\,d\tau,
\end{aligned}\tag{3.20}$$

where

$$B_{\eta,\mu,w}(\xi):=A_{\eta,w}(\xi)-2\mu-\frac{\varepsilon}{\eta}\big(e^{2\mu r}-1\big)b'\big(\phi(\xi)\big)\frac{w(\xi+cr)}{w(\xi)}\tag{3.21}$$

and

$$A_{\eta,w}(\xi) := -c\frac{w'(\xi)}{w(\xi)} + 2d_m - \frac{D_m}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 - \varepsilon\eta b'\big(\phi(\xi - cr)\big) - \frac{\varepsilon}{\eta}\frac{w(\xi + cr)}{w(\xi)} b'\big(\phi(\xi)\big). \quad (3.22)$$

Using (2.1) and (2.4) for $b(v) = b_1(v)$, or (2.2) and (2.5) for $b(v) = b_2(v)$, another straightforward computation shows

$$b_1''(\phi) = -paq\phi^{q-1}\big(2 - aq\phi^q\big)e^{-a\phi^q} \leqslant 0, \quad \text{or} \quad b_2''(\phi) = -\frac{paq\phi^{q-1}[1 + q - a(q-1)\phi^q]}{(1 + a\phi^q)^3} \leqslant 0$$

for $v_- \leqslant \phi \leqslant v_+$ with the corresponding constant equilibria $v_\pm$ given in (2.1) or (2.2). This implies

$$b''(\phi) \leqslant 0, \quad \text{for } v_- \leqslant \phi \leqslant v_+. \quad (3.23)$$

Thus, by (3.23) and the Taylor's formula, we have

$$Q(t - r, \xi - cr) = b(\phi + u) - b(\phi) - b'(\phi)u = \frac{b''(\tilde{\phi})}{2!}u^2 \leqslant 0, \quad (3.24)$$

where $\tilde{\phi}$ is some function between ϕ and $\phi + u = V^+(t, x)$. With the help of (3.24) and the fact that $w > 0$, $u > 0$, we immediately obtain

$$2\varepsilon \int_0^t \int_R e^{2\mu\tau} w(\xi)u(\tau, \xi)Q(\tau - r, \xi - cr)\, d\xi\, d\tau \leqslant 0. \quad (3.25)$$

Finally, substituting (3.25) into (3.20), we obtain

$$\begin{aligned}
&e^{2\mu t}\|u(t)\|^2_{L^2_w} + \int_0^t \int_R e^{2\mu\tau} B_{\eta,\mu,w}(\xi)w(\xi)u^2(\tau, \xi)\, d\xi\, d\tau \\
&\quad \leqslant \|u_0(0)\|^2_{L^2_w} + \frac{\varepsilon}{\eta}e^{2\mu r}\int_{-r}^0 \int_R e^{2\mu\tau} w(\xi + cr)b'\big(\phi(\xi)\big)u_0^2(s, \xi)\, d\xi\, d\tau \\
&\quad \leqslant C_1\left(\|u_0(0)\|^2_{L^2_w} + \int_{-r}^0 \|u_0(\tau)\|^2_{L^2_w}\, d\tau\right) \qquad (3.26)
\end{aligned}$$

for some positive constant C_1.

The most important step now is to prove $B_{\eta,\mu,w}(\xi) \geqslant C > 0$ for some constant C. For this we need the following key lemma.

Lemma 3.3. *Let $\eta = e^{-\lambda cr}$. Then*

$$A_{\eta,w}(\xi) \geqslant C_2 > 0, \quad \xi \in R, \quad (3.27)$$

for some positive constant C_2.

Proof. *Case 1*: $\xi \leqslant x_0$. From (2.12), we have $w(\xi) = e^{-2\lambda(\xi - x_0 - cr)}$ and $w(\xi + cr) = e^{-2\lambda(\xi - x_0)}$. Notice that, $b'(\phi)$ is decreasing for $\phi \in (v_-, v_+)$ (since $b''(\phi) < 0$ as shown in (3.23)) and $\phi(\xi)$ is increasing for $\xi \in (-\infty, \infty)$. Thus, $b'(\phi(\xi))$ is decreasing in ξ. This implies

$$0 < b'\big(\phi(x_0)\big) < b'\big(\phi(\xi)\big) < b'\big(\phi(\xi - cr)\big) < b'\big(\phi(-\infty)\big) = b'(v_-) = p. \tag{3.28}$$

Notice also $\eta = e^{-\lambda cr}$. We then obtain

$$\begin{aligned} A_{\eta,w}(\xi) &= 2\lambda c + 2d_m - \frac{D_m}{2}(-2\lambda)^2 - \varepsilon\eta b'\big(\phi(\xi - cr)\big) - \frac{\varepsilon}{\eta} b'\big(\phi(\xi)\big)e^{-2\lambda cr} \\ &\geqslant 2\lambda c + 2d_m - 2D_m\lambda^2 - \varepsilon\eta p - \frac{\varepsilon}{\eta} p e^{-2\lambda cr} \\ &= 2\big(c\lambda + d_m - D_m\lambda^2 - \varepsilon p e^{-\lambda cr}\big) \\ &=: C_3 > 0, \end{aligned} \tag{3.29}$$

where we used (2.11) for the last step.

Case 2: $x_0 < \xi \leqslant x_0 + cr$. In this case, we have $w(\xi) = e^{-2\lambda(\xi - x_0 - cr)}$ and $w(\xi + cr) = 1$. Since $b'(\phi(\xi))$ is decreasing in ξ, i.e., $b'(\phi(x_0 + cr)) < b'(\phi(\xi)) \leqslant b'(\phi(x_0)) < b'(\phi(\xi - cr)) \leqslant b'(\phi(x_0 - cr))$, and using the fact $e^{2\lambda(\xi - x_0 - cr)} \leqslant e^0 = 1$ due to $\xi - x_0 - cr \leqslant 0$, we then obtain

$$\begin{aligned} A_{\eta,w}(\xi) &= 2\lambda c + 2d_m - 2D_m\lambda^2 - \varepsilon\eta b'\big(\phi(\xi - cr)\big) - \frac{\varepsilon}{\eta} b'\big(\phi(\xi)\big)e^{2\lambda(\xi - x_0 - cr)} \\ &\geqslant 2\lambda c + 2d_m - 2D_m\lambda^2 - \varepsilon\eta b'\big(\phi(x_0 - cr)\big) - \frac{\varepsilon}{\eta} b'\big(\phi(x_0 - cr)\big) \\ &= 2\lambda c + 2d_m - 2D_m\lambda^2 - 2\varepsilon b'\big(\phi(x_0 - cr)\big)\left(\frac{\eta}{2} + \frac{1}{2\eta}\right) \\ &= 2\lambda c + 2d_m - 2D_m\lambda^2 - 2\varepsilon b'\big(\phi(x_0 - cr)\big)\cosh(\lambda cr) \\ &=: C_4. \end{aligned} \tag{3.30}$$

Now we need to prove $C_4 > 0$. In fact, since $b'(v_+) \ll 1$ (we assume it as a sufficient condition in Theorem 2.2, see also Remark 1 for the explanation), we may take

$$b'(v_+) < \frac{\min\{\lambda c + d_m - D_m\lambda^2, d_m\}}{\varepsilon\cosh(\lambda cr)}.$$

Let x_0 be sufficiently large such that

$$\big|b'(v_+) - b'\big(\phi(x_0 - cr)\big)\big| \ll 1.$$

In that case, we have

$$b'\big(\phi(x_0 - cr)\big) < \frac{\min\{\lambda c + d_m - D_m\lambda^2, d_m\}}{\varepsilon\cosh(\lambda cr)}. \tag{3.31}$$

With this help, we further obtain from (3.30) that

$$A_{\eta,w}(\xi) \geqslant 2\big[\lambda c + d_m - D_m\lambda^2 - \varepsilon b'\big(\phi(x_0 - cr)\big)\cosh(\lambda cr)\big] = C_4 > 0. \tag{3.32}$$

Case 3: $\xi > x_0 + cr$. In this case, $w(\xi) = w(\xi + cr) = 1$. Since $b'(\phi(\xi))$ is decreasing in ξ, we have

$$\begin{aligned} A_{\eta,w}(\xi) &= 2d_m - \varepsilon\eta b'\big(\phi(\xi - cr)\big) - \frac{\varepsilon}{\eta} b'\big(\phi(\xi)\big) \\ &\geqslant 2d_m - \varepsilon\eta b'\big(\phi(x_0 - cr)\big) - \frac{\varepsilon}{\eta} b'\big(\phi(x_0 - cr)\big) \\ &= 2d_m - 2\varepsilon b'\big(\phi(x_0 - cr)\big)\cosh(\lambda cr) \\ &=: C_5 > 0, \end{aligned} \tag{3.33}$$

because of (3.31).

Finally, let

$$C_2 := \min\{C_3, C_4, C_5\}. \tag{3.34}$$

Then (3.29), (3.32) and (3.33) imply (3.27). $\square$

Lemma 3.4. *It holds that*

$$B_{\mu,\eta,w}(\xi) > 0, \quad \xi \in (-\infty, \infty), \tag{3.35}$$

for $0 < \mu < \mu_1$, *where* $\mu_1 > 0$ *is the unique root of the following equation*

$$C_2 - 2\mu_1 - \frac{\varepsilon p}{\eta}\big(e^{2\mu_1 r} - 1\big) = 0. \tag{3.36}$$

Proof. As shown in Lemma 3.3, we have $0 < b'(\phi(\xi)) < p$ for $\xi \in (-\infty, \infty)$, and

$$\frac{w(\xi + cr)}{w(\xi)} = \begin{cases} e^{-2\lambda cr} < 1, & \text{for } \xi \leqslant x_0, \\ e^{2\lambda(\xi - x_0 - cr)} \leqslant e^0 = 1, & \text{for } x_0 < \xi \leqslant x_0 + cr, \\ 1, & \text{for } \xi > x_0 + cr. \end{cases}$$

It follows immediately that

$$\begin{aligned} B_{\mu,\eta,w}(\xi) &= A_{\eta,w}(\xi) - 2\mu - \frac{\varepsilon}{\eta}\big(e^{2\mu r} - 1\big) b'\big(\phi(\xi)\big)\frac{w(\xi + cr)}{w(\xi)} \\ &\geqslant C_2 - 2\mu - \frac{\varepsilon p}{\eta}\big(e^{2\mu r} - 1\big) > 0 \end{aligned} \tag{3.37}$$

for $0 < \mu < \mu_1$. $\square$

By dropping the positive term $\int_0^t \int_R e^{2\mu\tau} B_{\eta,\mu,w}(\xi) w(\xi) u^2(\tau, \xi)\, d\xi\, d\tau$ in (3.26), we obtain the first basic energy estimate.

Lemma 3.5. *It holds that*

$$e^{2\mu t}\|u(t)\|_{L_w^2}^2 \leqslant C_1\left(\|u_0(0)\|_{L_w^2}^2 + \int_{-r}^{0} \|u_0(\tau)\|_{L_w^2}^2\, d\tau\right), \quad t \geqslant 0. \tag{3.38}$$

Next, differentiating (3.13) with respect to ξ, and multiplying it by $e^{2\mu t}w(\xi)u_\xi(t,\xi)$, then integrating the resultant equation with respect to (t,ξ) over $[0,t]\times R$, and using the basic energy estimates (3.38) in Lemma 3.5, we can similarly prove the second energy estimate.

Lemma 3.6. *It holds that*

$$e^{2\mu t}\left\|u_\xi(t)\right\|^2_{L^2_w}\leqslant C_6\left(\left\|u_0(0)\right\|^2_{H^1_w}+\int_{-r}^{0}\left\|u_0(\tau)\right\|^2_{H^1_w}\,d\tau\right) \tag{3.39}$$

for some positive constant C_6.

Combining Lemmas 3.5 and 3.6, and noting $w(\xi)\geqslant 1$ for all $\xi\in R$, from the definition of w, we obtain the following decay rate.

Lemma 3.7. *It holds that*

$$\left\|u(t)\right\|_{H^1}\leqslant\left\|u(t)\right\|_{H^1_w}\leqslant C_7e^{-\mu t}\left(\left\|u_0(0)\right\|^2_{H^1_w}+\int_{-r}^{0}\left\|u_0(\tau)\right\|^2_{H^1_w}\,d\tau\right)^{\frac{1}{2}},\quad t\geqslant 0, \tag{3.40}$$

where $C_7=\max\{\sqrt{C_1},\sqrt{C_6}\}$.

Using Sobolev embedding theorem $H^1(R)\hookrightarrow C^0(R)$, we finally have the following stability result.

Lemma 3.8. *It holds that*

$$\sup_{x\in R}\left|V^+(t,x)-\phi(x+ct)\right|=\sup_{\xi\in R}\left|u(t,\xi)\right|\leqslant C_8e^{-\mu t},\quad t\geqslant 0, \tag{3.41}$$

for some positive constant C_8.

Step 2. *The convergence of $V^-(t,x)$ to $\phi(x+ct)$.*
Let $\xi=x+ct$ and

$$u(t,\xi)=\phi(x+ct)-V^-(t,x),\qquad u_0(s,\xi)=\phi(x+cs)-V_0^-(s,x). \tag{3.42}$$

As in Step 1, we can similarly prove that $V^-(t,x)$ converges to $\phi(x+ct)$.

Lemma 3.9. *It holds that*

$$\sup_{x\in R}\left|V^-(t,x)-\phi(x+ct)\right|=\sup_{\xi\in R}\left|u(t,\xi)\right|\leqslant C_9e^{-\mu t},\quad t\geqslant 0, \tag{3.43}$$

for some positive constant C_9.

Step 3. *The convergence of $v(t,x)$ to $\phi(x+ct)$.*
Finally, we prove that $v(t,x)$ converges to $\phi(x+ct)$ as follows.

Lemma 3.10. *It holds that*

$$\sup_{x\in R}\left|v(t,x)-\phi(x+ct)\right|\leqslant C_{10}e^{-\mu t},\quad t\geqslant 0, \tag{3.44}$$

for some positive constant C_{10}.

Proof. Since the initial data satisfy $V_0^-(x,s)\leqslant v_0(x,s)\leqslant V_0^+(x,s)$, from Lemma 3.2, it can be verified that the corresponding solutions of (1.7) and (1.2) satisfy

$$V^-(t,x)\leqslant v(t,x)\leqslant V^+(t,x),\quad (t,x)\in R_+\times R.$$

Thanks to Lemmas 3.8 and 3.9, we have the following convergence results,

$$\sup_{x\in R}\left|V^-(t,x)-\phi(x+ct)\right|\leqslant C_9e^{-\mu t},\qquad \sup_{x\in R}\left|V^+(t,x)-\phi(x+ct)\right|\leqslant C_8e^{-\mu t}.$$

Combining these inequalities, the squeeze technique gives

$$\sup_{x\in R}\left|v(t,x)-\phi(x+ct)\right|\leqslant C_{10}e^{-\mu t},\quad t>0,$$

for some positive constant C_{10}. This completes the proof. □

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