A new 2D discrete model applied to dynamic crack propagation in brittle materials

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ARTICLE INFO

Article history:
Received 25 November 2013
Received in revised form 16 July 2014
Available online 26 July 2014

Keywords:
Discrete models
Elastic behaviour
Crack branching
Dynamic fracture

ABSTRACT

In this work, a 2D discrete model (DM) applied to the dynamic crack propagation in brittle materials is developed and implemented. The proposed model is based on a particular discretization of Navier's equations, presenting similarities to the Born model, with the advantage that the constants appearing in it are explicitly related to the elastic properties. This model overcomes the limitations in the choice of Poisson's ratio present in other discrete models. Three numerical examples are presented to show the capability of this method in modelling wave propagation and dynamic fracture problems. The obtained results are in agreement with experimental and numerical results reported by other researchers.

1. Introduction

The finite element method (FEM) is the most widely used numerical procedure in the field of mechanics of solids and structural analysis. Although the classical FEM has proved to be a powerful tool in these areas, it has some limitations in the analysis of certain problems such as dynamic fracture (crack initiation, crack propagation, branching cracks with complex crack patterns) and fragmentation problems.

However, several developments have been done to apply the FEM framework to dynamic fracture. Nishioka (2001), Nishioka et al. (2001) and Khan et al. (2007) introduced the dynamic J-integral into a moving finite element mesh that needs a re-meshing algorithm with a very fine mesh around the crack tip. Belytschko and Black (1999) and Moes et al. (1999) suggested an extended FEM (XFEM) as a methodology for modelling cracks of arbitrary geometry in a FEM with minimal remeshing effort. The method was first applied to dynamic crack propagation by Belytschko et al. (2003). Subsequently, other applications and extensions were developed (Remmers et al., 2003; Hansbo and Hansbo, 2004; Rethore et al., 2005; Areias and Belytschko, 2006; Song et al., 2006; Fries and Belytschko, 2010; Campilho et al., 2011) among others.

In the cohesive finite element methods (CFEM) the crack is modelled as a separation along the element's edges. These methods are based on the cohesive zone concept, introduced by Dugdale (1960) and Barenblatt (1962), and its was originally formulated by Xu and Needleman (1994a) and further developed by other authors (Camacho and Ortiz, 1996; Pandolfi et al., 1999; Ruiz et al., 2000; Zhai et al., 2004; Mergheim et al., 2005; Zeng and Li, 2010; He and Li, 2012).

Other numerical techniques available to simulate problems of solid mechanics are the Meshfree or Meshless methods (MMs). These methods try to overcome part of the difficulties encountered when the approximation procedure is based on a mesh. There exist several kinds of MMs (Nguyen et al., 2008; Liu, 2010), all of which only involve nodes. The most popular methods for dynamic fracture are the Material Point Method (MPM) (Sulsky et al., 1994), Smoothed Particle Hydrodynamics (SPH) (Lucy, 1977) and specific versions of the Element-Free Galerkin Method (EFG) (Belytschko et al., 1994; Belytschko and Tabbara, 1996; Belytschko and Fleming, 1999; Rabczuk and Zi, 2007; Zi et al., 2007; Rabczuk et al., 2007; Bordas et al., 2008; Zhuang and Augarde, 2010; Peng et al., 2011).

All these models have increasingly added complexity that has led to the development of discrete models. These models, in addition to being simpler, are capable of solving complicated cracks propagation problems. Two different discrete approaches have been developed to simulate the mechanical behaviour of solids: particle-based and lattice-based models.

The particle modelling (PM) was introduced by Cundall (1971) and Cundall and Strack (1979), and further developed by Greenspan (1997), Meguro and Tagel-Din (2000), Öhâte et al. (2004) and Monaghan (2005). In PM each computational cell is considered as a particle with the mass lumped to its center, and its evolution in time is defined by dynamic equilibrium of forces acting on the particle. In this sense, these methods can be regarded
as an upscale MD, but applied to large length and time scales problems.

The peridynamics theory, first proposed by Silling (2000) inside this group can be considered. It is a nonlocal formulation of classical solid mechanics, in which every material point is connected to its neighbour inside a certain region (called “horizon”) through peridynamic bonds. In this way, instead of the divergence of stress terms in the classical equations, an integral over the horizon of the current point of forces per unit volume squared is used. Dynamic crack propagation using peridynamics theory has been investigated in recent years (Ha and Bobaru, 2010, 2011; Agwai et al., 2011).

The lattice-based approach (LM) represents the continuum by material particles interacting via the network elements. From the pioneer work of Hrennikoff (1941), different kinds of lattice models have been used in modelling solid problems (Nayfeh and Hefzy, 1978; Noor, 1988; Herrmann et al., 1989; Ostoj-Starzewski, 2002; Bolander and Sukumar, 2005; Berton and Bolander, 2006; Rinaldi and Lai, 2007; Rinaldi et al., 2008; Kosteski et al., 2012).

Based on the particle model proposed by Wang and Ostoj-Starzewski (2005), hybrid models (HLP) which combine some features of the particle-based and lattice-based have been recently developed (Wang et al., 2009). In the HLP methods the particle–particle interaction is derived from LM theory whereas the computational scheme follows the PM technique.

On the other hand, certain discretization techniques applied over the continuum mechanics laws lead to models that could be devised as discrete. Thus, Martín et al. (2000) construct a discrete lattice model from the continuum equations of elasticity. The discretization scheme is performed on a triangular lattice and the procedure leads to a formulation in which the nodes of the mesh can be considered as actual particles that interact with prescribed laws of force. The resulting force law coincides with the Born model (Born and Huang, 1954).

Most of the aforementioned discrete models have restrictions with the Poisson’s ratio to be considered. In some models, the restrictions are related to instability problems motivated by the negative stiffness obtained for certain values of Poisson’s ratio. That is the case of the models developed by Martin et al. (2000) and Zhao et al. (2010) which leads to an unstable behaviour for Poisson’s ratios greater than 0.25. In others models (Kosteski et al., 2012), the Poisson’s ratio should be exactly equal to 0.25 in 2D problems, in order to guarantee a consistent equivalence between the discrete model and the isotropic continuum.

In this paper, we propose a discrete model (DM) consisting of a discretization of Navier’s equations of elastodynamics once the total displacements (Landau and Lifshitz, 1986) has been decomposed into the sum of two terms corresponding to normal and transversal displacements. Taking advantage of the properties of the two kinds of displacements, the equations governing the evolution of each of them are discretized separately.

The method is used to solve some 2D benchmark problems related to wave travelling, as well as crack propagation and branching, showing that the method is stable in the whole range of Poisson’s ratio and its results are in good accordance with those obtained with other numerical and experimental procedures.

2. Description of the model

Since the proposed model consists in a special discretization of the equation of elastodynamics, firstly we present a discretization method previously developed by Martin et al. (2000). In the second part, we describe the model developed in this work highlighting its advantages. Finally, the fracture criterion applied in this work is presented.

2.1. Discretization of operators

The classical equation of elastodynamics for the displacement field \( \mathbf{u} \) in a linear elastic homogeneous materials is (Landau and Lifshitz, 1986),

\[
\ddot{\mathbf{u}} = c_i^2 \nabla^2 \mathbf{u} + (c_i^2 - c_j^2) \text{grad} \text{div} \mathbf{u},
\]

where the transverse \( c_t \) and longitudinal \( c_l \) sound speeds are material properties related to the mass density \( \rho \), Young’s modulus \( E \), and Poisson’s ratio \( \nu \). The case of plane strain, as corresponds to the situations considered in this work, the expressions for the transverse and longitudinal wave speeds are, respectively.

\[
c_t = \sqrt{\frac{E}{2\rho(1 + \nu)}}, \quad c_l = \sqrt{\frac{E(1 - \nu)}{\rho(1 + \nu)(1 - 2\nu)}}.
\]

Martín et al. (2000) have formulated the problem of discretizing equation (1) on an arbitrary mesh. In their work, \( f(r) \) is considered to be any scalar field and there exists a sample of points \( X \), distributed arbitrarily in the vicinity of a given point \( r \). The value of the scalar field at those points is denoted by \( f_j = f(r_j) \).

The procedure is formulated as an optimization problem and the solution essentially consists of finding the best paraboloid that fits points and contains the point \( r, f \). The equation of this paraboloid is

\[
P(r) = f_j + A \cdot (r - r_j) + \frac{1}{2} B \cdot (r - r_j)(r - r_j).
\]

A Taylor expansion of \( f(r) \) around \( r \), shows that \( A \) is an approximation for the gradient of \( f \) at \( r \), and \( B \) is an approximation for the matrix of second derivatives. We can calculate the corresponding square error function \( \phi \) as

\[
\phi(A, B) = \sum_{j=1}^{M} |P(r_j) - f_j|^2.
\]

By minimizing \( \phi \) with respect to \( A \) and \( B \), we will obtain the paraboloid that best fits the points \( r, f \).

\[
\frac{\partial \phi}{\partial A} = 2 \left( -F_1 + R_2 \cdot A + \frac{1}{2} R_3 \cdot B \right) = 0,
\]

\[
\frac{\partial \phi}{\partial B} = 2 \left( -F_2 + R_3 \cdot A + \frac{1}{2} R_4 \cdot B \right) = 0,
\]

where \( F_1 \) and \( F_2 \) are defined as follows

\[
F_1 = \sum_{j=1}^{M} (f_j - f_i)(r_j - r_i),
\]

\[
F_2 = \sum_{j=1}^{M} (f_j - f_i)(r_i - r_j)(r_j - r_i),
\]

and

\[
R_2 = \sum_{j=1}^{M} (r_j - r_i)(r_j - r_i),
\]

\[
R_3 = \sum_{j=1}^{M} (r_j - r_i)(r_j - r_i)(r_j - r_i),
\]

\[
R_4 = \sum_{j=1}^{M} (r_j - r_i)(r_j - r_i)(r_j - r_i)(r_j - r_i),
\]
where the variables \( \mathbf{R} \) are defined as a function of the initial lattice and the number of neighbours considered in the analysis. The condition of minimum provides

\[
\mathbf{F}_1 = \mathbf{R}_2 \cdot \mathbf{A} + \frac{1}{2} \mathbf{R}_3 \cdot \mathbf{B},
\]

\[
\mathbf{F}_2 = \mathbf{R}_3 \cdot \mathbf{A} + \frac{1}{2} \mathbf{R}_4 \cdot \mathbf{B}.
\]

This is a system of linear equations in which \( \mathbf{A} \) and \( \mathbf{B} \) are the unknowns. In their work, Martin et al. (2000) consider only near neighbours in a regular triangular lattice of spacing \( a \) in order to construct the discrete derivatives (see Fig. 1). In this case, \( \mathbf{A} \) and \( \mathbf{B} \) take the following form,

\[
\mathbf{A} = \nabla f_i = \frac{1}{3} \sum_{j=1}^{6} \frac{f_j - f_i}{a} \mathbf{r}_{ij},
\]

\[
\mathbf{B} = \nabla \nabla \mathbf{f}_i = \nabla \left[ \frac{2}{3} \sum_{j=1}^{6} \frac{f_j - f_i}{a^2} \right] - \frac{1}{3} \sum_{j=1}^{6} \frac{f_j - f_i}{a^2} \mathbf{I},
\]

where \( \mathbf{r}_{ij} = (\mathbf{r}_j - \mathbf{r}_i) / a \) and \( \mathbf{I} \) is the identity matrix. The Laplacian is given by the trace of \( \mathbf{B} \)

\[
\nabla^2 \mathbf{f}_i = \frac{2}{3} \sum_{j=1}^{6} \frac{f_j - f_i}{a^2}.
\]

Thus, these operators are discretized and can be applied to any set of partial differential equations, in particular to equations of elastodynamics. Substituting Eqs. (15) and (16) in Eq. (1) we obtain the following spatial discretization for a particular node \( i \),

\[
\mathbf{u}_i = \left[ c_l^2 - c_t^2 / 3 \right] \sum_{j=1}^{6} \frac{\mathbf{u}_j}{a^2} + \left[ 4(c_l^2 - c_t^2) / 3a^2 \right] \sum_{j=1}^{6} \mathbf{u}_{ij} \cdot \mathbf{n}_i \mathbf{n}_j,
\]

\( \mathbf{n}_i \) being the normal unit vector pointing from particle \( i \) to particle \( j \), and \( \mathbf{u}_{ij} \) is the relative displacement,

\[
\mathbf{u}_i = \mathbf{u}_j - \mathbf{u}_i.
\]

Eq. (17) can be interpreted as the equations of motion for a set of unit cell in the lattice, interacting with their nearest neighbours with a linear force law.

The applicability of this equation is limited by the fact that it becomes unstable for \( r > 0.25 \). This particular value of Poisson’s ratio is the one at which the coefficient of the first term of Eq. (17) changes sign. The limitations in the choice of Poisson’s ratio are also present in other discrete models (Wang and Ostoja-Starzewski, 2005; Kosteski et al., 2012; Wang et al., 2009).

### 2.2. The proposed DM model

To overcome the cited limitation in representing Poisson’s ratio, we propose a model based on a decomposition of the displacement field in normal and transversal components. A force proportional to an acceleration is generated from the interaction between the particles \( i \) and \( j \). This acceleration can be decomposed in two directions,

\[
\mathbf{u}_i = \mathbf{u}^n_i + \mathbf{u}^t_i.
\]

where \( \mathbf{u}^n_i \) corresponds to the direction of the vector \( \mathbf{n}_i \) (normal direction), and \( \mathbf{u}^t_i \) is a vector normal to it (transversal direction) (see Fig. 2).

Therefore, we can write the vector \( \mathbf{u} \) as the sum of a normal and transversal displacement, corresponding to the normal and transversal accelerations respectively. These two directions are defined from the initial lattice, and do not change throughout the analysis.

\[
\mathbf{u} = \mathbf{u}_n + \mathbf{u}_t.
\]

Substituting (20) and (19) in (1), we obtain

\[
\mathbf{u}_n + \mathbf{u}_t = c_l^2 \nabla^2 (\mathbf{u}_n + \mathbf{u}_t) + (c_l^2 - c_t^2) \nabla \nabla (\mathbf{u}_n).
\]

On the other hand normal accelerations do not involve any rotation in the unit cell and satisfy \( \nabla \mathbf{u}_n = 0 \), while transversal accelerations do not involve any change in volume in the unit cell and satisfy \( \nabla \mathbf{u}_t = 0 \).

Taking the divergence on both sides of Eq. (21), recalling that \( \nabla \mathbf{u}_n = 0 \), and using the identity \( \nabla \nabla = \nabla^2 \), we find

\[
\nabla \cdot \mathbf{u}_n = c_l^2 \nabla^2 \nabla (\mathbf{u}_n + \mathbf{u}_t) + (c_l^2 - c_t^2) \nabla^2 \nabla \mathbf{u}_n
\]

and recalling that \( \nabla \mathbf{u}_n = 0 \),

\[
\nabla \cdot (\mathbf{u}_n - c_l^2 \nabla^2 \mathbf{u}_n) = 0.
\]

The curl of the expression in parentheses is also zero, by \( \nabla \cdot \mathbf{u}_n = 0 \). If the curl and divergence of a vector both vanish in space, that vector must be zero identically. Therefore

\[
\frac{\partial^2 \mathbf{u}_n}{\partial t^2} - c_l^2 \nabla^2 \mathbf{u}_n = 0.
\]

Similarly, taking the curl of Eq. (21),

\[
\nabla \times \mathbf{u}_t = c_l^2 \nabla \times \nabla (\mathbf{u}_n + \mathbf{u}_t) + (c_l^2 - c_t^2) \nabla \times \nabla \mathbf{u}_n,
\]

recalling that the curls of \( \mathbf{u}_n \) and of any gradient are zero, we get that

\[
\nabla \times (\mathbf{u}_t - c_l^2 \nabla^2 \mathbf{u}_n) = 0.
\]

Since the divergence of the expression in parentheses is also zero, we obtain an equation for \( \mathbf{u}_t \) of the same form as Eq. (24),

\[
\frac{\partial^2 \mathbf{u}_t}{\partial t^2} - c_t^2 \nabla^2 \mathbf{u}_t = 0.
\]

![Fig. 1. A triangular lattice with a hexagonal unit cell.](image1)

![Fig. 2. Sketch of interactions between particles.](image2)
We can write the motion equation as the sum of Eq. (24),

\[ \ddot{u} = u_\text{n} + \ddot{u}_\text{t} = c_1^2 \nabla^2 u_\text{n} + c_1^2 \nabla^2 u_\text{t}. \]  

(28)

Applying Eq. (16) to Eq. (28), we obtain the equation of the motion in the case of plane strain discretized in a triangular lattice of spacing \( a \),

\[ \ddot{u}_i = \frac{2c_1^2}{3a^2} \sum_{j=1}^{6} u_{ij}^n + \frac{2c_1^2}{3a^2} \sum_{j=1}^{6} u_{ij}^t, \]  

(29)

where \( u_{ij}^n = (u_j - u_i)_n \) is the vector of normal displacement, and \( u_{ij}^t \) is the relative transversal displacement, which can be obtained as

\[ u_{ij}^t = u_j - u_i^n. \]  

(30)

Note that with Eq. (29) the problem of instability appearing for \( v \) values higher than 0.25 is solved.

As reported in the work of Martín et al. (2000), the discretized equation of motion could be obtained from the Born potential (Born and Huang, 1954), in which the energy of the bond between neighbouring sites \( i \) and \( j \) can be written as (Heino and Kaski, 1997),

\[ H_{ij} = \frac{\alpha}{2} \left( u_{ij}^n \right)^2 + \frac{\beta}{2} \left( u_{ij}^t \right)^2, \]  

(31)

where \( \alpha \) relates to tensile and \( \beta \) relates to bending stiffness of the Born model. In Heino and Kaski (1997), no equation is provided relating these constants to material properties, but their values are arbitrarily selected. Now the equation of motion for a mass site \( i \) can be written

\[ m_i \ddot{u}_i = \text{grad} \sum_J H_{ij} = 2 \sum_J u_{ij}^n + \beta \sum_J u_{ij}^t. \]  

(32)

Eq. (32), corresponding to the Born model, has the same shape as Eq. (29), developed in this work. Therefore, \( \alpha \) and \( \beta \) parameters are defined as a function of the material’s macroscopic elastic properties, such as Young’s modulus and Poisson’s ratio.

In order to integrate the equations of motion we use the Verlet algorithm (Allen and Tildesley, 1987), which provides a direct solution for second-order differential equations. This method does not involve velocities calculation, reducing the computational cost, but is based on positions \( u(t) \), accelerations \( \ddot{u}(t) \), and positions from the previous step \( u(t - \Delta t) \). The equation for temporal advancing positions reads

\[ u(t + \Delta t) = 2u(t) - u(t - \Delta t) + \Delta t^2 \ddot{u}(t) + O(\Delta t^4). \]  

(33)

If necessary to calculate energy variables, the speed can be calculated as

\[ v(t) = \frac{u(t + \Delta t) - u(t - \Delta t)}{2\Delta t} + O(\Delta t^2). \]  

(34)

To ensure that numerical errors do not grow rapidly in time, we apply the Courant–Friedrichs–Lewy criterion (Bathe, 1996), where the time increment depends on the discrete spacing and the material properties,

\[ \Delta t \leq \frac{a}{\sqrt{c_1}}. \]  

(35)

2.3. Fracture criterion

The fracture criteria proposed by Martín et al. (2000) and Vadluga and Kacianauskas (2007), gives rise to cracks that propagate through unbroken bonds, resulting in spurious breakage of
In this section, we study the capability of the presented model to solve three selected problems. The first one consists in the analysis of wave propagation through a strip for different values of Poisson's ratio with the finality to demonstrate the stability of our DM. The other problems show the ability of proposed DM to capture the main features of crack propagation process, determined by the crack advance speed and the fracture patterns.

3.1. Wave propagation through a strip

Wave propagation can be viewed as the transmission of dynamic loads through materials, an important research issue in dynamic failure study. Therefore we resort to the analysis of the elastodynamic behaviour of a strip with the aim of ensuring the stability of developed lattice model.

A two-dimensional rectangular solid domain is considered as a representative example. Its geometry is defined by two characteristic dimensions $H = 10.4 \text{ mm}$, $B = 200 \text{ mm}$ (see Fig. 3), and the lattice spacing $a = 1 \text{ mm}$. A linear elastic material is considered, characterized by its density $\rho = 2600 \text{ kg/m}^3$ and Young's modulus $E = 80 \text{ GPa}$.

Boundary conditions consist of a half cycling in both directions (amplitude 1 mm and frequency 50 kHz) at one end of the body, while the other three sides are straight and free. Two detection points $(50,5.2), R(100,5.2)$ are placed in the strip to record the wave propagation.

Simulation results are presented in Fig. 4, where longitudinal and transversal wave speed as a function of the Poisson's ratio are compared to the predictions from Eq. (2). Velocities are normalized with respect to Rayleigh's speed given by Landau and Lifshitz (1986),

$$c_R = \frac{(0.862 + 1.14\nu)}{(1 + \nu)}c_1. \tag{39}$$

The error being below 2.0% and presenting the same magnitude as those reported by other authors (Zhao et al., 2010). The results show that the method is stable and accurate for the entire range of values of Poisson's ratio, and the wave propagation speed are in good agreement with theoretical estimates independently of the value of \(\nu\) used.
3.2. Pre-notched glass sheet under tensile loading

Crack branching problem has been studied by many authors to verify different numerical methods (Xu and Needleman, 1994b; Belytschko et al., 2003; Rabczuk and Belytschko, 2004; Gupta et al., 2011). The description of the problem is depicted in Fig. 5, consisting of a finite rectangular plate of glass with an edge notch. A prescribed stress \( \sigma_y \) of 1 MPa is applied as a Heaviside function starting at \( t = 0 \). It is considered that the material exhibits a linear elastic behaviour with the following material properties: Young's Modulus \( E = 32 \) GPa, Poisson's ratio \( \nu = 0.20 \) and density \( \rho = 2450 \text{ kg/m}^3 \) (Song et al., 2008). A critical stretch value of \( 5.09 \times 10^{-4} \) is used for glass (Agwai et al., 2011). The notch is located in the middle of the plate and is modelled by cutting transversal bonds in the two central rows starting from the left boundary.

Fig. 6 shows the crack pattern obtained for different levels of discretization. In all cases the symmetry of the crack propagation pattern is in agreement with experimental results presented by Ramulu and Kobayashi (1985) (see Fig. 7(a)). It is also possible to see that the results for \( a = 1.00 \times 10^{-4} \text{ m} \) present a reasonable precision.

In Fig. 7(a) we can see the experimental crack pattern reported by Ramulu and Kobayashi (1985), while Fig. 7(b)–(d) shows the crack evolution predicted by our model for three different times. In this case the plate is discretized using a triangular lattice with distance between particles \( a = 1.00 \times 10^{-3} \text{ m} \). It is observed that both experimental and numerical results present similar fashion in crack growth prior to the major branching, with minor branches starting from the main crack, but only growing to small lengths. At around 30 \( \mu \text{s} \), the crack splits into two main branches which grow until they reach the right boundary of the specimen, at 51.3 \( \mu \text{s} \). We have observed that the proposed fracture criterion solves the problem of the dependence of crack propagation with the lattice orientation, and spurious breakage of bonds, present in other discrete models.

In order to analyse the mesh dependence of the model, we solve the aforementioned problem using with different lattice orientations.

The change in the lattice orientation consists in rotating the initial triangular lattice a certain angle with respect to the horizontal axis, generating the geometry of the solid according to this new orientation. The angles considered ranged from 0 to 50, and the selected distance between particles was \( a = 1.00 \times 10^{-3} \text{ m} \), corresponding to intermediate size mesh.
Fig. 8 shows the crack patterns obtained with the different tested meshes. In all cases, a branching in the crack propagation occurs. We can observe that the model has some dependence with the lattice orientation, and the symmetry with respect to the horizontal axis of cracks is lost when the angle of rotation is equal to 10° and 50°. However, in the other cases the symmetry is conserved.

Fig. 9 presents the crack propagation velocity in terms of the Rayleigh wave speed for glass, \( c_R \), as a function of time, predicted by different techniques. In these calculations, \( \alpha = 1.0 \times 10^{-4} \). Results obtained by applying the model developed in this work present a very good agreement with those obtained using other methods, such as peridynamic model by Agwai et al. (2011), XFEM and CZM by Song et al. (2008) and Cracking Node Method (XFEM-CNM) by Song and Belytschko (2009). Crack propagation starts at 17 \( \mu \)s, in agreement with the CZM. An increase in the velocity of crack propagation is observed when the branching starts, at 35 \( \mu \)s, in coincidence with both of the XFEM-based simulations. After branching, the crack tip speed becomes almost constant at 75% of the Rayleigh wave speed.

This problem has been solved with other numerical methods (Xu and Needleman, 1994b; Belytschko et al., 2003; Rabczuk and Belytschko, 2004; Gupta et al., 2011), which reliably predict the features of crack branching according to experimental evidence (Ramulu and Kobayashi, 1985; Sharon et al., 1995; Ravi-Chandar, 1998). The model proposed in this work produces similar results to those obtained with the aforementioned, but with much less complexity.

3.3. Edge–cracked plate under impulsive loading

These simulations concern the experiment reported by Kalthoff and Winkler (1987), in which a plate with two initial edge notches is impacted by a projectile as shown in Fig. 10.

The material is a maraging steel 18Ni1900 and its properties are Young’s Modulus \( E = 190 \) GPa, Poisson’s ratio \( \nu = 0.30 \) and density \( \rho = 8000 \) kg/m\(^3\) (Decker, 1979). It was adopted a maximum principal strain of \( 4.44 \times 10^{-3} \) (Belytschko et al., 2003).

In the experiment, two different failure modes were observed by modifying the projectile speed \( V_0 \) at high impact velocities. A brittle failure was observed to emanate from the notch at an angle of 70°, while a transversal band with a crack propagation angle of about –10° was observed at greater impact velocity. We have only studied the velocity range that resulted in a brittle fracture mode.

Fig. 10. Experimental set up for edge cracked plate under impulsive loading.
of the crack propagation shows lattice dependence. However, this is mitigated by reducing the distance between particles. Moreover, the proposed fracture criterion solves the problem of the spurious breakage of bonds, present in other discrete models. We have compared our numerical results not only with experimental results carried out by Kalthoff and Winkler (1987), but also with

**Fig. 11.** Final crack path: (a) Experimental results presented by Kalthoff and Winkler, 1987; (b)–(d) Present work’s DM for different cell sizes: (b) $a = 2.5 \times 10^{-3}$ m; (c) $a = 1.25 \times 10^{-2}$ m and (d) $a = 6.25 \times 10^{-4}$ m.

**Fig. 12.** Final crack path: (a) DM by present study; (b) XFEM by Song et al. (2008); DEM by Kosteski et al. (2012) and XFEM by Belytschko et al. (2003).
numerical ones obtained with DEM by Kosteski et al. (2012), XFEM by Song et al. (2008) and XFEM with loss of hyperbolicity criterion by Belytschko et al. (2003).

Fig. 12 shows that the model developed in this work fits very well the experimental results. In this case the plate is discretized using a lattice with distance between particles \( a = 6.25 \times 10^{-4} \text{ m} \). It should be highlighted that our model predicts an initial crack propagation angle of around 60°, which increases to 70° degrees at later times. Our predictions are in agreement with experimental results and other numerical predictions in which the angle of the crack is around 70° and 65°, respectively. It can also be seen that additional damage occurs at the end of the notch. This effect is not present in the experimental results. Therefore, further analysis are needed to avoid this spurious phenomenon.

Notice that our DM predicts a single progressing crack, which is in agreement with the experimental evidence presented by Kalthoff and Winkler (1987), who do not indicate the presence of the secondary crack predicted by Kosteski et al. (2012) and Belytschko et al. (2003), nor with the diffused damage in this region reported by Song et al. (2008).

Fig. 13 shows the crack patterns obtained with different lattice orientations, following the same procedure previously applied in the branching problem. The distance between particles is \( a = 1.25 \times 10^{-3} \text{ m} \), corresponding to intermediate mesh size. We can observe that the model has some dependence with the lattice orientation, but in all cases the slope of the cracks is consistent with experimental results. The dependence of our model with the lattice is stronger when the lattice is rotated 10° and 20°, while in the other cases the dependence with the lattice orientation is weak. In all cases the second spurious crack present in other models (Kosteski et al., 2012) is eliminated.

Fig. 14 displays the relative crack velocity \( v/c_R \) as a function of time, obtained by DM developed in the present work, as well as results from Kosteski et al. (2012) and Belytschko et al. (2003). In this calculations \( a = 6.25 \times 10^{-4} \text{ m} \). The time at which crack propagation begins (around 28 \( \mu \text{s} \)) is in the range obtained by the cited authors. The velocity at which the crack tip propagates is substantially lower than that reported by Kosteski et al. (2012) and it never exceeds the Rayleigh wave speed (2799.2 m/s).
4. Conclusions

In this work, we propose a model consisting of a discretization of Navier’s equations of elastodynamics once the total displacements (Landau and Lifshitz, 1986) has been decomposed into the sum of two terms corresponding to normal and transversal displacements. Taking advantage of the properties of the two kinds of displacements, the equations governing the evolution of each of them are discretized separately.

The model has the same form as the Born model (Born and Huang, 1954), with the advantage that the constants appearing in it are explicitly related to the elastic constants.

We have proposed a fracture criterion based on the calculation of the maximum principal strain from the strain tensor associated to each bond.

In order to evaluate the applicability of the proposed method, three numerical examples related to elastic waves propagation, branching and dynamic crack propagation, respectively, have been analysed for which experimental and numerical results are available.

The results of the first example show that the calculations of wave velocity are stable and accurate for the entire range of values of Poisson’s ratio, and that the wave propagation speeds are in good agreement with theoretical estimations.

The DM was adopted to model two different experimental dynamic fracture problems. Crack paths and crack propagation speed from DM simulations were compared against experimental findings and with predictions from XFEM, CZM, peridynamics, and other DM simulations of the same problem found in the literature. The temporal evolution of crack velocities reveals that results obtained with our model and those obtained with others are of the same order of magnitude. In the two problems analysed we have observed that crack speeds found never exceed the Rayleigh’s speed.

Moreover, we analyse the dependence of the direction of crack propagation with the lattice. Since our model only admits a regular triangular lattice, we proposed rotating its orientation a certain angle and generating the geometry based on this new configuration. The results of simulations show that our model has a dependence with the lattice orientation, but in all cases they are consistent with the experimental data.

The model developed in this work is able to obtain the same quality of solution as other much more complex models when applied to dynamic crack propagation problems in brittle materials.

Acknowledgments

The authors would like to thank the Comisión Interministerial de Ciencia y Tecnología of the Spanish Government for partial support of this work through the Research Project DPI2011 – 23191. The authors express sincere gratitude to Professor Ramon Zaera for helpful discussions.

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