Finite-size effects on the phase structure of the Nambu–Jona-Lasinio model

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Abstract

The Nambu–Jona-Lasinio (NJL) model is one of the most frequently used four-fermion models in the study of dynamical symmetry breaking. In particular, the NJL model is convenient for that analysis at finite temperature, chemical potential and size effects, as has been explored in the last decade. With this motivation, we investigate the finite-size effects on the phase structure of the NJL model in \( D = 3 \) Euclidean dimensions, in the situations that one, two and three dimensions are compactified. In this context, we employ the zeta-function and compactification methods to calculate the effective potential and gap equation. The critical lines that separate trivial and non-trivial fermion mass phases in a second order transition are obtained. We also analyze the system at finite temperature, considering the inverse of temperature as the size of one of the compactified dimensions.

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1. Introduction

The last decades witnessed significant investigations on the phase structure of quantum field theories, in particular on the chiral symmetry phase transitions in quantum chromodynamics (QCD). However, due to its complex structure, effective models have been largely employed to simplify that analysis. Among them, one of the most frequently used is the four-fermion theory known as Nambu–Jona-Lasinio (NJL) model [1]. The NJL model is specially convenient for the investigation of dynamical symmetries when the system is under certain conditions, like finite temperature, finite chemical potential, external gauge field, gravitation field and others [2–4].

Finite-size effects on the phase transitions of four-fermion models have also attracted a great interest [5–7]. This question emerges when the system has a finite size and it is not clear if it is large enough to apply the thermodynamic limit in a usual way; frequently it is necessary to take into account the fluctuations due to finite-size effects. In particular, Ref. [6] performed a numerical investigation of a three-dimensional four-fermion model in a finite-size scaling analysis, where the finite-size effects act as an external field. On the other hand, Ref. [7] studied the NJL model in the framework of the multiple reflection expansion, where in terms of a modified density of states finite-size effects are included (see also Ref. [8]). The critical temperature was suggested to decrease as the system is reduced.

In this Letter, we investigate finite-size effects on the dynamical symmetry breaking in a different way. We study the Euclidean three-dimensional NJL model in the framework of zeta-function and compactification methods [9]. This procedure in principle allows us to explore the mentioned model with one, two or three compactified dimensions with antiperiodic boundary conditions\(^1\) and compare their effects in the phase diagram of the model. With the choice of all dimensions being spatial, the system is considered

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\(^1\) For a discussion of periodicity of boundary conditions see [5].
confined between two parallel planes a distance \( L \) apart, confined to a cylinder having a square transversal section of area \( L^2 \), and to a cubic box of volume \( L^3 \), for one, two and three compactified coordinates, respectively. At finite temperature, we associate one of the compactified coordinates to the range \([0, \beta]\), where \( \beta \) is the inverse of the temperature \( T \). In this setting, we calculate and determine analytically the size-dependence of the effective potential and gap equation. Phase diagrams, where the symmetric and broken phases are separated by a size-dependent critical line, are obtained.

This Letter is organized as follows. In Section 2, we briefly review the NJL model and obtain the effective potential in mean-field approximation. In Section 3, we apply the zeta-function method to derive the effective potential. After, Section 4 is devoted to analyze the size-dependent gap equation. The phase diagrams are shown and discussed in Section 5. Finally, in Section 6 are given a summary and concluding remarks.

2. The formalism

2.1. NJL model as an effective theory of QCD

QCD is the theory of strongly interacting matter, which is constituted of quarks and gluons. In the study of dynamical symmetry breaking, the attractive interaction between quarks and antiquarks, coming from complicated processes of exchanges of gluons, are often replaced by an effective interaction between them. This interaction generates the quark–antiquark condensation in the vacuum, breaking, the attractive interaction between quarks and antiquarks, coming from complicated processes of exchanges of gluons, are often replaced by an effective interaction between them. This interaction generates the quark–antiquark condensation in the vacuum.

2.2. The effective potential

In the previous subsection we have heuristically derived the NJL model from a simplified version of QCD. To analyze With its dynamical symmetry phase diagram, we will begin by deriving its effective potential. For generality, we will consider the NJL model as an independent model by itself: the coupling constant is considered to be and arbitrary parameter, without any a priori relation with QCD aspects. Also, the dimension of the system is considered here arbitrary (\( 2 \leq D < 4 \)), as well as we limit ourselves to mesonic scalar and pseudoscalar modes.

We will treat with the colorless and massless NJL model,

\[
\mathcal{L} = \bar{q} i \gamma \eta q + \frac{G}{2} \sum_{i=0}^{N^2-1} \left[ (\bar{q} \lambda_i \gamma) q + (\bar{q} i \gamma_s \lambda_i) q \right],
\]

where the spinors \( q \) and \( \bar{q} \) here carry \( N \) flavors, and the matrices \( \lambda_i \) are the generators of the \( U(N) \), \( \lambda_i = (I/\sqrt{N}, \lambda^a); a = 1, \ldots, N^2 - 1 \).
In order to study the phase structure of this model, it is convenient to perform the bosonization. We introduce auxiliary fields $M^i$ and $\phi^i$ through the following way,

$$1 = \int D M^i \delta(M^i - \bar{q} \Gamma^i q) = \int D M^i D \phi^i \exp \left[ i \int d^D x (M^i - \bar{q} \Gamma^i q) \phi^i \right],$$

(5)

where $\phi^i = (\sigma^i, \pi^i)$ and $\Gamma^i = (\lambda^i, i \gamma^5 \lambda^i)$. In this sense, $\sigma^i$ and $\pi^i$ are fields corresponding to the scalar and pseudoscalar bilinear forms, respectively. The use of auxiliary fields in the generating functional of the model introduced in Eq. (4) yields

$$Z \propto \int \bar{q} D q D \phi^i \exp \left\{ i \int d^D x \left[ \bar{q} (i \partial - \Gamma^i \phi^i) q - \left( \frac{\phi^i}{2G} \right)^2 \right] \right\},$$

(6)

where the source terms have been omitted. Thus, the integration over the Grassmann variables $q$ and $\bar{q}$ results in

$$Z \propto \int D \phi^i \exp \left\{ i A_{\text{eff}} \right\},$$

(7)

where

$$A_{\text{eff}} = - \int d^D x \left( \frac{\phi^i}{2G} \right)^2 - i \text{Tr} \ln \left( \frac{i \partial - \Gamma^i \phi^i}{\mu} \right)$$

(8)

is the effective action associated to the model described in Eq. (4) with one-loop corrections; $\mu$ is a constant with dimension of mass. Here Tr means the trace over coordinate, Dirac and internal spaces.

In the above context we restrict $\phi$ to be independent of the spatial coordinates; moreover, we impose $\pi^i = 0$, $\sigma^a = 0$ ($a = 1, \ldots, N^2 - 1$); only the scalar part of $\phi$ which is diagonal in flavor space, $\sigma^0$, can assume non-vanishing values. Thus, it is convenient to define $\lambda^0 \sigma^0 \equiv \sigma$, with $\sigma$ assuming the role of dynamical fermion mass, its non-trivial value meaning that the system would be in the broken phase.

Taking into account the above considerations and performing the Wick rotation in the $x^0$-coordinate, we can use Eq. (8) to obtain the effective potential,

$$\frac{1}{N} U_{\text{eff}} = - \frac{A_{\text{eff}}}{V} = \frac{\sigma^2}{2G} + U_1(\sigma),$$

(9)

where $V$ is the volume, and

$$U_1(\sigma) = -h \int \frac{d^D k}{(2\pi)^D} \ln \left( \frac{k^2 + \sigma^2}{\mu^2} \right).$$

(10)

with $h$ being the dimension of the Dirac representation.

As our interest is to describe the finite-size effects on the phase structure of this model, it is convenient to use the zeta-function regularization techniques, described below.

3. The zeta-function regularization approach

3.1. The zeta-function

This method is based on the fact that one can define a generalized zeta-function from the eigenvalues $\alpha_i$ of a differential operator $A$ of order 2 on a $d$-dimensional compact manifold (a bounded manifold),

$$\zeta_A(s) = \sum_i \alpha_i^{-s},$$

(11)

valid for Re $s > \frac{d}{2}$. For the other values of $s$, we must perform an analytical continuation (for a review, see [9,11]).

In particular,

$$\zeta_A'(0) = \left. \frac{d \zeta_A(s)}{ds} \right|_{s=0} = - \sum_i \ln \alpha_i = - \text{Tr} \ln A.$$

(12)

Therefore one can associate the operator $A$ to the one appearing in the argument of the logarithm in Eq. (10), allowing to rewrite $U_1(\sigma)$ as

$$U_1(\sigma) = \frac{M}{2V} \left[ \zeta_A'(0) + \ln \mu^2 \zeta_A(0) \right].$$

(13)
3.2. Compactification

At this point we will take into account the finite size effects. Considering the system restricted to have $d \leq D$ compactified dimensions, we can rewrite the coordinate vectors as $x_E = (y, z)$, where

$$y = (y_1 \equiv x_{E1}^1, \ldots, y_n \equiv x_{En}^n), \quad z = (z_1 \equiv x_{E1}^{n+1}, \ldots, z_d \equiv x_{Ed}^D),$$

(14)

with the $z_i$ components defined in the domain $z_i \in [0, L_i]$ and $n = D - d$. The compactification of the coordinates makes the $k_z$-components of momenta to assume discrete values,

$$k_z^i \rightarrow \frac{2\pi}{L_i} (n_i + c_i), \quad n_i = 0, 1, 2, \ldots;$$

(15)

where $c_i = \frac{1}{2} (i = 1, 2, \ldots, d)$ for antiperiodic boundary conditions. Notice that we may associate a given $L_i$ to the inverse of temperature $\beta = 1/T$, i.e. $L_i \equiv \beta$; in this situation the chemical potential $\mu_0$ also could be introduced through the rule $c_i = \frac{1}{2} - \frac{\beta \mu_0}{2\pi}$.

The prescription given in Eq. (15) makes the zeta-function in (12) be given by

$$\zeta_A(s) = V_n \sum_{n_1,\ldots,n_d=-\infty}^{+\infty} \int \frac{d^n k_1}{(2\pi)^n} \left[ k_1^2 + k_1^2 + \sigma^2 \right]^{-s},$$

(16)

where $V_n$ is the $n$-dimensional volume. The techniques of dimensional regularization can be used to perform the integration over the $k_1$-components, yielding

$$\zeta_A(s; \{a_i\}, \{c_i\}) = \frac{V_n}{(4\pi)^{n/2}} \frac{\Gamma(s - \frac{n}{2})}{\Gamma(s)} Y_{d-2}^{\sigma^2} \left( s - \frac{n}{2}; \{a_i\}, \{c_i\} \right),$$

(17)

where $Y_{d-2}^{\sigma^2} (v; \{a_i\}, \{c_i\})$ is the generalized Epstein-zeta function, defined by

$$Y_{d-2}^{\sigma^2} (v; \{a_i\}, \{c_i\}) = \sum_{n_1,\ldots,n_d=-\infty}^{+\infty} \left[ a_1(n_1 + c_1)^2 + \cdots + a_d(n_d + c_d)^2 + \sigma^2 \right]^{-v},$$

(18)

being $a_i = \frac{4\pi^2}{L_i^2}$; notice that $Y_{d-2}^{\sigma^2}$ is valid for Re $v > \frac{d}{2}$.

3.3. The pole structure of the zeta-function

According to Eq. (13), in order to obtain the effective potential, we must perform the derivative of the $\zeta_A(s; \{a_i\}, \{c_i\})$ at $s = 0$. However, we note from Eq. (17) that there is a different pole structure coming from $\frac{\Gamma(s - \frac{n}{2})}{\Gamma(s)}$ at $s = 0$ for $n$ even or not. So, we are forced to consider these different situations. Furthermore, the function $Y_{d-2}^{\sigma^2}$ also has a different behavior for $d$ even or odd, as it is discussed in Ref. [11]. In the case of $d$ odd, $Y_{d-2}^{\sigma^2}$ has poles at $v = \frac{d}{2}, \frac{d}{2} - 1, \ldots, \frac{1}{2}$ and $-\frac{2l + 1}{2} (l \in \mathbb{N}_0)$, while for $d$ even they are at $v = \frac{d}{2}, \frac{d}{2} - 1, \ldots, 1$. Nevertheless, as it will be shown in the next section, for the study of the phase diagram we take $\sigma^2 \to 0$; in this limit, $Y_{d-2}^{\sigma^2}$ has pole only at $\frac{d}{2}$. This fact assures that for further calculations the pole structure of $Y_{d-2}^{\sigma^2}$ is not relevant.

Then, from Eq. (17) we have to deal with the following situation,

$$\zeta'_A(0; \{a_i\}, \{c_i\}) = \frac{V_n}{(4\pi)^{n/2}} \lim_{s \to 0} \left[ \frac{d}{ds} \left( \frac{\Gamma(s - \frac{n}{2})}{\Gamma(s)} \right) Y_{d-2}^{\sigma^2} \left( s - \frac{n}{2}; \{a_i\}, \{c_i\} \right) + \frac{\Gamma(s - \frac{n}{2})}{\Gamma(s)} \frac{d}{ds} \left( Y_{d-2}^{\sigma^2} \left( s - \frac{n}{2}; \{a_i\}, \{c_i\} \right) \right) \right].$$

(19)

After the necessary manipulations, we obtain

$$\zeta'_A(0; \{a_i\}, \{c_i\}) = \frac{V_n}{(4\pi)^{n/2}} \left( -\frac{1}{2} \Gamma \left( \frac{n}{2} \right) \left[ \zeta_{d-2}^{\sigma^2} \left( \frac{n}{2}; \{a_i\}, \{c_i\} \right) \right] \right)_{s=0} - Y_{d-2}^{\sigma^2} \left( \frac{n}{2}; \{a_i\}, \{c_i\} \right) \psi \left( \frac{n}{2} + 1 \right),$$

(20)

where $\gamma$ is the Euler–Mascheroni constant and $\psi(k)$ is the digamma function, and

$$\zeta'_A(0; \{a_i\}, \{c_i\}) = \frac{V_n}{(4\pi)^{n/2}} \Gamma \left( -\frac{n}{2} \right) Y_{d-2}^{\sigma^2} \left( -\frac{n}{2}; \{a_i\}, \{c_i\} \right),$$

(21)

Due to the discussion of the pole structure above, notice that Eq. (21) is valid for $d$ even or in the limit $\sigma^2 \to 0$. 
Now, coming back to Eq. (13), we have two possibilities for $U_1(\sigma)$: the use of Eq. (20) gives

$$U_1(\sigma; \{a_i\}, \{c_i\}) = \frac{\hbar}{2V_d(4\pi)^{n/2}} \left( \frac{-n}{2} \right)^{\sigma^2} \left[ \ln \mu^2 - \psi \left( \frac{n}{2} + 1 \right) \right], \quad \text{for } n \text{ even},$$

while with the help of Eq. (21) we obtain

$$U_1(\sigma; \{a_i\}, \{c_i\}) = \frac{\hbar}{2V_d(4\pi)^{n/2}} \Gamma \left( -\frac{n}{2} \right) Y^\sigma_{d/2} \left( -\frac{n}{2}; \{a_i\}, \{c_i\} \right), \quad \text{for } n \text{ odd},$$

valid for $d$ even or in the limit $\sigma^2 \to 0$.

Now, coming back to Eq. (13), we have two possibilities for $U_1(\sigma)$: the use of Mellin transforms and relations between Jacobi theta-functions [9], we find

$$Y^\sigma_{d/2}(\sigma; \{a_i\}, \{c_i\}) = \frac{\pi^{d/2}}{\sqrt{\prod_{j=1}^{d} a_j}} \left[ \frac{\Gamma \left( \frac{d}{2} - 2\nu \right)}{\Gamma \left( \frac{d}{2} \right)} \right] \sum_{\nu = 0}^{\infty} \left( \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - 2\nu \right)} \right) \left( \sum_{j=1}^{d} n_j^2 a_j / \prod_{j=1}^{d} a_j \right) \ln \mu^2 - \psi \left( \frac{n}{2} + 1 \right)$$

where $Z$ means the set of non-vanishing integers.

For small values of $\sigma^2$, $\sigma^2 \ll 1$, it is possible to use the binomial expansion for $Y^\sigma_{d/2}$ defined in Eq. (18) to expand it in powers of the $\sigma$-field, obtaining the expression

$$Y^\sigma_{d/2}(\sigma; \{a_i\}, \{c_i\}) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} T(q, j) Y_d(q + j; \{a_i\}, \{c_i\}) \sigma^{2j},$$

where $T(q, j) = \frac{\Gamma(\frac{d}{2} + j)}{\Gamma(\frac{d}{2})}$, and

$$Y_d(q; \{a_i\}, \{c_i\}) = \sum_{\nu = 0}^{\infty} \left( \sum_{j=1}^{d} n_j^2 a_j / \prod_{j=1}^{d} a_j \right) \ln \mu^2 - \psi \left( \frac{n}{2} + 1 \right)$$

is the homogeneous generalized Epstein zeta-function, with $Z_0$ being the set of integers.

4. The gap equation

4.1. Case without boundaries

In this section we focus our attention to the gap equation, which will allow us to analyze the phase structure of the model. It is obtained by minimizing the effective potential with respect to $\sigma$,

$$\frac{\partial}{\partial \sigma} U_{\text{eff}}(\sigma; \{a_i\}, \{c_i\}) \bigg|_{\sigma = m} = 0,$$

where $m$ is the dynamical mass of the fermion.

For completeness, we briefly describe the case without boundaries (for reviews see Refs. [12–14]). This situation means $d = 0$, and therefore $n = D$. So, it is more useful to use the formula (23) of $U_1$ for $n$ odd,

$$U_1(\sigma; d = 0) = -\frac{\hbar}{D(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \sigma^D.$$  

Notice that $U_1$ diverges for $D$ even. In particular, for $D = 2$, four-fermion theories are renormalizable, and the renormalization is implemented by imposing the following renormalization condition,

$$\frac{\partial^2}{\partial \sigma^2} U_{\text{eff}}(\sigma; d = 0) \bigg|_{\sigma = \mu} = \frac{1}{G_R},$$

2 It is possible to see that the employment of the minimal subtraction scheme in Eq. (28) generates the same expression for $U_1$ if we have used Eq. (22) with $d = 0$.  

which, with the help of Eqs. (9) and (29), yields

$$\frac{1}{G_R} = \frac{1}{G} - \frac{h(D-1)(4\pi)^{D/2}}{\mu D-2} \Gamma(1 - \frac{D}{2}) G_{\mu D-2}.$$

(30)

Thus, we can write the renormalized effective potential, valid for $2 \leq D < 4$, as

$$\frac{1}{N} U_{\text{eff}}(\sigma; d = 0) = \frac{\sigma^2}{2G_R} + \frac{h(D-1)(4\pi)^{D/2}}{\mu D-2} \Gamma \left(1 - \frac{D}{2}\right) G_{\mu D-2}$$

$$- \frac{h}{D(4\pi)^{D/2}} \Gamma \left(1 - \frac{D}{2}\right) \sigma^D.$$

(31)

Hence, the non-trivial solution of the gap equation is given by

$$\frac{1}{G_1} = \frac{1}{G_0} + \frac{h}{(4\pi)^{D/2}} \Gamma \left(1 - \frac{D}{2}\right) \left(\frac{m}{\mu}\right)^{D-2},$$

(32)

where we have defined the dimensionless coupling constant $G_1 = \mu^{D-2} G_R$, and

$$G_0 = \frac{(4\pi)^{D/2}}{h(1-D)\Gamma \left(1 - \frac{D}{2}\right)}.$$

(33)

From Eq. (32), we can infer that for $G_1 < G_0$ the allowed value of the mass $m$ is trivial, while for $G_1 > G_0$ the mass acquires a non-zero positive value. In this context, the constant $G_0$ acts as a critical value which $G_1$ must exceed to have dynamically generated fermion mass.

4.2. The presence of boundaries

In the case of presence of boundaries, we can use the same systematics as above, and therefore determine the finite-size contributions to the expression for $G_1$, with the only difference that we must analyze the gap equation in Eq. (32) with its last term in right-hand side in a different way,

$$\frac{1}{G_1} = \frac{1}{G_0} - \frac{h}{m^D \mu^{D-2}} \frac{\partial}{\partial \sigma} U_{\text{eff}}(\sigma; \{a_i\}, \{c_i\})\bigg|_{\sigma=m},$$

(34)

where $m$ is the fermion mass with the system in the presence of boundaries.

We can calculate the derivative of $U_{\text{eff}}$ with respect to $\sigma$ directly from Eq. (13), by acting it on the zeta-functions. Taking into account the different possibilities for $U_{\text{eff}}$, then the gap equation is written in the different situations,

$$\frac{1}{G_1} = \frac{1}{G_0} - \frac{h \mu^{2-D}}{V_d (4\pi)^{D/2}} \Gamma \left(1 - \frac{D}{2}\right) \left(\frac{n}{2} + 1; \{a_i\}, \{c_i\}\right)$$

$$+ Y_{\mu D} \left(\frac{n}{2} + 1; \{a_i\}, \{c_i\}\right) \left[\ln \mu^2 - \gamma - \psi \left(\frac{n}{2}\right)\right],$$

(35)

while

$$\frac{1}{G_1} = \frac{1}{G_0} + \frac{h \mu^{2-D}}{V_d (4\pi)^{D/2}} \Gamma \left(1 - \frac{n}{2}\right) Y_{\mu D} \left(\frac{n}{2} + 1; \{a_i\}, \{c_i\}\right),$$

(36)

(remembering that Eq. (36) is valid for $d$ even or in the limit $\sigma^2 \rightarrow 0$) and finally we have obtained,

$$\frac{1}{G_1} = \frac{1}{G_0} - \frac{h \mu^{2-D}}{V_d} F P Y_{\mu D} \left(1; \{a_i\}, \{c_i\}\right),$$

(37)

in which we have considered only the relevant terms for further calculations; $FP$ means the finite part of $Y_{\mu D}^n$.

Now, we are able to determine the critical value of the coupling constant $G_1$ with the corrections due to the presence of boundaries, just by taking the fermion mass approaching to zero in the gap equation. In this context, we can use the expansion in Eq. (25) for $m \rightarrow 0$; taking only the first term of this expansion, then $Y_{\mu D}^n$ reduces to the homogeneous generalized Epstein zeta-function $Y_d$. However, as it was remarked in the previous section, the pole structure of $Y_d$ is simpler than that of $Y_{\mu D}^n$; therefore Eq. (36) becomes valid for general $d$.

We can construct analytical continuations for $Y_d$ to write it in terms of Bessel and Riemann zeta functions, by considering a generalization of recurrence formulas in [9],

$$Y_d(v; \{a_i\}, \{c_i\}) = \frac{\Gamma(v-\frac{1}{2})}{\Gamma(v)} \sqrt{\pi} a_d Y_{d-1} \left(v - \frac{1}{2}; \{a_j\neq d\}, \{c_j\neq d\}\right) + \frac{4\pi^4}{\Gamma(v)} W_d \left(v - \frac{1}{2}; \{a_i\}, \{c_i\}\right),$$

(38)
where the set \( \{a_j \neq d\} \) means that the parameter \( a_d \) is excluded from it, and

\[
W_d(\eta; \{a_l\}; \{c_l\}) = \frac{1}{\sqrt{\lambda_d}} \sum_{\{n_j \neq d\} \in \mathbb{Z}_n} \sum_{n_d=1}^{\infty} \cos(2\pi n_d c_d) \left( \frac{n_d}{\sqrt{\lambda_d} X_{d-1}} \right)^{\eta} K_\eta \left( \frac{2\pi n_d}{\sqrt{\lambda_d} X_{d-1}} \right),
\]

(39)

with \( X_{d-1} = \sqrt{\sum_{k=1}^{d-1} a_k (n_k + c_k)^2} \) and \( K_\eta(z) \) being the modified Bessel function of second kind.

Notice from Eq. (38) that whatever the sum one chooses to perform firstly, when \( c_i = c_j \) the \( L_i \leftrightarrow L_j \) symmetry is lost [15]; in order to preserve this symmetry, we adopt here a symmetrized summation generalizing the prescription explicit in [16] for \( c_i = 0 \).

In the next section we will consider particular cases.

5. Phase boundary

In this section we will analyze the expressions (36), (37) for \( \bar{m} \to 0 \), which defines a critical surface \( L_i \)-dependent for the phase diagram of fermion mass. In the first three subsections, we consider the situations where all three dimensions are spatial coordinates. In the last subsection, we analyze the system at finite temperature, associating one of compactified dimension to the range of the inverse of temperature.

5.1. Case \( d = 1, n \) even

We now consider the simplest case of the compactification, namely when only one coordinate is compactified. This situation was in part studied in Ref. [12] considering \( L_1 \equiv \beta \); for completeness we will describe it here.

For \( d = 1 \), Eq. (26) is written as

\[
Y_1(v; a_1 = a, c_1 = c) = a^{-v} \left[ \xi(2v, c) + \xi(2v, 1 - c) \right],
\]

(40)

where \( \xi(s, c) = \sum_{k=0}^{\infty} \left[ k + c \right]^{-s} \) is the Hurwitz zeta-function. The use of this last equation in (35) for \( \bar{m} \cong 0 \) yields

\[
\frac{1}{G_1} = \frac{1}{G_0} - \frac{h \mu^2 - D}{L(4\pi)^{D-2}} \left[ D \frac{D-1}{2} \right] \left[ \zeta'(D - 3, c) + \zeta'(D - 3, 1 - c) \right]
\]

\[+ a \frac{D-1}{2} \left[ \ln \mu^2 - \psi \left( D - 1 \right) - \gamma - \ln a \right] \left[ \zeta(D - 3, c) + \zeta(D - 3, 1 - c) \right].
\]

(41)

Employing the expansion formula for the Hurwitz zeta-function,

\[
\zeta(v, c) \equiv \frac{1}{2} - c + v \left[ \ln \Gamma(c) - \frac{1}{2} \ln 2\pi \right],
\]

(42)

Eq. (41) becomes, for \( D = 3 \),

\[
\frac{1}{G_1} = \frac{1}{G_0} - \frac{1}{\pi L \mu} \ln \left( \frac{2\pi}{\gamma(1-c)} \right).
\]

(43)

Notice that for \( c = \frac{1}{2} \), Eq. (43) reduces to

\[
\frac{1}{G_1} = \frac{1}{G_0} - \frac{A_1}{L \mu},
\]

(44)

where \( A_1 = \frac{\ln^2 2}{\pi^2} \approx 0.221 \).

Noting that using (32) in (44), we obtain the formula \( L m = 2 \ln 2 \). It can be seen that the replacement of the length \( L \) by the inverse of temperature \( \beta \) gives the equation \( T_c = \frac{m}{2\ln 2} \), which can be identified as the critical temperature of a second order phase transition, as has been remarked in Ref. [5].

Furthermore, yet with \( L = \beta \) and \( c = \frac{1}{2} - \frac{1}{2} \imath \beta \mu_0 \), where \( \mu_0 \) is the chemical potential, Eq. (43) becomes

\[
\frac{1}{G_1} = \frac{1}{G_0} - \frac{1}{\pi L \mu} \ln \left[ 2 \cosh \left( \frac{\beta \mu_0}{2} \right) \right].
\]

(45)

The use of Eq. (32) in (45) generates the following formula for the critical temperature and the chemical potential, \( \frac{\beta m}{2} = \ln[2 \cosh \frac{\beta \mu_0}{2}] \), which agrees with that obtained in Ref. [12].
5.2. Case $d = 2$, $n$ odd

We now focus our interest in the case of two compactified coordinates. Taking $d = 2$ in Eq. (36) with $\bar{m} \equiv 0$ generates

$$
\frac{1}{G_1} = \frac{1}{G_0} + \frac{h \mu^{2-D}}{V_2(4\pi)^{D-2}} \Gamma\left(\frac{-D-4}{2}\right) Y_2\left(\frac{-D-4}{2}; a_1, a_2, c_1, c_2\right).
$$

(46)

The analytical continuation of this equation can be performed by using (38) in its symmetrized version, (40) and (42), allowing us to rewrite Eq. (46) for $D = 3$ as

$$
\frac{1}{G_1} = \frac{1}{G_0} - \frac{1}{\mu V_2 \pi} \left\{ \sum_{i=1}^{\infty} \ln \left( \frac{2\pi}{\Gamma(c_j) \Gamma(1-c_j)} \right) - \sum_{i=1}^{\infty} L_i \sum_{n_j=1}^{\infty} \frac{\cos(2\pi n_j c_i) K_0\left(2\pi \frac{L_i}{L_j} n_j \left| n_j + \frac{1}{2}\right.\right)}{\pi} \right\},
$$

(47)

with $j \neq i$.

Considering $c_i = \frac{1}{2}$, we obtain ($j \neq i$)

$$
\frac{1}{G_1} = \frac{1}{G_0} - \frac{1}{\mu V_2 \pi} \left\{ \frac{1}{2L_1 \mu} + \frac{1}{2L_2 \mu} \right\} \ln 2 - \frac{2}{L_1 \mu} \sum_{n_1=1}^{\infty} \sum_{n_2=\infty}^{\infty} (-1)^{n_1} K_0\left(2\pi \frac{L_1}{L_2} n_2 + \frac{1}{2}\right). \tag{48}
$$

In particular, for $L_1 = L_2 \equiv L$, we have

$$
\frac{1}{G_1} = \frac{1}{G_0} - \frac{A_2}{L \mu}, \tag{49}
$$

where

$$
A_2 = \frac{\ln 2}{\pi} - \frac{2}{\pi} \sum_{n_1=1}^{\infty} \sum_{n_2=\infty}^{\infty} (-1)^{n_1} K_0\left(2\pi n_2 + \frac{1}{2}\right) \approx 0.257. \tag{50}
$$

5.3. Case $d = 3$, $n = 0$

We now pay our attention to the case where three dimensions are compactified. Taking $d = 3$ in Eq. (37), we obtain for $\bar{m} \equiv 0$,

$$
\frac{1}{G_1} = \frac{1}{G_0} - \frac{2}{\mu V_3} F P Y_3(1; a_1, a_2, a_3, c_1, c_2, c_3). \tag{51}
$$

The analytical structure of the function $Y_3(1; a_1, a_2, a_3, c_1, c_2, c_3)$ can be obtained from the general symmetrized recurrence relation given by Eq. (38); explicitly, one has

$$
Y_3(1; a_1, a_2, a_3, c_1, c_2, c_3) = \frac{\pi}{3} \sum_{i,j,k=1}^{3} \frac{1}{2} \sum_{n_i} \sum_{n_j} \frac{1}{\Gamma(c_k) \Gamma(1-c_k)} Y_2\left(\frac{1}{2}; a_j, a_k, c_j, c_k\right) + \frac{4\pi}{3} W_3\left(\frac{1}{2}; a_1, a_2, a_3, c_1, c_2, c_3\right), \tag{52}
$$

where $\epsilon_{ijk}$ is the totally antisymmetric symbol. With the same expression for $Y_2$ as used in Eq. (46), we can rewrite (51) as

$$
\frac{1}{G_1} = \frac{1}{G_0} - \frac{1}{3 \pi V_3 \mu} \left\{ \sum_{i,j,k=1}^{3} \frac{1}{2} \sum_{n_i} \sum_{n_j} \cos(2\pi n_j c_i) K_0(2\pi L_i n_j T_k) + L_j \leftrightarrow L_k \right\}
$$

$$
- \sum_{i,j,k=1}^{3} \sum_{n_i} \sum_{n_j} \sum_{n_k} \left( \frac{n_i L_i}{T_{jk}} \right)^{\frac{1}{2}} K_1(2\pi n_i L_i T_{jk}) \right\}, \tag{53}
$$

where $T_{ij,...} = \sqrt{\frac{(n_i+\epsilon_{ijk})^2}{L_i^2} + \frac{(n_j+\epsilon_{ijk})^2}{L_j^2} + \cdots}$. 


Fig. 1. The critical coupling constant $G_1$ as a function of $x = (L\mu)^{-1}$ for the three situations: dashed, dotted and solid lines represent Eqs. (44), (50) and (55), which are the cases of $d = 1, 2, 3$ compactified dimensions, respectively, $L$ being the length in each case. For each situation the non-trivial mass phase corresponds to the region above the corresponding line.

If we restrict ourselves to the situation where $c_i = \frac{1}{2}$, then we obtain the following equation that describes the critical coupling,

$$
\frac{1}{G_1} = \frac{1}{G_0} - \frac{1}{3\pi V_3\mu} \left\{ \sum_{i,j,k=1}^{3} \frac{(1 + \varepsilon_{ijk})L_i}{2} \left[ \frac{L_j}{2} \ln 2 - L_j \sum_{n_j=1}^{\infty} \sum_{n_i=\infty}^{\infty} (-1)^{n_i} K_0(2\pi L_j n_j \tilde{T}_k) + (L_j \leftrightarrow L_k) \right] - \sum_{i,j,k=1}^{3} (1 + \varepsilon_{ijk})L_i \sum_{n_i=1}^{\infty} (-1)^{n_i} \sum_{n_j,n_k=\infty}^{\infty} \left( \frac{n_i L_i}{T_{jk}} \right) \frac{1}{2} K_{\frac{1}{2}}(2\pi n_i L_i \tilde{T}_{jk}) \right\},
$$

(54)

where $\tilde{T}_{ij...} = T_{ij...}|_{\varepsilon_i\varepsilon_j...=\frac{1}{2}}$.

In particular, for $L_1 = L_2 = L_3 \equiv L$, we get

$$
\frac{1}{G_1} = \frac{1}{G_0} - \frac{A_3}{L \mu},
$$

(55)

where

$$
A_3 = \frac{\ln 2}{\pi} - \frac{2}{\pi} \sum_{n_1=1}^{\infty} \sum_{n_2=\infty}^{\infty} (-1)^{n_1} K_0\left(2\pi n_1 \left| n_2 + \frac{1}{2} \right| \right)$$

$$
- \frac{2}{\pi} \sum_{n_1=1}^{\infty} (-1)^{n_1} \sum_{n_2,n_3=\infty}^{\infty} \left( \frac{n_1}{\sqrt{\sum_{i=2}^{3}(n_i + \frac{1}{2})^2}} \right) \frac{1}{2} K_{\frac{1}{2}}\left(2\pi n_1 \left| \sum_{i=2}^{3}(n_i + \frac{1}{2})^2 \right| \right) \approx 0.278.
$$

(56)

In Fig. 1 is plotted the critical coupling constant $G_1$ as a function of $x = (L\mu)^{-1}$ representing Eqs. (44), (50) and (55). These contexts correspond, respectively, to the system in the following scenarios: confined between two parallel planes a distance $L$ apart; confined to an infinitely long cylinder having a square transversal section of area $L^2$; and to a cubic box of volume $L^3$. We can see that as $L$ decreases, $G_1$ increases monotonically, meaning that to keep the non-trivial mass phase with compactified coordinates, it is necessary a stronger interaction than that in non-compactified space.

In addition, from Fig. 1 it is easy to see that there is a critical value $L_c$ below which there exists no more non-vanishing fermion mass, since the coupling constant tends to infinity at this point. We have the following values for $x_c = (L_c\mu)^{-1} = 1.44$ for $d = 1$, 1.24 for $d = 2$ and 1.14 for $d = 3$. Thus, the cubic box gives the greatest value for $L_c$, which means that the increasing of the number of compactified dimensions forces a stronger interaction to stay in the region of non-trivial mass.

### 5.4. System at finite temperature

Now we turn our attention to the system with compactified dimensions and at finite-temperature. First, let us look at the situation in Eq. (48), which has $d = 2$. Taking $L_1 \equiv L$ and $L_2 \equiv \beta$, we obtain the expression for $G_1$ which is dependent of the temperature.
Then, we obtain the following equation for critical line,

\[
\frac{-m}{\mu} + \left( \frac{1}{L\mu} + \frac{1}{\beta \mu} \right) \ln 2 - \frac{2}{L\mu} \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} (-1)^{n_1} K_0 \left( 2\pi \frac{L}{\beta} n_1 \left| n_2 + \frac{1}{2} \right| \right) = 0.
\]

(57)

The corresponding phase diagram in the \(x-T\) plane of Eq. (57) is plotted in Fig. 2, where \(x = (Lm)^{-1}\) and \(T = (\beta m)^{-1}\). Notice that the critical line separates the non-vanishing fermion mass phase below the line and the trivial fermion mass phase above the line. As we can see, the size decreasing reduces the critical temperature. Also, the diagram has a manifest \(x \leftrightarrow T\) symmetry.

Furthermore, in the \(L \to \infty\) limit, which in the diagram means \(x \to 0\), the variable \(T\) reaches the usual critical value \(T_c = 1/(2 \ln 2)\). Analytically, this results from Eq. (57), in which only the first term with Bessel function survives, and the summation over \(n_2\) is replaced by an integration, yielding

\[
\frac{2}{L\mu} \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} (-1)^{n_1} K_0 \left( 2\pi \frac{L}{\beta} n_1 \left| n_2 + \frac{1}{2} \right| \right) \xrightarrow{L \to \infty} \sum_{\mu=1}^{\infty} \frac{(-1)^n}{2\pi} \int_{0}^{\infty} dp K_0(n\beta p) = \frac{1}{\beta \mu} \lim_{s \to 1-} \zeta(s) = -\frac{1}{\beta \mu} \ln 2.
\]

(58)

Hence, the replacement of Eq. (58) in (57) in the limit \(L \to \infty\) reproduces the correct critical value \(T_c = 1/(2 \ln 2) = 0.72\). Moreover, since the diagram has a \(x \leftrightarrow T\) symmetry, the same argument is applied to \(L\) in the limit \(\beta \to \infty\).

Now we analyze the case with three compactified dimensions, by using Eq. (32) in Eq. (54) and taking \(L_1 = L_2 \equiv L\) and \(L_3 \equiv \beta\). Then, we obtain the following equation for critical line,

\[
\frac{-m}{\mu} + \frac{2}{3} \left( \frac{2}{L\mu} + \frac{1}{\beta \mu} \right) \ln 2 - \frac{4}{3L\mu} \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} (-1)^{n_1} K_0 \left( 2\pi \frac{L}{\beta} n_1 \left| n_2 + \frac{1}{2} \right| \right) = 0.
\]

(59)
The corresponding phase diagram of Eq. (59) in the $x$–$T$ plane is plotted in Fig. 3. As we can see, there is no $x \leftrightarrow T$ symmetry in the case of the critical line of Eq. (59). Besides, the critical temperature has a faster decreasing with the size reduction than the case of Fig. 2.

It should be noted that, by employing the same systematics used to get Eq. (58), the $x \to 0$ gives again $T_c = 1/(2 \ln 2) = 0.72$.

On the other hand, the $T \to 0$ limit generates $x_c \approx 0.62$.

6. Concluding remarks

In this Letter, we have discussed the finite-size effects on the phase structure of the Euclidean three-dimensional NJL model in the frame of zeta function and compactification methods.

In the leading order of the $1/N$-expansion, we analytically derived the expressions for the renormalized effective potential and gap equation for different situations of compactified coordinates without any further approximation.

Taking first the situation in which the dimensions are spatial and with antiperiodic boundary conditions, we estimated the strength of the critical coupling constant for the cases of one, two and three compactified dimensions with the same length $L$ and compare them. We found that increasing of the number of compactified dimensions requires the increasing of the strength of the interaction to remain in the region of non-trivial mass.

We also investigated the system at finite temperature, which formally is done by associating the size of one of compactified dimensions to $\beta = 1/T$. We have obtained the phase diagram in terms of temperature and inverse of the size and noted that the phase transition is easier (bigger values of $L$) for the situation with the largest number of compactified coordinates.

It is worthy to mention that we have obtained general expressions for the effective potential and gap equations (see Eqs. (23), (22) and (36), (37), for different numbers of total and compactified dimensions and also for different choices of the $c_i$-parameters that appear in Eq. (15). This approach in principle allows us to study other cases, as for example the phase diagram at finite chemical potential as well as non-trivial topologies.

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References