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Finite Element Method for Dissolved Oxygen and Biochemical Oxygen Demand in an Open Channel

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Abstract

The mathematical model for dissolved oxygen interaction with biochemical oxygen demand in an open channel flow is presented in this paper. We consider unsteady flow in one dimension. The model is in the form of partial differential equations which require to solve the dissolved oxygen coupled with the biochemical oxygen demand at the same corresponding position and time. For some specific flow cases, unsteady flows approach to certain stationary solutions which are possible to derive some corresponding exact solutions. However, for complicated flow problems, analytical solution cannot be derived. Efficient and accurate numerical method is required to solve numerically such the model. In this paper, we present the finite element method with linear basis function for solving the system. The accuracy of numerical solutions have been observed and compared with some existing exact solutions for some steady flow cases. To show the ability of our presented numerical scheme, the interaction between the dissolved oxygen and the biochemical oxygen demand is simulated for unsteady flow cases. Furthermore, the impacts of flow parameters such as flow velocity, kinetic reaction order, and diffusion coefficient have been simulated to observe the dissolved oxygen and biochemical oxygen demand dynamics.

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Keywords: BOD and DO; Finite Element Method; Diffusion; Advection

1. Introduction

Water pollution from human activities including social and economic development is an important problem in environmental management. Not only tiny creatures cannot survive, but also affects human health. Moreover pollutants from agricultural operations can be a significant contributor to the impairment

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of surface and groundwater quality [1]. Development of mathematical model enables us to get better understanding on how to control and predict water quality in river. There are many factors of water quality required to observe and use in the model such as dissolved oxygen level, water velocity, pollutant addition, and saturated oxygen concentration. The main consideration is aimed to predict the dissolved oxygen concentration and the biochemical oxygen demand along the river channel as well as the interaction of these two factors.

Recently, Pimpunchat B et al. [2] have presented a mathematical model of pollution in river, they presented a simple model and provided some analytical solutions at steady state flows. The model involving dissolved oxygen balance in river is proposed by Ramakar Jha et al. [3], and this model has been improved in various ways. Historically, the famous mathematical model used to predict water quality in river is proposed by Streeter HW, and Phelps EB in [4]. They developed the model equations to study how much the declination of dissolved oxygen concentration in a river at certain given points downstream. At steady state flow, the analytical solution can be derived and represented in the form of DO sag curve solution. However, for some extended and complicated models, the analytical solutions cannot be derived. It requires an efficient numerical method to find approximate solution instead.

Previously, various numerical methods have been presented for solving pollution model. For example, the finite element method for solving a one-dimensional problem involving convection-diffusion process and sources of pollution in a river is presented by Onyejekwe OO and Toolsi S in [5]. Bernardo Cockburn B, Lin SY and Shu CW in [6] proposed the discontinuous Galerkin method for solving the model equation in conservation law form. They constructed and analyzed the numerical method based on the class of TVB (total variation bounded) scheme. Tyagi B, Gakkhar S and Bhargava DS in [7] have presented an alternative numerical scheme for predicting the BOD variation in time at various successive distances from a wastewater outfall in a stream, which for the solution of differential equations incorporating the advection, dispersion, and biochemical decay. The significance of the stream self-purification is proposed by Wang LK, Vielkind D and Wang MH in [8]. The current stream quality standards on dissolved oxygen concentration, and the solubility of dissolved oxygen in water. Special emphasis is placed on the modelling of the biological reactions and the dissolved oxygen saturation concentration in fresh water. The BOD and DO sag models that describe water quality in river have been presented by Le TV, Roider EM, and Adrian DD in [9]. They provided an alternative approach for solving the first-order BOD equation in related with water quality models for DO. The DO differential equation is solved analytically by the application of Laplace transform method.

In this work, we present a mathematical model for dissolved oxygen concentration interaction with biochemical oxygen demand in an open channel flow. The model is in the same form as those equations presented in [2] but, in this study, we have considered and proposed a numerical method for studying arbitrary cross-sectional area case. Our numerical scheme is based on the finite element method with linear basis function. The dissolved oxygen concentration in coupled with the biochemical oxygen demand are approximated at the same corresponding channel position and time. Two test cases have been performed for checking the accuracy of our numerical scheme. These cases are zero dispersion and including dispersion. Furthermore, the effects of diffusion, advection and reaction kinetics are also investigated by including these terms in the model, and it is then solved numerically .

The mathematical models of water pollution and dissolved oxygen interaction with biochemical oxygen demand in an open channel flow are presented in section 2. The finite element approximation and time discretization are presented in section 3. Numerical examples and results are shown in sections 4 and 5 respectively. Finally, we have made some conclusions in section 6.

Nomenclature

$P(x, t)$	the concentration of water pollution ($kg\ m^{-3}$)
$X(x, t)$	the concentration of dissolved oxygen ($kg\ m^{-3}$)
D_p	the dispersion coefficient of pollutant in the x direction ($m^2\ day^{-1}$)
D_x	the dispersion coefficient of dissolved oxygen in the x direction ($m^2\ day^{-1}$)
l	is the polluted length of river (m)
v	is the water velocity in the x direction ($m\ day^{-1}$)
A	is the cross-section area of the river (m^2)
q	is the rate of pollutant addition along the river ($kg\ m^{-1}\ day^{-1}$)
K_1	is the degradation rate coefficient for pollutant (day^{-1})
K_2	is the de-aeration rate coefficient for dissolved oxygen (day^{-1})
k	is the half-saturated oxygen demand concentration for pollutant decay ($kg\ m^{-3}$)
α	is the mass transfer of oxygen from air to water ($m^2\ day^{-1}$)
S	is the saturated oxygen concentration ($kg\ m^{-3}$)
t	is time ($days$)
n	is number of elements

2. Mathematical Model

We consider flow in a river as being one-dimensional, using a single spatial variable $x(m)$ to describe the distance along the river. Quantities, such as pollutant or oxygen concentrations, are allowed to vary along the length of the river.

In this work, we consider flow quantities variation in time t (days). Main considering water qualities are water pollution concentration P ($kg\ m^{-3}$) and dissolved oxygen concentration X ($kg\ m^{-3}$) being change along the river.

Following [2], the mathematical model is composed of two coupled advection and dispersion equations. These equations account for the evolution of the pollutant and the dissolved oxygen concentrations. The rates of change of the concentration in position x and time t can be expressed as

$$\frac{\partial(AP)}{\partial t} = D_p \frac{\partial^2(AP)}{\partial x^2} - \frac{\partial(vAP)}{\partial x} - K_1 \frac{X}{X+k} AP + qH(x), (x > 0) \tag{1}$$

$$\frac{\partial(AX)}{\partial t} = D_x \frac{\partial^2(AX)}{\partial x^2} - \frac{\partial(vAX)}{\partial x} - K_2 \frac{X}{X+k} AP + \alpha(S - X), (x > 0). \tag{2}$$

In this presented model, we have generalized the model from [2] by allowing the cross sectional area A as a function of x . This means that this model can be applied to study in a river with arbitrary cross-sectional area.

3. The Finite Element Method

In this work, we present the finite element method to solve numerically the system of partial differential equations (1) and (2). There are four main steps as follows.

3.1. Spatial discretization

We begin by dividing the computational domain ($0 \leq x \leq l$) in n elements, thus length of an element is $L = \frac{l}{n}$. The points at $x = 0$ and $x = l$ refer to the upstream and downstream respectively.

3.2. Approximated function

The pollutant function $P(x)$ for each element is approximated by

$$P(x) = \sum_{i=1}^{n+1} \phi_i P_i = \phi_1 P_1 + \phi_2 P_2 = [\phi_1 \ \phi_2] \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = [\phi]_{1 \times 2} \{T\}_{2 \times 1}. \tag{3}$$

Similarly, the dissolved oxygen function $X(x)$ is approximated by

$$X(x) = [\phi]_{1 \times 2} \{X\}_{2 \times 1}, \tag{4}$$

where $\phi_1 = 1 - \frac{x}{L}$ and $\phi_2 = \frac{x}{L}$ represent the linear Lagrange polynomial basis functions in each element.

3.3. Weighted residual approach

We solve the systems (1) and (2) by the standard Galerkin finite element based on weighted residuals approach. We obtain

$$\int_0^L W \left(\frac{\partial(AP)}{\partial t} - D_p \frac{\partial^2(AP)}{\partial x^2} + \frac{\partial(vAP)}{\partial x} + K_1 \frac{X}{X+k} AP - qH(x) \right) dx = 0, \tag{5}$$

$$\int_0^L W \left(\frac{\partial(AX)}{\partial t} - D_x \frac{\partial^2(AX)}{\partial x^2} + \frac{\partial(vAX)}{\partial x} + K_2 \frac{X}{X+k} AP - \alpha(S - X) \right) dx = 0, \tag{6}$$

where W is a weighting function.

By the Galerkin approach, we choose the weighting function to be the same as the basis function. Hence the finite element formulation for equations (5) and (6) can be written in the form of

$$\int_0^L \phi_i \frac{\partial(AP)}{\partial t} dx - \int_0^L \phi_i D_p \frac{\partial^2(AP)}{\partial x^2} dx + \int_0^L \phi_i \frac{\partial(vAP)}{\partial x} dx + \int_0^L \phi_i K_1 \frac{X}{X+k} AP dx - \int_0^L \phi_i qH(x) dx = 0, \quad (7)$$

$$\int_0^L \phi_i \frac{\partial(AX)}{\partial t} dx - \int_0^L \phi_i D_x \frac{\partial^2(AX)}{\partial x^2} dx + \int_0^L \phi_i \frac{\partial(vAX)}{\partial x} dx + \int_0^L \phi_i K_2 \frac{X}{X+k} AP dx - \int_0^L \phi_i \alpha(S-X) dx = 0, \quad (8)$$

where i refers to the number of nodes in an element ($i = 1, 2$).

3.4. Time discretization

For a special case that the half-saturated oxygen demand concentration for pollutant decay k is negligible ($k \approx 0$), equations (7) and (8) are reduced to

$$\left[\int_0^L A(x) \phi_i \phi_j dx \right] \dot{P}_j + \left[D_p \int_0^L A(x) \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx \right] P_j + \left[v \int_0^L A(x) \phi_i \frac{\partial \phi_j}{\partial x} dx \right] P_j + \left[K_1 \int_0^L A(x) \phi_i \phi_j dx \right] P_j - qH(x) \int_0^L \phi_i dx = 0, \quad (9)$$

$$\left[\int_0^L A(x) \phi_i \phi_j dx \right] \dot{X}_j + \left[D_x \int_0^L A(x) \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx \right] X_j + \left[v \int_0^L A(x) \phi_i \frac{\partial \phi_j}{\partial x} dx \right] X_j + K_2 \int_0^L A(x) P(x) \phi_i dx - \alpha S \int_0^L \phi_i dx + \left[\alpha \int_0^L \phi_i \phi_j dx \right] X_j = 0, \quad (10)$$

where

$$\dot{P}_j = \frac{\partial P}{\partial t} = \frac{P^{n+1} - P^n}{\Delta t}, \quad P_j = \theta P^{n+1} + (1-\theta)P^n, \quad (11)$$

$$\dot{X}_j = \frac{\partial X}{\partial t} = \frac{X^{n+1} - X^n}{\Delta t}, \quad X_j = \theta X^{n+1} + (1-\theta)X^n. \quad (12)$$

Here θ is usually specified in the range of $0 \leq \theta \leq 1$ and that is used to weight approximate solution in time. The most commonly used value of θ is 0 (gives the explicit forward Euler scheme), 0.5 (gives a second-order central implicit method, or Crank-Nicolson-Galerkin), or 1 (leads to the fully implicit method).

Therefore, we can write equations (9) and (10) in matrix forms as

$$\left[[T] + \Delta t \theta \left([A_p] + [B] + K_1 [T] \right) \right] P_j^{n+1} = \left[[T] + \Delta t (\theta - 1) \left([A_p] + [B] + K_1 [T] \right) \right] P_j^n + \Delta t [G], \quad (13)$$

$$\left[[T] + \Delta t \theta \left([A_x] + [B] + [F2] \right) \right] X_j^{n+1} = \left[[T] + \Delta t (\theta - 1) \left([A_x] + [B] + [F2] \right) \right] X_j^n - \Delta t \left[[E] - [F1] \right], \quad (14)$$

where matrix $[T]$ can be expressed by

$$[T] = \int_0^L A(x) \phi_i \phi_j dx = \frac{L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix}, \quad (15)$$

the diffusion term can be approximated as

$$[A_p] = D_p \int_0^L A(x) \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx = \frac{D_p}{2L} \begin{bmatrix} A_1 + A_2 & -(A_1 + A_2) \\ -(A_1 + A_2) & A_1 + A_2 \end{bmatrix}, \quad (16)$$

and,

$$[A_x] = \frac{D_x}{2L} \begin{bmatrix} A_1 + A_2 & -(A_1 + A_2) \\ -(A_1 + A_2) & A_1 + A_2 \end{bmatrix}. \quad (17)$$

The advection term can be approximated as

$$[B] = v \int_0^L A(x) \phi_i \frac{\partial \phi_j}{\partial x} dx = \frac{v}{6} \begin{bmatrix} -(2A_1 + A_2) & 2A_1 + A_2 \\ -(A_1 + 2A_2) & A_1 + 2A_2 \end{bmatrix}. \quad (18)$$

The last term in equation (13) is

$$[G] = qH(x) \int_0^L \phi_i dx = qH(x) \frac{L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (19)$$

In a similar way, matrices $[F1]$, $[F2]$, and $[E]$ are given by

$$[F1] = \alpha S \int_0^L \phi_i dx = \alpha S \frac{L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (20)$$

$$[F2] = \alpha \int_0^L \phi_i \phi_j dx = \frac{\alpha L}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \quad (21)$$

$$[E] = K_2 \int_0^L A(x) P(x) \phi_i dx = K_2 \int_0^L \left(A_1 - \frac{x A_1}{L} + \frac{x A_2}{L} \right) \left(P_1 - \frac{x P_1}{L} + \frac{x P_2}{L} \right) \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} dx. \quad (22)$$

Finally, after assembling all elements together, we obtain the full form of the finite element equation system,

$$\begin{aligned} \left[[T]_{\text{sys}} + \Delta t \theta \left([A_p]_{\text{sys}} + [B]_{\text{sys}} + K_1 [T]_{\text{sys}} \right) \right] P_j^{n+1} \\ = \left[[T]_{\text{sys}} + \Delta t (\theta - 1) \left([A_p]_{\text{sys}} + [B]_{\text{sys}} + K_1 [T]_{\text{sys}} \right) \right] P_j^n + \Delta t [G]_{\text{sys}}, \quad (23) \end{aligned}$$

$$\begin{aligned} \left[[T]_{\text{sys}} + \Delta t \theta \left([A_x]_{\text{sys}} + [B]_{\text{sys}} + [F2]_{\text{sys}} \right) \right] X_j^{n+1} \\ = \left[[T]_{\text{sys}} + \Delta t (\theta - 1) \left([A_x]_{\text{sys}} + [B]_{\text{sys}} + [F2]_{\text{sys}} \right) \right] X_j^n - \Delta t \left[[E]_{\text{sys}} - [F1]_{\text{sys}} \right]. \quad (24) \end{aligned}$$

Subscript sys means the ensemble matrices. These two coupled systems can be solved iteratively by the Gauss-Seidel method. In all of our computations, we set tolerance in order of 10^{-12} to ensure completely converged solutions.

4. Numerical Examples

4.1. Zero dispersion

In this case, the exact solutions of downstream pollutant concentration and dissolved oxygen concentration at steady state can be derived in [2]. The close forms of these exact solutions are

$$\bar{P}(x) = \left(\frac{q}{K_1 A} \right) \left(1 - e^{-\frac{K_1 x}{v}} \right), \quad (26)$$

$$\bar{X}(x) = S - \frac{K_2 q}{K_1} \left(\frac{1}{\alpha} - \left(\frac{1}{\alpha - K_1 A} \right) e^{-\frac{K_1 x}{v}} \right) - \left(\frac{K_2 q A}{\alpha (\alpha - K_1 A)} \right) e^{\frac{\alpha x}{v}}. \quad (27)$$

To make some connections with our presented numerical method, we set the boundary conditions as

$$P(0) = 0, X(0) = S, \frac{\partial P(l)}{\partial x} = 0 \text{ and } \frac{\partial X(l)}{\partial x} = 0.$$

4.2. Including dispersion

When dispersion effect is included in the model, the exact solutions of downstream pollutant concentration and the dissolved oxygen concentration at steady-state are remain derived in [2]. The exact solutions are expressed as follows.

$$\bar{P}(x) = \left(\frac{q}{K_1 A} \right) \left(1 - \left(\frac{\delta + \beta}{2\beta} \right) e^{(\delta + \beta)x} \right), \quad (28)$$

$$\text{where } \delta = \frac{v}{2D_p} \text{ and } \beta = \frac{\sqrt{v^2 + 4D_p K_1}}{2D_p},$$

and,

$$\bar{X}(x) = S - \frac{K_2 q}{K_1 \alpha} + \frac{K_2 q}{K_1} \left[\left(\frac{\gamma + \eta}{2\eta \alpha} - \frac{\delta + \beta}{4\beta \eta A^*} + \frac{\delta - \beta}{4\beta \eta B^*} \right) e^{(\gamma + \eta)x} - \frac{\delta + \beta}{2\beta A^*} x e^{(\delta + \beta)x} \right], \quad (29)$$

$$\text{where } \gamma = \frac{v}{2D_x}, \quad \eta = \frac{\sqrt{v^2 + \frac{4\alpha D_x}{A}}}{2D_x}, \quad A^* = 2AD_x (\delta - \beta) - vA \text{ and } B^* = 2AD_x (\delta + \beta) - vA.$$

In this case, we set the initial and boundary conditions in our calculations as

$$P(0) = \frac{q}{K_1 A} \left(1 - \left(\frac{\delta + \beta}{2\beta} \right) \right), X(0) = S - \frac{K_2 q}{K_1 \alpha} + \frac{K_2 q}{K_1} \left(\frac{\gamma + \eta}{2\eta \alpha} - \frac{\delta + \beta}{4\beta \eta A^*} + \frac{\delta - \beta}{4\beta \eta B^*} \right),$$

$$\frac{\partial P(l)}{\partial x} = 0 \text{ and } \frac{\partial X(l)}{\partial x} = 0$$

$$\text{where } \gamma = \frac{v}{2D_x}, \quad \eta = \frac{\sqrt{v^2 + \frac{4\alpha D_x}{A}}}{2D_x}, \quad A^* = 2AD_x (\delta - \beta) - vA \text{ and } B^* = 2AD_x (\delta + \beta) - vA.$$

5. Numerical Results

In our simulations, we set the length of channel $L = 200\text{ m}$, the cross sectional area is uniform with $A = 1.5\text{ m}^2$, the degradation rate coefficient $K1 = 0.5\text{ day}^{-1}$, the de-aeration rate coefficient $K2 = 0.05\text{ day}^{-1}$, the half-saturated oxygen demand concentration for pollutant decay $k = 0\text{ kg m}^{-3}$, the saturated oxygen concentration $S = 2\text{ kg m}^{-3}$ and the final time simulated $t = 200\text{ days}$. Then, we vary the water velocity (v), the dispersion coefficient of pollutant in the x direction (D_p), the dispersion coefficient of dissolved oxygen in the x direction (D_x), rate of pollutant addition along the river (q) and the mass transfer of oxygen from air to water (α).

5.1. Zero dispersion

In this simulation, we give $v = 2\text{ m day}^{-1}$, $q = 0.2\text{ kg m}^{-1}\text{ day}^{-1}$, $D_p = 0\text{ m}^2\text{ day}^{-1}$, $D_x = 0\text{ m}^2\text{ day}^{-1}$ and $\alpha = 0.01\text{ m}^2\text{ day}^{-1}$.

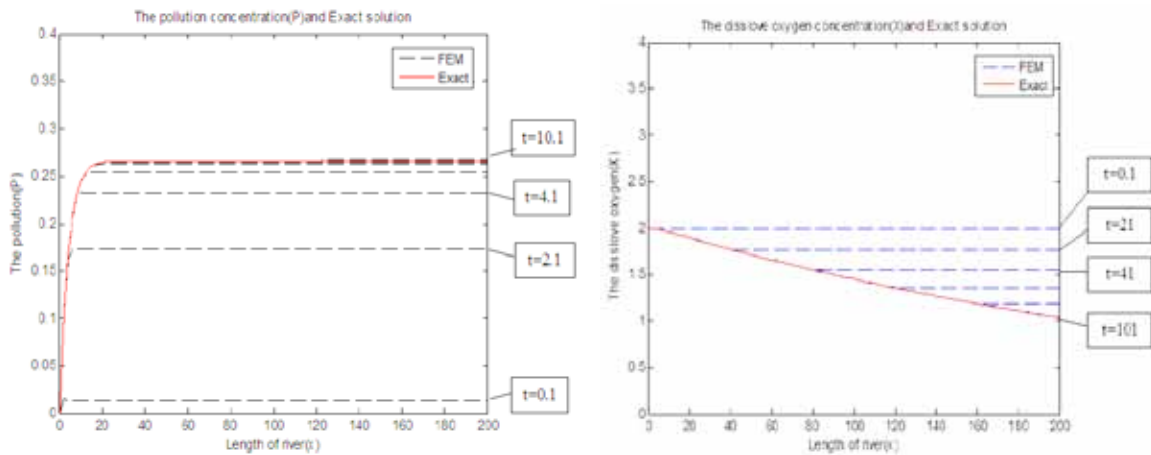


Fig.1. Zero dispersion case for $v = 2\text{ m day}^{-1}$, $q = 0.2\text{ kg m}^{-1}\text{ day}^{-1}$, $D_p = 0\text{ m}^2\text{ day}^{-1}$, $D_x = 0\text{ m}^2\text{ day}^{-1}$ and $\alpha = 0.01\text{ m}^2\text{ day}^{-1}$: (a) pollution concentration, $P(x, t)$ versus river length, (b) dissolve oxygen concentration, $X(x, t)$ versus river length.

From Fig. 1. (a), we set the initial pollution to be zero along the channel. It can be seen that the numerical solution of pollution concentration $P(x, t)$ increases dramatically to the steady exact solution in equation (26) as time increases. The pollution increases rapidly at the early upstream flow and then approaching to a constant value 0.25 when $L \approx 20\text{ m}$. The concentration of pollution is constant along the downstream channel due to the effect of constant point source along the downstream flow. For very large time simulated, the numerical results have in good agreement with the exact solution at steady state. By this numerical experiment, it confirms the accuracy of our presented numerical approximations.

The numerical results for the dissolved oxygen concentration $X(x, t)$ is shown in Fig. 1.(b). The initial value is set as $S = 2$. The dissolved oxygen concentration decreases along the downstream channel and its value approaches the steady exact solution in equation (27) for very large time simulated. In this case, the dissolved oxygen concentration decreases due to a constant point source effect adding along the channel flow.

The effects of increased flow velocity are shown in Fig. 2. (a) and 2. (b). We give $v = 5\text{ m day}^{-1}$ while other parameter values remain the same values as those values simulated in Fig. 1. (a). It can be shown that, as v increases, the increased pollution zone is larger than the smaller v case. It is approximately

$L \approx 80$ for $v = 5$. Similar behaviors can be made for dissolved oxygen concentration, its value decreases slower than the case of small flow velocity.

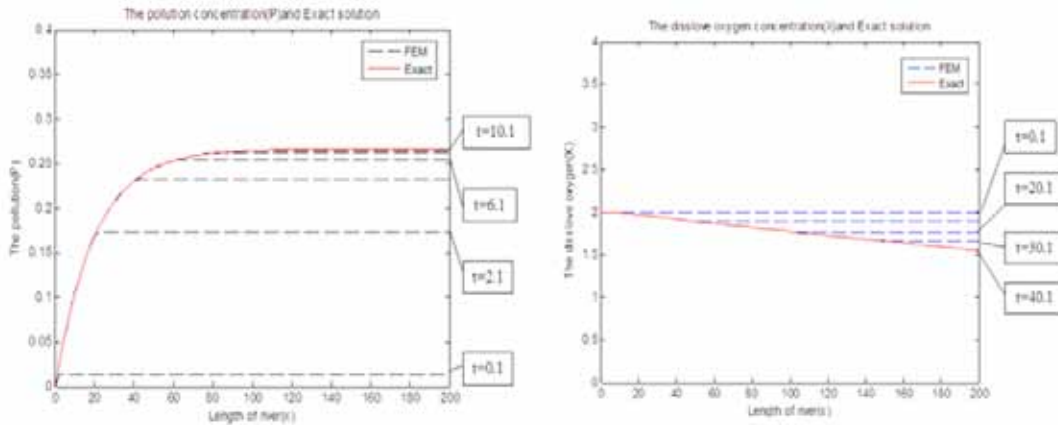


Fig.2. Zero dispersion case for $v = 5 \text{ m day}^{-1}$: (a) pollution concentration, $P(x,t)$ versus river length, (b) dissolve oxygen concentration, $X(x,t)$ versus river length.

5.2. Including dispersion

Numerical results when dispersion effect included is shown in figure 3. We set $v = 2 \text{ m day}^{-1}$, $q = 0.2 \text{ kg m}^{-1} \text{ day}^{-1}$, $D_p = 20 \text{ m}^2 \text{ day}^{-1}$, $D_x = 20 \text{ m}^2 \text{ day}^{-1}$, and $\alpha = 0.01 \text{ m}^2 \text{ day}^{-1}$.

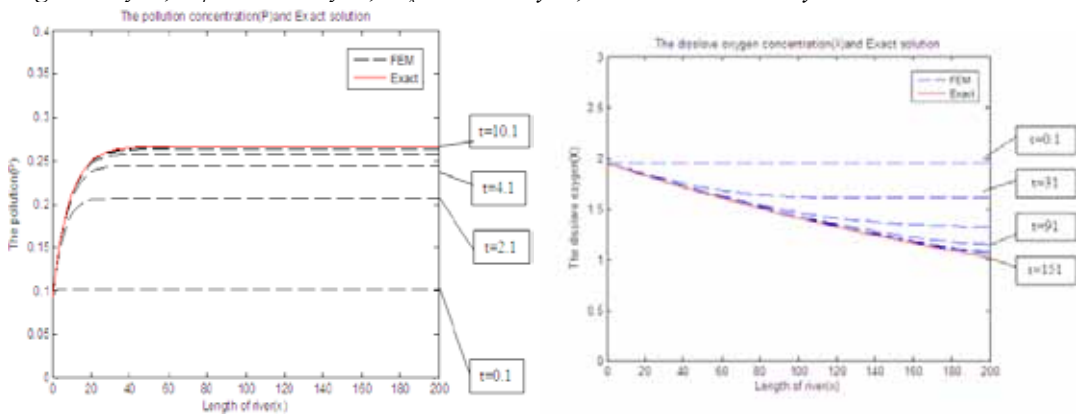


Fig. 3. Including dispersion effect for $v = 2 \text{ kg m}^{-1} \text{ day}^{-1}$, $q = 0.1 \text{ kg m}^{-1} \text{ day}^{-1}$, $D_p = 20 \text{ m}^2 \text{ day}^{-1}$, $D_x = 20 \text{ m}^2 \text{ day}^{-1}$, and $\alpha = 0.01 \text{ m}^2 \text{ day}^{-1}$: (a) pollution concentration versus length of river, (b) dissolve oxygen concentration versus length of river.

The results when diffusion effects included are shown in Fig. 3. Here we have set $D_p = 20 \text{ m}^2 \text{ day}^{-1}$ and $D_x = 20 \text{ m}^2 \text{ day}^{-1}$. Comparing Fig. 3. with Fig. 1, both pollution and dissolved oxygen concentration profiles have wider changing zone than the case of zero dispersion for the same flow velocity value. However, the pollution concentration remains constant because the constant point source effect is still added along the downstream flow. We have compared the numerical solutions by the finite element method with the exact solution at steady state. They are in good agreement for very large time simulated.

This again ensures the correctness of our numerical scheme when the dispersion matrix has been included in our calculations.

The simulations for the smaller rate of adding pollutant along the river are shown in Fig. 4. We reduce q from 0.2 to 0.1. At the steady state, it is found that the maximum pollution concentration level for smaller q is less than those values in the case of larger q , see Fig. 4 (a). It is reasonable because we have added smaller magnitude of point source that would produce smaller pollution concentration along the river. The pollution concentration directly affects to the dissolved oxygen concentration, see Fig. 4 (b). The dissolved oxygen level is larger than those value in the case of larger q .

The presented mathematical model in equation (2) can be used to study the effect of oxygen transfer coefficient from air to water in term of parameter α . We simulate by increasing the value of α to be 0.1 ($\alpha = 0.01$ in previous simulations). This should directly affect to the dissolved oxygen concentration along downstream. The unsteady numerical results and the exact solution at steady state are shown in

Fig 5. (b). The approximated oxygen concentration has small decreasing value and that approaches the steady state exact solution for small time simulation run. This shows the results of direct adding the rate of oxygen exchanged from air to water.

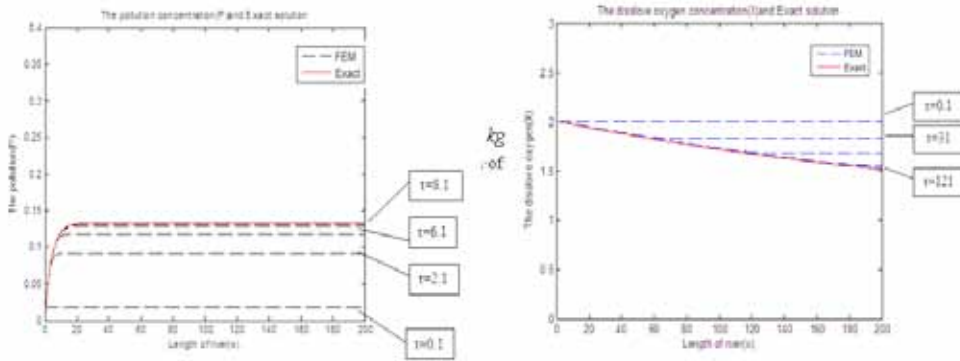


Fig. 4. Including dispersion effect for $v = 2 \text{ m day}^{-1}$, $q = 0.1 \text{ kg m}^{-1} \text{ day}^{-1}$, $D_p = 1 \text{ m}^2 \text{ day}^{-1}$, $D_x = 1 \text{ m}^2 \text{ day}^{-1}$, and $\alpha = 0.01 \text{ m}^2 \text{ day}^{-1}$: (a) pollution concentration versus length of river, (b) dissolve oxygen concentration versus length of river.

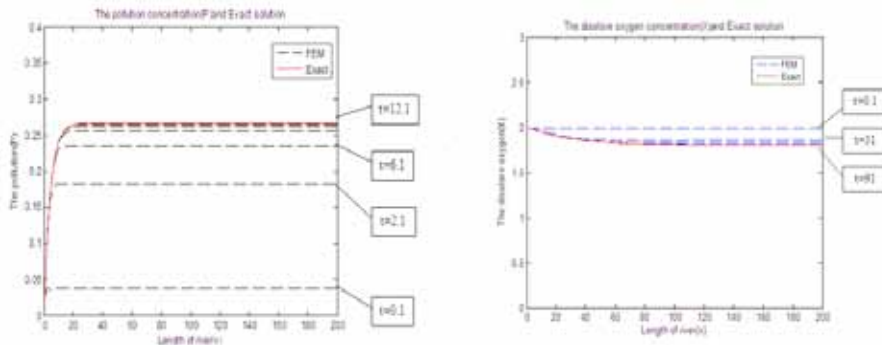


Fig. 5. Including dispersion effect for $v = 2 \text{ m day}^{-1}$, $q = 0.2 \text{ kg m}^{-1} \text{ day}^{-1}$, $D_p = 1 \text{ m}^2 \text{ day}^{-1}$, $D_x = 1 \text{ m}^2 \text{ day}^{-1}$, and $\alpha = 0.1 \text{ m}^2 \text{ day}^{-1}$: (a) pollution concentration versus length of river, (b) dissolve oxygen concentration versus length of river.

6. Conclusions

In this work, we have presented the mathematical model of dissolved oxygen concentration interaction with biochemical oxygen demand in an open channel flow. We consider one-dimensional and unsteady flow. The model is shown in the form of partial differential equations which is required to solve the dissolved oxygen concentration in coupled with biochemical oxygen demand at the same corresponding channel position and considering time. We have presented efficient numerical method based on the standard finite element with linear Lagrange basis function for solving the system model. Two test cases have been performed. These are zero dispersion and including dispersion effects. After simplifying the model and considering at the steady state flow, some analytical solutions are possible to derive. We have simulated these two test cases by setting some appropriate initial conditions. It is found that our numerical unsteady solutions approach the exact steady solutions for both two test cases. This observation ensures the correctness of the presented numerical scheme. Hence, the numerical method presented in this paper has the common applicability to solve the dynamics of surface water pollution in conjunction with the dissolved oxygen concentration along channel flow. Moreover, diffusion, advection and reaction kinetics are also included in the model. This method can be used to investigate each effect separately and this should be the future investigations for the model development.

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