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A complete proof of Viterbo's uniqueness theorem on generating functions

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Abstract

We present here the complete proof of a theorem of Claude Viterbo, stating a uniqueness property for quadratic at infinity generating functions. © 1999 Elsevier Science B.V. All rights reserved.

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In his paper [27], Claude Viterbo stated a uniqueness result about the quadratic at infinity generating functions which generate a Lagrangian submanifold hamiltonianly isotopic to the zero section, in the cotangent bundle of a closed manifold. This enabled him to define capacities for the open sets in $\mathbb{R}^{2n}$, with applications such as the non-squeezing theorem, the camel problem and the Weinstein conjecture.

The motivation for the present text is that the initial proof was a little too elliptic to be fully convincing for many readers. We have thus reworked every step, which lead us to change some parts—in particular in what we call the "invariance of the uniqueness property under isotopies" (Section 5) because an incorrect use of Sikorav's paper [21] was made in the original proof.

We also generalize both Sikorav's existence and Viterbo's uniqueness theorems to the non-exact case, using generating forms instead of functions.

This text is organized as follows. In Section 1, we recall the relevant basic definitions of symplectic geometry. Section 2 is concerned with generating functions and forms. We state Viterbo's result and the generalization to the non-exact case in Section 3, and we give there a brief sketch of the proofs. The remaining sections are devoted to the complete proofs.

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I undertook this work with the help of Claude Viterbo himself; I wish to thank him for his patience and constant kindness in answering all my—sometimes naive—questions. I also thank François Laudenbach for carefully reading the French version [23], and Jean-Claude Sikorav for useful remarks. And I am very grateful to Marc Chaperon, my thesis advisor, who taught me—and made me like—the difficult subject of generating functions; without his support, this work would not have come to existence.

1. Basic definitions

Throughout this paper, M^n will denote an n-dimensional connected smooth manifold, closed unless otherwise specified. All functions and maps will be assumed of class C^∞.

The *cotangent bundle* T^*M is equipped with its standard symplectic form $\omega_M = -\,\mathrm{d}\lambda_M$, where λ_M is the Liouville one-form. In local cotangent coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, we have

$$\lambda_M = y\,\mathrm{d}x = \sum_{i=1}^{n} y_i\,\mathrm{d}x_i \quad \text{and} \quad \omega_M = \mathrm{d}x \wedge \mathrm{d}y = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

A submanifold $L \subset T^*M$ is *Lagrangian* if $\dim L = \dim M$ and $i_L^*\omega_M = 0$, where i_L denotes the inclusion. It is moreover *exact* if $i_L^*\lambda_M$ is an exact one-form. For the sake of simplicity, we will consider *embedded* submanifolds only.

A *symplectic isotopy* of T^*M is a smooth path $(\varphi_t)_{t\in[0,1]}$ of diffeomorphisms of T^*M into itself, such that

$$\varphi_0 = \mathrm{id}_{T^*M} \quad \text{and} \quad \varphi_t^*\omega_M = \omega_M \quad \forall t \in [0, 1]. \tag{1}$$

Its infinitesimal generator $X = (X_t)_{t\in[0,1]}$ satisfies $L_{X_t}\omega_M = 0$, i.e.,

$$i_{X_t}\,\mathrm{d}\omega_M + \mathrm{d}(i_{X_t}\omega_M) = 0$$

by Cartan's formula. Since ω_M is closed, this simply means that $i_{X_t}\omega_M$ is a *closed* one-form. The isotopy is said to be *Hamiltonian* when $i_{X_t}\omega_M$ is globally exact. The image of a Lagrangian (respectively of an exact Lagrangian) submanifold under a symplectic (respectively Hamiltonian) isotopy remains Lagrangian (respectively exact Lagrangian).

An important part of symplectic geometry is devoted to intersection problems between Lagrangian submanifolds. More specifically, let L be a Lagrangian (respectively an exact Lagrangian) and $(\varphi_t)_{t\in[0,1]}$ be a symplectic (respectively Hamiltonian) isotopy: the question (see [1,2]) is to minorate the number of points in $L \cap \varphi_1(L)$, and also to measure how far $\varphi_1(L)$ is from L. Generating forms (respectively functions) can provide an answer to these questions when L is the zero section in T^*M, see [27]. For more on generating functions and their applications, see [6–11,14,21,24,23,26,27].

2. Generating functions and forms

Generating forms (respectively functions) provide a way of describing some specific Lagrangian (respectively exact Lagrangian) submanifolds in a cotangent bundle.

Let $\pi : E \to M$ be a vector bundle over M. In the case of a product bundle, i.e., $E = M \times \mathbb{R}^k$ with coordinates (x, v) and $\pi(x, v) = x$, we call x the *principal coordinate* and v the *auxiliary* one.

The coisotropic subbundle $W \subset T^*E$ is defined as the set of linear forms which vanish on vertical directions:

$$W = \left\{\xi \in T_e^* E;\ e \in E \text{ and } \xi = 0 \text{ on } T_e(\pi^{-1}(\pi e))\right\}.$$

Definition 2.1 [21].

(1) A closed 1-form $\alpha \in Z^1(E)$ is a *generating form* if, as a map from E to T^*E, it is transversal to W.
(2) A function $S : E \to \mathbb{R}$ is a *generating function* if its differential $\mathrm{d}S$ is a generating form.

Let $\alpha \in Z^1(E)$ be a generating form. Its *critical locus* Σ_α is the set $\alpha^{-1}(W)$, which is a submanifold of E with dimension $\dim M$. Next, a map $i_\alpha : \Sigma_\alpha \to T^*M$ is defined in the following way: to $e \in \Sigma_\alpha$ we associate an element of $T^*_{\pi(e)}M$ by the rule $\delta x \in T_{\pi(e)}M \mapsto \alpha(e).\delta e \in \mathbb{R}$, where δe is any vector in $T_e E$ such that $\mathrm{d}\pi(e).\delta e = \delta x$. Then i_α is a *Lagrangian immersion* of Σ_α in T^*M.

In the case of a generating function $S : E \to \mathbb{R}$, the critical locus will be denoted by Σ_S and the Lagrangian immersion—which is then exact—by i_S.

We will most often work in the product case. If S is a generating function on $M \times \mathbb{R}^k$, the transversality condition writes

$$0 \in (\mathbb{R}^k)^* \quad \text{is a regular value of} \quad \frac{\partial S}{\partial v} : M \times \mathbb{R}^k \to (\mathbb{R}^k)^*.$$

We then have

$$\Sigma_S = \left\{(x, v) \in M \times \mathbb{R}^k;\ \partial_v S(x, v) = 0\right\}$$

and

$$\begin{aligned} i_S : \Sigma_S &\to T^*M \\ (x, v) &\mapsto \big(x, \partial_x S(x, v)\big). \end{aligned}$$

Definition 2.2. Let L be an embedded Lagrangian submanifold in T^*M. We say that a generating form α (respectively function S) *generates* L if i_α (respectively i_S) is a diffeomorphism from Σ_α (respectively Σ_S) onto L.

A simple but basic property is that if α (respectively S) is a generating form (respectively function) for L, then the set of zeroes of α (respectively the set of critical points of S) is in bijection with $L \cap M$.

The intersection points in $L \cap M$ are geometrical objects whose existence—and then number—we wish to study. Since the first works of M. Chaperon [5,6], asymptotic conditions have been found to guarantee that the generating functions considered do have “sufficiently many” critical points (the case of generating forms is more complicated, see Remark 2.7).

Definition 2.3. Let $\pi : E \to M$ be a vector bundle over M.

(1) A *non-degenerate quadratic form* on E is a map $Q : E \to \mathbb{R}$ which is quadratic non-degenerate when restricted to the fibers of π.

(2) A generating form $\alpha \in Z^1(E)$ is *quadratic at infinity* if there exists a non-degenerate quadratic form $Q : E \to \mathbb{R}$ such that $\alpha - \partial_v Q : E \to E^*$ is bounded, where ∂_v denotes the vertical (fiber) derivative.

(3) A generating function $S : E \to \mathbb{R}$ is *quadratic at infinity* if its differential $\mathrm{d}S$ is quadratic at infinity. In case we have $S = Q$ outside a compact set, we will say that S is *exactly quadratic at infinity*.

(4) Following [21], an exactly quadratic at infinity function is *special* if the bundle $E \to M$ is a product bundle, and if the associated quadratic form Q is independent of the first coordinate of $E = M \times \mathbb{R}^k$.

Remark 2.4. If a generating form α satisfies $\alpha = \partial_v Q$ outside a compact set then it must be exact, so that there is no point in defining exactly quadratic at infinity generating forms in the non-exact case.

Notation 2.5. We will be interested in those generating functions (and forms) which are quadratic at infinity; we will use the abbreviation *gqi* for “generating and quadratic at infinity”.

Standard Lyusternik–Schnirelman theory (see [19] for instance) can be used to prove the following statement.

Theorem 2.6. *Let $S : E \to \mathbb{R}$ be a gqi function, where E is the total space of a vector bundle over the closed manifold M. Then the number of critical points of S is greater or equal to the cohomological cup-length of M.*

Remark 2.7. It is also possible to develop a theory for differential forms, see [15,16,18, 20].

We will consider the following *basic operations* on generating functions and forms.

Definition 2.8. Let $\pi : E \to M$ be a vector bundle. Let $\alpha \in Z^1(E)$ be a generating form, and $S : E \to \mathbb{R}$ a generating function.

(1) *Addition of a constant* (for functions only). If $c \in \mathbb{R}$, we set $S' := S + c : E \to \mathbb{R}$.

(2) *Diffeomorphism operation*. If $\pi' : E' \to M$ is another vector bundle and $\Phi : E' \to E$ is a diffeomorphism such that $\pi \circ \Phi = \pi'$, then we set $\alpha' := \Phi^* \alpha \in Z^1(E')$, and $S' := S \circ \Phi : E' \to \mathbb{R}$.

(3) *Stabilization*. If $E' \to M$ is another vector bundle endowed with a non-degenerate quadratic form Q', we set $\alpha' := \alpha \oplus \mathrm{d}Q' \in Z^1(E \oplus E')$, and $S' := S \oplus Q' : E \oplus E' \to \mathbb{R}$.

In all cases, the resulting α' (respectively S') is a generating form (respectively function) which generates the same Lagrangian as α (respectively S).

Remark 2.9. A *fiber diffeomorphism* between two vector bundles E and E' over M will be a diffeomorphism $\Phi : E' \to E$ such that $\pi' = \pi \circ \Phi$. In the case $E = E' = M \times \mathbb{R}^k$ it must be of the form $\Phi(x, v) = (x, \phi(x, v))$. We will also consider isotopies of such objects: a *fiber isotopy* of $M \times \mathbb{R}^k$ is a smooth path $(\Phi_t)_{t\in[0,1]}$ of fiber diffeomorphisms, starting at the identity. The infinitesimal generator of a fiber isotopy can obviously be identified with a map $(t, x, v) \mapsto X_t(x, v)$ from $[0, 1] \times M \times \mathbb{R}^k$ to $\mathbb{R}^k$, defined by the relation

$$X_t\big(\phi_t(x, v)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x, v) \quad \forall (t, x, v) \in [0, 1] \times M \times \mathbb{R}^k.$$

Definition 2.10. Two generating forms (respectively functions) are *equivalent* if they can be made equal after a succession of basic operations (cf. Definition 2.8).

Note that the sets of critical values of equivalent generating functions differ only by an additive constant.

Remark 2.11. The succession of a diffeomorphism operation then stabilization can be reversed. Thus, if S' is obtained from S through a succession of basic operations, it can be obtained through *one* stabilization followed by *one* diffeomorphism.

The property of being quadratic at infinity is preserved by addition of a constant, by stabilization, and by some diffeomorphism operations (in particular, if the diffeomorphism is compactly supported). However, that of being special (see Definition 2.3) is preserved by none of our basic operations. Since it is obviously easier to work with special functions, the following fact is quite useful.

Proposition 2.12. *Any gqi function is equivalent to a* special *one.*

Proof. Let $\pi : E \to M$ be a vector bundle, and $S : E \to \mathbb{R}$ be a gqi function.

(1) S is equivalent to a gqi function defined on a product bundle. Indeed, there is a vector bundle E' over M such that $E \oplus E'$ is trivial; choose any non-degenerate quadratic form Q' on E', and set

$$S' := S \oplus Q' : E \oplus E' \to \mathbb{R}.$$

From now on we suppose that $E = M \times \mathbb{R}^k$ and π is the first projection.

(2) Let $Q : M \times \mathbb{R}^k \to \mathbb{R}$ be a non-degenerate quadratic form on a product bundle. The Euclidean structure on $\mathbb{R}^k$ allows to define, for each $x \in M$, the stable and unstable spaces E_x^+ and E_x^- associated to $Q_x : \mathbb{R}^k \to \mathbb{R}$. We obtain two vector bundles E^+ and E^- over M, whose Whitney sum is trivial. *If they are trivial, then Q is isomorphic to a quadratic*

form independent of the principal coordinate x. Indeed, in case E^- is trivial there exist linearly independent sections $e_1, \dots, e_i : M \to E^-$, i being the index of the forms Q_x. By the Gram–Schmidt process, we may suppose that these sections are actually orthonormal. Similarly, we consider orthonormal sections $e_{i+1}, \dots, e_k$ for E^+. Then

$$Q(\alpha_1 e_1 + \cdots + \alpha_k e_k) = -\left(\alpha_1^2 + \cdots + \alpha_i^2\right) + \left(\alpha_{i+1}^2 + \cdots + \alpha_k^2\right)$$

which means that if Φ is the fiber linear automorphism of $M \times \mathbb{R}^k$ defined by

$$\Phi(x; \alpha_1, \dots, \alpha_k) = (x; \alpha_1 e_1 + \cdots + \alpha_k e_k),$$

then the form $Q \circ \Phi$ does not depend on the base point x.

(3) We apply point 2 to the quadratic form Q associated to the gqi function S. Let E^+ (respectively E^-) be its stable (respectively unstable) bundle as above. We stabilize S by the opposite of Q, obtaining the gqi function $S' = S \oplus (-Q)$, defined on $M \times \mathbb{R}^k \times \mathbb{R}^k$. The quadratic form associated to S' is $Q' = Q \oplus (-Q)$, and its stable (respectively unstable) bundle is $E^+ \oplus E^-$ (respectively $E^- \oplus E^+$). These are trivial bundles, so we may apply point 2.

(4) Now we suppose that $S : M \times \mathbb{R}^k \to \mathbb{R}$ is a gqi function with quadratic form Q independent of the first coordinate (so that we may write $Q : \mathbb{R}^k \to \mathbb{R}$ as well). In [23] we prove that, through a diffeomorphism operation, we can have $S = Q$ outside a compact set; furthermore, we may ask that the diffeomorphism be trivial on any prescribed compact set. □

3. Existence and uniqueness statements

Let M be a closed manifold, and $(\varphi_t)_{t \in [0,1]}$ a Hamiltonian isotopy of T^*M.

Sikorav's Existence Theorem 3.1 [21]. *Let L be a closed Lagrangian submanifold in T^*M, which admits a gqi function. Then $\varphi_1(L)$ also has a gqi function.*

The zero section M obviously has a gqi function, so Theorem 3.1 asserts that $\varphi_1(M)$ always has a gqi function. According to Theorem 2.6, this implies that $\varphi_1(M) \cap M$ has "many" points. These intersection points correspond to critical points of a generating function, and it was Viterbo's idea to use the critical values of a gqi function to define symplectic invariants attached to $\varphi_1(M)$—and also to φ_1 itself. To do this, one should be able to compare the various gqi functions of a given Lagrangian.

Viterbo's Uniqueness Theorem 3.2 [27]. *The gqi functions of $\varphi_1(M)$ are all equivalent.*

We will prove in Section 7 that these results can be generalized to symplectic isotopies.

Theorem 3.3 (the non-exact case). *Both Theorems* 3.1 *and* 3.2 *remain true when the isotopy is only symplectic and we consider generating forms instead of functions.*

Proof. The proof of Theorem 3.2 is actually divided in two parts, as follows.

(1) *The set of closed Lagrangian submanifolds whose gqi functions are all equivalent is stable under Hamiltonian isotopies*. We will show this by restating Sikorav's existence theorem as a fibration statement: the map sending a gqi function to its associated Lagrangian is a Serre fibration in some weak sense (Theorem 4.2). As a consequence, the set of gqi functions of the Lagrangian submanifold we consider is arc-connected "up to equivalence" (this means that before connecting them, we may have to perform basic operations on the functions, see Lemma 5.2). We have thus reduced our problem into proving that if a path of gqi functions generates a constant Lagrangian, then all the functions are equivalent, which is done at the end of Section 5.

(2) *The uniqueness property holds for the zero section M*. It immediately turns out that this is a purely differential problem, so that there is no symplectic geometry in this part.

Examples of gqi functions for M are the non-degenerate quadratic forms on a vector bundle over M (Definition 2.3) and we will prove that any gqi function S is equivalent to one of them (it is easy to see that non-degenerate quadratic forms are all equivalent). According to Proposition 2.12 we may suppose that S is a *special* function, defined on $M \times \mathbb{R}^k$. We consider it as a family of functions $S_x : \mathbb{R}^k \to \mathbb{R}$ with parameter $x \in M$.

Each S_x has only one critical point, which is non-degenerate, and we may suppose that it is located at the origin $0 \in \mathbb{R}^k$. We will eventually prove that S is equivalent to the quadratic form

$$q(x, v) = q_x(v) = \frac{1}{2} d^2 S_x(0).(v, v).$$

To get a feeling of what is going on, assume first that $x \in M$ is fixed. By the Morse Lemma, we may already think that $S_x = q_x$ on a neighborhood of the origin. Now, a simple argument shows (Lemma 6.4) that to have S_x and q_x globally equivalent, it is enough to find a diffeomorphism between $S_x^{-1}(-\varepsilon)$ and $q_x^{-1}(-\varepsilon)$ for any fixed $\varepsilon > 0$. We know that these spaces are diffeomorphic for each x, but what we need is a whole family of diffeomorphisms for varying $x \in M$. Since $q_x = S_x$ near 0, we have the inclusion of a piece of $q_x^{-1}(-\varepsilon)$ into $S_x^{-1}(-\varepsilon)$. We will consider the set $\mathcal{P}_x$ of all diffeomorphisms which extend the inclusion; forming the union

$$\mathcal{P} = \bigsqcup_{x \in M} \mathcal{P}_x,$$

we prove (Theorems 6.5 and 6.7) that the obvious projection $\mathcal{P} \to M$ is a locally trivial fibration with contractible fibers. This means that it admits a global cross-section, i.e., that we can solve our extension problem.

We deal with the non-exact case in Section 7. The idea is to reduce the situation to the exact case, where we can use existence and uniqueness results; the point is that every Lagrangian submanifold considered can be deformed into an exact one, and every symplectic isotopy can be homotoped to a Hamiltonian one. □

4. Sikorav's existence theorem

M is now a *closed* manifold, L a closed Lagrangian submanifold of T^*M, and $(\varphi_t)_{t\in[0,1]}$ a Hamiltonian isotopy of T^*M. We restate Sikorav's original theorem, and show it can be formulated as a true fibration theorem.

The original statement 4.1 [21]. *Assume that L admits a special gqi function $S : M \times \mathbb{R}^k \to \mathbb{R}$. Then there exists an integer $l \geqslant 0$ and a path $(S_t)_{t\in[0,1]}$ of special functions defined on $M \times \mathbb{R}^k \times \mathbb{R}^{2l}$ such that*

(1) *$S_0(x, v, w) = S(x, v) + Q(w)$, where Q is a non-degenerate quadratic form on $\mathbb{R}^{2l}$ with signature (l, l)*;
(2) *$S_t = S_0$ outside a compact set*;
(3) *S_t generates $\varphi_t(L)$.*

Let $\mathcal{L} = \mathcal{L}(T^*M)$ be the set of those Lagrangians of T^*M which are diffeomorphic to M and admit a gqi function. If Y is a manifold, we will say that a map $f : Y \to \mathcal{L}$ is *smooth* when there is a differentiable map $\tilde{f} : Y \times M \to T^*M$ such that for every $y \in Y$ the map $\tilde{f}(y, \cdot)$ is a Lagrangian embedding of M into T^*M, with image $f(y)$.

For every integer $k \geqslant 0$, we call $\mathcal{F}_k$ the set of gqi functions defined on $M \times \mathbb{R}^k$: it is an open subset of $C^\infty(M \times \mathbb{R}^k, \mathbb{R})$ endowed with the Whitney strong topology [13]. If Y is a manifold, we will say that a map $F : Y \to \mathcal{F}_k$ is *smooth* when F is continuous for the strong topology and there is a differentiable function $\widetilde{F} : Y \times M \times \mathbb{R}^k \to \mathbb{R}$ such that $\widetilde{F}(y, \cdot, \cdot) = F(y)$ for every $y \in Y$.

When we want to consider such a map from a (compact) space Y into one of the $\mathcal{F}_k$'s without specifying the integer k, we will talk about "a smooth map from Y to $\mathcal{F}$".

Recall that for F to be (strongly-) continuous at $y_0 \in Y$, it is necessary that there is a neighborhood U of y_0 and a compact set K of $M \times \mathbb{R}^k$, such that the functions $F(y)$ all coincide outside K, when y is in U. In particular, we see that the map $t \mapsto S_t$ appearing in the above theorem is continuous—this is precisely why we choose the strong topology.

In the following, Δ_n denotes the standard n-simplex in $\mathbb{R}^{n+1}$.

Theorem 4.2. *The map $\pi : \mathcal{F} \to \mathcal{L}$ is a smooth Serre fibration, up to equivalence. More precisely, if the smooth map $f : \Delta_n \to \mathcal{L}$ has a smooth lift $F : \Delta_n \to \mathcal{F}$ and if $(f_t : \Delta_n \to \mathcal{L})_{t\in[0,1]}$ is a smooth homotopy of $f = f_0$, then there is a smooth homotopy $(F_t : \Delta_n \to \mathcal{F})_{t\in[0,1]}$ such that $F_0 = F$ up to equivalence, and $\pi \circ F_t = f_t$ for every $t \in [0, 1]$.*

To prove Theorem 4.2 we begin with a lemma which enables us to restrict to the case $\Delta_n = \Delta_0 = \{0\}$.

Lemma 4.3.

(1) *Let $f : \Delta_n \times M \to T^*M$, $(u, x) \mapsto f_u(x)$ be a differentiable map such that each f_u is an exact Lagrangian embedding of M in T^*M. Then there is a map*

$$v : \Delta_n \times M \to (\mathbb{R}^n)^*$$

such that

$$L := \{(u, v(u,x);\, f_u(x)) \in T^*\Delta_n \times T^*M\}$$

is an exact Lagrangian submanifold in $T^*(\Delta_n \times M) \cong T^*\Delta_n \times T^*M$. *Two maps* v *and* w *realizing this condition are linked by a relation* $w = v + \mathrm{d}c$ *for some function* $c : \Delta_n \to \mathbb{R}$.

Furthermore, if $S_u : M \times \mathbb{R}^k \to \mathbb{R}$ *is a smooth family of gqi functions of* $L_u := f_u(M)$, *then the total function* $S : \Delta_n \times M \times \mathbb{R}^k \to \mathbb{R}$ *is a gqi function for a Lagrangian* L *as above.*

(2) *Conversely, if* $F : \Delta_n \times M \to T^*(\Delta_n \times M)$ *is an exact Lagrangian embedding, transversal to the spaces* $W_u := \{u\} \times (\mathbb{R}^n)^* \times T^*M$ *and having a gqi function* $S : \Delta_n \times M \times \mathbb{R}^k \to \mathbb{R}$, *then the symplectic reduction of its image* L *with respect to* W_u *gives a Lagrangian* L_u *in* T^*M, *having as gqi function the restriction* S_u *of* S *to* $\{u\} \times M \times \mathbb{R}^k$.

Proof. (1) To any map $v : \Delta_n \times M \to (\mathbb{R}^n)^*$ we associate the embedding $F(u,x) = (u, v(u,x);\, f_u(x))$. We have

$$F^*\lambda_{\Delta_n \times M}(u,x).(\delta u, \delta x) = f_u^*\lambda_M(x).\delta x + \lambda_M(f_u(x)) \circ \frac{\partial}{\partial u} f(u,x).\delta u + v(u,x).\delta u.$$

Each f_u is exact by assumption; consequently, there is a smooth function $\alpha : \Delta_n \times M \to \mathbb{R}$ such that $\mathrm{d}\alpha_u = f_u^*\lambda_M$ for all u in Δ_n. To get an *exact* embedding F—i.e., to have $F^*\lambda_{\Delta_n \times M}$ exact as a 1-form—it is sufficient to set

$$v(u,x) = \partial_u\alpha(u,x) - \lambda_M(f_u(x)) \circ \frac{\partial}{\partial u} f(u,x).$$

If v and w are two maps whose associate embeddings F_v and F_w are exact, then the 1-form $(w - v)(u,x).\delta u$ has to be exact too; this implies the existence of a function $c : \Delta_n \to \mathbb{R}$ such that $w - v = \mathrm{d}c$.

(2) This point is straightforward. □

Proof of Theorem 4.2. Let $(f_t : \Delta_n \to \mathcal{L})_{t\in[0,1]}$ be a smooth homotopy, and assume that f_0 has a lift $S : \Delta_n \to \mathcal{F}$. According to the first part of Lemma 4.3, we construct a path

$$\bar{f} : [0,1] \to \mathcal{L}(T^*(\Delta_n \times M)),$$

whose initial point $\bar{f}_0$ admits a generating function $\overline{S}_0$. According to the second part of the Lemma, it is now sufficient to prove that we can lift the path $\bar{f}$ from the initial point $\overline{S}_0$. In other words, we have reduced our problem to a path-lifting property, which is precisely the content of Theorem 4.1. □

Remark 4.4. We are using here Sikorav's existence theorem in the case of the cotangent bundle of a manifold *with boundary,* namely $\Delta_n \times M$. Here is one way to resolve this additional difficulty.

Let M be a manifold with boundary. Consider a smooth path $(f_t)_{t\in[0,1]}$ of exact Lagrangian embeddings of M into T^*M such that $f_t(\partial M) \subset T^*_{\partial M}M$ for every t (this condition is clearly fulfilled in our setting). Then there is a Hamiltonian isotopy $(\varphi_t)_{t\in[0,1]}$ on T^*M such that

$$\varphi_t \circ f_0 = f_t \quad \text{and} \quad \varphi_t\big(T^*_{\partial M}M\big) = T^*_{\partial M}M \quad \forall t \in [0,1].$$

Consider next the double N of M: it is a closed manifold in which M is included. We can extend $(\varphi_t)_{t\in[0,1]}$ to get a Hamiltonian isotopy $(\psi_t)_{t\in[0,1]}$ on T^*N—just by extending the Hamiltonian function. We can also extend the embedding $f_0 : M \to T^*M$ to $F_0 : N \to T^*N$—by extending to $N \times \mathbb{R}^k$ the generating function S_0 of f_0, defined on $M \times \mathbb{R}^k$. Then we may use Theorem 3.1 in T^*N, and restrict to $M \times \mathbb{R}^k$ the generating functions thus obtained.

5. Invariance of the uniqueness property under isotopies

A given Lagrangian $L \in \mathcal{L}$ is said to have the uniqueness property if all its gqi functions are equivalent. We will prove the following result.

Theorem 5.1. *Suppose that $L_0 \in \mathcal{L}$ has the uniqueness property, and $(\varphi_t)_{t\in[0,1]}$ is a Hamiltonian isotopy of T^*M. Then $L_1 = \varphi_1(L)$ also has the uniqueness property.*

Let S and S' be two gqi functions of L_1. Recall that $\pi : \mathcal{F} \to \mathcal{L}$ is the fibration defined in Section 4.

Lemma 5.2. *Up to equivalence, S and S' can be connected by a path in $\pi^{-1}(L_1)$.*

Proof. By assumption, L_0 and L_1 are connected by a smooth path $(L_t)_{t\in[0,1]}$ in $\mathcal{L}$. By the path lifting property, we find smooth paths $(S_t)_{t\in[0,1]}$ and $(S'_t)_{t\in[0,1]}$ such that $S_1 = S$, $S'_1 = S'$ (up to equivalence), and such that S_t and S'_t generate L_t for every t in $[0,1]$. Then S_0 and S'_0 both generate L_0, so they are equivalent: L_0 has the uniqueness property. By performing basic operations on the whole paths $(S_t)_{t\in[0,1]}$ and $(S'_t)_{t\in[0,1]}$, we may suppose that $S_0 = S'_0$.

Consider now the loop γ in $\mathcal{L}$, which goes from L_1 to L_0 along $(L_t)_{t\in[0,1]}$, then backwards to L_1. It has the following lift $\tilde{\gamma}$ in $\mathcal{F}$: the path starting at $S = S_1$ and going to S_0 along $(S_t)_{t\in[0,1]}$, then from $S_0 = S'_0$ to $S' = S'_1$ along $(S'_t)_{t\in[0,1]}$. The loop γ is obviously homotopic to the constant loop based at L_1. According to Theorem 4.2, we may find in $\mathcal{F}$ a smooth homotopy of paths starting at $\tilde{\gamma}$ and covering the contraction of γ to the constant loop. The time 1 of this lifted homotopy gives the desired path in $\pi^{-1}(L) = \pi^{-1}(L_1)$. □

Consequently, to prove Theorem 5.1 it suffices to prove that if $(S_t)_{t\in[0,1]}$ is a smooth path of gqi functions generating the constant Lagrangian L, then S_0 and S_1 are equivalent.

Let $M \times \mathbb{R}^k$ be the space on which the gqi functions are defined. We will seek a fiber isotopy $(x, v) \mapsto \Phi_t(x, v) = (x, \phi_t(x, v))$ of $M \times \mathbb{R}^k$ such that

$$S_t\big(x, \phi_t(x, v)\big) = S_0(x, v) \quad \forall (t, x, v) \in [0, 1] \times M \times \mathbb{R}^k. \tag{2}$$

Derivating (2) with respect to t and denoting by X_t the infinitesimal generator of the fiber isotopy (see Remark 2.9) we find

$$\frac{\partial S_t}{\partial t}(x, v) + \frac{\partial S_t}{\partial v}(x, v).X_t(x, v) = 0 \quad \forall (t, x, v) \in [0, 1] \times M \times \mathbb{R}^k. \tag{3}$$

Now let Σ_t be the critical locus of S_t, and $i_t : \Sigma_t \to L$ be the Lagrangian embedding. Note that the resolution of Eq. (3) is problematic at points of Σ_t only: outside Σ we can set

$$X_t(x, v) = -\frac{1}{\|\partial_v S_t(x, v)\|^2} \frac{\partial S_t}{\partial t}(x, v). \tag{4}$$

Lemma 5.3. *Up to fiber isotopy and addition of constants, we may assume that*

(1) *Σ_t and i_t do not depend on t—so we will write Σ and i, respectively, and*

(2) $(\partial S_t / \partial t) = 0$ *on Σ.*

Proof. See [23]. The idea is that $t \mapsto \Sigma_t$ is an isotopy of Σ_0 in $M \times \mathbb{R}^k$, that can be extended to a fiber isotopy of $M \times \mathbb{R}^k$ with compact support. This global isotopy is then used to obtain $\Sigma_t = \Sigma_0$ and $i_t = i_0$. Next, a simple computation shows that $S_t(z) - S_0(z)$ does not depend on the point $z \in \Sigma$. Subtracting this real constant to S_t, we obtain $S_t = S_0$—and thus $\partial_t S_t = 0$—on Σ. □

We are now in position to solve Eq. (2) near Σ. Indeed, the Hadamard Lemma tells us that—Σ being the inverse image of the regular value $0 \in \mathbb{R}^k$ for $\partial_v S_t$, and $\partial_t S_t$ vanishing on Σ—there exists a smooth map X_t defined on a neighborhood U of Σ in $M \times \mathbb{R}^k$ with values in $\mathbb{R}^k$, such that $\partial_t S_t = -\partial_v S_t . X_t$ on U. It remains to paste the two solutions (the one just constructed near Σ and solution (4) outside Σ) with the help of a bump function with (compact) support in U and equal to 1 near Σ.

Remark 5.4. The isotopy $(\Phi_t)_{t \in [0,1]}$ is compactly supported since we have $S_t = S_0$ outside a compact set, see Theorem 3.1.

This concludes the proof of Theorem 5.1.

6. The uniqueness result for the zero section

This section is devoted to the following result.

Theorem 6.1. *Let M be a closed manifold. Then the gqi functions of the zero section of T^*M are all equivalent.*

As we have already noticed, we may suppose that we consider a *special* gqi function

$$S : M \times \mathbb{R}^k \to \mathbb{R},$$

with associated quadratic form $Q_\infty : \mathbb{R}^k \to \mathbb{R}$. Now, since S generates the zero section, it is readily verified that its critical locus Σ_S projects well onto M, and is thus the graph of a map $v_S : M \to \mathbb{R}^k$. We define a fiber diffeomorphism of $M \times \mathbb{R}^k$ by $\Phi(x, v) = (x, v + v_S(x))$, with the property that the critical locus of $S \circ \Phi$ is now $M \times \{0\}$. A straightforward application of the generalized Morse Lemma [13] is now:

Lemma 6.2. *Any gqi function of the zero section is equivalent to a gqi function $S : M \times \mathbb{R}^k \to \mathbb{R}$ such that for each $x \in M$, setting $S_x := S(x, \cdot) : \mathbb{R}^k \to \mathbb{R}$:*

(1) *S_x is equal to a non-degenerate quadratic form q_x in a neighborhood of* $0 \in \mathbb{R}^k$;
(2) *S_x is equal to the non-degenerate quadratic form Q_∞ outside of a compact set*;
(3) *S_x has only one—non-degenerate—critical point on $\mathbb{R}^k$, at the origin.*

Remark 6.3. We have now *two* non-degenerate quadratic forms on the bundle $M \times \mathbb{R}^k$: q and Q_∞, the latter being independent of the base point. Because the functions S_x do not have critical points outside the origin, we conclude that the forms q_x and Q_∞ have the same index, which we denote by i in the following.

Note that our problem is now clearly to prove a kind of *global Morse Lemma with parameter*: each function will be shown to be globally equivalent to the quadratic form q_x—which it equals near the origin—and this must be made in a smooth way with respect to the "parameter" $x \in M$.

Let $\varepsilon > 0$ be small: S and q being equal on a neighborhood U of $M \times \{0\}$ in $M \times \mathbb{R}^k$, we have an injection j of $q^{-1}(-\varepsilon) \cap U$ into $S^{-1}(-\varepsilon)$.

Lemma 6.4. *If the injection j extends to a fiber diffeomorphism $q^{-1}(-\varepsilon) \to S^{-1}(-\varepsilon)$, then S and q are equivalent through a fiber diffeomorphism of $M \times \mathbb{R}^k$.*

Proof. Suppose j can be extended: we then have a fiber diffeomorphism

$$J : q^{-1}(-\varepsilon) \cup U \xrightarrow{\sim} S^{-1}(-\varepsilon) \cup U.$$

Choosing for instance the Euclidean metric on $\mathbb{R}^k$, we define the vertical gradients

$$X(x, v) = \big(0, \nabla_v q(x, v)\big) \quad \text{and} \quad Y(x, v) = \big(0, \nabla_v S(x, v)\big)$$

whose flows are obviously complete.

The X-orbit of any $z \in M \times \mathbb{R}^k$ meets $q^{-1}(-\varepsilon) \cup U$ in at least one point, say z'. Then $Z' = j(z')$ is defined, and on its Y-orbit there is a unique point Z such that $S(Z) = q(z)$. It is easy to check that Z does not depend on the particular z' chosen, so that we can set $J(z) := Z$, defining a global fiber diffeomorphism of $M \times \mathbb{R}^k$ as required. □

The fibration $\mathcal{P} \to M$. Recall that U is the neighborhood of $M \times \{0\}$ on which S and q coincide. For each $x \in M$, we will set $U_x = U \cap (\{x\} \times \mathbb{R}^k)$. By restricting them if necessary, we may choose U and ε so that the pair $(U_x, U_x \cap S^{-1}(-\varepsilon))$ is diffeomorphic to $(D^{k-i} \times D^i, D^{k-i} \times S^{i-1})$.

Since $S = Q_\infty$ outside a compact set, we have a fiber diffeomorphism between $S^{-1}(-\infty)$ and $M \times \mathbb{R}^{k-i} \times S^{i-1}$, where $i = \operatorname{ind}(Q_\infty)$. Since the vertical gradient of S induces a fiber diffeomorphism between $S^{-1}(-\infty)$ and $S^{-1}(-\varepsilon)$, we identify the embedding j with a fiber embedding of $q^{-1}(-\varepsilon) \cap U$ into $M \times \mathbb{R}^{k-i} \times S^{i-1}$.

For every $x \in M$, we note j_x the restricted embedding of $q_x^{-1}(-\varepsilon) \cap U_x$ into $\{x\} \times \mathbb{R}^{k-i} \times S^{i-1}$, and we set

$$\mathcal{P}_x := \left\{ f_x : q_x^{-1}(-\varepsilon) \to \{x\} \times \mathbb{R}^{k-i} \times S^{i-1} \text{ diffeomorphism extending } j_x \right\},$$

then

$$\mathcal{P} := \bigsqcup_{x \in M} \mathcal{P}_x \quad \text{the disjoint union of the } \mathcal{P}_x, \tag{5}$$

$$\pi : \mathcal{P} \to M \quad \text{the projection which associate } x \in M \text{ to } f_x \in \mathcal{P}_x. \tag{6}$$

This set $\mathcal{P}$ can be seen as a subset of the total space of the principal fiber bundle associated to the bundle $q^{-1}(-\varepsilon) \to M$, with structural group $G := \operatorname{Diff}(\mathbb{R}^{k-i} \times S^{i-1})$ endowed with the weak C^∞ topology (uniform convergence on compact subsets, see [13]).

Theorem 6.5. *The fibration $\pi : \mathcal{P} \to M$ is locally trivial.*

Recall that if E and B are two topological spaces, a "fibration"—i.e., a continuous map—$\pi : E \to B$ is said to be *locally trivial* if for every $b \in B$ there exist a neighborhood U of b and a homeomorphism h from $\pi^{-1}(U)$ to $U \times \pi^{-1}(b)$ such that $\pi(e) = \mathrm{pr}_1 \circ h(e)$ for all $e \in \pi^{-1}(U)$.

It is easy to show—see for instance the Appendix of [4]—that a fibration is locally trivial when for every $b_0 \in B$ there is a topological group G acting continuously on B and E, in such a way that

- the G-actions are compatible with π, i.e., $\pi(ge) = g\pi(e)$ for all $e \in \pi^{-1}(U)$;
- the map $g \in G \mapsto gb_0 \in B$ has a local cross-section near b_0.

Let Diff be the group of diffeomorphisms of $\mathbb{R}^{k-i} \times S^{i-1}$ endowed with the weak C^∞ topology (uniform convergence on compact sets), and Pl be the space of embeddings from $D^{k-i} \times S^{i-1}$ into $\mathbb{R}^{k-i} \times S^{i-1}$ endowed with the strong C^∞ topology (uniform convergence).

Lemma 6.6. *The restriction map*

$$p : \mathrm{Diff} \to \mathrm{Pl}$$

$$\varphi \mapsto \varphi|_{D^{k-i} \times S^{i-1}}$$

is a locally trivial fibration.

Proof. We apply the remarks above, taking $G = E = \mathrm{Diff}$, $B = \mathrm{Pl}$ and $\pi : E \to B$ the restriction map. In [3]—and [17] in a slightly simpler case—we find the existence, for every

$f_0 \in B$, of a local continuous section $(B, b_0) \to (G, \mathrm{id}_G)$ for the map $\varphi \in G \mapsto \varphi \circ f_0 \in B$, when the spaces are endowed with the Whitney strong topology. This change of topology—recall that we use weak topologies—concerns only $G = E = \mathrm{Diff}(\mathbb{R}^{k-i} \times S^{i-1})$. In particular, since the weak topology is contained in the strong, it does not alter the *continuous* character of the local section. □

Proof of Theorem 6.5. We consider the fibration $q^{-1}(-\varepsilon) \to M$.

- In case it is a product fibration, i.e., $q^{-1}(-\varepsilon) = M \times \mathbb{R}^{k-i} \times S^{i-1}$ and $q^{-1}(-\varepsilon) \cap U = M \times D^{k-i} \times S^{i-1}$, then the embedding j induces a continuous map from M to Pl, still denoted by $x \mapsto j_x$. The fibration π is then identified with the pull-back $j^*(\mathrm{Diff} \to \mathrm{Pl})$, and is thus locally trivial.
- In the general case, $q^{-1}(-\varepsilon) \to M$ is locally diffeomorphic to a product fibration, so that we may locally conclude as before. □

Theorem 6.7. *The fibers of π are contractible for the weak topology if $k > \frac{3}{2}i + 1$.*

Proof. For any $x \in M$, we consider the embedding $j_x : D^{k-i} \times S^{i-1} \to \mathbb{R}^{k-i} \times S^{i-1}$. It is easy to see that it induces a degree 1 map between the spheres S^{i-1}. The idea is now to consider the fibration $p : \mathrm{Diff} \to \mathrm{Pl}$ from Lemma 6.6 and to say: the set of embeddings satisfying the degree 1 condition is arcwise connected, so the fibers of p corresponding to these embeddings are all homeomorphic; then it suffices to study the fiber of a "simple" enough embedding.

We will not actually show that the above embeddings form an arcwise connected set, only that each such embedding is homotopic to an embedding of the form

$$h(z, u) = \big(A(u).z, u\big),$$

where $A : S^{i-1} \to \mathrm{GL}(\mathbb{R}^{k-i})$ is continuous.

Let us first show that the fiber over such a h is indeed contractible for the weak C^∞ topology. Let $H(z, u) = (a(z, u), b(z, u))$ be a diffeomorphism of $\mathbb{R}^{k-i} \times S^{i-1}$ extending h. For $0 < t \leqslant 1$, we define $H_t(z, u) = (t^{-1}a(tz, u), b(tz, u))$. It is easy to check that we have thus defined a continuous path in Diff, converging to $H_0(z, u) := (A(u).z, u)$ when $t \to 0$.

Now we provide a homotopy from j_x to an h as before.

Let $\bar{g}(u) := (0, u)$ be the standard embedding of S^{i-1} into $\mathbb{R}^{k-i} \times S^{i-1}$, and let $\bar{j}_x$ be the restriction of j_x to $\{0\} \times S^{i-1}$, seen as another embedding of S^{i-1} into $\mathbb{R}^{k-i} \times S^{i-1}$.

When $k > \frac{3}{2}i + 1$, two elements of $\mathrm{Pl}(S^{i-1}, \mathbb{R}^{k-i} \times S^{i-1})$ are homotopic (in Pl) if and only if they induce the same degrees on the spheres. This fact is indeed true for $C^0(S^{i-1}, \mathbb{R}^{k-i} \times S^{i-1})$: homotopy classes of $C^0(S^{i-1}, S^{i-1})$ are characterized by the degree and $\mathbb{R}^{k-i}$ is contractible. Furthermore, two embeddings of a manifold X into a manifold Y which are homotopic through continuous maps are actually homotopic through embeddings when $1 + \dim Y > 2(1 + \dim X)$, see [12]. In our situation, this condition writes $k > \frac{3}{2}i + 1$.

Consequently, $\bar{g}$ and $\bar{j}$ are homotopic in $\mathrm{Pl}(S^{i-1}, \mathbb{R}^{k-i} \times S^{i-1})$ if $k > \frac{3}{2}i + 1$.

Then we use the fact that the restriction map

$$\mathrm{Pl}\big(D^{k-i} \times S^{i-1}, \mathbb{R}^{k-i} \times S^{i-1}\big) \to \mathrm{Pl}\big(S^{i-1}, \mathbb{R}^{k-i} \times S^{i-1}\big),$$
$$f \to \bar{f} := f|_{\{0\}} \times S^{i-1}$$

is a locally trivial fibration (see [3]), so that it satisfies the path lifting property. Consequently, there is a homotopy f_t in $\mathrm{Pl}(D^{k-i} \times S^{i-1}, \mathbb{R}^{k-i} \times S^{i-1})$ connecting $f_0 = j_x$ to an embedding in the fiber over $\bar{g}$.

Writing $g(z,u) = (a(z,u), b(z,u))$, we set $g_t(z,u) := (t^{-1}a(tz,u), b(tz,u))$ for $0 < t \leqslant 1$, thus defining a continuous path of embeddings, converging to $h(z,u) := (\partial_z a(0,u).z, u)$ as $t \to 0$. It is easy to check that $\partial_z a(0,u)$ is invertible for every u. □

We now have a locally trivial fibration $\pi : \mathcal{P} \to M$ with contractible fibers, thus admitting a *global cross-section*. We would like to say that this implies that the hypothesis of Lemma 6.4 is satisfied, thus finishing the proof of Theorem 6.1. However, we have to pay a little attention to differentiability: a global cross-section is a map $J : M \to \mathcal{P}$, i.e., for each $x \in M$ we are given a diffeomorphism J_x between $q_x^{-1}(-\varepsilon)$ and $S_x^{-1}(-\varepsilon) \cong \{x\} \times \mathbb{R}^{k-i} \times S^{i-1}$. What we also need, and have not proved, is that the "total map" $q^{-1}(-\varepsilon) \to S^{-1}(-\varepsilon)$ be globally smooth—not only when restricted to the fibers $q_x^{-1}(-\varepsilon)$.

For this we recall some simple ideas, see [22]. Let X, Y, Z and X', Y' be smooth finite-dimensional manifolds. A map $f : Z \to C^\infty(X,Y)$ is said to be *quasi-differentiable* if the "total map" $\bar{f} : Z \times X \to Y$ is smooth. Now a map $g : C^\infty(X,Y) \to Z'$ is said to be quasi-differentiable if, for any map $f : Z \to C^\infty(X,Y)$, the composition $g \circ f : Z \to Z'$ is smooth. Finally, a map $h : C^\infty(X,Y) \to C^\infty(X',Y')$ is said to be quasi-differentiable if for every quasi-differentiable map $f : Z \to C^\infty(X,Y)$, the composition $h \circ f : Z \to C^\infty(X',Y')$ is quasi-differentiable.

For instance, the map $p : \mathrm{Diff} \to \mathrm{Pl}$ of Lemma 6.6 is quasi-differentiable. Furthermore, it is a *quasi-differentiable fibration* in the sense that there exist quasi-differentiable local trivializations. This follows from the fact that the local sections of $(B, f_0) \to (G, \mathrm{id})$—see the proof of Lemma 6.6—are constructed with the Whitney extension theorem, which extends functions in a quasi-differentiable way with respect to additional parameters. See [3, pp. 290–293], for the construction of local sections, and [25] for the Whitney extension theorem.

Consequently, $\pi : \mathcal{P} \to M$ is also a quasi-differentiable fibration.

Corollary 6.8. *The fibration $\pi : \mathcal{P} \to M$ has a global cross-section. Hence the spaces $q^{-1}(-\varepsilon)$ and $S^{-1}(-\varepsilon)$ are diffeomorphic as bundles over M, and there exists a diffeomorphism J from $q^{-1}(-\varepsilon)$ to $S^{-1}(-\varepsilon)$ extending the initial embedding j from $q^{-1}(-\varepsilon) \cap U$ into $S^{-1}(-\varepsilon)$.*

We are now in a position to apply Lemma 6.4. This means that we have proved Theorem 6.1.

7. Generating forms

The material of this section comes from a fruitful discussion with Denis Sauvaget.

Let L be a Lagrangian submanifold in T^*M. We denote by $p : T^*M \to M$ the bundle projection, by $i_L : L \to T^*M$ the inclusion, and by $p_L : L \to M$ the projection of L to M. *We will always assume that p_L induces an isomorphism between $H^1(L)$ and $H^1(M)$—or equivalently, that i_L induces an isomorphism between $H^1(L)$ and $H^1(T^*M)$.* Consequently, we can choose a closed form $\bar{\alpha} \in Z^1(M)$ such that $p_L^*(\bar{\alpha})$ and $i_L^*(\lambda_M)$ are cohomologous. Now, for $(x, y) \in T^*M$, let $\mu_t(x, y) = (x, y - t\bar{\alpha}(x))$: this defines a symplectic isotopy in T^*M. It is easy to check that $\mu_1(L)$ is now an *exact* Lagrangian submanifold.

Suppose now that L has a gqi form $\alpha \in Z^1(E)$, where $\pi : E \to M$ is a vector bundle. We claim that $\pi^*\bar{\alpha}$ and α are cohomologous (since these are closed forms, it is enough to check that they have identical periods; we leave this verification to the reader). Hence there is a function $S : E \to \mathbb{R}$ such that $\mathrm{d}S = \alpha - \pi^*\bar{\alpha}$. It is clear that S is a gqi function, which generates the exact Lagrangian $\mu_1(L)$.

Next, let $(\varphi_t)_{t\in[0,1]}$ be a symplectic isotopy of T^*M. We choose a smooth family of closed forms $\beta_t \in Z^1(M)$ such that $p^*\beta_t$ and $\varphi_t^*(\lambda_M)$ are cohomologous, and we define the symplectic isotopy $\eta_t(x, y) = (x, y - \beta_t(x))$ of T^*M. It is designed to make the pull-back $(\eta_t \circ \varphi_t)^*\lambda_M$ *exact* for each t, which means that the isotopy $(\eta_t \circ \varphi_t)_{t\in[0,1]}$ is *Hamiltonian*.

We are now ready to prove the existence result.

Theorem 7.1. *If L is a closed Lagrangian submanifold with a gqi form and $(\varphi_t)_{t\in[0,1]}$ is a symplectic isotopy, then $\varphi_1(L)$ also has a gqi form.*

Proof. According to the preceding discussion we may compose the symplectic isotopy $(\varphi_t \circ \mu_1^{-1})_{t\in[0,1]}$ to obtain a Hamiltonian isotopy $(\eta_t \circ \varphi_t \circ \mu_1^{-1})_{t\in[0,1]}$. We know that $\mu_1(L)$ has a gqi function, so—by the existence theorem in the exact case—$\eta_1 \circ \varphi_1 \circ \mu_1^{-1}(\mu_1(L)) = \eta_1 \circ \varphi_1(L)$ also has a gqi function, say $S_1 : E \to \mathbb{R}$. Then there is a closed form $\gamma \in Z^1(M)$ such that $\mathrm{d}S_1 - \pi^*\gamma$ is a gqi form for $\varphi_1(L)$, which concludes. □

Theorem 7.2. *If $(\varphi_t)_{t\in[0,1]}$ is a symplectic isotopy of T^*M, the gqi forms of $\varphi_1(M)$ are all equivalent.*

Proof. We reduce the proof to the exact case, handled in Sections 4–6. Since the zero section is exact, its gqi forms are simply differentials of gqi functions, so they are all equivalent by Theorem 3.2. We are thus left with the invariance of the uniqueness property under symplectic isotopies.

Suppose that the Lagrangian L has the uniqueness property, i.e., that all its gqi forms are equivalent. Let $(\varphi_t)_{t\in[0,1]}$ be a symplectic isotopy of T^*M. We set

$$L_t = \varphi_t(L), \quad i_t : L \hookrightarrow T^*M \quad \text{and} \quad p_t : L \to M,$$

as before. We can choose a smooth path of closed 1-forms $\bar{\alpha}_t \in Z^1(M)$ such that $i_t^*\lambda_M = p_t^*\bar{\alpha}_t$ for all $t \in [0, 1]$. We use it to deform the path $(L_t)_{t\in[0,1]}$ into a path $(\widetilde{L}_t)_{t\in[0,1]}$ of exact Lagrangian submanifolds, as in the beginning of the section. It is clear that, since L_0 has the uniqueness property, so does $\widetilde{L}_0$. Since this property is stable under a Hamiltonian isotopy (or through an isotopy into the space of exact Lagrangian submanifolds), the end point $\widetilde{L}_1$ has this property for its gqi functions. This in turn implies the uniqueness property for the gqi forms of L_1. □

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