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Invertible and nilpotent matrices over antirings

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Abstract

In this paper, we characterize invertible matrices over an arbitrary commutative antiring S with 1 and find the structure of $GL_n(S)$. We find the number of nilpotent matrices over an entire commutative finite antiring. We prove that every nilpotent $n \times n$ matrix over an entire antiring can be written as a sum of $\lceil \log_2 n \rceil$ square-zero matrices and also find the necessary number of square-zero summands for an arbitrary trace-zero matrix to be expressible as their sum.
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1. Introduction

A *semiring* is a set S equipped with binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity element 0 and $(S, \cdot)$ is a monoid with identity element 1. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates S. A semiring is *commutative* if $ab = ba$ for all $a, b \in S$.

A semiring S is called an *antiring* if it is zerosumfree, i.e. if the condition $a + b = 0$ implies that $a = b = 0$ for all $a, b \in S$.

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An antiring is called *entire* if $ab = 0$ implies that either $a = 0$ or $b = 0$.

For example, the set of nonnegative integers with the usual operations of addition and multiplication is a commutative entire antiring. Boolean algebras and distributive lattices are commutative (but not entire) antirings.

A set $\{a_1, a_2, \ldots, a_r\} \subseteq S$ of nonzero elements is called an *orthogonal decomposition of* 1 *of length* r in S if $a_1 + a_2 + \cdots + a_r = 1$ and $a_i a_j = 0$ for all $i \neq j$.

A matrix $A \in M_n(S)$ is *an orthogonal combination* of matrices $A_1, A_2, \ldots, A_r$ if there exists an orthogonal combination $\{a_1, a_2, \ldots, a_r\}$ of 1, such that $A = \sum_{i=1}^{r} a_i A_i$.

Let us denote by $U(S)$ the group of all invertible elements in S, i.e. $U(S) = \{a \in S; ab = ba = 1 \text{ for some } b \in S\}$.

Tan [4] characterized the invertible matrices over a commutative antiring. He proved that for a commutative antiring S, where $U(S) = \{1\}$, a matrix $A \in M_n(S)$ is invertible if and only if A is an orthogonal combination of some $n \times n$ permutation matrices [4, Prop. 3.7]. Here, we generalize this result to an arbitrary commutative antiring (see Theorem 1) and we prove that $GL_n(S) \simeq U(S)^n \rtimes (S_n)^k$, where k is the maximal length of an orthogonal decomposition of 1.

Tan [5] characterized the nilpotent matrices in terms of principal permanental minors and permanental adjoint matrices. In Section 3, we give two results on nilpotent matrices. In Theorem 8, we find the number of all nilpotent matrices over an entire commutative finite antiring. Next, we develop a result similar to [6] and prove that every nilpotent $n \times n$ matrix can be written as a sum of $\lceil \log_2 n \rceil$ square-zero matrices. We also find the number of square-zero matrices needed for an arbitrary trace-zero matrix to be expressible as their sum.

2. Invertible matrices over S

In this section, we give the characterization of invertible matrices over a commutative antiring and thus generalize [4, Prop. 3.7].

Theorem 1. *If S is a commutative antiring, then $A \in M_n(S)$ is invertible if and only if*

$$A = D \sum_{\sigma \in S_n} a_\sigma P_\sigma,$$

where D is an invertible diagonal matrix, P_σ is a permutation matrix and $\sum_{\sigma \in S_n} a_\sigma = 1$ is an orthogonal decomposition of 1.

Proof. Let $A = [a_{ij}]$ be an arbitrary invertible matrix. From [4, Theorem 3.1], we know that the matrices AA^{T} and $A^{\mathrm{T}}A$ are (not necessarily equal) invertible diagonal matrices. Then $a_{ij}a_{ik} = a_{ji}a_{ki} = 0$ for all $j \neq k$. Denote by L_i the entry $(AA^{\mathrm{T}})_{ii} = \sum_{k=1}^{n} a_{ik}^2 \in U(S)$.

It can be easily seen that $(\sum_{k=1}^{n} a_{ik})^2 = L_i$, so we know that $l_i = \sum_{k=1}^{n} a_{ik}$ is invertible for each i.

We write $L = \prod_{i=1}^{n} \sum_{k=1}^{n} a_{ik} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \in U(S)$. Let $a_\sigma = L^{-1} \prod_{i=1}^{n} a_{i\sigma(i)}$ and verify that $\sum_{\sigma \in S_n} a_\sigma = 1$. Since $a_{ik}a_{jk} = 0$ for $j \neq i$, we have $a_\sigma a_\tau = 0$ for $\sigma \neq \tau$ and $a_\sigma^2 = a_\sigma$. This gives us an orthogonal decomposition of 1 to a sum of idempotents. (Note that a_σ may be 0 for some $\sigma \in S_n$.) For $a_\sigma \neq 0$, the matrix $a_\sigma A$ has exactly one nonzero element in each row (and column). Since $a_{ij} l_i = a_{ij}^2$ for every i, there exists a permutation matrix P_σ such that $a_\sigma A = a_\sigma D P_\sigma$, where $D = \mathrm{Diag}(l_1, l_2, \ldots, l_n)$. Thus, $A = (\sum_{\sigma \in S_n} a_\sigma) A = D \sum_{\sigma \in S_n} a_\sigma P_\sigma$.

Now, let $A = D\sum_{\sigma\in S_n} a_\sigma P_\sigma$, where $D = \mathrm{Diag}(d_1, d_2, \ldots, d_n)$ is an invertible diagonal matrix, P_σ a permutation matrix and $\sum_{\sigma\in S_n} a_\sigma = 1$ an orthogonal decomposition of 1. Let us write $B = \sum_{\sigma\in S_n} a_\sigma \mathrm{Diag}(d^{-1}_{\sigma^{-1}(1)}, d^{-1}_{\sigma^{-1}(2)}, \ldots, d^{-1}_{\sigma^{-1}(n)})P_\sigma^{\mathrm{T}}$. Since a_σ are orthogonal idempotents, one can easily verify that $AB = I$. By [4, Lemma 2.1], it follows that $BA = I$ and thus A is invertible. □

Corollary 2. *If S is a commutative antiring, then the group $GL_n(S)$ of invertible $n \times n$ matrices over S is isomorphic to the group $U(S)^n \rtimes (S_n)^k$, where k is the maximal length of an orthogonal decomposition of* 1.

Proof. Let $1 = a_1 + a_2 + \cdots + a_k$ be an orthogonal decomposition of 1 of the maximal length. If $1 = b_1 + b_2 + \cdots + b_r$ is another orthogonal decomposition of 1, then by multiplying these two equations, for each i we get $\sum_{i=1}^{k}\sum_{j=1}^{r} a_i b_j = 0$. Suppose that $a_{i_1} b_j \neq 0$ for at least two j. Since the longest orthogonal decomposition of 1 is of length k, it follows that for some i_2 all products $a_{i_2} b_j$ are equal to 0. Thus $a_{i_2} = a_{i_2}\sum_{j=1}^{r} b_j = 0$, which contradicts the definition of an orthogonal decomposition. So, denote by $\sigma(i)$ the only index such that $a_i b_{\sigma(i)} \neq 0$ and notice that $a_i b_{\sigma(i)} = a_i$.

Now, for any l, we have $b_l = b_l(\sum_{i\in\sigma^{-1}(l)} a_i) = \sum_{i\in\sigma^{-1}(l)} a_i$, so all the summands of the second orthogonal sum are actually sums of some of the summands of the first sum.

Since an invertible diagonal matrix is exactly a matrix with invertible diagonal elements, we can use Theorem 1 to prove the corollary. Conjugation with a permutation matrix preserves diagonal matrices, therefore, the group of invertible diagonal matrices is a normal subgroup of the group of all invertible matrices. So, the group of invertible matrices is indeed isomorphic to a semidirect product of diagonal matrices and sums of permutation matrices. (This is, of course, not a direct product unless $n = 1$.) □

Corollary 3. *If S is an entire commutative antiring, then $GL_n(S) \simeq U(S)^n \rtimes S_n$.*

Proof. Let S be an entire antiring and $1 = a_1 + a_2 + \cdots + a_r$ an orthogonal decomposition of 1 of length r. By definition, $a_i a_j = 0$ for $i \neq j$ and since S is entire, it follows that either $a_i = 0$ or $a_j = 0$, which implies that $r = 1$. □

3. Nilpotent matrices over S

In [5], Tan characterized the nilpotent matrices over a commutative antiring by studying acyclic directed graphs. Here, we use his notation and develop some further results on nilpotent matrices.

For a matrix $A \in M_n(S)$ we denote by $A(i, j)$ the entry in the ith row and the jth column of the matrix A.

Definition. For a matrix $A \in M_n(S)$ we define a directed graph (or simply a *digraph*) $D(A)$ with vertices $\{1, 2, \ldots, n\}$. A pair (i, j) is an edge of $D(A)$ if and only if $A(i, j) \neq 0$. A *path* in the digraph $D(A)$ (of length k) is a sequence of edges $(i_0, i_1), (i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ such that $A(i_0, i_1)A(i_1, i_2)A(i_2, i_3)\cdots A(i_{k-1}, i_k) \neq 0$. If $i_0 = i_k$, then the path is called a *cycle*. An edge (i, i) is called a *loop*. A digraph is called *acyclic* if it does not contain cycles of any length.

We will assume that S has no nonzero nilpotent elements.

By Tan [5, Prop. 3.4], we know that A is nilpotent if and only if the digraph $D(A)$ is acyclic. As is well known (not only in the theory of antirings), digraphs are a useful alternative way of considering nilpotent matrices. For example, we have the following:

Lemma 4. *Suppose that S is an entire commutative antiring and let $\iota(A)$ denote the index of nilpotency of a nilpotent matrix $A \in M_n(S)$. Then the longest path in the digraph $D(A)$ is equal to $\iota(A) - 1$.*

Proof. Since $A^{\iota(A)} = 0$ and $A^{\iota(A)-1} \neq 0$, there exists a sequence of integers $i_1, i_2 \ldots, i_{\iota(A)-2}$ such that $A^{\iota(A)-1}(i, j) = A(i, i_1)A(i_1, i_2) \cdots A(i_{\iota(A)-3}, i_{\iota(A)-2})A(i_{\iota(A)-2}, j) \neq 0$. Therefore, the length of the longest path in $D(A)$ is greater than or equal to $\iota(A) - 1$.

Suppose that there exists a path in $D(A)$ of length $\iota(A)$ that contains the edges (j_0, j_1), $(j_1, j_2), (j_2, j_3), \ldots, (j_{\iota(A)-1}, j_{\iota(A)})$. Since S is entire, it follows that $A^{\iota(A)}(j_0, j_{\iota(A)}) \neq 0$, which contradicts the definition of $\iota(A)$. $\square$

Similarly as in Corollary 2, we would like to describe the set of nilpotent matrices over a commutative antiring. Unfortunately, the set of nilpotent matrices is not closed under addition and under multiplication. Let us start by giving some examples of nilpotent matrices over finite antirings.

Example 5. Let R be the lattice of all idempotents of a commutative artinian ring and let $e_1, e_2, \ldots, e_m$ be the minimal idempotents in R. Thus, R consists of 2^m elements.

Since R has no nonzero nilpotent elements, the nilpotent matrices over R must have all the diagonal entries equal to 0. Thus, all the 2×2 nilpotent matrices over R are of the form $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$, where $ab = 0$.

Since all the elements of R are the sums of minimal idempotents, it follows that $a = e_{\pi(1)} + e_{\pi(2)} + \cdots + e_{\pi(k)}$ and $b = e_{\sigma(1)} + e_{\sigma(2)} + \cdots + e_{\sigma(l)}$, for some $0 \leqslant k, l \leqslant m$ and permutations π and σ of the set $\{1, 2, \ldots, m\}$. Since $ab = 0$, it follows that $\sigma(i) \notin \{\pi(1), \pi(2), \ldots, \pi(k)\}$ for $i = 1, 2, \ldots, l$. Thus, we can set b to be equal to any sum of minimal idempotents that is not represented in a. Since there are 2^{m-k} such sums, the number of nilpotent 2×2 matrices over R is equal to $\sum_{k=0}^{m} \binom{m}{k} 2^{m-k} = 3^m$.

Example 6. It is easy to find the number of nilpotent 2×2 matrices over a finite entire commutative antiring with q elements.

Since the diagonal entry of a nilpotent matrix must be equal to 0, and the digraph of the nilpotent matrix is acyclic, a nonzero nilpotent 2×2 matrix is either of the form $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$. Since $a, b \in S - \{0\}$ are arbitrary, it follows that there are $2q - 1$ nilpotent 2×2 matrices.

Example 7. There are exactly $6q^3 - 6q^2 + 1$ nilpotent 3×3 matrices over a finite entire commutative antiring S with q elements.

Namely, the nonzero nilpotent 3×3 matrices over S have either 1, 2, or 3 nonzero entries. And thus, their digraphs are isomorphic to one of the digraphs D_1, D_2, D_3, D_4 and D_5 in Fig. 1.

Note that there are $6(q - 1)$ nilpotent matrices over S with the digraph isomorphic to D_1, $6(q - 1)^2$ nilpotent matrices over S with the digraph isomorphic to D_2, $3(q - 1)^2$ nilpotent

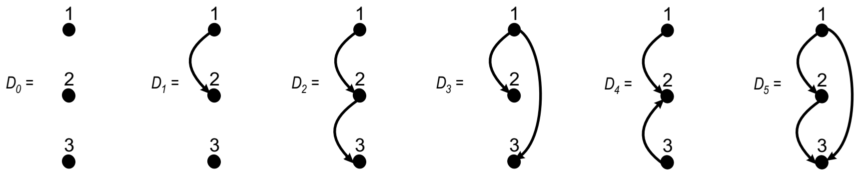

Fig. 1. Nonisomorphic labeled acyclic digraphs with three vertices.

matrices with the digraph isomorphic to D_3, $3(q-1)^2$ nilpotent matrices with the digraph isomorphic to D_4 and $6(q-1)^3$ nilpotent matrices with the digraph isomorphic to D_5. Thus, there are $6q^3 - 6q^2$ nonzero nilpotent 3×3 matrices over S.

These examples give rise to a problem (and give an idea) on how to find all nilpotent matrices over a finite antiring. The following theorem gives us the answer for a finite entire antiring.

Theorem 8. *Let S be a finite entire commutative antiring with $|S| = q$. Then, the number of nilpotent $n \times n$ matrices over S is equal to*

$$\sum_{\substack{\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \\ \mu_1+\mu_2+\cdots+\mu_k=n}} (-1)^{n-k} \frac{n!}{\mu_1!\mu_2!\cdots\mu_k!} q^{\frac{1}{2}\left(n^2-\sum_{i=1}^{k}\mu_i^2\right)}.$$

Proof. By the definition of the digraph $D(A)$, we know that every acyclic directed graph D corresponds to a set of nilpotent matrices A (with the same pattern), such that $D = D(A)$. Namely, if D has r edges, then nilpotent matrices A, such that $D(A) = D$, have r nonzero entries.

Since S is entire, it follows that the number $A_{n,r}$ of all acyclic digraphs on n vertices with r edges is equal to the number of nilpotent $n \times n$ matrices over S with exactly r nonzero entries. By the main theorem of combinatorics, there are exactly $(q-1)^r$ such matrices.

Thus, there are exactly $A_n(q-1) = \sum_{r=0}^{\infty} A_{n,r}(q-1)^r$ nilpotent $n \times n$ matrices, where $A_n(x)$ denotes the generating function for all labeled acyclic digraphs of order n, i.e. $A_n(x) = \sum_{r=0}^{\infty} A_{n,r}x^r$. Rodionov [3] proved that

$$A_n(x) = \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} (1+x)^{m(n-m)} A_{n-m}(x)$$

or explicitly [3, Corollary]

$$A_n(x) = \sum_{\substack{\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \\ \mu_1+\mu_2+\cdots+\mu_k=n}} (-1)^{n-k} \frac{n!}{\mu_1!\mu_2!\ldots\mu_k!} (1+x)^{\frac{1}{2}\left(n^2-\sum_{i=1}^{k}\mu_i^2\right)}.$$

Thus, the theorem holds. $\square$

Note that we cannot omit the condition that the antiring S is entire. (See Example 5.)

Example 9. Note that the leading term of the polynomial $A_n(q-1)$ is equal to $n!q^{\binom{n}{2}}$. Moreover, we have

n	Number of $n \times n$ nilpotent matrices over an entire antiring with q elements
1	1
2	$2q - 1$
3	$6q^3 - 6q^2 + 1$
4	$24q^6 - 36q^5 + 6q^4 + 8q^3 + 1$
5	$120q^{10} - 240q^9 + 90q^8 + 60q^7 - 20q^6 - 10q^4 + 1$
6	$720q^{15} + 1800q^{14} + 390q^{12} - 360q^{11} - 79q^9 + 30q^8 + 12q^5 - 1$

In [6], Wang and Wu characterized all matrices over a field, that can be written as a sum of two square-zero matrices. They showed that a matrix T is a sum of two square-zero matrices if and only if it is similar to $-T$. Fong and Sourour [1] showed that a matrix is a sum of two nilpotent matrices if and only if its trace is equal to zero. Using this result, Wang and Wu showed that any trace-zero matrix is a sum of four square-zero matrices.

Note that over an antiring, an arbitrary matrix with its trace equal to 0 (equivalently, all the diagonal entries of the matrix are equal to 0) can be written as a sum of two nilpotent matrices (one being its strictly upper triangular part and the other being its strictly lower triangular part). However, if a matrix over an antiring with no nonzero nilpotent elements has a nonzero diagonal entry, then it cannot be written as a sum of nilpotent matrices (since its corresponding digraph contains a loop).

Here we prove that every nilpotent $n \times n$ matrix can be written as a sum of $\lceil \log_2 n \rceil$ square-zero matrices (and that this bound is sharp). This implies that every trace-zero matrix over an antiring can be written as a sum of at most $2\lceil \log_2 n \rceil$ square-zero matrices. However, this bound is not sharp (see Example 12) and we find the exact upper bound in Theorem 13.

Definition. If Γ_1 and Γ_2 are digraphs with vertices $\{1, 2, \ldots, n\}$, we denote by $\Gamma_1 \uplus \Gamma_2$ the digraph on vertices $\{1, 2, \ldots, n\}$, where (i, j) is an edge of $\Gamma_1 \uplus \Gamma_2$ if (i, j) is an edge of Γ_1 or Γ_2.

We can easily see that the following lemma holds.

Lemma 10. *Let $A = A_1 + A_2 + \cdots + A_k$ be an $n \times n$ matrix over an antiring S. Then $D(A) = D(A_1) \uplus D(A_2) \uplus \cdots \uplus D(A_k)$.*

This enables us to prove the following theorems.

Theorem 11. *Let A be an $n \times n$ nilpotent matrix over an entire antiring S. Then A can be written as*

$$A = \sum_{i=1}^{\lceil \log_2 n \rceil} B_i,$$

where $B_i \in M_n(S)$ is a square-zero matrix (i.e. $B_i^2 = 0$).

Moreover, for every n there exists a nilpotent matrix $A \in M_n(S)$ such that it cannot be written as a sum of k square-zero matrices, where $k < \log_2 n$.

Proof. Without any loss of generality, we can assume that a nilpotent $A \in M_n(S)$ is a strictly uppertriangular matrix (see [5, Lemma 4.1]). By Lemma 4, all paths in the digraph corresponding to a square-zero matrix are of length at most 1.

Let $\chi(\Gamma)$ be the least number of colors needed to color the edges of a graph Γ such that no vertex is a source and a sink of two edges of the same color. Equivalently, every path in Γ has no two incident edges of the same color.

Let us denote by $\widetilde{A}_n$ an arbitrary strictly uppertriangular $n \times n$ nilpotent matrix with $\widetilde{A}_n(i, j) \neq 0$ for $1 \leqslant i < j \leqslant n$, and let $\widetilde{\Gamma}_n$ be its digraph. Such a digraph $\widetilde{\Gamma}_n$ is called a *transitive tournament*. Note that $\chi(D(A)) \leqslant \chi(\widetilde{\Gamma}_n)$.

Arc colorings of some special digraphs (including transitive tournaments) were studied by Harner and Entringer [2] (and recently also by Zwonek in [7]). By [2, Theorem 4], it follows that $\chi(\widetilde{\Gamma}_n) = \lceil \log_2 n \rceil$ and thus $\chi(D(A)) \leqslant \lceil \log_2 n \rceil$. By Lemma 10, it follows that every nilpotent $n \times n$ matrix can be written as a sum of at most $\lceil \log_2 n \rceil$ square-zero matrices and $\widetilde{A}_n$ cannot be written as a sum of less than $\lceil \log_2 n \rceil$ square-zero matrices. □

Example 12. The theorem immediately implies that every trace-zero matrix over an antiring can be written as a sum of at most $2\lceil \log_2 n \rceil$ square-zero matrices.

However, consider an arbitrary 3×3 trace-zero matrix A over an antiring S. Clearly,

$$A = \begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & f & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ e & 0 & 0 \end{bmatrix},$$

so A can be written as a sum of $3 < 4 = 2\lceil \log_2 3 \rceil$ square-zero matrices.

Theorem 13. *Let A be an $n \times n$ trace-zero matrix over an antiring S without nonzero nilpotent elements. Then A can be written as*

$$A = \sum_{i=1}^{N(n)} B_i,$$

where $B_i \in M_n(S)$ is a square-zero matrix and $N(n)$ is the smallest integer, such that $n \leqslant \binom{N(n)}{\lceil \frac{N(n)}{2} \rceil}$.

Moreover, for every n there exists a trace-zero matrix $A \in M_n(S)$ such that it cannot be written as a sum of k square-zero matrices, where $k < N(n)$.

Proof. Again, every path in the digraph corresponding to a square-zero matrix is of length at most 1.

Recall the definition of $\chi(\Gamma)$ from the proof of the Theorem 11. Let us denote by $\widetilde{C}_n$ an arbitrary $n \times n$ trace-zero matrix with $\widetilde{C}_n(i, j) \neq 0$ for all $i \neq j$ and let $\widetilde{\Delta}_n$ be its digraph. Such a digraph $\widetilde{\Delta}_n$ is called a *complete digraph* on n vertices and it was proved in [7, Corollary 4] that $\chi(\widetilde{\Delta}_n) = N(n)$. Thus, it follows that $\chi(D(A)) \leqslant \chi(\widetilde{\Delta}_n) = N(n)$.

By Lemma 10, it follows that every trace-zero $n \times n$ matrix can be written as a sum of at most $N(n)$ square-zero matrices and $\widetilde{C}_n$ cannot be written as a sum of less than $N(n)$ square-zero matrices. □

Remark 14. Observe that by Stirling's formula, $\binom{m}{\lceil \frac{m}{2} \rceil} \sim \frac{2^m}{\sqrt{m}}$. If we denote $\frac{2^m}{\sqrt{m}}$ by $f(m)$, then every trace-zero $n \times n$ matrix can be written as a sum of approximately $f^{-1}(n)$ square-zero matrices. Or equivalently, the largest dimension of matrices, such that every trace-zero matrix can be written as a sum of at most n square-zero matrices is approximately $\frac{2^n}{\sqrt{n}}$. Compare this result with the result that the largest dimension of matrices, such that every nilpotent matrix can be written as a sum of at most n square-zero matrices is 2^n.

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