The compact-$G_δ$-open topology on $C(X)$

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1. Introduction

The set $C(X)$ of all continuous real-valued functions on a Tychonoff space $X$ has a number of natural topologies. The idea of topologizing $C(X)$ arose from the notion of convergence of sequences of functions. Also continuous functions and Baire measures on Tychonoff spaces are linked by the process of integration. A number of locally convex topologies on spaces of continuous functions have been studied in order to clarify this relationship. They enable the powerful duality theory of locally convex spaces to be profitably applied to topological measure theory.

Two commonly used topologies on $C(X)$ are the compact-open topology $k$ and point-open topology $p$. The point-open topology is also known as the topology of pointwise convergence. The study of pointwise convergence of sequences of functions is as old as the calculus. The compact-open topology made its appearance in 1945 in a paper by Ralph H. Fox (see [1]), and soon after was developed by Richard F. Arens in [2] and Arens and Dugundji in [3]. This topology was shown in [4] to be the proper setting to study sequences of functions which converge uniformly on compact subsets.

It is easily seen that $k = p$ if and only if the compact subsets of $X$ are finite. This condition is quite extreme in nature. So there is a considerable gap between these two topologies. So it is quite natural to fill up this gap by considering some...
new topologies on \( C(X) \) lying between \( p \) and \( k \) from the viewpoint of topology as well as of topological measure theory. Two such topologies, weak and support-open topologies on \( C(X) \) have been studied in [5] and [6]. But it is also interesting to study topologies on \( C(X) \) which are weaker than \( k \), but not necessarily stronger than \( p \) from the viewpoint of topology as well as topological measure theory.

The primary concern of this work is to study another natural topology, the compact-\( G_δ \)-open topology on \( C(X) \), in detail from the topological point of view. While studying the set-open topologies on \( C(X) \), we only consider certain family of subsets of \( X \). But we do not bring the elements of \( C(X) \), that is, the continuous functions themselves directly into the picture in order to define more meaningful and natural topologies on \( C(X) \).

Obviously there is no better way than considering the zero sets in \( X \) in order to bring the elements of \( C(X) \) while defining meaningful topologies on \( C(X) \). So the family of all compact zero sets on \( X \) is naturally a prime candidate for defining a topology on \( C(X) \) from this viewpoint. Here one should note that a compact subset of \( X \) is a zero set if and only if it is a \( G_δ \)-set in \( X \). So by considering the family of all compact \( G_δ \)-sets, we are going to define a close relative of the compact-open topology \( k \) on \( C(X) \), which is, at the same time, substantially different from \( k \).

There is also a measure-theoretic aspect for the need of considering the compact-\( G_δ \)-open topology \( kz \) on \( C(X) \). In measure theory occasionally the \( σ \)-algebra or \( σ \)-ring generated by the compact \( G_δ \)-subsets of a space \( X \) has been considered; see Chapter X of [7] and Chapter 13 of [8]. If \( M_{kz}(X) \) is the space of all closed regular Borel measures on \( X \) with the compact-\( G_δ \)-support, then what will be a locally convex topology \( τ \) on \( C(X) \) so that the dual space of \( (C(X), τ) \) will be precisely \( M_{kz}(X) \)? The compact-\( G_δ \)-open topology on \( C(X) \) is likely to be one such \( τ \). But we would like to study this aspect in the future in another paper.

In Section 2, we define formally the compact-\( G_δ \)-open topology on \( C(X) \) and show that this topology can be viewed in three different ways. In Section 3, we compare this topology with two well-known topologies \( p \) and \( k \) on \( C(X) \) in order to have a better understanding of the compact-\( G_δ \)-open topology. In the last section, we study the metrizability, separability and uniform completeness of the compact-\( G_δ \)-open topology on \( C(X) \).

Throughout the rest of the paper, we use the following conventions. All spaces are completely regular Hausdorff, that is, Tychonoff (though we may specify that these spaces have additional properties). Unless mentioned otherwise, we assume that each of these spaces has at least one nonempty compact zero subset. If \( X \) and \( Y \) are two spaces with the same underlying set, then we use \( X = Y \), \( X \leq Y \) and \( X < Y \) to indicate, respectively, that \( X \) and \( Y \) have the same topology, that the topology on \( Y \) is finer than or equal to the topology on \( X \) and that the topology on \( Y \) is strictly finer than the topology on \( X \). The symbols \( \mathbb{R} \) and \( \mathbb{N} \) denote the space of real numbers and natural numbers, respectively. Finally the constant zero-function in \( C(X) \) is denoted by \( 0 \).

2. The compact-\( G_δ \)-open topology on \( C(X) \): Different views

A subset \( A \) of a space \( X \) is said to be a zero set if there exists a real-valued continuous function \( f \) on \( X \) such that \( A = \{ x : f(x) = 0 \} \). In addition if \( A \) is compact, we say that \( A \) is a compact zero set. Note that a compact set is a zero set if and only if it is a \( G_δ \)-set. See [9] for a reference involving compact \( G_δ \)-sets. Every zero set is a \( G_δ \)-set. In a normal space every closed \( G_δ \)-set is a zero set. The zero sets in \( X \) are closed under the formation of finite unions and countable intersections. Consequently, a countable intersection of compact zero sets is a compact zero set. Also it can be easily seen that the inverse image of a zero set under a continuous function is a zero set. Note that a closed subset of a zero set need not be a zero set. If \( X \) is a noncompact Tychonoff space, then no point of \( βX \setminus X \) is a \( G_δ \)-point (see 3.6.G(a), p. 181 in [10]). Consequently no point of \( βX \setminus X \) is a zero set, though \( βX \) is compact.

For any compact zero set \( A \) in \( X \) and any open subset \( V \) of \( \mathbb{R} \), we define

\[
[A, V] = \{ f \in C(X) : f(A) \subseteq V \}.
\]

Now let \( KZ(X) \) be the set of all compact zero sets in \( X \). For the compact-\( G_δ \)-open topology on \( C(X) \), we take as subbase, the family

\[
\{ [A, V] : A \in KZ(X), V \text{ is open in } \mathbb{R} \};
\]

and we denote the corresponding space by \( C_{kz}(X) \).

Now we define the topology of uniform convergence on compact zero sets. For each \( A \in KZ(X) \) and \( ε > 0 \), let

\[
A_ε = \{ (f, g) \in C(X) \times C(X) : |f(x) - g(x)| < ε \text{ for all } x \in A \}.
\]

Then it can be easily verified that the collection \( \{ A_ε : A \in KZ(X), ε > 0 \} \) is a base for some uniformity on \( C(X) \). We denote the space \( C(X) \) with the topology induced by this uniformity by \( C_{kz,u}(X) \). This topology is called the topology of uniform convergence on \( KZ(X) \). For each \( f \in C(X) \), \( A \in KZ(X) \) and \( ε > 0 \), let

\[
\langle f, A, ε \rangle = \{ g \in C(X) : |f(x) - g(x)| < ε \text{ for all } x \in A \}.
\]

Then for each \( f \in C(X) \), the collection \( \{ \langle f, A, ε \rangle : A \in KZ(X), ε > 0 \} \) forms a neighborhood base at \( f \) in \( C_{kz,u}(X) \). Actually the collection \( \{ \langle f, A, ε \rangle : f \in C(X), A \in KZ(X), ε > 0 \} \) forms a base for the topology of uniform convergence on \( KZ(X) \). In particular, each such set \( \langle f, A, ε \rangle \) is open in \( C_{kz,u}(X) \).
Now for each $A \in \mathcal{K}(X)$, we define the seminorm $p_A$ on $C(X)$ by

$$p_A(f) = \sup_{x \in A} |f(x)|.$$ 

Also for each $A \in \mathcal{K}(X)$ and $\epsilon > 0$, let

$$V_{A,\epsilon} = \{ f \in C(X) : p_A(f) < \epsilon \}.$$

Let $V = \{ V_{A,\epsilon} : A \in \mathcal{K}(X), \epsilon > 0 \}$. It can be easily shown that for each $f \in C(X)$, $f + V = \{ f + V : V \in V \}$ forms a neighborhood base at $f$. The space $C(X)$ with this topology generated by a collection of seminorms is a locally convex space. But it need not be Hausdorff.

In the first result, we show that the three topologies on $C(X)$, defined above, are the same.

**Theorem 2.1.** For any space $X$, the compact-$G_δ$-open topology on $C(X)$ is same as the topology of uniform convergence on the compact $G_δ$-subsets of $X$, that is, $C_{kz}(X) = C_{kz,u}(X)$. Moreover $C_{kz}(X)$ is a locally convex space.

**Proof.** Let $[A, V]$ be a subbasic open set in $C_{kz}(X)$ and let $f \in [A, V]$. Since $f(A)$ is compact, there exist $z_1, z_2, \ldots, z_n \in f(A)$ such that $f(A) \subseteq \bigcup_{i=1}^n (z_i - \epsilon_i, z_i + \epsilon_i) \subseteq \bigcup_{i=1}^n (z_i - 2\epsilon_i, z_i + 2\epsilon_i) \subseteq V$. Choose $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$. Now if $g \in (f, A, \epsilon)$ and $x \in A$, then $|g(x) - f(x)| < \epsilon$ and there exists an $i$ such that $|f(x) - z_i| < \epsilon_i$. Hence $|g(x) - z_i| < 2\epsilon_i$ and thus $g(x) \in V$. So $g(A) \subseteq V$, that is, $g \in [A, V]$. But this means $(f, A, \epsilon) \subseteq [A, V]$. Now let $W = \bigcap_{i=1}^k A_i$ be a basic neighborhood of $f$ in $C_{kz}(X)$. Then there exist positive real numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ such that $f \in (f, A_i, \epsilon_i) \subseteq [A_i, V_i]$ for each $i = 1, 2, \ldots, k$. If $A = \bigcup_{i=1}^k A_i$ and $\epsilon = \min_{1 \leq i \leq k} \epsilon_i$, then $f \in (f, A, \epsilon) \subseteq W$. This shows $C_{kz}(X) \subseteq C_{kz,u}(X)$.

Now let $(f, A, \epsilon)$ be a basic neighborhood of $f$ in $C_{kz,u}(X)$. Since $f(A)$ is compact, there exist $z_1, z_2, \ldots, z_n \in f(A)$ such that $f(A) \subseteq \bigcup_{i=1}^n (z_i - \epsilon_i, z_i + \epsilon_i)$ and $A_1 = A \cap f^{-1}(\{z_i - \epsilon_i, z_i + \epsilon_i\})$. Since every closed set in the real line is a zero set, $f^{-1}(\{z_i - \epsilon_i, z_i + \epsilon_i\})$ is a zero set. Since intersection of two zero sets is again a zero set, $A_1$ is a zero set in $X$. Being a closed subset of a compact set $A$, $A_1$ is also compact. Consequently $A_1$ is a compact zero set, that is, $A_1 \subseteq \mathcal{K}(X)$. Note that $A = \bigcup_{i=1}^k A_i$ and $W = \bigcap_{i=1}^k A_i$. Now we show that $f \in \bigcap_{i=1}^k [A_i, W_i] \subseteq (f, A, \epsilon)$. It is clear that $f \in \bigcap_{i=1}^k [A_i, W_i]$. Let $g \in \bigcap_{i=1}^k [A_i, W_i]$ and $x \in A$. Then there exists an $i$ such that $x \in A_i$ and consequently $f(x) \in [z_i - \epsilon_i, z_i + \epsilon_i]$. Since $g(x) \in (z_i - \epsilon_i, z_i + \epsilon_i)$, then $f(x) - g(x) < \epsilon$. So $g \in (f, A, \epsilon)$ and consequently $C_{kz,u}(X) \subseteq C_{kz}(X)$.

Note that for each $f \in C(X)$, we have $f + V_{A,\epsilon} \subseteq (f, A, \epsilon)$ and $(f, A, \epsilon) \subseteq f + V_{A,\epsilon}$ for all $A \in \mathcal{K}(X)$. This shows that the topology of uniform convergence on the compact zero sets is same as the topology generated by the collection of seminorms $(p_A : A \in \mathcal{K}(X))$. Hence $C_{kz}(X) = C_{kz,u}(X)$ is a locally convex space. □

**Theorem 2.2.** For a space $X$, $C_{kz}(X)$ is Hausdorff if and only if $\bigcup\{A : A \subseteq \mathcal{K}(X)\}$ is dense in $X$. In this case, $C_{kz}(X)$ is a locally convex Hausdorff space.

**Proof.** By Proposition 2.3 of [11], $C_{kz,u}(X)$ is Hausdorff if and only if $\bigcup\{A : A \subseteq \mathcal{K}(X)\}$ is dense in $X$. But by Theorem 2.1, $C_{kz,u}(X) = C_{kz}(X)$. □

In the collection of subbasic open sets $\{[A, V] : A \subseteq \mathcal{K}(X), V \subseteq \mathbb{R}\}$ in $C_{kz}(X)$, the open set $V$ can always be taken as a bounded open interval. The precise statement follows.

**Theorem 2.3.** For any space $X$, the collection $\{[A, V] : A \subseteq \mathcal{K}(X), V \subseteq \mathbb{R}\}$ forms a subbase for $C_{kz}(X)$.

**Proof.** The proof is quite similar to that of $C_{kz,u}(X) \subseteq C_{kz}(X)$. Let $[A, V]$ be a subbasic open set in $C_{kz}(X)$. Here $A$ is a compact zero subset of $X$ and $V$ is open in $\mathbb{R}$. Let $f \in [A, V]$. Since $f(A)$ is compact, there exist $z_1, z_2, \ldots, z_n$ in $f(A)$ and positive real numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that $f(A) \subseteq \bigcup_{i=1}^n (z_i - \epsilon_i, z_i + \epsilon_i) \subseteq \bigcup_{i=1}^n (z_i - 2\epsilon_i, z_i + 2\epsilon_i) \subseteq V$. Let $W_i = (z_i - 2\epsilon_i, z_i + 2\epsilon_i)$ and $A_i = A \cap f^{-1}(\{z_i - \epsilon_i, z_i + \epsilon_i\})$ for each $i = 1, 2, \ldots, n$. Note that each $A_i \subseteq \mathcal{K}(X)$, $A = \bigcup_{i=1}^n A_i$ and $f \in \bigcap_{i=1}^n [A_i, W_i]$. It can be easily verified that $\bigcap_{i=1}^n [A_i, W_i] \subseteq [A, V]$. □

We end this section with the result that the subset $C^*(X)$ of bounded members of $C(X)$ is dense in $C_{kz}(X)$.

**Theorem 2.4.** For any space $X$, $C^*(X)$ is dense in $C_{kz}(X)$.

**Proof.** Let $(f, A, \epsilon)$ be a basic neighborhood of $f$ in $C_{kz}(X)$. Since $A$ is compact, $f(A) \subseteq [a, b]$ for some $a, b \in \mathbb{R}$. Now take a continuous extension $f_1$ of $f|_A$ from $X$ to $[a, b]$. Now note that, $f_1 \in (f, A, \epsilon) \cap C^*(X)$. Thus $C^*(X)$ is dense in $C_{kz}(X)$. □

### 3. Comparison of topologies

In this section, we compare the compact-$G_δ$-open topology with the compact-open topology and the point-open topology. Let us recall the definitions of these latter two topologies.
Given a subset $A$ of a space $X$, $f \in C(X)$ and $\epsilon > 0$, define, as before, $(f, A, \epsilon) = \{g \in C(X): |f(x) - g(x)| < \epsilon, \forall x \in A\}$. If $K(X)$ is the collection of all compact subsets of $X$, then for each $f \in C(X)$, the collection $\{(f, K, \epsilon): K \in K(X), \epsilon > 0\}$ forms a neighborhood base at $f$ in the compact-open topology $k$ on $C(X)$. When $C(X)$ is equipped with the compact-open topology $k$, we denote the corresponding space by $C_k(X)$. Also it can be shown that each $(f, K, \epsilon)$, where $K$ is compact in $X$, is actually open in $C_k(X)$. If $F(X)$ is the collection of all finite subsets of $X$, then for each $f \in C(X)$, the collection $\{(f, F, \epsilon): F \in F(X), \epsilon > 0\}$ forms a neighborhood base at $f$ in the point-open topology $p$ on $C(X)$. When $C(X)$ is equipped with the point-open topology $p$, we denote the corresponding space by $C_p(X)$. Also it can be shown that each $(f, F, \epsilon)$, where $F$ is a finite subset of $X$, is actually open in $C_p(X)$.

Since $KZ(X) \subseteq K(X)$, we have the following result.

**Theorem 3.1.** For any space $X$, $C_{kz}(X) \subseteq C_k(X)$.

A space $X$ is said to be of point pseudocountable type if each point in $X$ is contained in a compact $G_\delta$-subset of $X$.

**Theorem 3.2.** For any space $X$, $C_p(X) \subseteq C_{kz}(X)$ if and only if the set $X = \bigcup\{A: A \in KZ(X)\}$, that is, if and only if $X$ is of point pseudocountable type.

**Proof.** Let $C_p(X) \subseteq C_{kz}(X)$. If possible, suppose there exists $x \in X \setminus \bigcup\{A: A \in KZ(X)\}$. Since $\{x\}, \{x\} \subseteq \{-1, 1\}$ is a (subbasic) open set in $C_p(X)$, $\{x\}, \{-1, 1\}$ is open in $C_{kz}(X)$ also. Hence there exists $A \in KZ(X)$ and $\epsilon > 0$ such that $0 \in (0, A, \epsilon) \subseteq \{x\}, \{-1, 1\}$. Since $x \notin A$, there exists a continuous function $f: X \to \mathbb{R}$ such that $f(a) = 0, \forall a \in A$ and $f(x) = 1$. But then $f \in (0, A, \epsilon) \subseteq \{x\}, \{-1, 1\}$. We arrive at a contradiction. Hence $X = \bigcup\{A: A \in KZ(X)\}$.

Conversely, suppose that $X = \bigcup\{A: A \in KZ(X)\}$. We will show that every subbasic open set in $C_p(X)$ is open in $C_{kz}(X)$. So let $[(x), V]$ be a subbasic open set in $C_p(X)$, where $x \in X$ and $V$ is an open set in $\mathbb{R}$. Let $f \in [(x), V]$. So $(f(x)) \in V$. Since every point in $\mathbb{R}$ is a zero set and an inverse image of a zero set under a continuous map is a zero set, $(f(x))$ is a zero set in $X$. Since $X = \bigcup\{A: A \in KZ(X)\}$, there exists $A \in KZ(X)$ such that $x \in A$. Let $B = f^{-1}(f(x)) \cap A$. Note that $B$ is a compact zero set. It is easy to see that $f \in [B, V] \subseteq [(x), V]$.

**Corollary 3.3.** If every point of $X$ is a $G_\delta$-set, that is, if $X$ is of countable pseudocharacter, then $C_p(X) \subseteq C_{kz}(X)$.

**Theorem 3.4.** For any space $X$, $C_{kz}(X) \subseteq C_p(X)$ if and only if every compact zero set is finite.

**Proof.** If every compact zero set is finite, then it is easy to see that $C_{kz}(X) \subseteq C_p(X)$.

Conversely, let $C_{kz}(X) \subseteq C_p(X)$. Suppose $A$ is a compact zero set. So $(0, A, 1)$ is open in $C_{kz}(X)$ and consequently there exist a finite subset $F$ of $X$ and $\epsilon > 0$ such that $(0, F, \epsilon) \subseteq (0, A, 1)$. If possible, let $x \in A \setminus F$. Then there exists a continuous function $g: X \to [0, 1]$ such that $g(x) = 1$ and $g(y) = 0, \forall y \in F$. Note that $g \in (0, F, \epsilon) \setminus (0, A, 1)$ and we arrive at a contradiction. Hence $A \subseteq F$ and consequently $A$ is finite.

**Corollary 3.5.** For any space $X$, $C_{kz}(X) = C_p(X)$ if and only if every compact zero set is finite and $X = \bigcup\{A: A \in KZ(X)\}$.

A subset $S$ of a space $X$ is said to have countable character if there is a sequence $\{W_n: n \in \mathbb{N}\}$ of open subsets in $X$ such that $S \subseteq W_n$ for each $n$ and if $W$ is any open set containing $S$, then $W_n \subseteq W$ for some $n$.

A space $X$ is said to be of countable type if each compact set is contained in a compact set having countable character.

A space $X$ is said to be of pseudocountable type if each compact subset of $X$ is contained in a compact $G_\delta$-subset of $X$.

Clearly a space of countable type is of pseudocountable type. Čech-complete spaces are of countable type and hence so are locally compact spaces. Also metrizable spaces are of countable type. See [12] for details on spaces of pseudocountable type.

**Theorem 3.6.** For every space $X$, $C_k(X) = C_{kz}(X)$ if and only if $X$ is of pseudocountable type.

**Proof.** Suppose $X$ is of pseudocountable type. Let $[A, V]$ be a subbasic open set in $C_k(X)$. Then $f(A) \subseteq V$. Note that $A \subseteq f^{-1}(f(A))$ and $f^{-1}(f(A))$ is a zero set. Since $X$ is of pseudocountable type, there exists a compact zero set $B$ such that $A \subseteq B$. Let $C = B \setminus f^{-1}(f(A))$. Note that $C$ is a compact zero set and $A \subseteq C$. Also $f \in [C, V] \subseteq [A, V]$. Consequently, $C_k(X) \subseteq C_{kz}(X)$. Hence $C_k(X) = C_{kz}(X)$.

Suppose that $C_k(X) = C_{kz}(X)$ and let $A$ be any compact subset of $X$. So $(0, A, 1)$ is open in $C_k(X)$ and consequently there exist a compact zero subset $Z$ of $X$ and $\epsilon > 0$ such that $(0, Z, \epsilon) \subseteq (0, A, 1)$. If possible, let $x \in A \setminus Z$. Then there exists a continuous function $g: X \to [0, 1]$ such that $g(x) = 1$ and $g(y) = 0, \forall y \in Z$. Note that $g \in (0, Z, \epsilon) \setminus (0, A, 1)$ and we arrive at a contradiction. Consequently, $A \subseteq K$.

**Corollary 3.7.** For a compact Hausdorff space $X$, we have $C_k(X) = C_{kz}(X)$. 


We now give some examples to illustrate the relations between the above-mentioned topologies on $C(X)$.

**Example 3.8.** For a discrete space $X$, we have

$$C_p(X) = C_{kz}(X) = C_k(X).$$

A space $X$ is said to be perfectly normal if every closed set is a $G_δ$-set. Every metrizable space is perfectly normal. Obviously, every perfectly normal space is of pseudocountable type.

**Example 3.9.** For a perfectly normal space $X$ having an infinite compact subset, we have

$$C_p(X) < C_{kz}(X) = C_k(X).$$

In particular for $\mathbb{R}$, we have

$$C_p(\mathbb{R}) < C_{kz}(\mathbb{R}) = C_k(\mathbb{R}).$$

**Example 3.10.** Consider the Fortissimo space $F$ (Example 25, p. 53 in [13]), which is uncountable. Every compact subset of $F$ is finite. So we have $C_{p}(F) = C_k(F)$. There exists a point $x$ in $F$ which is not contained in any compact $G_δ$-subset of $F$. So $F$ is not of pseudocountable type. So for this space $F$, we have

$$C_{kz}(F) < C_k(F) = C_p(F).$$

Now we give an example where the $p$ and $kz$ are incomparable.

**Example 3.11.** Let $Y = F \oplus \mathbb{R}$ where $F$ is the space mentioned in Example 3.10. For this space $X$, we have neither $C_p(X) \leq C_{kz}(X)$ nor $C_{kz}(X) \leq C_p(X)$.

The following example is due to Prof. Ronnie Levy.

**Example 3.12.** Let $X$ be the remainder of the Dedekind completion, call it $C$, of an $\aleph_1$ set, call it $Q$. For $\aleph_1$ set, see [14]. Then

(a) $C$ is countably compact, and, in particular, every increasing sequence converges to a point of $C$.
(b) Every nonempty zero set of $X$ has nonempty interior.
(c) $Q$ is dense in $C$.
(d) Every point of $Q$ is a $P$-point of $C$, so no point of $Q$ is a limit of a strictly increasing sequence of $C$.

Now we show that $X$ has no nonempty compact zero sets. To see this, if $Z$ is a nonempty zero set, then by (b) $Z$ contains a set of the form $(a, b) \cap X$, where $(a, b)$ is an interval in $C$. But by (c), the interval $(a, b)$ contains an element $q$ of $Q$. Then the open cover $\{((-\infty, r) \cup (s, \infty)) \cap Z : r < q < s, \ r, s \in C\}$ of $Z$ has no finite subcover.

Let $(x_n)$ be a strictly increasing sequence in $X$. By (a), this sequence converges to a point of $C$, but by (d), $x$ is not in $Q$, so $x$ is in $X$. Therefore, $\{x_n : n = 1, 2, \ldots\} \cup \{x\}$ is an infinite compact subset of $X$. Let $D$ be a discrete topological space. Let $E = X \oplus D$. For the space $E$, we have

$$C_{kz}(E) < C_p(E) < C_k(E).$$

We would like to have examples of spaces $X$, for which we will have

(i) $C_p(X) < C_{kz}(X) < C_k(X)$, and
(ii) $C_p(X) = C_{kz}(X) < C_k(X)$.

If we have an example of a space $Y$ satisfying (ii), then for $Z = X \oplus Y$, where $X$ is the space mentioned in Example 3.9, we will have $C_p(Z) < C_{kz}(Z) < C_k(Z)$.

So we really need to find a space $X$ satisfying (ii), that is, we need to find a space $X$ of point pseudocountable type such that the compact $G_δ$-subsets of $X$ are finite, but $X$ has an infinite compact subset. We are yet to find such a space. Actually, we would like to know if such a space exists.
4. Additional properties: Metrizability, separability and completeness

In this section, our goal is to study the metrizability of $C_kz(X)$. It is well-known that a locally convex Hausdorff space is metrizable if and only if it is first countable. But we will show that even some properties of $C_kz(X)$ weaker than first countability are equivalent to the metrizability of $C_kz(X)$. We also show that several other topological properties are equivalent to the metrizability of $C_kz(X)$.

So we first define these topological properties.

**Definition 4.1.** A space $X$ is said to be of pointwise countable type if each point is contained in a compact set having countable character.

A $\pi$-base for a space $X$ is a family of nonempty open sets in $X$ such that every nonempty open set in $X$ contains a member of this family. A point $x \in X$ is said to have a countable local $\pi$-base, if there exists a countable collection $B_x$ of nonempty open sets in $X$ such that each neighborhood of $x$ contains some member of $B_x$. If each point of $X$ has a countable $\pi$-base, then $X$ is said to have countable $\pi$-character.

A map $d : X \times X \rightarrow \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ is called a semimetric on $X$ if $d$ satisfies (i) $d(x, y) = 0$ if and only if $x = y$ and (ii) $d(x, y) = d(y, x)$. Like a metric, a semimetric generates a topology on $X$. A space $X$ is semimetrizable if $X$ admits a semimetric compatible with its topology.

A space $X$ is called locally metrizable if each point $x$ in $X$ has a neighborhood which is metrizable. A space $X$ is an $r$-space if each point of $X$ has a sequence $\{V_n : n \in \mathbb{N}\}$ of neighborhoods with the property that if $x_n \in V_n$ for each $n$, then the set $\{x_n : n \in \mathbb{N}\}$ is contained in a compact subset of $X$. A property weaker than being an $r$-space is that of being a $q$-space. A space $X$ is a $q$-space if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of $x$ such that $x_n \in U_n$ for each $n$, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point. Another property stronger than being a $q$-space is that of being an $M$-space, which can be characterized as a space that can be mapped onto a metric space by a quasi-perfect map (a continuous closed map in which inverse images of points are countably compact).

A space of pointwise countable type is an $r$-space and a metrizable space is of countable type.

For more details on the properties discussed above, see [15–18].

In order to relate the metrizability of $C_kz(X)$ with the topological properties discussed above, we need the following known result, the proof of which is omitted.

**Lemma 4.2.** Let $D$ be a dense subset of a space $X$, $A$ be a compact subset of $D$ and $x \in D$. Then

(i) $x$ has a countable local $\pi$-base in $D$ if and only if $x$ has a countable local $\pi$-base in $X$.

(ii) $D$ has a countable $\pi$-base if and only if $X$ has a countable $\pi$-base.

(iii) $A$ has countable character in $D$ if and only if $A$ has countable character in $X$.

Note that $C_kz(X)$ is a locally convex space and hence, in particular, a homogeneous space. A space $X$ is called homogeneous if for every pair of points $x, y \in X$, there exists a homeomorphism of $X$ onto $X$ itself which carries $x$ to $y$. So the previous lemma can be used to prove the following result.

**Proposition 4.3.** For any space $X$,

(a) $C_kz(X)$ has countable $\pi$-character if and only if $C_kz(X)$ contains a dense subspace which has countable $\pi$-character.

(b) $C_kz(X)$ is of pointwise countable type if and only if $C_kz(X)$ has a dense subspace of pointwise countable type.

A space $X$ is called hemi-$kz$ if there exists a sequence of compact zero sets $\{A_n : n \in \mathbb{N}\}$ in $X$ such that for any compact zero subset $A$ of $X$, $A \subseteq A_n$ holds for some $n$. This concept of hemi-$kz$ space is an analog of that of hemicompact space, which is used to characterize the metrizability of $C_k(X)$, and which has the same definition except that the compact sets are used instead of compact zero sets.

**Theorem 4.4.** Suppose $C_kz(X)$ is Hausdorff, that is $X = \bigcup\{A : A \in KZ(X)\}$. Then following assertions are equivalent.

(a) $C_kz(X)$ is metrizable.

(b) $C_kz(X)$ is first countable.

(c) $C_kz(X)$ has countable $\pi$-character.

(d) $C_kz(X)$ contains a dense subspace which has a countable local $\pi$-base.

(e) $C_kz(X)$ is semimetrizable.

(f) $C_kz(X)$ is locally metrizable.

(g) $C_kz(X)$ contains a nonempty open metrizable subspace.

(h) $C_kz(X)$ is of countable type.
(i) $C_{kz}(X)$ is of pointwise countable type.

(ii) $C_{kz}(X)$ has a dense subspace of pointwise countable type.

(iii) $C_{kz}(X)$ is an $M$-space.

(iv) $C_{kz}(X)$ is an $r$-space.

(v) $C_{kz}(X)$ is a q-space.

(vi) $X$ is hemi-kr.

**Proof.** From the earlier discussions, we have $(a) \Rightarrow (h) \Rightarrow (i) \Rightarrow (l) \Rightarrow (m)$ and $(a) \Rightarrow (k) \Rightarrow (m)$. Also $(a) \Rightarrow (e) \Rightarrow (b), (f) \Rightarrow (g), (a) \Rightarrow (f) \Rightarrow (b)$ are all immediate.

By Proposition 4.3, (c) $\Leftrightarrow$ (d) and (i) $\Leftrightarrow$ (j) are equivalent. Since $C_{kz}(X)$ is a topological group with respect to addition, by Birkhoff–Kakutani theorem (see Theorem 3.3.12 in [19], p. 155), $(a) \Rightarrow (b)$. Also by Proposition 5.2.6 in [19, p. 298], $(b) \Leftrightarrow (c)$.

(m) $\Rightarrow$ (n). Suppose that $C_{kz}(X)$ is a q-space. Hence there exists a sequence $\{U_n: n \in \mathbb{N}\}$ of neighborhoods of the zero-function 0 in $C_{kz}(X)$ such that if $f_n \in U_n$ for each $n$, then $\{f_n: n \in \mathbb{N}\}$ has a cluster point in $C_{kz}(X)$. Now for each $n$, there exist a compact zero subset $A_n$ of $X$ and $\epsilon_n > 0$ such that $0 \in (0, A_n, \epsilon_n) \subseteq U_n$.

Let $A$ be a compact zero subset of $X$. If possible, suppose that $A$ is not a subset of $A_n$ for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists a continuous function $f_n: X \to [0, 1]$ such that $f_n(a_n) = n$ and $f_n(x) = 0$ for all $x \in A_n$. It is clear that $f_n$ is $0, A_n, \epsilon_n).$ By hypothesis, $\{f_n: n \in \mathbb{N}\}$ has a cluster point $f$ in $C_{kz}(X)$. Then for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $f_n \in (f, A, 1).$ For all $k \in \mathbb{N}$, $f_n(a_n) > f_n(a_n) - 1 = n_k - 1 \geq k.$ But this means that $f$ is unbounded on the compact set $A$. So the sequence $\{f_n: n \in \mathbb{N}\}$ cannot have a cluster point in $C_{kz}(X)$ and consequently $C_{kz}(X)$ fails to be a q-space. Hence $X$ must be hemi-kr.

(g) $\Rightarrow$ (f). Let $W$ be a nonempty open set in $C_{kz}(X)$ such that $W$ is metrizable. Let $h \in W$ and $f \in C_{kz}(X)$. Consider the map $\psi: C_{kz}(X) \to C_{kz}(X)$ defined by $\psi(g) = g + f - h, \forall g \in C(X)$. Then $\psi$ is a homeomorphism and $f \in \psi(W)$. But $W$ being metrizable and open in $C_{kz}(X)$, $\psi(W)$ is also metrizable and open in $C_{kz}(X)$. Hence $C_{kz}(X)$ is locally metrizable.

(n) $\Rightarrow$ (a). By Theorem 2.2, the topology generated by the family of seminorms $\{p_A: A \in KZ(X)\}$ is a locally convex space. Here we need the well-known result which says that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable (see p. 119 in [20]). Now the locally convex topology on $C(X)$ generated by the countable family of seminorms $\{p_A: A \in KZ(X)\}$ is metrizable and weaker than the $kz$-open topology. But since for each compact zero set $A$ in $X$, there exists $A_n$ such that $A \subseteq A_n$, the locally convex topology generated by the family of seminorms $\{p_A: A \in KZ(X)\}$, that is, the $kz$-open topology is weaker than the topology generated by the family of seminorms $\{p_A: n \in \mathbb{N}\}$. Hence $C_{kz}(X)$ is metrizable. \[\square\]

**Theorem 4.5.** Suppose $X = \bigcup\{A: A \in KZ(X)\}$. Then following assertions are equivalent.

(a) $C_{kz}(X)$ is separable.

(b) $C_{kz}(X)$ is separable.

(c) $C_{kz}(X)$ is separable.

(d) $X$ has a weaker separable metrizable topology.

**Proof.** First note that by Corollary 4.2.2 in [21], (a), (c) and (d) are equivalent. Since by Theorem 3.2, $C_{kz}(X) \subseteq C_{kz}(X)$, we have (b) $\Rightarrow$ (a). Since $C_{kz}(X) \subseteq C_{kz}(X)$, we have (c) $\Rightarrow$ (b). \[\square\]

The topology of uniform convergence on the compact zero subsets of $X$ is actually generated by the uniformity of uniform convergence on these subsets. When this uniformity is complete, $C_{kz}(X)$ is said to be **uniformly complete**. This uniform completeness can also be seen as the completeness of a topological group. A topological group $E$ is called complete provided that every Cauchy net in $E$ converges to some element in $E$, where a net $(x_\alpha)$ in $E$ is Cauchy if for every neighborhood $U$ of 0 in $E$, there exists $\alpha_0$ such that $x_\alpha \to x_\alpha \in U$ for all $\alpha, \alpha_2 \geq \alpha_0$ (for $E$ additive). One can check that $C_{kz}(X)$ is uniformly complete if and only if it is complete as an additive topological group.

A space $X$ is called a $kz$-space if every function $f: X \to \mathbb{R}$, which is continuous on compact $G_\delta$-subsets of $X$, is continuous.

**Theorem 4.6.** Suppose $X = \bigcup\{A: A \in KZ(X)\}$. Then the space $C_{kz}(X)$ is uniformly complete if and only if $X$ is a $kz$-space.

**Proof.** See Theorem 4.6 in [11]. \[\square\]

**References**


