Applied Mathematics Letters 22 (2009) 642–645

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Attractors for a deconvolution model of turbulence

Roger Lewandowski [a,*], Yves Preaux [b]

[a] IRMAR, UMR 6625, Université Rennes 1, Campus Beaulieu, 35042 Rennes cedex, France
[b] Lycée du Puy de Lôme, rue du Puy de Lôme, 29 200 Brest, France

ARTICLE INFO

Article history:
Received 20 May 2008
Accepted 20 May 2008

Keywords:
Navier–Stokes equations
Large eddy simulation
Deconvolution models

ABSTRACT

We consider a deconvolution model for 3D periodic flows. We show the existence of a global attractor for the model.

1. Introduction

This note is concerned with the deconvolution model of order N introduced in [6] (model (2.7)) for 3D periodic flows. This model takes inspiration from the class of the so called α-models (see [2,4] and references therein) and also from the class of ADM models (see in [7]). We are interested in the question of the existence of a global attractor for this model.

The question of attractors has already been considered for the alpha model (see [1]) corresponding to the case $N = 0$. We prove in this work the existence of an attractor for each N (see Theorem 3.1).

In order to make this work self-contained, we describe carefully how the deconvolution model is constructed. Next, we recall basic notions for the attractors, notions that can be found in the book of Temam (see [8]). Finally we prove the existence of the attractor. The question of its dimension is under examination.

2. The deconvolution model

2.1. Function spaces

For $s \in \mathbb{R}$, let us define the space function

$$\mathbf{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k}} \widehat{\mathbf{w}} e^{i\mathbf{k}\cdot\mathbf{x}},\ \nabla \cdot \mathbf{w} = 0,\ \widehat{\mathbf{w}}(\mathbf{0}) = \mathbf{0}, \quad \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}(\mathbf{k}, t)|^2 < \infty \right\}. \tag{2.1}$$

We define the $\mathbf{H}_s$ norms by

$$\|\mathbf{w}\|_s^2 = \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}(\mathbf{k}, t)|^2, \tag{2.2}$$

* Corresponding author. Tel.: +33 2 23 23 58 64; fax: +33 2 23 23 67 90.
E-mail addresses: Roger.Lewandowski@univ-rennes1.fr (R. Lewandowski), yves.preaux@free.fr (Y. Preaux).
URL: http://perso.univ-rennes1.fr/roger.lewandowski/ (R. Lewandowski).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2008.05.007

where of course $\|\mathbf{w}\|_0^2 = \|\mathbf{w}\|^2$. It can be shown that when s is an integer, $\|\mathbf{w}\|_s^2 = \|\nabla^s \mathbf{w}\|^2$ (see [3]).

We denote by P_L the Helmholtz–Leray orthogonal projection of $(L^2)^3$ onto $\mathbf{H}_0$ and by A the Stokes operator defined by $A = -P_L\triangle$ on $D(A) = \mathbf{H}_0 \cap (H^2)^3$. We note that in the space-periodic case, $A\mathbf{w} = -\triangle\mathbf{w}$ for all $\mathbf{w} \in D(A)$.

The operator A^{-1} is a self-adjoint positive definite compact operator from $\mathbf{H}_s$ onto $\mathbf{H}_s$, for $s = 1$ and $s = 2$ (see [5]). We denote as λ_1 the smallest eigenvalue of A.

We introduce the trilinear form b, defined by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{\mathbf{i},\mathbf{j}} \int_\Omega u_i \partial_i v_j w_j \mathrm{d}x. \tag{2.3}$$

wherever the integrals make sense. Note that $b(\mathbf{u}, \mathbf{w}, \mathbf{w}) = 0$ when $\nabla \cdot \mathbf{u} = 0$.

2.2. The filter and the deconvolution process

Let $\mathbf{w} \in \mathbf{H}_0$ and $\overline{\mathbf{w}} \in \mathbf{H}_1$ be the unique solution to the following Stokes problem with periodic boundary conditions:

$$-\delta^2\triangle\overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla r = \mathbf{w} \quad \text{in } \mathbb{R}^3, \quad \nabla \cdot \overline{\mathbf{w}} = 0, \quad \int_\Omega \overline{\mathbf{w}} = \mathbf{0}. \tag{2.4}$$

We denote the filtering operation by G, so $\overline{\mathbf{w}} = G\mathbf{w}$. Writing $\mathbf{w}(\mathbf{x}, t) = \sum_{\mathbf{k}} \widehat{\mathbf{w}}(\mathbf{k}, t)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$, it is easily seen that $\nabla r = 0$ and $\overline{\mathbf{w}}(\mathbf{x}, t) = \sum_{\mathbf{k}} \frac{\widehat{\mathbf{w}}(\mathbf{k},t)}{1+\delta^2|\mathbf{k}|^2}\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$. Then writing $\overline{\mathbf{w}} = G(\mathbf{w})$, we see that in the corresponding spaces of the type $\mathbf{H}_s$, the transfer function of G, denoted by $\widehat{G}$, is the function $\widehat{G}(\mathbf{k}) = \frac{1}{1+\delta^2|\mathbf{k}|^2}$, and we also can write on the $\mathbf{H}_s$ type spaces

$$-\delta^2\triangle\overline{\mathbf{w}} + \overline{\mathbf{w}} = \mathbf{w} \quad \text{in } \mathbb{R}^3, \quad \nabla \cdot \overline{\mathbf{w}} = 0, \quad \int_\Omega \overline{\mathbf{w}} = \mathbf{0}. \tag{2.5}$$

The procedure of deconvolution by the Van Citter approximation is described in [6]. This yields the operator $D_N\mathbf{w} = \sum_{n=0}^N (I - G)^n\mathbf{w}$.

Definition 2.1. The truncation operator $H_N : \mathbf{H}_s \to \mathbf{H}_s$ is defined by $H_N\mathbf{w} := D_N\overline{\mathbf{w}} = (D_N \circ G)\mathbf{w}$. ■

Note that for any $s \geqslant 0$ we have the following properties (see [6]):

$$\|H_N\mathbf{w}\|_s \leqslant \|\mathbf{w}\|_s, \quad \|H_N\mathbf{w}\|_{s+2} \leqslant C(\delta, N)\,\|\mathbf{w}\|_s. \tag{2.6}$$

2.3. The model

Let $\mathbf{u}_0 \in \mathbf{H}_0, f \in \mathbf{H}_{-1}$. For $\delta > 0$, let the averaging be defined by (2.4). The problem that we consider is the following: for a fixed $T > 0$, find $(\mathbf{w}, q)$ where

$$\begin{cases} \mathbf{w} \in L^2([0,T], \mathbf{H}_1) \cap L^\infty([0,T], \mathbf{H}_0), \quad \partial_t\mathbf{w} \in L^2([0,T], \mathbf{H}_{-1}) \\ q \in L^2([0,T], L^2_{\mathrm{per},0}), \\ \partial_t\mathbf{w} + (H_N(\mathbf{w}) \cdot \nabla)\,\mathbf{w} - \nu\triangle\mathbf{w} + \nabla q = H_N(\mathbf{f}) \quad \text{in } \mathcal{D}'([0,T] \times \mathbb{R}^3), \\ \mathbf{w}(\mathbf{x}, 0) = H_N(\mathbf{u}_0) = \mathbf{w}_0. \end{cases} \tag{2.7}$$

where $L^2_{\mathrm{per},0}$ denotes the scalar fields in $L^2_{loc}(\mathbb{R}^3)$, 2π-periodic with zero mean value. We prove in [6] the following result.

Theorem 2.1. *Problem (2.7) admits a unique solution $(\boldsymbol{w}, q)$, $\boldsymbol{w} \in L^\infty([0,T], \mathbf{H}_1) \cap L^2([0,T], \mathbf{H}_2)$, and the following energy equality holds:*

$$\frac{1}{2}\,\|\boldsymbol{w}(t)\|^2 + \nu\int_0^t\int_\Omega |\nabla\boldsymbol{w}|^2\mathrm{d}\mathbf{x}\mathrm{d}t' = \frac{1}{2}\,\|H_N(\boldsymbol{u}_0)\|^2 + \int_0^t\int_\Omega H_N(\mathbf{f}).\boldsymbol{w}\,\mathrm{d}\mathbf{x}\mathrm{d}t'. \quad ■ \tag{2.8}$$

3. Main result

3.1. Recalling basic notions for attractors

We denote by $\mathbf{w}(t,\cdot) = S(t)(\mathbf{w}_0)$ the (unique) solution of system (2.7) at time t. We recall the definitions of a global attractor and an absorbing set (see in [8]).

Definition 3.1. We say that $\mathcal{A} \subset \mathbf{H}_0$ is a global attractor for the dynamical system (2.7) if and only if
(P1) $\mathcal{A}$ is compact in the space $\mathbf{H}_0$,
(P2) $\forall\, t \in \mathbb{R}, S(t)(\mathcal{A}) \subset \mathcal{A}$,
(P3) for every bounded subset $B \subset \mathbf{H}_0$, $\rho(S(t)(B), \mathcal{A})$ goes to zero when t goes to infinity, where $\rho(S(t)(B), \mathcal{A}) = \sup_{v\in B} \inf_{u\in\mathcal{A}} \|u - v\|$. ■

Definition 3.2. 1. A set $A \subset \mathbf{H}_0$ is an absorbing set if and only if for every bounded subset $B \subset \mathbf{H}_0$ there exists $t_1 > 0$ such that for all $t \geq t_1$ one has $S(t)(B) \subset A$.
2. We say that the semi-group $S(t)$ is uniformly compact if and only if for every bounded subset $B \subset \mathbf{H}_0$ there exists $t_2 = t_2(B)$ such that $\overline{\bigcup_{t\geq t_2} S(t)(B)}$ is compact.
3. We denote by $\omega(A)$ the set $\omega(A) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)(A)}$. ■

Proposition 3.1. *Assume that there exists an absorbing bounded set A and that the semi-group $S(t)$ is uniformly compact; then $\mathcal{A} = \omega(A)$ is the global attractor for the dynamical system defined by $S(t)$.*

See the proof in [8].

3.2. Existence of a global attractor

We are now in a position to state and prove the main result of this note.

Theorem 3.1. *The system (2.7) has a global attractor.* ■

Proof. Thanks to Proposition 3.1, it remains to prove that system (2.7) has an absorbing set and that $S(t)$ is uniformly compact, in the sense of Definition 3.2. Both results are derived from basic estimates that we detail in the following.

Absorbing set in $\mathbf{H}_0$: We take the inner product of the first equation of system (2.7) with $\mathbf{w}$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|^2 + b(H_N(\mathbf{w}), \mathbf{w}, \mathbf{w}) + \nu\|\mathbf{w}\|_1^2 = (H_N(\mathbf{f}), \mathbf{w}). \tag{3.1}$$

Observing that $b(H_N(\mathbf{w}), \mathbf{w}, \mathbf{w}) = 0$ due to $\nabla \cdot H_N(\mathbf{w}) = 0$, applying the Young inequality and the Poincare' inequality $\|\mathbf{w}\| \leqslant \lambda_1^{-\frac{1}{2}}\|\mathbf{w}\|_1$, and using (2.6), we are left with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|^2 + \nu\lambda_1\|\mathbf{w}\|^2 \leqslant \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2. \tag{3.2}$$

So, noting $\rho_0 = \frac{1}{\nu\lambda_1}\|\mathbf{f}\|$ and applying the Gronwall lemma, we obtain

$$\|\mathbf{w}\|^2 \leqslant \|\mathbf{w}_0\|^2 \mathrm{e}^{-\nu\lambda_1 t} + \rho_0^2(1 - \mathrm{e}^{-\nu\lambda_1 t}). \tag{3.3}$$

Considering $\mathbf{w}_0$ included in a ball $B(0, R)$ and choosing $\rho_0' > \rho_0$, the previous inequality implies that, for $t > T_0$,

$$\|\mathbf{w}(t)\|^2 < {\rho_0'}^2, \quad \text{with } T_0 = \frac{1}{\nu\lambda_1}\ln\frac{R^2}{{\rho_0'}^2 - {\rho_0}^2}. \tag{3.4}$$

Since each bounded set of $\mathbf{H}_0$ is included in a ball $B(0, R)$, one deduces that $B(0, \rho_0')$ is an absorbing set in $\mathbf{H}_0$.
Moreover, as an alternative to (3.2) we may obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|^2 + \nu\|\mathbf{w}\|_1^2 \leqslant \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2. \tag{3.5}$$

Integrating between t and $t + r$, we observe that, for $\mathbf{u}_0 \in B(0, R)$, $\rho_0' > \rho_0$ and $t > T_0$ $\left(\text{with } T_0 = \frac{1}{\nu\lambda_1}\ln\frac{R^2}{{\rho_0'}^2 - {\rho_0}^2}\right)$,

$$\int_t^{t+r}\|\mathbf{w}(s)\|_1^2\,\mathrm{d}s \leqslant \frac{r}{\nu^2\lambda_1}\|\mathbf{f}\|^2 + \frac{{\rho_0'}^2}{\nu}. \tag{3.6}$$

Absorbing set in $\mathbf{H}_1$: We take now the inner product of the first equation of system (2.7) with $A\mathbf{w}$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_1^2 + b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w}) + \nu\|A\mathbf{w}\|^2 = (H_N(\mathbf{f}), A\mathbf{w}), \tag{3.7}$$

leading to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_1^2 + +\nu\|A\mathbf{w}\|^2 \leqslant \frac{1}{\nu}\|H_N(\mathbf{f})\|^2 + \frac{\nu}{4}\|A\mathbf{w}\|^2 + |b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})|, \tag{3.8}$$

The trilinear form b satisfies the following inequality (see [6]):

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leqslant c'\|\mathbf{u}\|^{1/4}\|\mathbf{u}\|_1^{3/4}\|\mathbf{v}\|_1^{1/4}\|A\mathbf{v}\|^{3/4}\|\mathbf{w}\|. \tag{3.9}$$

Therefore, one has

$$|b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})| \leqslant c'\|H_N(\mathbf{w})\|^{1/4}\|H_N(\mathbf{w})\|_1^{3/4}\|\mathbf{w}\|_1^{1/4}\|A\mathbf{w}\|^{7/4}. \tag{3.10}$$

Using (2.6) we have $\|H_N(\mathbf{w})\|_1 \leqslant \|H_N(\mathbf{w})\|_2 \leqslant C(\delta, N)\|\mathbf{w}\|$ and using (2.6),

$$|b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})| \leqslant C'(\delta, N)\|\mathbf{w}\|\,\|\mathbf{w}\|_1^{1/4}\|A\mathbf{w}\|^{7/4}. \tag{3.11}$$

By the Young inequality we obtain

$$|b(H_N(\mathbf{w}), \mathbf{w}, A\mathbf{w})| \leqslant \frac{\nu}{4}\|A\mathbf{w}\|^2 + \frac{C_1(\delta, N)}{2}\|\mathbf{w}\|^8\|\mathbf{w}\|_1^2, \tag{3.12}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_1^2 + \nu\|A\mathbf{w}\|^2 \leqslant \frac{2}{\nu}\|H_N(\mathbf{f})\|^2 + C_1(\delta, N)\|\mathbf{w}\|^8\|\mathbf{w}\|_1^2 \tag{3.13}$$

We now use a Gronwall type proposition (see the proof in [8]): □

Proposition 3.2. *Assume that y, g and h are positive, locally integrable functions on $]t_0, +\infty[$, and that for $t \geqslant t_0$,*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leqslant gy + h, \qquad \int_t^{t+r} y(s)\mathrm{d}s \leqslant k_1, \qquad \int_t^{t+r} g(s)\mathrm{d}s, \leqslant k_2, \qquad \int_t^{t+r} h(s)\mathrm{d}s \leqslant k_3,$$

where r, k_1, k_2, k_3 are four positive constants; then

$$y(t+r) \leqslant \left(\frac{k_1}{r} + k_3\right)\mathrm{e}^{k_2}, \quad \forall t \geqslant t_0. \quad \blacksquare$$

We can now finish the proof. Thanks to (3.4) and (3.6), using this lemma with $y = \|\mathbf{w}\|_1^2$, $g = C_1(\delta, N)\|\mathbf{w}\|^8$ and $h = \frac{2}{\nu}\|H_N(\mathbf{f})\|^2$, we obtain

$$\|\mathbf{w}(t)\|_1^2 \leqslant \left(\frac{k_1}{r} + k_3\right)\mathrm{e}^{k_2}, \quad \forall t \geqslant T_0 + r, \tag{3.14}$$

with $k_1 = \frac{r}{\nu^2\lambda_1}\|\mathbf{f}\|^2 + \frac{1}{\nu}{\rho'_0}^2$, $k_2 = C_1(\delta, N){\rho'_0}^8$, $k_3 = \frac{2r}{\nu}\|\mathbf{f}\|^2$.

Thus, after a time $T_1 = T_1(\|\mathbf{w}_0\|, \|\mathbf{f}\|, \nu)$, $\mathbf{w}$ is included in a ball of radius $R = R(\|\mathbf{f}\|, \nu, \delta, N)$. One deduces that there exists an absorbing set in $\mathbf{H}_1$.

Let B be a bounded set in $\mathbf{H}_1$. Estimate (3.14) implies that $\bigcup_{t\geq T_0+r} S(t)B$ is a bounded set in $\mathbf{H}_1$ which is compactly imbedded in $\mathbf{H}_0$, so $S(t)$ is uniformly compact. Estimate (3.14) also implies the existence of an absorbing bounded set since k_1, k_2 and k_3 are independent of $\mathbf{w}_0$. Thanks to (3.1), this achieves the proof of the theorem. ■

References

[1] V.V. Chepyzhov, E.S. Titi, M.I. Vishik, On the convergence of the Leray-alpha model to the trajectory attractor of the 3D Navier–Stokes system, Matematicheskii Sbornik 12 (2007) 3–36.
[2] A. Cheskidov, D.D. Holm, E. Olson, E.S. Titi, On a Leray-α model of turbulence, Royal Society London, Proceedings, Series A, Mathematical, Physical and Engineering Sciences 461 (2005) 629–649.
[3] C. Doering, J. Gibbon, Applied Analysis of the Navier–Stokes Equations, Cambridge University Press, 1995.
[4] C. Foias, D.D. Holm, E.S. Titi, The Navier–Stokes-alpha model of fluid turbulence, Physica D 152 (2001) 505–519.
[5] C. Foias, O. Manley, R. Rosa, R. Temam, Navier–Stokes Equations and Turbulence, Cambridge University Press, 2001.
[6] W. Layton, R. Lewandowski, A high accuracy Leray-deconvolution model of turbulence and its limiting behavior, Analysis and Applications 6 (2008) 1–27.
[7] S. Stolz, N.A. Adams, L. Kleiser, An approximate deconvolution model for large-eddy simulation with application to incompressible wall-bounded flows, Physics of Fluids 13 (2001) 997–1015.
[8] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, 1988.