We introduce a new category of Banach algebras, \( l^1 \)-Munn algebras which we use as a tool in the study of semigroup algebras. Then we characterize amenable \( l^1 \)-Munn algebras and also semisimple ones in this category. Applying these results to the semigroup algebras provides some characterizations of amenable semigroup algebras. We also provide a counter example to a conjecture of Duncan and Paterson.

1. INTRODUCTION

In [7, 8, 13, 14, 24] the authors studied the amenability of certain weighted semigroup algebras. In the process of characterizing amenable semigroup algebras we discovered a new category of Banach algebras which we call \( l^1 \)-Munn algebras. Indeed a very special type of these algebras that are finite dimensional, without any topological structure on them, was introduced by Munn [28]. He used these algebras to interpret the algebraic semigroup algebra of a finite Rees matrix semigroup in terms of a matrix algebra over a finite group and since then his technique has been used in the study of algebraic semigroup rings, see, for example, [33]. In [8, p. 145] Duncan and Paterson observed that Munn’s technique can be used in the study of semigroup algebras of completely 0-simple semigroups with a finite number of idempotents. Besides, some special \( l^1 \)-Munn algebras have certain relations and interactions with some well known algebras. These applications provide strong reasons to study these algebras as abstract objects.

This paper is organized as follows: In Section 2 we introduce our notations and in Section 3 we show some basic facts about the structure of \( l^1 \)-Munn algebras, in particular characterizing those with bounded approximate identities. In Section 4 we characterize amenable \( l^1 \)-Munn algebras by explicit construction of approximate diagonals. Then we consider the semisimplicity of these algebras and use the results to show
the relation between semisimplicity and amenability in the concrete case of semigroup algebras in Section 5. We provide some characterizations of amenable semigroup algebras which show that the amenability problem of the semigroup algebras is reduced to the completely (0-)simple case. This was done only for inverse semigroups in [7, Theorem 7]. We also give a counter example to the conjecture of Duncan and Paterson [8, p. 145]. In Section 6 we give some examples of the algebras that appeared in the previous sections plus some interesting counter examples that were promised earlier. Further study of the structure of the \( \ell^1 \)-Munn algebras, in particular their duals and topological center, representation theory and their multiplier algebras, is done in [9, 10, 11].

2. NOTATIONS

Let \( \mathcal{A} \) be a Banach algebra. Throughout by \( \mathcal{A} \) module we mean Banach \( \mathcal{A} \) bimodule. We denote the projective tensor product of two \( \mathcal{A} \) modules \( X \) and \( Y \) by \( X \otimes \mathcal{A} Y \). A short exact sequence

\[
0 \rightarrow X \xrightarrow{\pi} Y \xrightarrow{\epsilon} Z \rightarrow 0
\]

of \( \mathcal{A} \) modules and bounded \( \mathcal{A} \) module homomorphisms is called admissible (split) if \( \pi \) has a bounded linear \( \mathcal{A} \) module homomorphism left inverse. A bounded net \( \{e_\alpha\} \) in \( \mathcal{A} \) is called an approximate diagonal if \( e_\alpha a - ae_\alpha \to 0 \) and \( \pi(e_\alpha a) \to a \) for every \( a \in \mathcal{A} \). Let \( X \) be a Banach \( \mathcal{A} \) bimodule. We will denote the set of all bounded (inner) derivations from \( \mathcal{A} \) into \( X \) by \( \mathcal{A} \mathcal{D} X \) (\( B^1(\mathcal{A}, X) \)). Also set \( \mathcal{H}^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/B^1(\mathcal{A}, X) \). \( \mathcal{A} \) is called amenable if \( \mathcal{H}^1(\mathcal{A}, X^*) = 0 \) for every dual \( \mathcal{A} \) bimodule \( X^* \), or equivalently if \( \mathcal{A} \) has an approximate diagonal [20, Lemma 1.2 and Theorem 1.3].

In the algebraic notations for semigroups mainly we follow [5]. Throughout, \( S \) (\( G \)) is a semigroup (group) and \( E_S \) is the set of idempotent elements of \( S \). If \( T \) is an ideal of \( S \) then the Rees factor semigroup \( ST \) is the result of collapsing \( T \) into a single element 0 and retaining the identity of elements of \( S \setminus T \). We make the convention that \( S/\emptyset = S \). If \( S \) has an identity then \( S^1 = S \); otherwise \( S^1 = S \cup \{1\} \) where 1 is the identity joined to \( S \). For \( a \in S \), \( J(a) \) is the principal ideal \( S \langle a \rangle \) and \( J_a \) is the set of elements \( b \in J(a) \) such that \( J(b) = J(a) \). The inclusion among the principal ideals induces the following order among the equivalence classes \( J_a \): \( J_a \subseteq J_b \) if \( J(a) \subseteq J(b) \) \( (J_b \supseteq J_a \text{ if } J(a) \supseteq J(b)) \). By \( I(a) \) we mean the ideal \( \{b \in J(a) : J_b < J_a \} \), i.e., \( I(a) = J(a) \setminus J_a \). On \( E_S \) we have a usual order: \( e, f \in E_S, \ e \leq f \) if \( ef = fe = e \). An idempotent \( e \in E_S \) is called primitive if it is nonzero and
is minimal in the set of nonzero idempotents. A semigroup $S$ with zero is 0-simple if $\{0\}$ and $S$ are the only ideals of $S$. $S$ is called completely (0-)simple if it is (0-)simple and contains a nonzero primitive idempotent. The factors $J(a)/I(a)$, $a \in S$ are called the principal factors of $S$. Each principal factor of $S$ is either 0-simple, simple, or null, i.e., the product of any two elements is zero [5, Lemma 2.39]. If every principal factor of $S$ is 0-simple or simple, we say that $S$ is semisimple.

A (relative) ideal series $S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \emptyset$ that has no proper refinement is called a principal (composition) series. If $S$ has a principal series as above, then the factors of this series are isomorphic in some order to the principal factors of $S$ [5, Theorem 2.40].

A semigroup $S$ is called regular if for every $a \in S$ there is $b \in S$ such that $a = aba$. $S$ is an inverse semigroup if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$.

Let $G$ be a group, $I$ and $J$ be arbitrary nonempty sets, and $G^0 = G \cup \{0\}$ be the group with zero arising from $G$ by adjunction of a zero element. An $I \times J$ matrix $A$ over $G^0$ that has at most one nonzero entry $a = A(i, j)$ is called a Rees $I \times J$ matrix over $G^0$ and is denoted by $(a)_{ij}$. Let $P$ be a $J \times I$ matrix over $G$. The set $S = G \times I \times J$ with the composition $(a, i, j) \cdot (b, l, k) = (aP_{jl}b, i, k)$, $(a, i, j), (b, l, k) \in S$ is a semigroup that we denote by $\mathcal{A}(G, P)$ [16, p. 68]. Similarly if $P$ is a $J \times I$ matrix over $G^0$, then $S = G \times I \times J \cup \{0\}$ is a semigroup under the following composition operation:

$$(a, i, j) \cdot (b, l, k) = \begin{cases} (aP_{jl}b, i, k) & \text{if} \quad P_{jl} \neq 0 \\ 0 & \text{if} \quad P_{jl} = 0 \end{cases}$$

$$(a, i, j) \cdot 0 = 0 \cdot (a, i, j) = 0 \cdot 0 = 0.$$

This semigroup which is denoted by $\mathcal{A}(G, P)$ also can be described in the following way: The set of all Rees $I \times J$ matrices over $G^0$ form a semigroup under the binary operation $A \cdot B = APB$, which is called the Rees $I \times J$ matrix semigroup over $G^0$ with the sandwich matrix $P$ and is isomorphic to $\mathcal{A}(G, P)$ [16, pp. 61–63]. An $I \times J$ matrix $P$ over $G^0$ is called regular (invertible) if every row and every column of $P$ contains at least (exactly) one nonzero entry.

For $f \in \ell^\infty(S)$ the left and right translations of $f$ by $s \in S$ will be denoted by $fs$ and $sf$ respectively. $S$ is called amenable if there exists $m \in \ell^\infty(S)^*$ such that $m \geq 0$, $m(1_s) = 1$ and $m(xs) = m(xf) = m(fx)$, for every $x \in S$, $f \in \ell^\infty(S)$. As usual the semigroup algebra of $S$ is $\ell^1(S)$ with the convolution product. If $S$ has a zero, then we call the algebra $\ell^1(S)/\ell^1(\{0\})$ the contracted semigroup algebra of $S$, where $\ell^1(\{0\}) = \ell^1(\{0\})$.

Throughout $\mathcal{A}(H)$ means the algebra of compact operators on the Hilbert space $H$. 
3. THE $\ell^1$-MUNN ALGEBRAS

In this section we introduce the $\ell^1$-Munn algebras and compare them with some other well-known algebras. Then we investigate some of their basic structural properties.

**Definition 3.1.** Let $\mathcal{A}$ be a unital Banach algebra, $I$ and $J$ be arbitrary index sets, and $P$ be a $J \times I$ nonzero matrix over $\mathcal{A}$ such that $\|P\|_{\infty} = \sup \{ \|P_{ij}\| : i \in I, j \in J \} \leq 1$. Let $\mathcal{L}(\mathcal{A}, P)$ be the vector space of all $I \times J$ matrices $A$ over $\mathcal{A}$ such that $\|A\|_1 = \sum_{i \in I, j \in J} \|A_{ij}\| < \infty$. Then it is easy to check that $\mathcal{L}(\mathcal{A}, P)$ with the product $A \cdot B = APB$, $A, B \in \mathcal{L}(\mathcal{A}, P)$, and the $\ell^1$-norm is a Banach algebra that we call the $\ell^1$-Munn $I \times J$ matrix algebra over $\mathcal{A}$ with sandwich matrix $P$ or, briefly, the $\ell^1$-Munn algebra.

When $I = J$ and $P$ is the identity $J \times J$ matrix over $\mathcal{A}$, we denote $\mathcal{L}(\mathcal{A}, P)$ by $\mathcal{M}_J(\mathcal{A})$. Also we denote $\mathcal{L}(\mathcal{A}_J(\mathcal{C})$ simply by $\mathcal{L}(\mathcal{A}_J)$. In particular when $|J| = m < \infty$, $\mathcal{L}(\mathcal{A})$ is the algebra $\mathcal{M}_m$ of $m \times m$ complex matrices.

**Convention.** (i) From now on we use $\mathcal{A}$ for an arbitrary unital Banach algebra, $I$ and $J$ for index sets, and $P$ for the sandwich matrix, exclusively.

(ii) Throughout, $\{e_{ij} : i \in I, j \in J\}$ is the standard matrix unit system of the matrix algebra under discussion.

(iii) Unless otherwise stated we assume henceforth that nonzero entries of $P$ are invertible, that $P$ has no zero rows or columns, and $\|P\|_\infty \leq 1$. These conditions are satisfied in the applications given below.

The following lemma is in a sense a generalization of [3, Lemma 4, p. 231] which can be proved with a similar argument.

**Lemma 3.2.** Every $u \in \mathcal{L}(\mathcal{A})$ has a unique expression of the form $u = \sum_{i, j \in J} e_{ij} \otimes a_{ij}$, $a_{ij} \in \mathcal{A}$.

The next lemma is well-known for the case that $J$ is finite, see [31, p. 4]. The general case can be proved with the same technique and using Lemma 3.2.

**Lemma 3.3.** $\mathcal{L}(\mathcal{A})$ is isometrically algebra isomorphic to $\mathcal{L}(\mathcal{A}) \hat{\otimes} \mathcal{A}$.

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Using Tanbay’s notation [35, p. 710], let $A$ be a $\mathbb{N} \times \mathbb{N}$ matrix with complex entries for which there is an $m \in \mathbb{R}^+$ such that $\sum_{i \in \mathbb{N}} |A_{ij}| \leq m$ for all $j \in \mathbb{N}$ and $\sum_{j \in \mathbb{N}} |A_{ij}| \leq m$ for all $i \in \mathbb{N}$. Let $\mathcal{M}_\theta$ be the collection of all
such matrices under the natural operations. Then $\mathcal{M}_0$ is a self adjoin sub-algebra of $\mathcal{B}(H)$. Define the norm $\| \cdot \|$ on $\mathcal{M}_0$ by

$$\|A\| = \inf \left\{ m : \sum_{i \in \mathbb{N}} |A_{ik}| \leq m \sum_{j \in \mathbb{N}} |A_{ij}| \leq m \right\}.$$ 

Then for every $A \in \mathcal{M}_0$ we have $\|A\| \leq \|\|A\|\|$, where $\|\cdot\|$ is the operator norm [35, p. 710]. Let $\mathcal{M} = \mathcal{M}_0^{-1}$. With these notations we have:

**Proposition 3.4.** (i) $\mathcal{L}.\mathcal{M}_0$ is a proper ideal in $\mathcal{M}$ and for every $T, U, A \in \mathcal{L}.\mathcal{M}_0$ we have $\|TAU\|_1 \leq \|T\|_1 \|A\|_1 \|U\|_1$. 

(ii) $\mathcal{L}.\mathcal{M}_0 \subseteq \mathcal{L}.\mathcal{M}_0^{-1} = \mathcal{K}(H) \subseteq \mathcal{M}$.

**Proof.** (i) Let $A \in \mathcal{L}.\mathcal{M}_0, T, U \in \mathcal{M}_0$. Then it is easy to check that $\|TAU\|_1 \leq \|T\|_1 \|A\|_1 \|U\|_1$. Combination of these two relations with the fact that the identity matrix is in $\mathcal{M} \setminus \mathcal{L}.\mathcal{M}_0$, proves the first part.

(ii) Let $T \in \mathcal{L}.\mathcal{M}_0$. There is a sequence $T_n$ of matrices, each with a finite number of nonzero entries, such that $T_n \xrightarrow{1} T$. Now by part (i) and the fact preceding the proposition, $\|T\|_1 \leq \|T\|_1 \|A\|_1$ which implies that $T_n \xrightarrow{1} T$ and since $T_n \in \mathcal{K}(H)$, then $T \in \mathcal{K}(H)$. Also the diagonal matrix $\text{diag}(1/n)$ is in $\mathcal{K}(H) \setminus \mathcal{L}.\mathcal{M}_0$ which shows that the inclusion $\mathcal{L}.\mathcal{M}_0 \subseteq \mathcal{L}.\mathcal{M}_0^{-1} = \mathcal{K}(H)$ is proper. On the other hand every finite rank operator is the (operator norm) limit of matrices with a finite number of nonzero entries. So the middle equality holds.

The inclusion $\mathcal{K}(H) \subseteq \mathcal{M}$ was shown by Tanbay [35, p. 710].

**Lemma 3.5.** Suppose $I$ and $J$ are finite and $V (W)$ is an invertible $J \times J$ ($I \times I$) matrix over $\mathcal{M}$. Let $\mathcal{B} = \mathcal{L}.\mathcal{M}(\mathcal{A}, P)$ and $\mathcal{C} = \mathcal{L}.\mathcal{M}(\mathcal{A}, VP)$ ($\mathcal{C} = \mathcal{L}.\mathcal{M}(\mathcal{A}, PW)$). Then $\mathcal{B}$ and $\mathcal{C}$ are topologically algebra isomorphic.

**Proof.** Define the map $\phi: \mathcal{B} \rightarrow \mathcal{C}$ by $\phi(A) = AV^{-1}$ ($\phi(A) = W^{-1}A$). It is easy to check that $\phi$ is an onto algebra isomorphism. Let $A \in \mathcal{B}$. Then,

$$\|\phi(A)\| \leq \sum_{i \in I} \sum_{j \in J} \sum_{k \in J} \|A_{ik}\| \|V_{kj}\| \leq \|A\| \|V^{-1}\|.$$ 

So by the open mapping theorem $\phi$ is a topological algebra isomorphism.
Lemma 3.6. Let $I$ and $J$ be finite of orders $m$ and $n$, respectively. Then there are invertible matrices $V$ and $W$ over $\mathcal{A}$ of size $n \times n$ and $m \times m$, respectively, a natural number $k$, $k \leq \min(m, n)$, and a $(n-k) \times (m-k)$ matrix $E$ over $\mathcal{A}$ such that

\[
VPW = \begin{bmatrix}
I_k & 0 \\
0 & E
\end{bmatrix}.
\]

Note that $E$ need not satisfy condition (iii) of the convention preceding Lemma 3.2.

Proof. It is easy to show that each of the following linear algebraic operations is equivalent to multiplying $P$ on the left (right) by an (invertible) elementary matrix:

(i) Multiplying a row (column) of $P$ by an invertible element $a$ of $\mathcal{A}$.

(ii) Adding a row (column) of $P$ to another row (column) of $P$.

(iii) Interchanging two rows (columns) of $P$.

Since every nonzero $a \in \mathcal{A}$ can be written as $a = 2 \lVert a \rVert ((a/2 \lVert a \rVert - 1) + 1)$ which is a difference of two invertible elements, then we can combine parts (i) and (ii) to get:

(iv) Adding a nonzero multiple of one row (column) of $P$ to another row (column).

Now we can do a finite sequence of the above operations to $P$ to get:

\[
\begin{bmatrix}
I_k & 0 \\
0 & E
\end{bmatrix}
\]

which is the result of multiplying $P$ on the left and right by appropriate invertible matrices $V$ and $W$ respectively.

Lemma 3.7. The following conditions are equivalent:

(i) $\mathcal{L}(\mathcal{A}, P)$ has an identity,

(ii) $\mathcal{L}(\mathcal{A}, P)$ has a bounded approximate identity,
(iii) $I$ and $J$ are finite and $\mathcal{L}(\mathcal{A}, P)$ has a left and a right approximate identity.

(iv) $I$ and $J$ are finite and $P$ is invertible.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) We need only to show that the index sets are finite. Let $\{E^\gamma : \gamma \in \Gamma\}$ be a bounded approximate identity for $\mathcal{L}(\mathcal{A}, P)$, $E^\gamma = [e^\gamma_{ij}]$ and $\|E^\gamma\| \leq M$ for all $\gamma \in \Gamma$. Then for given $k \in I$, $l \in J$ we have:

\[
0 = \lim_{\gamma} \|E^\gamma e_{kl} - e_{kl}\| = \lim_{\gamma} \left\| \sum_{l,j} c^\gamma_{lj} e_{kj} - e_{kl}\right\| = \lim_{\gamma} \left\| \sum_{l,j} c^\gamma_{lj} P_{jk} e_{kl} - e_{kl}\right\| = \lim_{\gamma} \left( \sum_{l} \sum_{j} c^\gamma_{lj} P_{jk} - 1 \right).
\]

So $\lim_{\gamma} \|\sum_{l} c^\gamma_{lk} P_{jk} - 1\| = 0$. Let $\epsilon > 0$ be given and for every $k \in I$, $\gamma_k \in \Gamma$ be such that for every $\gamma \geq \gamma_k$, $1 - \epsilon < \sum_{l,j} \|c^\gamma_{lj}\|$. Now if $I$ is infinite, choose $N \in \mathbb{N}$ such that $(1 - \epsilon)N > M$, then choose distinct $k_1, \ldots, k_N \in I$ and $\gamma \geq \gamma_k$, $i = 1, \ldots, N$. We have:

\[
M < (1 - \epsilon)N < \sum_{i=1}^{N} \sum_{j} \|c_{kj}\| \leq \sum_{l,j} \|c^\gamma_{lj}\| \leq M
\]

which is a contradiction. So $I$ is finite. Similarly if we apply $E^\gamma$ to the right, we conclude that $J$ must be finite.

(iii) $\Rightarrow$ (iv) Suppose $P$ is not invertible and $\{E^\gamma : \gamma \in \Gamma\}$ be a left approximate identity for $\mathcal{L}(\mathcal{A}, P)$. By Lemma 3.6 there are invertible matrices $V$ and $W$, a necessarily non-invertible matrix $E$ and a positive integer $k$ such that:

\[
Q = VPW = \begin{bmatrix} I_k & 0 \\ 0 & E \end{bmatrix}.
\]

We assume $n = |J| \leq |I| = m$. The argument for the other case is similar. By induction on $m$ we can show that there is a nonzero column matrix $Y$ in $\mathcal{A}^m$ such that $QY = 0$ and hence there is a nonzero column matrix $X$ in $\mathcal{A}^m$ such that $PX = 0$. Now if $B \in \mathcal{L}(\mathcal{A}, P)$ is the matrix that all of its columns are equal to $X$, then $B = \lim E^\gamma PB = 0$ which is a contradiction. So $P$ must be invertible.

(iv) $\Rightarrow$ (i) By Lemma 3.5 $\mathcal{L}(\mathcal{A}, P) \cong \mathcal{L}(\mathcal{J}, \mathcal{A})$ and since $\mathcal{L}(\mathcal{J}, \mathcal{A})$ is unital, then so is $\mathcal{L}(\mathcal{A}, P)$. \[\blacksquare\]
4. AMENABILITY AND SEMISIMPLICITY OF $\ell^1$-MUNN ALGEBRAS

In this section we use the results of the previous section to characterize the amenability and semisimplicity of $\ell^1$-Munn algebras.

**Theorem 4.1.** The following conditions are equivalent:

(i) $\mathcal{L}\mathcal{M}(\mathcal{A}, P)$ is amenable,

(ii) $\mathcal{A}$ is amenable, $I$ and $J$ are finite, and $P$ is invertible.

**Proof.** (i) $\Rightarrow$ (ii) Since $\mathcal{L}\mathcal{M}(\mathcal{A}, P)$ has a bounded approximate identity, then by Lemma 3.7, $I$ and $J$ are finite and $P$ is invertible. So by using Lemma 3.5 and then Lemma 3.4 we conclude that $\mathcal{L}\mathcal{M}(\mathcal{A}, P)$ is topologically algebra isomorphic to $\mathcal{M}_m \hat{\otimes} \mathcal{A}$ where $m = |I| = |J|$. Since $\mathcal{M}_m \hat{\otimes} \mathcal{A}$ is amenable, it has an approximate diagonal $\{e_x : x \in I\}$ which by Lemma 3.2 can be represented in the form:

\[
\left\{ \sum_{k=1}^{\infty} \left( \sum_{i,j=1}^{m} e_{ij} \otimes a_{ij}^{sk} \right) \otimes \left( \sum_{r,l=1}^{m} e_{rl} \otimes b_{rl}^{sk} \right) : x \in I \right\}.
\]

Let $u = \sum_{i,t=1}^{m} e_{it} \otimes x_{it} \in \mathcal{M}_m \hat{\otimes} \mathcal{A}$. Then,

\[
u = \lim_{n} \pi(e_n u) = \lim_{n} \left( \sum_{k=1}^{\infty} \left( \sum_{i,j=1}^{m} e_{ij} \otimes a_{ij}^{nk} \right) \otimes \left( \sum_{r,l=1}^{m} e_{rl} \otimes b_{rl}^{nk} \right) \right)
\]

\[
\otimes \left( \sum_{a,t=1}^{m} e_{at} \otimes x_{at} \right)
\]

\[
= \lim_{n} \sum_{k=1}^{\infty} \left( \sum_{i,j,l,t=1}^{m} e_{ij} \otimes a_{ij}^{nk} b_{jl}^{nk} x_{lt} \right)
\]

\[
= \sum_{i,t=1}^{m} \left( e_{it} \otimes \lim_{n} \sum_{k=1}^{\infty} \sum_{j,l=1}^{m} a_{ij}^{nk} b_{jl}^{nk} x_{lt} \right).
\]

Therefore we have:

\[
\sum_{i,t=1}^{m} e_{it} \otimes \left( x_{it} - \lim_{n} \sum_{k=1}^{\infty} \sum_{j,l=1}^{m} a_{ij}^{nk} b_{jl}^{nk} x_{lt} \right) = 0. \quad (1)
\]
On the other hand,

\[ 0 = \lim \limits_{\rightarrow} (e_n u - w e_n) \]

\[ = \lim \limits_{\rightarrow} \left( \left( \sum_{k=1}^{\infty} \left( \sum \sum_{r,t=1}^{m} e_{rt} \otimes b_{rt} \right) \right) \left( \sum_{k=1}^{\infty} e_{kl} \otimes x_{kl} \right) \right) \]

\[ - \left( \sum_{k=1}^{\infty} \left( \sum_{k=1}^{\infty} \left( \sum_{r,t=1}^{m} e_{rt} \otimes x_{rt} \right) \right) \right) \]

\[ = \lim \limits_{\rightarrow} \left( \sum_{k=1}^{\infty} \left( \sum_{k=1}^{\infty} \left( \sum_{r,t=1}^{m} e_{rt} \otimes x_{rt} \right) \right) \right) \]

Let \( \psi \) be the onto linear isometry:

\[ \psi : (\mathcal{M}_m \otimes \mathcal{A}) \otimes (\mathcal{M}_m \otimes \mathcal{A}) \rightarrow (\mathcal{M}_m \otimes \mathcal{M}_m) \otimes (\mathcal{A} \otimes \mathcal{A}) \]

\[ \psi((c \otimes x) \otimes (d \otimes y)) = (c \otimes x \otimes d \otimes y), \quad c, d \in \mathcal{M}_m, \quad x, y \in \mathcal{A}. \]

If we apply \( \psi \) to the above identity, we get:

\[ 0 = \lim \limits_{\rightarrow} \sum_{k=1}^{\infty} \left( \sum_{r,t=1}^{m} e_{rt} \otimes x_{rt} \otimes a^{ab}_{rt} \otimes b^{st}_{rt} x_{st} \right) \]

\[ - \sum_{r,t=1}^{m} e_{rt} \otimes x_{rt} \otimes a^{ab}_{rt} \otimes b^{st}_{rt} \]

So for every \( i, j, r, t \leq m \) we have:

\[ \lim \limits_{\rightarrow} \sum_{k=1}^{\infty} \sum_{r,t=1}^{m} (a^{ab}_{rt} \otimes b^{st}_{rt} x_{st} - x_{rt} a^{ab}_{rt} \otimes b^{st}_{rt}) = 0. \quad (2) \]

Suppose \( x \in \mathcal{A}, \ x_{11} = x \ \text{and} \ x_{ii} = 0 \ \text{if} \ i \neq 1 \ \text{or} \ t \neq 1. \ By \ the \ relation \ (1) \ \text{we} \ \text{have:} \]

\[ x = \lim \limits_{\rightarrow} \sum_{k=1}^{\infty} \sum_{j=1}^{m} a^{ab}_{kj} b^{st}_{jk} x = \lim \limits_{\rightarrow} \pi \left( \sum_{k=1}^{\infty} \sum_{j=1}^{m} a^{ab}_{kj} \otimes b^{st}_{jk} \right) x. \quad (3) \]
Suppose \( j = r \) and \( i = t = 1 \). Under the assumptions preceding relation (3), we conclude from the relation (2) that:

\[
\lim_{x} \sum_{k=1}^{\infty} \left( a_{ij} b_{kj} x - x a_{ij} b_{kj} \right) = 0.
\]

Taking sum over \( j \), we get:

\[
\lim_{x} \left( \sum_{k=1}^{\infty} \sum_{j=1}^{m} a_{ij} b_{kj} x - x \sum_{k=1}^{\infty} \sum_{j=1}^{m} a_{ij} b_{kj} \right) = 0. \tag{4}
\]

Relations (3) and (4) together with the boundedness of \( \{ e_{n} : x \in I \} \) imply that:

\[
\left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{m} a_{ij} b_{kj} : x \in I \right\}
\]

is an approximate diagonal for \( \mathcal{A} \). Therefore \( \mathcal{A} \) is amenable.

(ii) \( \Rightarrow \) (i) As in the previous part \( \mathcal{L}(\mathcal{A}, P) \) is topologically algebra isomorphic to \( \mathcal{M}_{m} \otimes \mathcal{A} \) for some \( m \in \mathbb{N} \). So by \([19, Proposition 5.4] \), \( \mathcal{L}(\mathcal{A}, P) \) is amenable.

Remark 4.2. In the proof of the above Theorem we constructed an approximate diagonal for \( \mathcal{A} \) from an approximate diagonal for \( \mathcal{M}_{m} \otimes \mathcal{A} \) which is the converse of \([19, Proposition 5.4] \) for this particular algebra. This is the only partial converse for that result known to the author. This constructive method can also be used to provide an alternate proof for \([19, Proposition 5.4] \) without involvement of derivations and their extensions.

Theorem 4.3. The following conditions are equivalent:

(i) \( I \) and \( J \) are finite and \( \mathcal{L}(\mathcal{A}, P) \) is semisimple,

(ii) \( \mathcal{A} \) is semisimple and \( \mathcal{L}(\mathcal{A}, P) \) has a bounded approximate identity.

Proof. (i) \( \Rightarrow \) (ii) First we show that the semisimplicity of \( \mathcal{L}(\mathcal{A}, P) \) implies that \( P \) is invertible. Suppose \( P \) is not invertible. Then, as in the proof of (iii) \( \Rightarrow \) (iv) of Lemma 3.7, there is a nonzero matrix \( B \) in \( \mathcal{L}(\mathcal{A}, P) \) such that \( PB = 0 \). Thus the ideal generated by \( B \) has square zero and so \( B \in \text{Rad}(\mathcal{L}(\mathcal{A}, P)) \) which is a contradiction. Therefore \( P \) is invertible and by Lemma 3.7, \( \mathcal{L}(\mathcal{A}, P) \) has a bounded approximate identity. Now let \( |I| = m \) and \( \phi \) be the topological algebra isomorphism
from \( \mathcal{L}(\mathcal{A}, P) \) onto \( \mathcal{M}(\mathcal{A}) \), as in Lemma 3.5. By [29, Proposition 4.3.12],
\[
\mathcal{M}(\text{Rad}(\mathcal{A})) = \text{Rad}(\mathcal{M}(\mathcal{A})) = \phi(\text{Rad}(\mathcal{L}(\mathcal{A}, P))) = 0.
\]
Therefore \( \mathcal{A} \) is semisimple.

(ii) \( \Rightarrow \) (i) By Lemma 3.7 and Lemma 3.5 we need only to show that \( \mathcal{M}(\mathcal{A}) \) is semisimple, which can be done similar to the previous part, by applying [29, Proposition 4.3.12].

5. APPLICATIONS TO THE SEMIGROUP ALGEBRAS

In this section we apply the results of Sections 3, 4, and 5 to semigroup algebras mainly to characterize the amenable ones and also find their topological center. Most of the results of this section can be proved for weighted semigroup algebras with the same argument, but for simplicity we consider just the unweighted case.

Without any topological assumptions, for finite semigroups part (iii) of the following lemma is due to Munn [28, 3.1].

**Lemma 5.1.** Let \( T \) be an ideal of \( S \).

(i) \( \ell^1(T) \) is isometrically algebra isomorphic to a closed complemented ideal of \( \ell^1(S) \).

(ii) If \( S \) has a zero element, then \( \ell^1(S) \) is topologically isomorphic to \( \ell^1(S) / \ell^1(0) \).

(iii) \( \ell^1(S) / \ell^1(T) \) is isometrically algebra isomorphic to \( \ell^1(S/T) / \ell^1(0) \).

**Proof.**

(i) Straightforward.

(ii) Consider the following short exact sequence of \( \ell^1(S) \) modules and module homomorphisms:
\[
0 \to \ell^1(0) \to \ell^1(S) \to \ell^1(S) / \ell^1(0) \to 0
\]
where \( i \) is the inclusion map and \( \tau \) is the canonical map. Define the map \( \psi: \ell^1(S) \to \ell^1(0) \) by \( \psi(f) = f * \delta_0 \), \( f \in \ell^1(S) \). For every \( f = \sum_{s \in S} f(s) \delta_s \), \( h = \sum_{s \in S} h(s) \delta_s \in \ell^1(S) \) we have,
\[
\psi(f * h) = f * \left( \sum_{s \in S} h(s) \delta_s * \delta_0 \right) = f * \left( \delta_0 * \sum_{s \in S} h(s) \delta_s \right) = \psi(f) * h.
\]
Similarly $\psi(f*h) = f \star \psi(h)$. So $\psi$ is a bounded $\ell^1(S)$ module homomorphism and since $\psi$ is a left inverse for $i$, then the sequence splits. Now as in the argument of [17, Theorem IV.1.18] the map

$$\phi: \ell^1(S) \to (\ell^1(S)/\ell^1(0)) \oplus \ell^1(0), \phi(f) = (\pi(f), \psi(f))$$

is an $\ell^1(S)$-module isomorphism. Moreover for every $f, h \in \ell^1(S)$ we have:

$$\phi(f*h) = (f \star h, (f \star \delta_0) \star (h \star \delta_0)) = \phi(f) \phi(h).$$

Thus $\phi$ is a bounded algebra isomorphism and hence it is a topological algebra isomorphism, by the open mapping Theorem.

(iii) Define the map $\theta: \ell^1(S) \to \ell^1(S/T)/\ell^1(0)$ by $\theta(f) = h + \ell^1(0)$ where

$$h(\bar{s}) = \begin{cases} f(s) & \text{if } \bar{s} \neq 0 \\ 0 & \text{if } \bar{s} = 0. \end{cases}$$

One can show that $\theta$ is an onto algebra homomorphism with kernel $\ell^1(T)$. So the map

$$\rho: \ell^1(S)/\ell^1(T) \to \ell^1(S/T)/\ell^1(0)$$

$$\rho(f + \ell^1(T)) = \theta(f)$$

is an algebra isomorphism. Since $\|\rho(f + \ell^1(T))\| = \|\theta(f)\| = \|f + \ell^1(T)\|$, then $\rho$ is an isometrical algebra isomorphism. □

The following lemma is more or less known [8, p. 145]. The finiteness of index sets is based on the observation that every nonzero entry $(P_k)_{i,j}$ of the sandwich matrix $P_k$ produces a nonzero idempotent $((P_k)_{i,j}^{-1}, i, j)$ in the $k$th principal factor and consequently in $S$. Also note that for regular semigroups there is no distinction between principal and composition series, since regular semigroups are semisimple. By now the reader is seeing the use of invertibility condition (iii) of the convention after Definition 3.1.

**Lemma 5.2.** If $S$ is a regular semigroup with $E_S$ finite, then $S$ has a principal series $S = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_m \supseteq S_{m+1} = \phi$. Moreover for every $k = 1, \ldots, m - 1$ there are natural numbers $n_k, l_k$, a group $G_k$ and a regular $l_k \times n_k$ matrix $P_k$ on $G_k^\omega$ such that $S_k/S_{k+1} = \hat{A}(G_k, P_k)$. Also $S_m = \hat{A}(G_m, P_m)$ for some $l_m \times n_m$ matrix $P_m$ over a group $G_m$.

**Proposition 5.3.** For a semigroup $S$, $\ell^1(S)$ is amenable if and only if $S$ has a principal series $S = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_m \supseteq S_{m+1} = \phi$ and $\ell^1(T)$ is amenable for every principal factor $T$ of $S$. 

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Proof. Suppose $\ell^1(S)$ is amenable. [8, Theorem 2] and Lemma 5.2 imply that $S$ has a principal series $S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \emptyset$ [19, Proposition 5.1] and [6, Theorem 3.7] together with Lemma 5.1 imply that $\ell^1(S_{k+1}/S_k)/\ell^1(0)$, $k = 1, \ldots, m$, is amenable. Since the factors $S_k/S_{k+1}$, $k = 1, \ldots, m$ are the principal factors of $S$ [5, Theorem 2.40], then $\ell^1(T)$ is amenable for every principal factor $T$ of $S$.

Conversely suppose $S$ has a principal series $S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \emptyset$ and $\ell^1(T)$ is amenable for every principal factor $T$ of $S$. By [5, Theorem 2.40], inductive application of Lemma 5.1, and [19, Proposition 5.1] one can conclude that $\ell^1(S)$ is amenable. \[\]

In order to provide a counter example to the conjecture of Duncan and Paterson [8, p. 145] we need the following Lemma.

Lemma 5.4. Suppose $S$ is a semigroup which admits a principal series $S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \emptyset$ such that $S_m$ is an inverse semigroup and every Rees factor semigroup $S_k/S_{k+1}$, $k = 1, \ldots, m$ is completely (0)-simple with an invertible sandwich matrix. Then $S$ is an inverse semigroup.

Proof. First suppose $T = \mathcal{A}(G, P)$ where $P$ is invertible. In particular $|I| = |J|$. Let $Q$ be the identity $I \times I$ matrix on $G^0$, i.e., $Q_{ii} = 1$ and $Q_{ij} = 0$ for $i \neq j$. [5, Corollary 3.12] with the identity map on $G^0$ as $\alpha$, $U = P$, and $V = Q$ implies that $\mathcal{A}(G, P) \cong \mathcal{A}(G, Q)$. Now by [5, Theorem 3.9], $T$ is an inverse semigroup.

In the rest of the proof we use the fact that if $x, y \in S_k \setminus S_{k+1}$ are such that $\bar{x} = \bar{y}$ where $\bar{x}$ and $\bar{y}$ are the equivalence classes of $x$ and $y$ in $S_k/S_{k+1}$, respectively, then $x = y$. By the previous part, $S_k/S_{k+1}$ is an inverse semigroup, $k = 1, \ldots, m$. Let $x \in S_k \setminus S_{k+1}$ for some $k \leq m$ and $\bar{x}^*$ be the inverse of $\bar{x}$ in $S_k/S_{k+1}$. Then $\bar{x}^* \neq 0$, because otherwise we would have $\bar{x} = \bar{x}x^*\bar{x} = 0$. Thus $xx^*x = x$ and $x^*xx^* = x^*$ and hence $x$ has an inverse $x^*$ in $S$. If $y$ is another inverse for $x$, then $\bar{x}^*\bar{y} = \bar{x}^*$ and $\bar{y}^*\bar{x} = \bar{y}^*$. So $\bar{y} = \bar{x}^*$ which implies $y = x^*$ as $\bar{y}^* = x^* \neq 0$. Therefore $x^*$ is the unique inverse of $x$ in $S \setminus S_{k+1}$. If $k > 1$, then by a similar argument we can see that $x$ has no inverse in $S \setminus S_k$. Therefore $S$ is an inverse semigroup. \[\]

Remark. 5.5. With the notations of Lemma 5.2, Duncan and Paterson [8, p. 145] have conjectured that if $\ell^1(S)$ is amenable, then $G_k$ is amenable for every $k \leq m$, $S_m = G_m$, and $P_k$ is invertible for every $k = 1, \ldots, m - 1$. If this conjecture is true, then amenability of $\ell^1(S)$ implies that $S$ is an inverse semigroup, which is not the case as we will see in Example 6.3.

Now we characterize those regular Rees matrix semigroups $S$ for which $\ell^1(S)$ is amenable. In order to do this, we need the following proposition. The algebraic version of the first part for finite semigroups without any analytical assumption is due to Munn [28, 3.8]. Also in [8, p. 145] the
authors have used it for the case of finite index sets, without proof. However the general case can be proved directly by showing that the following map is an isometrical algebra isomorphism, where $S = \mathcal{A}(G, P)$:

$$
\phi: \mathcal{L}(\ell^1(G), P) \to \ell^1(S)/\ell^1(0)
$$

$$
\phi([f_{ij}]) = \left( \sum_{i \in I} \sum_{j \in J} f_{ij}(g) \delta_{i, j, g, P} \right) + \ell^1(0).
$$

This method is different from Munn’s proof. Indeed Munn’s proof is based on the finite dimensionality of $\ell^1(S)$.

**Proposition 5.6.** Suppose $S = \mathcal{A}(G, P)$. Identify the zero of $G^0$ with the zero of the $\ell^1$-Munn algebra $\mathcal{L}(\ell^1(G), P)$, where $P$ is considered as a matrix over $\ell^1(G)$. Then $\ell^1(S)/\ell^1(0)$ is isometrically algebra isomorphic to $\mathcal{L}(\ell^1(G), P)$.

A similar statement holds for $S = \mathcal{A}(G, P)$.

Now we can give an alternate proof of the main result of [8] for a special case.

**Corollary 5.7.** [8, Theorem 2] Suppose $S$ is a regular semigroup that admits a principal series. Then amenability of $\ell^1(S)$ implies that $E_S$ is finite.

**Proof.** We use the notations of Lemma 5.2. By Propositions 5.3 and 5.6, $\mathcal{L}(\ell^1(G_k), P_k)$ is amenable and hence by Theorem 4.1 the index sets of every principal factor are finite. Now using regularity of sandwich matrices of the principal factors and the fact that every nonzero entry of the sandwich matrices corresponds to a nonzero idempotent, we conclude that $E_S$ is finite.

**Remarks 5.8.**

(i) Existence of a principal series is a crucial assumption in the second part of Proposition 5.3 and cannot be dropped, as we will see in Example 6.2.

(ii) Let $S$ be a semigroup such that $\ell^1(S)$ is amenable and $S = S_1 \supset S_2 \supset \cdots \supset S_m \supset \phi$ be a principal series for $S$. Duncan and Paterson [8, p. 141] have asked which of the ideals $S_k$ have an amenable semigroup algebra. Proposition 5.3 answers this question.

(iii) Proposition 5.3 reduces the amenability problem to the completely (0-)simple case which was done only for inverse semigroups in [7, Theorem 7].

(iv) Existence of an identity in $\ell^1(S)$ does not imply that $S$ has an identity even in the special case of regular Rees matrix semigroups. One can check that a regular Rees matrix semigroup $S = \mathcal{A}(G, P)$ ($S = \mathcal{A}(G, P)$) has an identity if and only if $|I| = |J| = 1$. 

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Theorem 5.9. With the notations of Lemma 5.2, the following conditions are equivalent:

(i) \( \ell^1(S) \) is amenable,

(ii) \( \mathcal{L}M(\ell^1(G_k), P_k) \) has an identity and \( \ell^1(G_k) \) is amenable, \( k = 1, \ldots, m \).

Proof. (i) \( \Rightarrow \) (ii) Since by Proposition 5.3 \( \ell^1(S_k/(S_k\backslash S_{k+1})) \) is algebra isomorphic with \( \mathcal{L}M(\ell^1(G_k), P_k) \) which is amenable (Lemma 3.7 and Theorem 4.1), then \( \ell^1(T) \) is amenable for every principal factor \( T \) of \( S \). Therefore \( \ell^1(S) \) is amenable, by Proposition 5.3.

(ii) \( \Rightarrow \) (i) Similar to the previous part, the result follows from Propositions 5.3, 5.6, Theorem 4.1, and Lemma 3.7.

The following lemma for the case of inverse semigroups has been proved in [7, Theorem 8], by a technical method. Here we present an elementary proof for the general case.

Lemma 5.10 Under the assumptions and notations of Lemma 5.2, the maximal subgroups of \( S \) (up to isomorphism) are precisely \( G_k, k = 1, \ldots, m \), and the trivial group \( \{0\} \) (in the case that \( P_k \) has a zero entry for some \( k \leq m - 1 \)).

Proof. Let \( G \) be a maximal subgroup of \( S \). Suppose \( G \cap (S_k\backslash S_{k+1}) \neq \phi \) for some \( k \leq m \). If \( G \cap S_{k+1} \neq \phi \), then choose \( x \in G \cap (S_k\backslash S_{k+1}) \) and \( y \in G \cap S_{k+1} \). We have \( x = (xy^{-1})y \in S_{k+1} \) which is a contradiction. Therefore \( G \subseteq (S_k\backslash S_{k+1}) \) for some \( k \leq m \). For simplicity in the rest of the proof we denote \( P_k \) by \( P \).

Case I. Suppose \( G \subseteq S_m \) and let \( (f, i, j) \in S_m = \mathcal{H}(G_m, P) \) be the identity of \( G \). Then for every \((h, r, t) \in G \) we have \( r = i \) and \( t = j \). Now define the map \( \psi : G \to G_m \) by \( \psi((h, i, j)) = hp \). Then \( \psi \) is a group homomorphism. Moreover if \( (h, i, j) = 0 \), then \( (h, i, j) = (p, i, j) = (f, i, j) \). Therefore \( \psi \) is a group homomorphism. On the other hand the set \( H = \{(h, i, j) \mid h \in G_m \} \subseteq G \), under the product of \( S \), forms a subgroup of \( S_m \) which is isomorphic to \( G_m \) by a similar argument. Indeed this shows that \( S \) has at least one subgroup isomorphic to \( G_m \). Now since \( G \) is maximal, then \( H = G \). Therefore \( G \) is isomorphic to \( G_m \).

Case II. Suppose \( G \subseteq (S_k\backslash S_{k+1}) \) for some \( k < m \). \( G \approx G \) is the image of \( G \) in \( S_k/\mathcal{S}_{k+1} \) and \( (f, i, j) \) is the identity of \( G \). Then as in the previous case, we can show that all of the elements of \( G \) are of the form \((h, i, j) \). Now if \( P \neq 0 \), then similar to the first case we can show that \( G \approx G_k \) and \( S \) has at least one maximal subgroup of this kind. If \( P = 0 \) then every subgroup \( H \) of \( S \) in \( S_k\backslash S_{k+1} \) that \( H \approx H \subseteq \{(h, i, j) \mid h \in G_k \} \), is the trivial group.
{0}, since the product of $H$ is zero. Moreover any zero entry of $P$ gives the trivial group $\{0\}$ as a maximal subgroup as we showed.

It is well known that $\ell^1(S)$ is semisimple for every inverse semigroup \cite[Theorem 2]{36}. So the following theorem is the general form of \cite[Theorem 8]{7}.

**Theorem 5.11.** Let $S$ be a regular semigroup with a finite number of idempotents. The following conditions are equivalent:

(i) $\ell^1(S)$ is amenable,

(ii) every maximal subgroup of $S$ is amenable and $\ell^1(T)$ is semisimple for every principal factor $T$ of $S$.

In particular if $\ell^1(S)$ is amenable, then it is semisimple.

**Proof.** Throughout the proof we use the notations of Lemma 5.2.

(i) $\Rightarrow$ (ii) Since $\mathcal{LM}(\ell^1(G_k), P_k)$ is amenable by Propositions 5.3 and 5.6, $k = 1, \ldots, m$, then amenability of maximal subgroups of $S$ follows from Lemma 5.10. On the other hand, Theorem 4.3 implies that $\mathcal{LM}(\ell^1(G_k), P_k)$ is semisimple, $k = 1, \ldots, m$. Thus by \cite[Theorem 4.3.2(c)]{29} and Lemma 5.1(iii), $\text{Rad}(\ell^1(S_k/S_{k+1})) = \text{Rad}(\ell^1(0)) = 0$, $k = 1, \ldots, m - 1$. Therefore $\ell^1(T)$ is semisimple for every principal factor $T$ of $S$.

(ii) $\Rightarrow$ (i) By Lemma 5.10 $\ell^1(G_k)$ is amenable, $k = 1, \ldots, m$. Also Lemma 5.1(ii) implies that $\ell^1(S_k/S_{k+1})/\ell^1(0)$ is an ideal of $\ell^1(S_k/S_{k+1})$, $k = 1, \ldots, m - 1$. So, by \cite[Theorem 4.3.2(a)]{29},

$$\text{Rad}(\ell^1(S_k/S_{k+1})/\ell^1(0)) = (\ell^1(S_k/S_{k+1})/\ell^1(0)) \cap \text{Rad}(\ell^1(S_k/S_{k+1})) = 0.$$ 

Therefore $\mathcal{LM}(\ell^1(G_k), P_k)$ is semisimple. Similarly semisimplicity of $\ell^1(S_m)$ implies that $\mathcal{LM}(\ell^1(G_m), P_m)$ is semisimple. So by Lemma 3.7 $\mathcal{LM}(\ell^1(G_k), P_k)$ is unital, $k = 1, \ldots, m$. Now, Theorems 4.3 and 5.9 imply that $\ell^1(S)$ is amenable.

For the last statement it is enough to show that if $\ell^1(T)$ is semisimple for every principal factor $T$ of $S$, then $\ell^1(S)$ is semisimple. As in the previous part we can check that $\ell^1(S_k)/\ell^1(S_{k+1})$ is semisimple, $k = 1, \ldots, m - 1$. Now \cite[Theorem 4.3.2(c)]{29} implies that $\text{Rad}(\ell^1(S_{m-1})) = \text{Rad}(\ell^1(S_m)) = 0$. By doing this process repeatedly we conclude that $\text{Rad}(\ell^1(S)) = 0$.

**Conjecture.** In Theorem 5.11 the condition of semisimplicity of $\ell^1(T)$ for every principal factor $T$ of $S$ can be replaced with the weaker condition, semisimplicity of $\ell^1(S)$. 


6. EXAMPLES AND COUNTEREXAMPLES

Example 6.1. Let $\mathcal{A}$ be an arbitrary unital Banach algebra, $I = J$ be finite of order $n$ and

$$P = \begin{bmatrix}
1 & & & 0 \\
-1 & \ddots & & \\
& \ddots & \ddots & \\
0 & & \ddots & -1 \\
& & & 1
\end{bmatrix}.$$

Let $\mathcal{D}$ be the set of all lower triangular elements $[a_{ij}]$ of $\mathcal{M}(\mathcal{A}, P)$ such that in every column all entries on and under the main diagonal are equal, i.e., there is a subset $\{a_1, ..., a_n\}$ of $\mathcal{A}$ such that $a_{ij} = a_j$ if $i \geq j$ and $a_{ij} = 0$ otherwise. Clearly $\mathcal{D}$ is a closed subalgebra of $\mathcal{M}(\mathcal{A}, P)$ and one can check that multiplication of $\mathcal{D}$ which is inherited from $\mathcal{M}(\mathcal{A}, P)$ coincides with Shur (i.e., componentwise) multiplication. In the case that $\mathcal{A} = L^1(\mu)$ for a probability measure $\mu$, $\mathcal{D}$ is a special algebra of triangular arrays of random variables, of interest to experimental scientists as the data of previous experiments are usually reused in the new (bigger) samples.

Example 6.2. Let $S = \mathbb{N}$ be the natural numbers with the binary operation $m \cdot n = \min(m, n)$. Then,

(i) $\ell^1(S)$ is not amenable as $E_S$ is infinite.

(ii) $\ell^1(T)$ is amenable for every principal factor $T$ of $S$, since for every $a \in S$, $J(a)/J(a) \simeq \{0, 1\}$ with the usual product which has amenable semigroup algebra.

(iii) $S$ has no principal series. Indeed $\{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \cdots$ is a chain of ideals of $S$.

Example 6.3. Let $G = \{1\}$ be the trivial group,

$$P = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}$$

and $S = \mathcal{M}_0(G, P)$.

Then,
\(\ell^1(S)\) is amenable. Indeed Proposition 4.11 implies that 
\(\ell^1(S)/\ell^1(0)\) is isometrically algebra isomorphic to \(L^M(\ell^1(G), \hat{P})\) which is amenable.

(ii) \(S\) is not an inverse semigroup, since \((1, 1, 1)\) has two different “inverses”: itself and \((1, 1, 2)\).

**Example 6.4.** Let \(m\) and \(n\) be natural numbers, \(G_1 = \{1\}, G_2, \ldots, G_{m-1}\) be groups such that all except \(G_2\) are amenable, and \(G_m = \{0\}\). Let \(T_i = \mathcal{M}(G_i, P_i)\) \(1, \ldots, m - 1\) where \(P_i\) is the identity \(n_i \times n_i\) matrix over \(G_i\) and \(T_m = \{0\}\). We identify the zeros of all of these semigroups with \(0 \in G_m\). Suppose \(S\) is the direct union of \(T_1, \ldots, T_m\), i.e., \(S = T_1 \cup \cdots \cup T_m\) with the product \(a \cdot b = ab\) if \(a, b \in T_i\) for some \(i \leq m\) and \(a \cdot b = 0\) otherwise. Then,

(i) \(S\) is regular as every \(T_i\) is an inverse semigroup [5, Theorem 3.9] and \(E_S = E_{T_1} \cup E_{T_2} \cup \cdots \cup E_{T_m}\); and

(ii) \(S\) is amenable as \(S\) has a zero element, but \(\ell^1(S)\) is not amenable, since the maximal subgroup \(G_2\) of \(S\) is not amenable.

**Example 6.5.** Suppose \(S\) is the semigroup of Example 6.4. Then,

(i) \(\ell^1(T)\) is semisimple for every principal factor \(T\) of \(S\), since \(T_1, \ldots, T_m\) are the principal factors of \(S\) and each \(T_i\) is an inverse semigroup [36, Theorem 2];

(ii) at least one maximal subgroup of \(S\) is not amenable.

**ACKNOWLEDGMENTS**

This work is a part of the author's Ph.D. thesis under the supervision of Professor A. T.-M. Lau. The author is very grateful to him for his supervision, encouragement, and invaluable suggestions. This work was supported by a scholarship from MCHE, Iran and also partially by the Department of Mathematical Sciences, University of Alberta. The author thanks both of these agencies for their kind support.

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