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On Minimally Subspace-Comparable F-Spaces

L. DREWNOWSKI

Institute of Mathematics, A. Mickiewicz University, Poznań, Poland

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An F-space (complete metric linear space) is minimal if it admits no strictly weaker linear Hausdorff topology, and quotient (q-) minimal if all of its Hausdorff quotients are minimal. Two F-spaces are (q-minimally) minimally s-comparable if they have no isomorphic (q-) nonminimal closed linear subspaces. It is proved that if X, Y are q-minimally (resp., minimally) s-comparable F-subspaces of an arbitrary topological linear space E (resp., with $X \cap Y = \{0\}$), then $X + Y$ is an F-subspace of E. Also, if $X_1, \ldots, X_n$ are F-subspaces of E, then $X_1 + \cdots + X_n$ is an F-subspace of E, provided that X_i/F and X_j/G are minimally s-comparable whenever F and G are closed minimal subspaces of X_i and X_j, $i \neq j$. These are analogs of some results due to Gurariĭ and Rosenthal concerning totally incomparable Banach spaces.

1. INTRODUCTION

When the topological linear spaces X and Y are called *comparable*, we mean that one of them is isomorphic (that is, linearly homeomorphic) to a subspace of the other; otherwise they are *incomparable*. It may even happen, and does indeed, that X and Y have no isomorphic closed subspaces of infinite dimension; then X and Y are called *totally incomparable*, after Rosenthal [15].

Gurariĭ [8] and Rosenthal [15] proved that if X, Y are totally incomparable Banach subspaces of a Banach (or normed) space E, then their algebraic sum $X + Y$ is a Banach subspace of E. This result was then improved by Diestel and Lohman [4] in that they allowed the underlying space E to be merely locally convex (so it was not even known a priori whether $X + Y$ was metrizable in its relative topology).

The purpose of this paper is to give further results in this direction. Originally, we proved a Gurariĭ–Rosenthal-type theorem for totally incomparable F-subspaces (i.e., metrizable complete subspaces) X and Y of an arbitrary topological linear space E. We observed then, however, that the core of what we needed to prove the assertion that $X + Y$ is an F-subspace, was that X and Y contain no isomorphic closed subspaces which are nonminimal, in a certain sense. This led us to *minimal* and *quotient-minimal* spaces, and to the corresponding

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notions of *minimally subspace-comparable* and *quotient-minimally subspace-comparable* spaces (see Section 3). Consequently, in Section 4, which contains our main results, we prove two extensions of the Gurariĭ–Rosenthal theorem, one for minimally subspace-comparable F-subspaces X and Y with $X \cap Y = \{0\}$, and the other one for quotient-minimally subspace-comparable F-subspaces X and Y with, in general, $X \cap Y \neq \{0\}$ (Theorem 4.1).

Following Rosenthal, consider these two sets of properties of the F-spaces X and Y:

(I) X and Y are minimally subspace-comparable;

(II) for every topological linear space E and every continuous one-to-one linear mapping $T: X \times Y \to E$, if $T \mid X$ and $T \mid Y$ are isomorphisms (into), then T is an isomorphism (into), and

($\mathrm{I_q}$) X and Y are quotient-minimally subspace-comparable;

($\mathrm{II_q}$) for every topological linear space E and every continuous linear mapping $T: X \times Y \to E$, if $T \mid X$ and $T \mid Y$ are isomorphisms, then T is relatively open, i.e., the associated mapping $\hat{T}$ of $(X \times Y)/T^{-1}(0)$ onto $T(X \times Y)$ is an isomorphism.

The two extensions of the Gurariĭ–Rosenthal theorem given in Theorem 4.1 are equivalent to the implications (I) $\Rightarrow$ (II) and ($\mathrm{I_q}$) $\Rightarrow$ ($\mathrm{II_q}$). We show in Theorem 3.3 that the converse implications (II) $\Rightarrow$ (I) and ($\mathrm{II_q}$) $\Rightarrow$ ($\mathrm{I_q}$) are also valid, even for X and Y not necessarily supposed to be F-spaces. The arguments we use to prove this are quite simple, based only upon some very general ideas of constructing weaker linear topologies. (Thus the proof of a similar implication in [15, Theorem 2] is unnecessarily complicated.)

In the class of Banach spaces, minimal and quotient-minimal spaces are nothing else but finite dimensional spaces, so that minimal subspace-comparability equals quotient-minimal subspace-comparability which equals total incomparability. If we pass to locally convex F-spaces (i.e., Fréchet spaces), then, again, the classes of minimal and quotient-minimal spaces coincide, but this time they also include the spaces isomorphic to the space of all scalar sequences, so that minimal subspace-comparability equals quotient-minimal subspace-comparability which is implied by total incomparability.

Unfortunately, it still remains an open question of whether or not there are any nonlocally convex minimal F-spaces, and neither could we prove nor disprove that minimal equals quotient-minimal in general.

In order to extend Theorem 4.1 to finite families of subspaces intersecting, possibly, not only at 0, we had to introduce a still stronger version of subspace-incomparability: *minimal* sq-*comparability*. It requires that not only the spaces themselves but also their quotients by minimal subspaces be minimally subspace-comparable. Again, if we restrict our attention to Fréchet spaces, this property coincides with minimal subspace-comparability, but we do not know what the status is in the nonlocally convex setting.

The paper depends crucially upon the fundamental results of Kalton [9] on the existence of *basic* sequences in F-spaces. Actually, we use only strongly regular (Markushevich) M-*basic* sequences; therefore we asked whether the apparently weaker assertions concerning the existence of M-basic sequences could be proved directly, without deducing them from the much stronger results about basic sequences, as is the case in [9; 11]. This turned out to be possible but, not without a surprise, we found also that it was enough to extend our proof by one short argument to arrive at Kalton's result on extracting basic sequences. Nevertheless, we think that our approach has some interesting points, and therefore we decided to present it in Section 2. This section also contains a "stability" result for strongly regular minimal (in particular M-basic) sequences, Theorem 2.8; it is of key importance for our arguments in Section 4.

Throughout, we consider only *Hausdorff* topological linear spaces (tls), over the fixed field $\mathbf{K}$ of either real or complex scalars. A tls E with topology τ is denoted (E, τ) or E^τ, or simply E. Usually, we shall deal in our theorems with a basic (underlying) tls, say $E = E^\tau$, on which some other linear topologies, not necessarily Hausdorff, may be considered as well, but we assume the convention that all topological concepts involving E (closures of sets, neighborhoods, induced topologies, and so forth), whenever used with respect to an unspecified topology on E, shall be referred to the basic topology τ.

A *subspace* of a tls E^τ means a linear subspace F of E endowed with the topology $\tau \mid F$ induced by τ on F; the resulting tls is written (F, τ) or F^τ.

An F-*space* is a metrizable complete tls, nonlocally convex in general. When we say that F is an F-*subspace* of a tls E^τ, it means that F^τ is an F-space. Terms like "normed space," "Banach space," etc., are used in this paper in their topological-linear-space-category sense, so that for instance "a normed space" means a locally bounded locally convex tls. F-*norms* and F-*seminorms* are used in the sense defined in [9].

2. Minimal and Basic Sequences

First we recall some definitions and facts, essentially found in [9, 11, and 17]. A sequence (x_n) in a tls E is *minimal* (*semibasic* in [9; 11]), if $x_n \notin \overline{\operatorname{lin}}(x_k)_{k>n}$ for each $n \in N$ or, equivalently, if there exists a (necessarily unique) sequence (f_n) of continuous linear functionals on the subspace $X = \overline{\operatorname{lin}}(x_n)$ of E, called *biorthogonal* to (x_n), such that $f_i(x_j) = \delta_{ij}$. If, moreover, (f_n) is total on X, i.e., $f_n(x) = 0$ for all $n \in N$, implies $x = 0$, then (x_n) is M-*basic*. Suppose (x_n) is minimal (M-basic); then it is called *strongly regular* if the sequence (f_n) is equicontinuous on X, in other words if $\| x \| = \sup | f_n(x)|$ is a continuous seminorm (norm) on X. We say (x_n) is *basic*, if it is linearly independent and the linear operators S_k: $\operatorname{lin}(x_n) \to \operatorname{lin}(x_n)$ defined by $S_k(x) = \sum_{i=1}^{k} t_i x_i$ for $x = \sum_{i=1}^{\infty} t_i x_i$, (t_i) finitely nonzero, are equicontinuous, $k \in N$. Equivalently:

if it is minimal and the operators $S_k\colon X \to X$ defined by $S_k(x) = \sum_{i=1}^{\infty} f_i(x)x_i$ are equicontinuous. Then $\lim S_k(x) = x$ for every $x \in X$. If the space E is metrizable and $\tilde{E}$ denotes its completion, then (x_n) is basic iff every x in the closed linear span of (x_n) in $\tilde{E}$ has a unique expansion in the form $x = \sum_{i=1}^{\infty} t_i x_i$. We say (x_n) is ***regular***, if there is a neighborhood U of 0 in E such that $x_n \notin U$ for all $n \in N$.

Evidently, regular basic $\Rightarrow$ strongly regular M-basic $\Rightarrow$ regular M-basic, and all the properties introduced above, except for being basic, are preserved while strengthening the topology of E. Note also that while the properties of being basic, minimal, or strongly regular are in fact properties of the sequence (x_n) with respect to the subspace $\operatorname{lin}(x_n)$ in its relative topology, to be an M-basic sequence essentially depends on the space E in which $\operatorname{lin}(x_n)$ is embedded.

PROPOSITION 2.1. *A sequence (x_n) is strongly regular* M-*basic in the* tls E^τ *iff there exists on E a linear Hausdorff topology $\rho \leqslant \tau$ such that (x_n) is a regular basic sequence in the* tls E^ρ.

Proof. "Only if": We have already noticed that $\| x \| = \sup | f_n(x)|$ is a norm defining on $X = \overline{\operatorname{lin}}(x_n)$ a topology $\nu \leqslant \tau \mid X$. Then the topology $\rho = \tau \wedge \nu$ (see Section 3) is as required.

If ρ is a linear topology on a tls E^τ, we say τ is *ρ-polar* if τ has a base of ρ-closed neighborhoods of 0. Then ρ is necessarily Hausdorff, and τ is also $(\tau \wedge \rho)$-polar, so that we may often additionally assume $\rho \leqslant \tau$.

THEOREM 2.2. *Suppose E^τ is a* tls *and ρ a metrizable linear topology on E such that τ is ρ-polar. Then every sequence $(x_n) \subset E$ such that $x_n \to 0(\rho)$ but $x_n \not\to 0(\tau)$ has a subsequence (z_n) which is a strongly regular* M-*basic sequence in E^τ.*

Proof. Let $| \cdot |$ be an F-norm defining ρ; we may assume that $\sum | x_n | < \infty$. Since $x_n \not\to 0(\tau)$, the set $\{x_n\colon n \in N\}$ is not τ-precompact. Hence there exists a ρ-closed balanced τ-neighborhood U of 0 such that for every τ-compact set K in E we have

$$\{x_n\colon n \in N\} \not\subset K + U.$$

In particular this holds for all the sets $K = \{\sum_{i \in e} t_i x_i\colon | t_i | \leqslant 1\}$, where e is any finite subset of N. It follows easily that there exists a subsequence (y_n) of (x_n) such that for every $n \in N$:

$$\sum_{i=1}^{n} t_i y_i \notin U \qquad \text{provided that} \quad | t_i | \leqslant 1 \qquad \text{and} \quad | t_n | = 1. \tag{1}$$

We claim that for each n there is an $m > n$ such that

$$\sum_{i=1}^{n} t_i y_i + \sum_{i=m}^{r} t_i y_i \notin U \quad \text{whenever } |t_i| \leqslant 1 \text{ and } |t_n| = 1; m \leqslant r. \quad (2)$$

Suppose this is false. Then we can find a sequence $n + 1 = m_1 < m_2 < \cdots$, a sequence $(t_i)_{i>n}$ with $|t_i| \leqslant 1$, and for each $k \in N$ a finite sequence $(t_{ki})_{i=1}^{n}$ with $|t_{ki}| \leqslant 1$ and $|t_{kn}| = 1$, such that

$$w_k = \sum_{i=1}^{n} t_{ki} y_i + \sum_{i \in e_k} t_i y_i \in U \qquad \text{for all} \quad k \in N,$$

where $e_k = \{m_k, m_k + 1, \ldots, m_{k+1} - 1\}$. We may assume that each sequence $(t_{ki})_{k \in N}$ converges, say to t_i; clearly $|t_i| \leqslant 1$ and $|t_n| = 1$; $i = 1, \ldots, n$. Since

$$\left| \sum_{i \in e_k} t_i y_i \right| \leqslant \sum_{i \in e_k} |y_i| \to 0 \qquad \text{as} \quad k \to \infty,$$

the sequence (w_k) is ρ-convergent to $w = \sum_{i=1}^{n} t_i y_i$. But U is ρ-closed, so $w \in U$, which contradicts (1).

Applying (2) we easily define a subsequence (z_n) of (y_n) such that if

$$u = \sum_{n=1}^{m} t_n z_n \in Z = \mathrm{lin}(z_n) \qquad \text{and} \qquad \max |t_n| = 1, \qquad \text{then } u \notin U.$$

It follows immediately that (z_n) is linearly independent and the coefficient functionals $g_n(u) = t_n$ are τ-equicontinuous on Z; hence their continuous extensions f_n to $F = \bar{Z}$ are τ-equicontinuous as well. It remains to check that (f_n) is total on F. Suppose $u \in F$ and $f_n(u) = 0$ for all n. Let (u_α) be a net in Z which τ-converges to u. Then $\sup_n |f_n(u_\alpha)| \to 0$, and this together with $\sum |z_n| < \infty$ implies that $u_\alpha \to 0(\rho)$. Hence $u = 0$.

Remarks 2.3. (a) The above proof combines some ideas found in [5, 6, and 10], and is particularly close to the proof of Theorem 3.3 in [10]; see also [7, 13, and the references given therein] for some variations of the method.

The relevance to results on operators acting from ℓ_∞ or its subspaces is not incidental. In fact, let $c_{00} = c_{00}(N)$ be the linear space of finitely nonzero scalar sequences (t_i) endowed with the usual norm $\|(t_i)\| = \sup |t_i|$. Then, since $\sum |x_i| < \infty$, the linear mapping $T: (t_i) \mapsto \sum t_i x_i$ of c_{00} into E^ρ is continuous (and exhaustive, in the sense of [10]). Our proof is precisely a construction of an infinite subset M of N such that the restriction S of T to the subspace $c_{00}(M) = \{(t_i) \in c_{00}: t_i = 0 \text{ for all } i \notin M\}$ of c_{00} is one-to-one and S^{-1}: $(T(c_{00}(M)), \tau) \to c_{00}(M)$ is continuous (but not an isomorphism, in general).

(b) The same proof shows that, given any finite linearly independent sequence $u_1, \ldots, u_m$ in E, the subsequence (z_n) can be selected so that $u_1, \ldots, u_m$,

$z_1, z_2, \ldots,$ be a strongly regular M-basic sequence in E^τ. A similar remark applies to the forthcoming Theorem 2.4.

The main assertion, (d), of the theorem below is due to Kalton [9, Theorem 3.2]. We have exposed the implicit reduction to a "metrizable" situation in his original proof, mainly because of its relevance to some questions discussed in Section 3. This reduction, combined with Theorem 2.2, enables us to give a somewhat less technical proof of his result, although we follow his ideas quite closely.

THEOREM 2.4. *Suppose E^τ is a metrizable* tls *and α, ρ are linear topologies on E such that $\alpha \leqslant \rho \leqslant \tau$, α is semimetrizable and τ is ρ-polar. Let $(x_a)_{a \in A}$ be a net in E such that $x_a \to 0(\rho)$ but $x_a \not\to 0(\tau)$. Then there exists a strictly increasing sequence $(a(n))$ in A and a semimetrizable linear topology γ on E such that, denoting $z_n = x_{a(n)}$ and $F = \overline{\operatorname{lin}}(z_n)$, we have:*

(a) $\alpha \leqslant \gamma \leqslant \rho$;

(b) *$z_n \to 0(\gamma)$ but $z_n \not\to 0(\tau)$; hence $\gamma \mid F < \tau \mid F$;*

(c) *$\tau \mid F$ is $(\gamma \mid F)$-polar, so $\gamma \mid F$ is Hausdorff;*

(d) *(z_n) is a regular basic sequence in E^τ.*

Proof. Since τ is metrizable and ρ-polar, there exists an upward directed family P of ρ-continuous F-seminorms on E such that the formula

$$\| x \| = \sup_{p \in P} p(x)$$

defines an F-norm determining the topology τ (Kalton [9, Proposition 2.1]). We may assume that each $p \in P$ determines a topology stronger than α. Since $x_a \not\to 0(\tau)$, we may also suppose $\inf \| x_a \| > 0$.

It is an easy consequence of Dini's theorem and the fact that finite dimensional spaces admit a unique linear Hausdorff topology, that

$$\text{If } K \text{ is a finite-dimensional compact subset of } E, \text{ then for each } r > 0 \text{ there exists } p \in P \text{ such that } p(x) \leqslant \| x \| \leqslant p(x) + r \quad \text{for all } x \in K. \tag{3}$$

Using (3) and the assumption $p(x_a) \to 0$ for every $p \in P$, we easily find a strictly increasing sequence $(a(n))$ in A and a sequence $p_1 \leqslant p_2 \leqslant \cdots$, in P such that

$$p_n(x) \leqslant \| x \| \leqslant p_n(x) + 2^{-n-2} \qquad \text{for all } x \in nK_n, \tag{4}$$

where $K_n = \{\sum_{i=1}^n t_i z_i : |t_i| \leqslant 1\}$, $z_i = x_{a(i)}$, and

$$p_n(x_a) \leqslant 2^{-n-2} \qquad \text{for} \quad a \geqslant a(n+1); \quad n \in N. \tag{5}$$

Then for $x \in \operatorname{lin}(z_n)$ we have $\| x \| = \sup p_n(x)$ and $p_k(z_n) \to 0$ as $n \to \infty$, for all $k \in N$. Hence the semimetrizable topology γ determined by (p_n) satisfies (a), (b), and (c) on $\text{lin}(z_n)$, hence also on its τ-closure F.

In view of Theorem 2.2, by passing to subsequences of (z_n) and (p_n), we may assume that (z_n) is a strongly regular M-basic sequence in F^τ, with (4) and (5) continuing to hold. Let U be a neighborhood of 0 in F^τ such that $x \in U \Rightarrow \sup |f_n(x)| \leqslant 1$, where (f_n) is the sequence biorthogonal to (z_n). We are going to show that (z_n) is basic in F^τ, i.e., that the partial sum operators $S_n(x) = \sum_{i=1}^n f_i(x) z_i$ on $\text{lin}(z_n)$ are equicontinuous; evidently, each S_n *is* continuous.

Let $x \in U$. Then

$$\begin{aligned} \| S_n(x)\| &\leqslant p_n(S_n(x)) + 2^{-n-2} \\ &\leqslant p_n(S_{n+1}(x)) + p_n(f_{n+1}(x) z_{n+1}) + 2^{-n-2} \\ &\leqslant \| S_{n+1}(x)\| + 2^{-n-1}, \end{aligned}$$

and hence, after a finite number of iterations, we obtain

$$\| S_n(x)\| \leqslant \| x \| + 2^{-n}.$$

This implies quickly that S_n are τ-equicontinuous.

COROLLARY 2.5. *Suppose E^ν is a metrizable* tls *and ρ is a linear Hausdorff topology on E such that $\rho < \nu$. Then if ν is ρ-polar or if E^ν is an* F*-space, then there exists a separable closed subspace F of E and a metrizable linear topology γ on F such that $\gamma \leqslant \rho \mid F$ and $\gamma < \nu \mid F$.*

Proof. The case when ν is ρ-polar is clear. Suppose E^ν is an F-space. Then the linear topology τ on E, whose base at 0 is formed by ρ-closures of ν-neighborhoods of 0, is metrizable, ρ-polar, and satisfies $\rho < \tau \leqslant \nu$, and so we may apply Theorem 2.4.

Remarks 2.6. (a) Even if E^ν is a Banach space and $\rho < \nu$ is a locally convex topology on E such that ν is ρ-polar, it need not be true that $\rho \mid F < \nu \mid F$ on a separable subspace F of E^ν. *Counterexample*: Let T be an uncountable set. Then let E be the space of all bounded scalar-valued functions on T with at most countable supports, ν is the topology of uniform convergence on T, and ρ is the topology of uniform convergence on countable subsets of T.

(b) If, in Theorem 2.4, we omit the assumption $\rho \leqslant \tau$, then we can only insist that $z_n \to 0(\alpha)$ but not that $\alpha \leqslant \gamma$ (cf. [9, Corollary 3.3].)

In Section 4 we shall need only the following corollary to Theorem 2.4 (or to Theorem 2.2, in case ρ is metrizable); cf. [9, Corollary 3.4; 11, Theorem 2.1].

COROLLARY 2.7. *Let E^ν be an* F-*space. Then, for every linear Hausdorff topology ρ on E with $\rho < \nu$ and every ρ-continuous* F-*seminorm $|\cdot|$, there exists a strongly regular M-basic sequence (z_n) in E^ν such that $\sum_{n=1}^{\infty} |z_n| < \infty$.*

Now we are going to establish a "stability" result for strongly regular minimal (in particular M-basic) sequences; it will play a crucial role in our arguments in Section 4. For analogous results in Banach spaces see Singer [17, Chap. I, Sect. 10].

Let (x_n), (y_n) be sequences in tl spaces X and Y, respectively. Then, following Singer ([17, Chap. I, Sect. 8]), we say that (x_n) *dominates strictly* (y_n), $(x_n) \gg (y_n)$, if there exists a continuous linear mapping $T\colon \operatorname{lin}(x_n) \to \operatorname{lin}(y_n)$ such that $T(x_n) = y_n$ for all $n \in N$. If $(x_n) \gg (y_n)$ and $(y_n) \gg (x_n)$, then we write $(x_n) \approx (y_n)$ and say that these two sequences are *strictly equivalent*; in this case the mapping T is an isomorphism of $\operatorname{lin}(x_n)$ onto $\operatorname{lin}(y_n)$. Clearly, if X and Y are complete, then T extends uniquely to a continuous linear mapping (resp., an isomorphism) of $\overline{\operatorname{lin}}(x_n)$ into (resp., onto) $\overline{\operatorname{lin}}(y_n)$.

THEOREM 2.8. *Suppose X and Y are subspaces of a* tls *E^τ and $|\cdot|$ is an* F-*seminorm defining a topology ρ on E such that*

$$\rho \leqslant \tau, \qquad \rho \mid X = \tau \mid X, \qquad \rho \mid Y = \tau \mid Y. \tag{$*$}$$

Suppose further that $(x_n) \subset X$ and $(y_n) \subset Y$ are sequences such that

$$\sum_{n=1}^{\infty} |x_n - y_n| < \infty. \tag{$**$}$$

Then if any of the sequences (x_n), (y_n), and $(z_n) \equiv (x_n - y_n)$ is a strongly regular minimal sequence in E, there exists an $m \in N$ such that

$$(x_n)_{n \geqslant m} \approx (y_n)_{n \geqslant m} \qquad \textit{and} \qquad (x_n)_{n \geqslant m} \gg (z_n)_{n \geqslant m}.$$

Proof. If (u_n), $(v_n) \subset E$ and (u_n) is linearly independent, then we shall call *natural* the mapping of $\operatorname{lin}(u_n)$ onto $\operatorname{lin}(v_n)$ which associates with each finite combination $\sum t_i u_i$ the corresponding combination $\sum t_i v_i$.

From $(*)$ it is clear that

if $(u_n) \subset X$ and $(v_n) \subset Y$, then any two among the relations

$$u_n \to 0(\tau), \qquad u_n - v_n \to 0(\rho), \qquad v_n \to 0(\tau) \tag{6}$$

imply the remaining one and imply $u_n - v_n \to 0(\tau)$.

Denote $X_n = \operatorname{lin}(x_k)_{k \geqslant n}$, $Y_n = \operatorname{lin}(y_k)_{k \geqslant n}$, $Z_n = \operatorname{lin}(z_k)_{k \geqslant n}$. First suppose (x_n) is a strongly regular minimal sequence in E; let (f_n) be the sequence of functionals biorthogonal to (x_n); then $\|u\| = \sup |f_n(u)|$ is a continuous norm on X_1. This together with $(**)$ easily implies continuity of the natural mapping $P\colon (X_1, \|\cdot\|) \to (Z_1, \rho)$. Hence by (6) the mappings $P\colon (X_1, \tau) \to (Z_1, \tau)$ and $Q\colon (X_1, \tau) \to (Y_1, \tau)$, where $Q(u) = u - P(u)$, are continuous as well. Thus $(x_n) \gg (z_n)$ and $(x_n) \gg (y_n)$.

Now we show that there exists an $m \in N$ and $a > 0$ such that if

$$\left| \sum_{i=m}^{n} t_i y_i \right| < a, \qquad \text{then} \quad \left\| \sum_{i=m}^{n} t_i x_i \right\| = \max | t_i | < 1 \qquad (m \leqslant n).$$

Suppose it is not so. Then we can find a sequence $1 = m_1 < m_2 < \cdots$, of indices and a sequence (t_i) of scalars such that

$$| v_k | \to 0 \qquad \text{and} \qquad \| u_k \| \geqslant 1 \qquad \text{for all } k \in N,$$

where

$$v_k = \sum_{i \in e_k} t_i y_i, \qquad u_k = \sum_{i \in e_k} t_i x_i, \qquad e_k = \{m_k, \ldots, m_{k+1} - 1\}.$$

Evidently, we may assume $\| u_k \| = 1$ for all k. Then $| t_i | \leqslant 1$ for all i and hence, by (**) and monotonicity of the F-seminorm $| \cdot |$,

$$| u_k - v_k | \leqslant \sum_{i \in e_k} | z_i | \to 0 \qquad \text{as} \quad k \to \infty.$$

Hence, by (6), $u_k \to 0(\tau)$ and so $\| u_k \| \to 0$; a contradiction.

Thus (7) holds indeed. Now (7) implies that the sequence $(y_n)_{n \geqslant m}$ is linearly independent and the natural mapping $R\colon (Y_m, \tau) \to (X_m, \|\cdot\|)$ is continuous. Then its composition $P \circ R$ with continuous $P\colon (X_1, \|\cdot\|) \to (Z_1, \rho)$ maps continuously (Y_m, τ) onto (Z_m, ρ), and $P \circ R$ is the natural mapping of Y_m onto Z_m. In virtue of (6), $R\colon (Y_m, \tau) \to (X_m, \tau)$ is continuous and, obviously, $R = (Q \mid X_m)^{-1}$. Thus $Q \mid X_m$ is an isomorphism of (X_m, τ) onto (Y_m, τ) with $Q(x_n) = y_n$ for $n \geqslant m$, and so $(x_n)_{n \geqslant m} \approx (y_n)_{n \geqslant m}$.

The proof in the case (z_n) is supposed to be a strongly regular minimal sequence in E is quite analogous. First we define the norm $\|\cdot\|$ on Z_1 by $\| w \| = \sup | f_n(w) |$, where (f_n) is the sequence biorthogonal to (z_n). Then we establish an assertion of type (7), and deduce from it that (for some $m \in N$) the natural mappings of (X_m, τ) and (Y_m, τ) onto $(Z_m, \|\cdot\|)$ are continuous. Since $\| \ \|$, as easily seen, is stronger than ρ, these mappings remain continuous when Z_m is endowed with the topology ρ. Hence, using (6), we obtain that the natural mapping of (X_m, τ) onto (Y_m, τ) is an isomorphism, and that of (X_m, τ) onto (Z_m, τ) is continuous. This completes the proof.

Remark 2.9. The result of Kalton [9, Lemma 4.3], concerning stability of basic sequences, is easily seen to be a consequence of Theorem 2.8.

3. Minimal Spaces and Minimal Subspace-Comparability

Let E^τ be a tls, F its closed subspace, and suppose we have on F a linear topology $\gamma \leqslant \tau \mid F$. Then we define $\tau \wedge \gamma$, the ***infimum*** of τ and γ, to be the strongest among the linear topologies ξ on E such that $\xi \leqslant \tau$ and $\xi \mid F \leqslant \gamma$. (This is clearly a very special case of inductive topologies, cf. [16].) If U, V range over some neighborhood bases of 0 for τ and γ, respectively, then the sets $U + V$ form a base at 0 for $\tau \wedge \gamma$. Hence $\tau \wedge \gamma$ induces γ on F, F is closed in $(E, \tau \wedge \gamma)$, and if γ is Hausdorff, then so is $\tau \wedge \gamma$.

If τ and γ are metrizable (normed) [locally convex], then so is $\tau \wedge \gamma$, and if $|\cdot|_\tau$ and $|\cdot|_\gamma$ are F-norms (norms) determining τ and γ, then the F-norm (norm) $\|\cdot\|$ given by

$$\| x \| = \inf\{| x - y |_\tau + | y |_\gamma : y \in F\}, \qquad x \in E,$$

determines $\tau \wedge \gamma$.

Given a class $\mathscr{A}$ of tl spaces, we say that a tls X^ξ is ***minimal with respect to*** $\mathscr{A}$, shortly $\mathscr{A}$***-minimal***, if every continuous one-to-one linear mapping of X onto a space in $\mathscr{A}$ is an isomorphism, i.e., if α is a linear topology on X such that $\alpha \leqslant \xi$ and $X^\alpha \in \mathscr{A}$, then $\alpha = \xi$. We also introduce a stronger (at least formally) version of minimality: we say that X is ***quotient*** $\mathscr{A}$***-minimal***, if every quotient of X by its closed subspace (i.e., Hausdorff quotient) is $\mathscr{A}$-minimal. Equivalently: if every continuous linear mapping of X onto a space in $\mathscr{A}$ is open. Evidently, every Hausdorff quotient of a quotient $\mathscr{A}$-minimal space is quotient $\mathscr{A}$-minimal.

Further, we say that the tl spaces X and Y are (only) $\mathscr{A}$***-minimally subspace-comparable*** (resp., ***quotient*** $\mathscr{A}$***-minimally subspace-comparable***) if, whenever E and F are isomorphic closed subspaces of X and Y, respectively, then E and F are $\mathscr{A}$-minimal (resp., quotient $\mathscr{A}$-minimal). When $\mathscr{A}$ is the class of all tl spaces, we speak simply on ***minimal*** (***quotient-minimal***) spaces and ***minimally*** (***quotient-minimally***) ***subspace-comparable*** spaces. In the following we abbreviate quotient-minimal(ly) to q-minimal(ly) and subspace-comparable to s-comparable.

We first collect together a few less or more known facts about minimal spaces.

Proposition 3.1. (a) *Let* $\mathscr{A}$ *denote any of the following four classes*: ***all*** tl ***spaces***, ***all locally convex*** tl ***spaces***, ***all metrizable*** tl ***spaces***, ***all normed spaces***. *Then*: *A space in* $\mathscr{A}$ *is* (*quotient-*) $\mathscr{A}$***-minimal*** ***iff every closed subspace of it is*** (*quotient-*) $\mathscr{A}$***-minimal***.

(b) *A locally convex space is* (*quotient-*) ***minimal with respect to locally convex spaces iff it is isomorphic with a product*** $\mathbf{K}^I$ ***of the scalar field*** $\mathbf{K}$.

(c) *A locally convex* F*-space is* (*quotient-*) ***minimal iff it is finite dimensional or isomorphic with*** $\omega = \mathbf{K}^N$.

(d) *A normed space is minimal with respect to normed spaces iff it is finite dimensional.*

(e) *A separable metrizable* tls *is (quotient-) minimal iff it is (quotient-) minimal with respect to metrizable* tl *spaces.*

(f) *An* F-*space is not minimal iff it contains a regular basic sequence; iff it contains a strongly regular* M-*basic sequence, iff it has an infinite dimensional closed subspace which admits a continuous norm.*

(f′) *An* F-*space is minimal iff it is minimal with respect to metrizable* tl *spaces.*

(g) *An* F-*space is not quotient-minimal iff it contains a strongly regular minimal sequence iff there exists a continuous linear mapping of a closed subspace of it onto an infinite dimensional normed space.*

Proof. (a) is easily proved by taking infima of suitable topologies. For example, suppose a closed subspace Y of a tls X^{ξ} is not q-minimal. Then, for a closed subspace Z of Y, the quotient Y/Z admits a linear Hausdorff topology η strictly weaker than the quotient topology. Since Y/Z is a closed subspace of $X/Z = (X/Z, \hat{\xi})$, $\hat{\xi} \wedge \eta$ is a linear Hausdorff topology $<\hat{\xi}$, and so X^{ξ} is not q-minimal.

(b) and (c) are well known (cf. [16, I.3.2 and Exer. 6, p. 191; see also 3]), except the fact that ω is minimal, which is a recent result due to Kalton ([9, Proposition 4.1]); see Remark 3.2 below for another proof.

Although (d) is also known (cf. [1; 2, p. 108]), we wish to give a proof of it. Suppose $(E, \|\cdot\|)$ is an infinite dimensional normed space; we must find a norm strictly weaker than $\|\cdot\|$. By (a) it will suffice to consider the case when $(E, \|\cdot\|)$ is separable; then there exists a sequence (f_n) of linear functionals such that $\|x\| = \sup |f_n(x)|$ for $x \in E$. Then the formula $|x| = \sum_{i=1}^{\infty} 2^{-i} |f_i(x)|$ defines a norm on E satisfying $|x| \leqslant \|x\|$ for every $x \in E$, and these two norms cannot be equivalent. In fact, choosing for every n an element x_n in $f_1^{-1}(0) \cap \cdots \cap f_n^{-1}(0)$ with $\|x_n\| = 2^n$, we obtain a sequence bounded in $(E, |\cdot|)$, but not in $(E, \|\cdot\|)$.

(e) follows from the fact that if γ is any weaker linear Hausdorff topology on a separable metrizable tls, then there always exists a metrizable linear topology below γ (cf. [12]).

(f) is due to Kalton and Shapiro [11, Theorem 3.2], with the last equivalence contained implicitly in their proof. Omitting the statement concerning basic sequences, (f) is a consequence of Corollary 2.7 and the assertions (a) and (d).

(f′) follows from (f) by taking infimum of suitable topologies; instead of (f), Corollary 2.5 may be used as well.

(g) is an easy consequence of (f).

Remark 3.2. Applying (f), we can show minimality of ω as follows. Suppose F is a closed subspace of ω that admits a continuous norm. Then, similarly as in the proof of [4, Proposition 3] or by a simple indirect argument, we can find m such that the natural projection $(x(n))_{n\in N} \to (x(n))_{n\geqslant m}$ of F into $\mathbf{K}^m$ is one-to-one. Hence $\dim F < \infty$, and so ω is minimal.

It is an open question, raised by Kalton in [9], whether or not there exist any minimal nonlocally convex F-spaces (cf. also [11]). To be more accurate, the present author does not know of any other minimal spaces, metrizable or not, than those listed in (c) above.

The next theorem is our main result in this section. When $\mathscr{A}$ is specified to be the class of all tls, it yields immediately the implications (II) $\Rightarrow$ (I) and $(\mathrm{II_q}) \Rightarrow (\mathrm{I_q})$ of Section 1. When $\mathscr{A}$ is the class of normed spaces, its consequence is the "if" part of Rosenthal's Theorem 2 in [15].

THEOREM 3.3. *Let $\mathscr{A}$ have the same meaning as in Proposition* 3.1(a), *let X^ξ, $Y^\eta \in \mathscr{A}$, and let π denote the topology of the product $X^\xi \times Y^\eta$.*

(a) *If X^ξ and Y^η are $\mathscr{A}$-nonminimally* s-*comparable, then there exists on $X \times Y$ a topology ζ such that $(X \times Y, \zeta) \in \mathscr{A}$ and $\zeta < \pi$, $\zeta \mid X = \xi$, $\zeta \mid Y = \eta$.*

(b) *If X^ξ and Y^η are quotient $\mathscr{A}$-nonminimally* s-*comparable, then there exists a closed subspace M of $(X \times Y, \pi)$ and a topology $\hat{\zeta}$ on $\hat{Z} = (X \times Y)/M$ such that $(\hat{Z}, \hat{\zeta}) \in \mathscr{A}$, $\hat{\zeta} < (\pi/M)$ and the restrictions $Q \mid X$ and $Q \mid Y$ of the quotient mapping $Q\colon (X \times Y, \pi) \to (\hat{Z}, \hat{\zeta})$ are isomorphisms.*

Proof. We shall consider only the case when $\mathscr{A}$ is the class of all tls. By assumption, there exist closed subspaces E in X and F in Y which are isomorphic and not minimal (respectively, q-minimal); let g be an isomorphism of F onto E. Then the graph $G = \{(g(y), y): y \in F\}$ of g is a closed subspace of $(X \times Y, \pi)$, isomorphic to both the subspaces E and F. Hence in case (a) we can find on G a linear Hausdorff topology $\gamma < \pi \mid G$, and in case (b) a linear topology γ on G such that $\gamma < \pi \mid G$ and $(\gamma/M) < ((\pi \mid G)/M)$, where $M = \overline{\{0\}}^\gamma$. Define ζ to be the topology $\pi \wedge \gamma$ on $X \times Y$; it is strictly weaker than π, and is Hausdorff in case (a). We are going to show that $\zeta \mid X = \xi$.

Given any neighborhood U of 0 in X, choose a balanced neighborhood U_1 of 0 in X with $U_1 + U_1 \subset U$. Since $g\colon F \to E$ is continuous, we find then a neighborhood V_1 of 0 in Y such that $g(V_1 \cap F) \subset U_1$. Finally, let W be any neighborhood of 0 in G^γ. Then $P = (U_1 \times V_1) + W$ is a ζ-neighborhood of $(0, 0)$ in $X \times Y$, and we claim that $P \cap (X \times \{0\}) \subset U \times \{0\}$. Let $(x, 0) \in P$; then $(x, 0) = (u_1, v_1) + (g(y), y)$, where $u_1 \in U_1$, $v_1 \in V_1$, and $(g(y), y) \in W$. Hence $v_1 = -y \in V_1 \cap F$, so that $(x, 0) = (u_1, 0) + (g(y), 0) \in U_1 \times \{0\} + U_1 \times \{0\} \subset U \times \{0\}$. This proves that ξ is weaker than $\zeta \mid X$, and thus they must be identical. Similarly, η and $\zeta \mid Y$ are identical. This completes the proof in case (a). To finish the proof in case (b), it suffices to put $\hat{\zeta} = (\zeta/M)$.

The forthcoming Corollary 3.5 shows that, for Banach spaces, a (nonmetrizable) topology ζ satisfying the assertion of Theorem 3.3(a) can always be chosen between the product topology and the associated weak topology.

PROPOSITION 3.4. *Let ζ be a linear Hausdorff topology on a Banach space E^ν such that $\zeta < \nu$. Then $\zeta \leqslant \gamma = \sup\{\zeta, \sigma(E, E')\} < \nu$, where E' is the dual space of E^ν.*

Proof. Suppose $\gamma = \nu$. Then there is a neighborhood U of 0 in E^ζ and a $\sigma(E, E')$-neighborhood $V = \{z \in E: |z_i'(z)| \leqslant 1, i = 1,..., k\}$ of 0 in E such that $U \cap V \subset B$, where B is a bounded neighborhood of 0 in E^ν. Let $F = \{z \in E: z_i'(z) = 0, i = 1,..., k\}$; F is a finite-codimensional closed subspace of E^ν and $U \cap F = (U \cap V) \cap F \subset B$. It follows that $\zeta | F = \nu | F$, so that F^ζ is a Banach space, and hence a closed finite-codimensional subspace of E^ζ. Consequently, if G denotes any algebraic complement to F in E then, in both the spaces E^ζ and E^ν, G endowed with its unique linear Hausdorff topology is also a topological complement to F (cf. [16, 1.3.5]). Hence $E^\zeta = E^\nu$, a contradiction.

From Theorem 3.3 and Proposition 3.4 we now easily derive:

COROLLARY 3.5. *If X^ξ and Y^η are Banach spaces which are not totally incomparable, then there exists a locally convex linear topology γ on $E = X \times Y$ such that $\sigma(E, E^{\nu\prime}) < \gamma < \nu = \xi \times \eta$ and $\gamma | X = \xi$, $\gamma | Y = \eta$. Moreover, if ζ is any locally convex Hausdorff topology on E satisfying conditions $\zeta < \nu$, $\zeta | X = \xi$, and $\zeta | Y = \eta$, then γ may be chosen so that $\zeta \leqslant \gamma$.*

We conclude this section with a simple result concerning quasicomplements which are not complements. Recall that closed subspaces X, Y of an F-space Z are *quasicomplements* (resp., *complements*) provided $X \cap Y = \{0\}$ and $X + Y$ is dense in Z (resp., $X + Y = Z$). Our result has its origin in Lohman's negative answer [14] to the question, suggested to him by the extension of Rosenthal's result given in [4], of whether the sum of two normed subspaces is always normed in the relative topology. (In view of Theorem 3.3 and its proof, it is clear that the answer is "no": it is enough to take as γ the weak topology of (G, π) to produce a nonmetrizable topology $\zeta = \pi \wedge \gamma$ on $X \times Y$ whose restrictions to X and Y are normed.) In his counterexample Lohman constructs a nonmetrizable nonbarreled locally convex Hausdorff topology on the sum $X + Y$ of suitably chosen closed subspaces X and Y of the Banach space $\ell_\infty(\Gamma)$ of scalar-valued bounded functions defined on an uncountable set Γ, with $X \cap Y = \{0\}$, $X + Y$ not closed in $\ell_\infty(\Gamma)$, and the constructed topology inducing the original topologies of X and Y. Thus X and Y in this counterexample are quasicomplements of the Banach space $\overline{X + Y}$.

Now, by a direct application of Proposition 3.4, we show that the possibility of such a construction is not an incidental property of quasicomplements which are not complements.

COROLLARY 3.6. *Let X, Y be quasicomplements of a Banach space Z^τ, which are not complements, and let ν denote the direct sum topology of $\tau \mid X$ and $\tau \mid Y$ on $E = X + Y$. Then $\gamma = \sup\{\tau \mid E, \sigma(E, E^{\nu'})\}$ is a nonmetrizable, nonbarreled,..., locally convex Hausdorff topology on E satisfying $\sigma(E, E^{\nu'}) < \gamma < \nu$ and $\tau \mid E < \gamma$, hence also $\gamma \mid X = \tau \mid X$ and $\gamma \mid Y = \tau \mid Y$.*

(Of course, by the Gurariĭ–Rosenthal theorem, these subspaces X and Y are not totally incomparable, but we did not need this to prove the corollary.)

4. THE SUM OF MINIMALLY AND q-MINIMALLY s-COMPARABLE F-SUBSPACES

In this section we establish our main results.

THEOREM 4.1. *Let X and Y be two* F*-subspaces of a* tls *E^τ. Then $X + Y$ is an* F*-subspace of E in either of the following two cases:*

(a) *X and Y are* q*-minimally* s*-comparable, or*

(b) *X and Y are minimally* s*-comparable and $X \cap Y = \{0\}$ (or, more generally, $X \cap Y$ is complemented in one of the spaces X, Y).*

Proof. We may assume that $X + Y = E$. Let ν denote the strongest linear topology on E such that $\nu \mid X = \tau \mid X$ and $\nu \mid Y = \tau \mid Y$; in other words, the strongest linear topology on E such that the mapping $(x, y) \to x - y$ of the F-space $X^\tau \times Y^\tau$ onto E is continuous. Clearly, $\tau \leqslant \nu$ and E^ν is an F-space.

Let $| \cdot |$ be an F-seminorm determining a topology ρ on E such that $\rho \leqslant \tau$, $\rho \mid X = \tau \mid X$, $\rho \mid Y = \tau \mid Y$; it is easily seen that such a topology ρ does exist.

We wish to prove that E^τ is an F-space or, equivalently, that $\tau = \nu$.

Suppose $\tau < \nu$. Then by Corollary 2.7 there exists a strongly regular M-basic sequence (z_n) in E^ν with $\sum | z_n | < \infty$. For each $n \in N$ choose x_n in X and y_n in Y so that $z_n = x_n - y_n$. Then, by Theorem 2.8, there is an $m \in N$ such that the sequences $(x_n)_{n \geqslant m}$ and $(y_n)_{n \geqslant m}$ are strictly equivalent and dominate strictly the sequence $(z_n)_{n \geqslant m}$ (in E^ν). Hence the subspaces $\overline{\text{lin}}(x_n)_{n \geqslant m}$ and $\overline{\text{lin}}(y_n)_{n \geqslant m}$ of X and Y are isomorphic and not q-minimal. This proves the theorem in case (a).

In case (b), where we have $X \cap Y = \{0\}$(or may easily reduce to this situation that more general condition placed in parantheses in (b)), X and Y are complements for E^ν, and therefore the natural projection of $\text{lin}(z_n)_{n \geqslant m}$ onto $\text{lin}(x_n)_{n \geqslant m}$ is continuous. It follows that $(x_n)_{n \geqslant m}$ is strictly equivalent to $(z_n)_{n \geqslant m}$, and so $\overline{\text{lin}}(x_n)_{n \geqslant m}$ is nonminimal, by Proposition 3.1(f).

THEOREM 4.2. *Let X and Y be two closed subspaces of an* F*-space E^ν, and suppose there exists on Y a metrizable linear topology $\eta < \nu \mid Y$ such that $(\nu \wedge \eta) \mid X < \nu \mid X$. Then X and Y are nonminimally* s*-comparable.*

Proof. Let $|\cdot|$ and $\|\cdot\|$ be any F-norms determining ν and η, respectively. Then $\nu \wedge \eta$ is determined by the F-norm $|||z||| = \inf\{|z - y| + \|y\| : y \in Y\}$, $z \in E$. By assumption $(\nu \wedge \eta)| X < \nu | X$, hence Corollary 2.7 gives us a strongly regular M-basic sequence (x_n) in X^ν such that $\sum |||x_n||| < \infty$. It follows that there is a sequence (y_n) in Y with $\sum(|x_n - y_n| + \|y_n\|) < \infty$, and we need only $\sum |x_n - y_n| < \infty$ to apply Theorem 2.8 and see that X and Y are non-minimally s-comparable.

Remark 4.3. Under the additional assumption $X \cap Y = \{0\}$, Theorem 4.2 would be an easy corollary to Theorem 4.1, case (b). In fact, if X and Y were only minimally s-comparable, then $Z = X + Y$ would be an F-subspace of E with X and Y as complements for Z. It would then follow that $\nu \wedge \eta = (\nu | X) \oplus \eta$, and hence $(\nu \wedge \eta)| X = \nu | X$.

COROLLARY 4.4. *Suppose E^τ, X, and Y are as in Theorem* 4.1 *and either* (a) *or* (b) *of Theorem* 4.1 *is satisfied, so that $Z = X + Y$ is an* F*-subspace of E. Then if G is a nonminimal* F*-subspace of Z, then either X and G or Y and G are nonminimally* s*-comparable.*

Proof. Since G is nonminimal, there is a metrizable linear topology η on G with $\eta < \tau | G$ (see Proposition 3.1(f′)). Suppose the topology $\alpha = (\tau | Z) \wedge \eta$ on Z coincides with τ on both the subspaces X and Y of Z. Then X and Y are F-subspaces of the tls Z^α, hence by Theorem 4.1, Z^α is an F-space. It follows that $\alpha = \tau | Z$, and so $\eta = \tau | G$, a contradiction. Thus either on X or on Y the topology α must be strictly weaker than τ, and we may apply Theorem 4.2.

We shall say that the F-spaces X and Y are *minimally* sq-*comparable*, if for any minimal closed subspaces F and G of X and Y, respectively, the quotient F-spaces X/F and Y/G are minimally s-comparable. This is clearly equivalent to: For all F-spaces X_1 and Y_1, if there exist continuous linear mappings $T_1 : X \to X_1$ and $T_2 : Y \to Y_1$ which are onto and have minimal kernels $T_1^{-1}(0)$ and $T_2^{-1}(0)$, then X_1 and Y_1 are minimally s-comparable. It is easily seen that minimally sq-comparable $\Rightarrow$ q-minimally s-comparable.

PROPOSITION 4.5. *Let T be a continuous linear mapping of an* F*-space X onto another* F*-space Y such that its kernel $F = T^{-1}(0)$ is a minimal subspace of X. Then, for every minimal closed subspace H of Y, its inverse image $T^{-1}(H)$ is a minimal subspace of X.*

Proof. Suppose it is not so. Then, with no loss of generality, we may assume that Y is minimal, while X is not. Since Y is isomorphic to X/F, we may further assume that $Y = X/F$, and T is the quotient mapping of X onto X/F. Let $\|\cdot\|_1$ be an F-norm defining topology of X. By our assumption and Proposition 3.1 (f′), there exists an F-norm $\|\cdot\|_2$ on X which is strictly weaker than $\|\cdot\|_1$. On X/F, however, the corresponding "quotient" F-norms $\|x - F\|_i^* =$

$\inf\{\| x - f \|_i : f \in F\}$ are equivalent. Let $\| x_n \|_2 \to 0$. Then $\| T(x_n)\|_1^* \to 0$, and so there exist $f_n \in F$ such that $\| x_n - f_n \|_1 \to 0$. Then $\| x_n - f_n \|_2 \to 0$, and hence $\| f_n \|_2 \to 0$. Since $\| \cdot \|_2$ and $\| \cdot \|_1$ are equivalent on F, we have $\| f_n \|_1 \to 0$. But also $\| x_n - f_n \|_1 \to 0$, and so finally $\| x_n \|_1 \to 0$. Thus $\| \cdot \|_2$ is stronger than $\| \cdot \|_1$; a contradiction.

COROLLARY 4.6. *If X is a nonminimal* F-*space and F is its closed minimal subspace, then X/F is nonminimal.*

From Proposition 4.5 we easily deduce:

PROPOSITION 4.7. *Suppose X and Y are minimally* sq-*comparable* F-*spaces, and let F and G be minimal closed subspaces of X and Y, respectively. Then the quotient spaces X/F and Y/G are minimaly* sq-*comparable.*

We are already prepared to extend Theorem 4.1 to finite families of F-subspaces.

THEOREM 4.8. *Let $X_1, \ldots, X_n$ be* F-*subspaces of a* tls *E, and suppose one of the following two conditions is satisfied* :

(a) *$X_1, \ldots, X_n$ are pairwise minimally* sq-*comparable, or*

(b) *$X_1, \ldots, X_n$ are pairwise minimally* s-*comparable and $X_{m+1} \cap \sum_{i=1}^{m} X_i = \{0\}$ for $m = 1, \ldots, n-1$.*

Then $X_1 + \cdots + X_n$ is an F-*subspace of E and, for $m = 1, \ldots, n-1$, the* F-*subspaces $X_1 + \cdots + X_m$ and $X_{m+1} + \cdots + X_n$ are minimally* sq-*comparable in case* (a), *and minimally* s-*comparable in case* (b).

Proof. We will consider case (a) only, as the proof in case (b) presents no problems.

It is clear that the proof will be easily completed after we have proved that: *If X, Y, Z are pairwise minimally* sq-*comparable* F-*subspaces of E, then $X + Y$ and Z are also minimally* sq-*comparable.*

First note that $W = X + Y$ is an F-subspace of E, by Theorem 4.1. Let F, G be minimal closed subspaces of W and Z, respectively, and consider the quotient F-spaces $\hat{W} = W/F$ and $\hat{Z} = Z/G$. Let Q be the quotient mapping of W onto $\hat{W}$. It is easy to see that $\hat{X} = Q(X)$ is isomorphic to $X/(X \cap F)$, and $\hat{Y} = Q(Y)$ to $Y/(Y \cap F)$. Hence, by Proposition 4.7, $\hat{X}$ and $\hat{Y}$ are minimally sq-comparable F-subspaces of $\hat{W}$; in particular, they are q-minimally s-comparable.

Now suppose $\hat{W}$ and $\hat{Z}$ are nonminimally s-comparable. Hence there exists a closed nonminimal subspace H of $\hat{W} = \hat{X} + \hat{Y}$, which is isomorphic to a subspace of $\hat{Z}$. By Corollary 4.4, H is nonminimally s-comparable with either $\hat{X}$ or $\hat{Y}$. Hence either $\hat{X}$ and $\hat{Z}$ or $\hat{Y}$ and $\hat{Z}$ are nonminimally s-comparable. A contradiction.

COROLLARY 4.9. *The sum X of any finite family $X_1, \ldots, X_n$ of* q*-minimal* F*-subspaces of a* tls *E is a* q*-minimal* F*-subspace of E.*

Proof. First let us note that an F-space Z is q-minimal iff, it is minimally sq-comparable with every F-space Y.

Now, given any F-space Y, we have that $X_1, \ldots, X_n$, Y are pairwise minimally sq-comparable F-subspaces of $E \times Y$. Hence, by Theorem 4.8, X is an F-space, and X and Y are minimally sq-comparable. This implies that X is q-minimal.

COROLLARY 4.10. *The product of any finite family of minimal (respectively,* q*-minimal)* F*-spaces is minimal (respectively,* q*-minimal).*

Proof. This follows immediately from Theorem 4.8 as well as from Corollary 4.6 (resp., from Corollary 4.9).

Our final result resembles the well-known fact that finite dimensional spaces have a unique linear Hausdorff topology; it is a direct consequence of Theorem 3.3 and Theorem 4.8, case (b).

COROLLARY 4.11. *Let $X_1, \ldots, X_n$ be* F*-spaces. Then the following are equivalent:*

(i) *The product topology on $X = X_1 \times \cdots \times X_n$ is the unique linear Hausdorff topology on X which induces on each factor space X_i its original topology.*

(ii) *$X_1, \ldots, X_n$ are pairwise minimally* s*-comparable.*

Remark 4.12. Some modifications of the minimal sq-comparability may be useful as well. For example, call the F-spaces X and Y q-(*totally incomparable*) if X/F and Y/G are totally incomparable, whenever F and G are finite dimensional subspaces of X and Y, respectively. Then the corresponding version of Theorem 4.8 (a) will hold for q-(totally incomparable) F-subspaces, too, under the additional assumption that $\dim(X_i \cap X_j) < \infty$ if $i \neq j$.

Remark 4.13. It is an open question whether the above results can be extended to countable families of F-spaces. For instance, suppose $(X_i)_{i \in I}$ is a family of pairwise minimally s-comparable F-spaces and J, K are disjoint countable subsets of I. Is it then true that the product spaces $\prod_{i \in J} X_i$ and $\prod_{i \in K} X_i$ are minimally s-comparable?

Postscript. After the present version of this paper had been typed, the author learned from Dr. N. J. Kalton that:

(a) *The product $\mathbf{K}^I$ of the scalar field $\mathbf{K}$ is a minimal* tls *for any I, and*

(b) *Every minimal* tls *E is necessarily complete.*

The assertion (a) follows easily from the minimality of $\mathbf{K}$. with respect to locally convex spaces (Proposition 3.1(b)) and the following simple result: *Let E^τ be a locally convex* tls *in its weak topology, and suppose $\alpha \leqslant \tau$ is a linear*

Hausdorff topology on E. Then $(E, \alpha)'$ *separates the points of E.* In fact, then every α-neighborhood of 0 contains a closed finite-codimensional subspace of E^{τ}. (This is also a consequence of Proposition 1.2 in V. Klee, Exotic topologies for linear spaces, Proc. Symposium on General Topology and its Relations to Modern Algebra, Prague 1961).

To prove (b), suppose E is different from its completion $\tilde{E}$, and pick an x in $\tilde{E}\backslash E$. Then the restriction to E of the quotient mapping Q: $\tilde{E} \to \tilde{E}/\text{lin}\,(x)$ is one-to-one, but *not* an isomorphism. Hence E is not minimal.

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