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Monotone versions of countable paracompactness

Chris Good, Lylah Haynes *

School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK

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Abstract

One possible natural monotone version of countable paracompactness, MCP, turns out to have some interesting properties. We investigate various other possible monotone versions of countable paracompactness and how they are related.

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1. Introduction

Countably paracompact spaces were introduced by Dowker [3] and Katětov [14] and their importance lies in Dowker's classic result: $X \times [0,1]$ is normal if and only if X is both normal and countably paracompact. Countable paracompactness occurs with normality in a number of other results concerning both separation and the insertion of continuous functions (see [4,7]). Moreover, it turns out that a number of set-theoretic results concerning the separation of closed discrete collections in normal spaces have direct analogues for countably paracompact spaces. For example, Burke [1] modifies Nyikos's 'provisional' solution to the normal Moore space problem by showing that countably paracompact, Moore spaces are metrizable assuming PMEA and Watson [23] shows that, assuming $V = L$, first countable, countably paracompact spaces are collectionwise Hausdorff. When normality is strengthened to monotone normality, pathology is reduced and the need for set-theory in such results is generally avoided. It turns out that the same is often true for one possible monotone version of countable paracompactness, MCP (see below for a definition), introduced in [7] and also in [22,20]. For example, every MCP Moore space is metrizable and every first countable or locally compact MCP space is collectionwise Hausdorff. Interestingly, although every monotonically normal space is collectionwise Hausdorff, if there is an MCP space which fails to be collectionwise Hausdorff, then there is a measurable cardinal, and if there are two measurable cardinals, then there is an MCP space that is not collectionwise Hausdorff [6].

Since there is a reasonable amount one can say about MCP spaces and since there is a large number of characterizations of countable paracompactness, it makes sense to ask about other monotone versions of countable

* Corresponding author.
E-mail addresses: c.good@bham.ac.uk (C. Good), haynesl@maths.bham.ac.uk (L. Haynes).

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paracompactness. In this paper we look at some of these possible properties. Our notation is standard as used in [4,15], where any undefined terms may be found. All spaces are T_1 and regular.

The characterizations of countable paracompactness fall, broadly, into four categories relating to covering properties, perfect normality (see conditions (2)–(4) of Theorem 1), interpolation theorems (conditions (5)–(7)), and normality in products (8)–(10). It seems harder to make sensible, useful and widely satisfied definitions of monotone covering properties and so we do not consider countable paracompactness as a covering property in this paper; neither do we consider $\aleph_0$-expandability.

Recall that a subset D of X is a regular G_δ if, for each $n \in \omega$, there are open sets U_n containing D such that $D = \bigcap_{n\in\omega} \overline{U}_n$. We say that a space is δ-normal if and only if every pair of disjoint closed sets, one of which is a regular G_δ, can be separated by disjoint open sets. (In [2] δ-normality refers to a distinct property.)

Theorem 1. *For T_3 spaces the following are equivalent*:

(1) *X is countably paracompact.*
(2) *For every decreasing sequence $(D_n)_{n\in\omega}$ of closed sets satisfying $\bigcap_{n\in\omega} D_n = \emptyset$ there exists a sequence $(U_n)_{n\in\omega}$ of open sets such that $D_n \subseteq U_n$ for $n \in \omega$ and $\bigcap_{n\in\omega} \overline{U}_n = \emptyset$ (Ishikawa [13]).*
(3) *For every countable increasing open cover $\{U_n\colon n \in \omega\}$ there is an open refinement $\{V_n\colon n \in \omega\}$ such that $\overline{V}_n \subseteq U_n$ for all $n \in \omega$.*
(4) *For every decreasing sequence $(D_n)_{n\in\omega}$ of closed, nowhere dense sets satisfying $\bigcap_{n\in\omega} D_n = \emptyset$ there exists a sequence $(U_n)_{n\in\omega}$ of open sets such that $D_n \subseteq U_n$ for $n \in \omega$ and $\bigcap_{n\in\omega} \overline{U}_n = \emptyset$ (Hardy and Juhasz [10]).*
(5) *For every increasing dense open cover there is a countable closed refinement of sets whose interiors cover X.*
(6) *For every lower semicontinuous, real-valued function $g > 0$ on X there is an upper semicontinuous h such that $0 < h < g$ (Dowker [3]).*
(7) *For every lower semicontinuous, real-valued function $g > 0$ on X there exist real-valued functions u and v where u is upper semicontinuous and v is lower semicontinuous and such that $0 < u \leqslant v \leqslant g$ (Mack [17]).*
(8) *For every locally bounded, real-valued function g on X there exists a locally bounded, lower semicontinuous function h such that $|g| \leqslant h$ (Mack [16]).*
(9) *$X \times [0,1]$ is δ-normal (Mack [17]).*
(10) *If C is a closed subset of $X \times [0,1]$ and D is a closed subset of $[0,1]$ such that $C \cap (X \times D) = \emptyset$, then there are disjoint open sets separating C and $X \times D$ (Tamano [21]).*

One possible natural monotone version of countable paracompactness, introduced independently in [7,22,20], is MCP (that the definition stated below is equivalent to the original is shown in [9]).

Definition 2. A space X is MCM if and only if there is an operator U assigning to each $n \in \omega$ and each closed set D an open set $U(n, D)$ such that

(1) $D \subseteq U(n, D)$,
(2) if $E \subseteq D$, then $U(n, E) \subseteq U(n, D)$ and
(3) if $(D_n)_{n\in\omega}$ is a decreasing sequence of closed sets with empty intersection, then $\bigcap_{n\in\omega} U(n, D_n) = \emptyset$.

X is MCP if, in addition, U satisfies

(3′) if $(D_n)_{n\in\omega}$ is a decreasing sequence of closed sets with empty intersection, then $\bigcap_{n\in\omega} \overline{U(n, D_n)} = \emptyset$.

MCM (monotonically countably metacompact) spaces are precisely the β-spaces (see [7]) and MCP is closely related to the wN property and stratifiability, which can be characterized (see [9]) by conditions (1), (2) and

(3″) if $(D_n)_{n\in\omega}$ is a decreasing sequence of closed sets, then
$\bigcap_{n\in\omega} \overline{U(n, D_n)} = \bigcap_{n\in\omega} D_n$.

It turns out that many of the natural monotone versions of the properties listed in Theorem 1 are equivalent to MCP.

Theorem 3. *For a T_3 space, the following are equivalent*:

(1) *X is MCP.*
(2) *There is an operator V assigning to each decreasing sequence $(D_j)_{j\in\omega}$ of closed sets with empty intersection, a sequence of open sets $(V(n,(D_j)_{j\in\omega}))_{n\in\omega}$ such that*: *$D_n \subseteq V(n,(D_j)_{j\in\omega})$ for each $n\in\omega$; if $D_n \subseteq E_n$ for each $n\in\omega$, then $V(n,(D_j)_{j\in\omega}) \subseteq V(n,(E_j)_{j\in\omega})$; $\bigcap_{n\in\omega}\overline{V(n,(D_j)_{j\in\omega})} = \emptyset$.*
(3) *Suppose that $\mathbb{H}$ is any partially-ordered set and F is any map from $\omega\times\mathbb{H}$ to the closed subsets of X such that both $F(\cdot,h)$ and $F(n,\cdot)$ are order-reversing, and $\bigcap_{n\in\omega}F(n,h)=\emptyset$ for all $h\in\mathbb{H}$. There is a map G from $\omega\times\mathbb{H}$ to the open subsets of X such that $F(n,h)\subseteq G(n,h)$, for each $n\in\omega$ and $h\in\mathbb{H}$, both $G(\cdot,h)$ and $G(n,\cdot)$ are order-reversing, and $\bigcap_{n\in\omega}\overline{G(n,h)}=\emptyset$ for all $h\in\mathbb{H}$.*
(4) *There is an operator W assigning to each $n\in\omega$ and each open set U an open set $W(n,U)$ such that*: *$\overline{W(n,U)}\subseteq U$; if $U\subseteq U'$, then $W(n,U)\subseteq W(n,U')$; and if $(U_i)_{i\in\omega}$ is an increasing open cover of X, then $(W(n,U_n)^\circ)_{n\in\omega}$ is a refinement of $(U_i)_{i\in\omega}$.*
(5) *There is an operator φ assigning to each lower semicontinuous, real-valued function $0<g$ on X, an upper semicontinuous $\varphi(g)$ such that*: *$0<\varphi(g)<g$; and $\varphi(g)\leqslant\varphi(g')$ whenever $g\leqslant g'$.*
(6) *There is an operator χ assigning to each lower semicontinuous, real-valued function $0<g$ on X and each $i=0,1$, real-valued functions $\chi(g,0)$ and $\chi(g,1)$ such that*: *$\chi(g,0)$ is lower semicontinuous, $\chi(g,1)$ is upper semicontinuous*; *$0<\chi(g,0)\leqslant\chi(g,1)\leqslant g$ (equivalently $0<\chi(g,0)<\chi(g,1)<g$); and $\chi(g,i)\leqslant\chi(g',i)$, $i=0,1$, whenever $g\leqslant g'$.*
(7) *There is an operator ψ assigning to each locally bounded, real-valued function g on X, a locally bounded, lower semicontinuous, real-valued $\psi(g)$ such that*: *$|g|\leqslant\psi(g)$ (equivalently $|g|<\psi(g)$); and $\psi(g)\leqslant\psi(g')$ whenever $|g|\leqslant|g'|$.*
(8) *There is an operator π assigning to each locally bounded, real-valued function g on X and each $i=0,1$, locally bounded, real-valued functions $\pi(g,0)$ and $\pi(g,1)$ such that*; *$\pi(g,1)$ is lower semicontinuous, $\pi(g,0)$ is upper semicontinuous*; *$\pi(g,0)\leqslant g\leqslant\pi(g,1)$ (equivalently $\pi(g,0)<g<\pi(g,1)$); and $\pi(g,i)\leqslant\pi(g',i)$, $i=0,1$, whenever $g\leqslant g'$.*
(9) *$X\times[0,1]$ is MCP.*
(10) *There is an operator H assigning to each pair (C,D), where C is closed in $X\times[0,1]$ and D is closed in $[0,1]$ such that $C\cap(X\times D)=\emptyset$, an open set $H(C,D)$ such that*:
 (a) *$C\subseteq H(C,D)\subseteq\overline{H(C,D)}\subseteq X\times([0,1]\setminus D)$*;
 (b) *if $C\subseteq C'$ and $D'\subseteq D$, then $H(C,D)\subseteq H(C',D')$.*

Proof. The equivalences of (1), (2), (3), (5), (9), and (7 for $\leqslant$) are proved in [7,9]. That (4) is equivalent to (1) follows easily by de Morgan's Laws. Clearly (6, 7, 8 for $<$) imply (6, 7, 8 for $\leqslant$).

(6) $\to$ (7 for $<$): Suppose $g:X\to\mathbb{R}$ is locally bounded. Define $h=1/(|g|+1)^*$ (where f^* denotes the upper limit of f). Then h is lower semicontinuous and strictly positive. By (6) there exists a real-valued upper semicontinuous function $\chi(h,1)$ such that $0<\chi(h,1)\leqslant h$. Let $\psi(g)=1/\chi(h,1)$. Then $\psi(g)$ is lower semicontinuous and $|g|<(|g|+1)^*\leqslant\psi(g)$. It is routine to show that $\psi(g)\leqslant\psi(g')$ whenever $|g|\leqslant|g'|$.

(7) $\to$ (8 for $<$): If g is a locally bounded function, then $g^+=\max\{g,0\}$, $g^-=\min\{g,0\}$ and $-g^-$ are locally bounded and $g^-\leqslant g\leqslant g^+$. By (7), there are locally bounded, lower semicontinuous functions $\psi(g^+)$ and $\psi(-g^-)$ such that $|g^+|=g^+<\psi(g^+)$ and $|g^-|=-g^-<\psi(-g^-)$. But then $-\psi(-g^-)<g^-\leqslant g\leqslant g^+<\psi(g^+)$, $\psi(g^+)$ is locally bounded and lower semicontinuous, and $-\psi(-g^-)$ is locally bounded and upper semicontinuous. Moreover, if $g\leqslant g'$, then $g^+\leqslant(g')^+$, so that $\psi(g^+)\leqslant\psi((g')^+)$, and $0\leqslant-(g')^-\leqslant-g^-$, so that $-\psi(-g^-)\leqslant-\psi(-(g')^-)$.

(8) $\to$ (7): Suppose $g:X\to\mathbb{R}$ is locally bounded. Define $h=(|g|+1)^*$. Then h is upper semicontinuous and strictly positive and is therefore locally bounded. By (8), there exists a locally bounded, lower semicontinuous function $\pi(h,1)$ such that $h\leqslant\pi(h,1)$. Let $\psi(g)=\pi(h,1)$, then $|g|<h\leqslant\psi(g)$. If $|g|\leqslant|g'|$, then $h\leqslant h'$ and so $\psi(g)\leqslant\psi(g')$.

(10) $\to$ (6 for $<$): Suppose that $0<g$, where g is a lower semicontinuous function. We follow the argument used in the proof of Theorem 1.6 of [8]. $B_g=\{(x,r)\colon r\geqslant g(x)\}$ and $X\times\{0\}$ are disjoint closed subsets of $X\times[0,1]$, so

by (10), there exists an open set V_g such that $X \times \{0\} \subseteq V_g \subseteq \overline{V}_g \subseteq (X \times [0,1]) \smallsetminus B_g$ with the property that $V_g \subseteq V_{g'}$ whenever $g \leqslant g'$. Let

$$u_g(x) = \sup\bigl\{r\colon\ (x,s) \in \overline{V}_g \text{ for all } s < r\bigr\}$$
$$l_g(x) = \sup\bigl\{r\colon\ (x,s) \in V_g \text{ for all } s < r\bigr\}.$$

As in [8], $u_g < g$, $u_g \leqslant u_{g'}$, whenever $g \leqslant g'$, and u_g is upper semicontinuous. Similarly $0 < l_g \leqslant u_g$ and $l_g \leqslant l_{g'}$, whenever $g \leqslant g'$. Monotonicity is also clear.

It remains to prove that l_g is lower semicontinuous, to which end we show that $l_g^{-1}(r,\infty)$ is open for any $r \in \mathbb{R}$. Suppose, then that $x \in l_g^{-1}(r,\infty)$ so that $r < l_g(x)$. Choose some $r < t < l_g(x)$, then for each $s \in [0,t]$, $(x,s) \in V_g$. For each such s, choose an open neighbourhood W_s of x and an $\varepsilon_s > 0$ such that $W_s \times (s - \varepsilon_s, s + \varepsilon_s) \subseteq V_g$. By compactness, there are finitely many $s_0, \dots, s_n$ such that $\bigcup_{i \leqslant n}(W_{s_i} \times (s_i - \varepsilon_{s_i}, s_i + \varepsilon_{s_i}))$ covers $\{x\} \times [0,t]$. Hence $(x,r) \in (\bigcap_{i \leqslant n} W_{s_i}) \times (0,t)$, so for all $y \in \bigcap_{i \leqslant n} W_{s_i}$, $l_g(y) > r$ as required.

(1) $\to$ (10): Finally, let U be an MCP operator on X such that $U(n+1,E) \subseteq U(n,E)$ for each closed E and $U(n,\emptyset) = \emptyset$. Suppose that D is any closed subset of $[0,1]$ and C is any closed subset of $X \times [0,1]$ disjoint from $X \times D$.

Let $V(n,D) = \{s \in [0,1]\colon\ (\exists d \in D)\ |s - d| < 1/n\}$ and let $C(n,D)$ be the projection onto X of $C \cap \overline{X \times V(n,D)}$. Then $(V(n,D))_{n\in\omega}$ is a decreasing sequence of open sets containing D such that $D = \bigcap_{n\in\omega} \overline{V(n,D)}$ and $(C(n,D))_{n\in\omega}$ is a decreasing sequence of closed subsets of X with empty intersection. Furthermore, $\bigcap_{n\in\omega} \overline{U(n,C(n,D))} = \emptyset$ and $U(m+1,C(n,D)) \subseteq U(m,C(n,D))$ for each $m \in \omega$.

Now, for $r \in [0,1]$, let $\varepsilon_{r,D} = \inf\{|r-d|\colon\ d \in D\}/2$ and, if $r \notin D$, let $n_{r,D}$ be the least natural number such that $1/n_{r,D} < \varepsilon_{r,D}$. Let C_r be the closed set $\{x \in X\colon\ (x,r) \in C \cap (X \times \{r\})\}$. Then $C_r \subseteq U(n_{r,D}, C_r)$ and $U(n_{r,D}, C_r) \subseteq U(n_{r,D}, C'_r)$, whenever $C_r \subseteq C'_r$.

Define $H(C,D) = \bigcup_{r\in[0,1]} U(n_{r,D}, C_r) \times (r - \varepsilon_{r,D}, r + \varepsilon_{r,D})$. Clearly $H(C,D)$ is open and contains C. Suppose $C \subseteq C'$ and $D' \subseteq D$. For any $r \notin D$, $n_{r,D'} \leqslant n_{r,D}$. Hence, by monotonicity, $U(n_{r,D}, C_r) \subseteq U(n_{r,D'}, C'_r)$, from which it follows that $H(C,D) \subseteq H(C',D')$.

It remains to show that $\overline{H(C,D)}$ and $X \times D$ are disjoint. To this end, let $(x,d) \in X \times D$. From above, there is some $n_x \in \omega$ such that x is not in $\overline{U(n_x, C(n_x,D))}$. Let W be an open neighbourhood of x disjoint from $U(n_x, C(n_x,D))$. For any $r \in V(n_x,D)$, $n_{r,D} > 2n_x$ and $C_r \subseteq C(n_x,D)$. Hence $U(n_{r,D}, C_r) \subseteq U(n_{r,D}, C(n_x,D)) \subseteq U(n_x, C(n_x,D)))$ and W is disjoint from $U(n_{r,D}, C_r)$ for any $r \in V(n_x,D)$.

Now, if $r \notin V(n_x,D)$, then $J_d = (d - \frac{1}{2n_x}, d + \frac{1}{2n_x})$ and $(r - \varepsilon_{r,D}, r + \varepsilon_{r,D})$ are disjoint, so that

$$(W \times J_d) \cap \bigl(U(n_{r,D}, C_r) \times (r - \varepsilon_{r,D}, r + \varepsilon_{r,D})\bigr) = \emptyset$$

for any $r \in [0,1]$. Thus $W \times J_d$ is an open neighbourhood of (x,d), disjoint from $U(C,D)$. $\square$

We note that essentially the same proof shows that conditions (9) and (10) hold for any compact metric space in place of $[0,1]$.

The natural monotone version of δ-normality seems to be the following.

Definition 4. A space X is *monotonically δ-normal* (mδn) iff there is an operator H assigning to each pair of disjoint closed sets C and D, at least one of which is a regular G_δ-set, an open set $H(C,D)$ such that

(1) $C \subseteq H(C,D) \subseteq \overline{H(C,D)} \subseteq X \setminus D$;
(2) if $C \subseteq C'$ and $D' \subseteq D$, then $H(C,D) \subseteq H(C',D')$.

Replacing $H(C,D)$ with $H(C,D) \smallsetminus \overline{H(D,C)}$, if necessary, one may assume that $H(C,D) \cap H(D,C) = \emptyset$.

We discuss mδn in more detail in [5], for the present we restrict our attention to the relationship between MCP and mδn.

Theorem 5. *If $X \times [0,1]$ is mδn, then X is both mδn and MCP. Moreover, if $X \times [0,1]$ is mδn, then $X \times [0,1]$ is MCP.*

Proof. If C and D are disjoint closed subsets of X, one of which is a regular G_δ, then $C \times \{0\}$ and $D \times \{0\}$ are disjoint closed subsets of $X \times [0,1]$, one of which is a regular G_δ. Furthermore, if E is a closed subset of $[0,1]$, then $X \times E$ is a regular G_δ-subset of $X \times [0,1]$. It follows that, if $X \times [0,1]$ is mδn, then X is mδn and satisfies condition (10) of Theorem 3. The last statement of the theorem follows by condition (9) of Theorem 3. □

Example 6. There exists an MCP space that is not mδn, hence its product with $[0,1]$ is not mδn.

Proof. Let $\mathcal{A}$ be a maximal almost disjoint family of subsets of ω, then Mrówka's Ψ [19] is the locally compact, locally countable, zero-dimensional, non-metrizable Moore space, $\Psi = \omega \cup \mathcal{A}$, in which each $n \in \omega$ is isolated and basic (clopen) neighbourhoods of $a \in \mathcal{A}$ take the form $\{a\} \cup (a \cap (k,\omega))$ for some $k \in \omega$.

Let X be the one-point compactification of Ψ, so that X is MCP (see [7]). We claim that X is not mδn. Observe that if C and D are disjoint closed subsets of X, at least one contains at most a finite subset of $\mathcal{A}$. It is easy to show that any closed subset of X containing at most finitely many points of $\mathcal{A}$ is a regular G_δ-subset of X. It follows that, if X were mδn, it would be monotonically normal, and hence Ψ would be monotonically normal, which it is not (it is a non-metrizable Moore space).

By Theorem 5, the product $X \times [0,1]$ is not mδn. □

Both the Sorgenfrey and Michael lines are monotonically normal, but not MCP [7]. Hence we have the following.

Example 7. There exists an mδn, in fact a monotonically normal, space X such that X is not MCP and $X \times [0,1]$ is not mδn.

Example 8. There is a space X that is both mδn and MCP such that $X \times [0,1]$ is not mδn.

Proof. The Alexandroff duplicate of the unit interval $[0,1]$ is compact, therefore MCP, and monotonically normal (see Theorem 24 in [7]) but not stratifiable (as it is not perfect). However, it is first countable and regular, so by a result in [5], if $X \times [0,1]$ were mδn, it would be monotonically normal and therefore X would be stratifiable. □

The appropriate monotonization of Theorem 1(4) is clearly the following.

Definition 9. A space X is nMCM (nowhere dense MCM) if and only if there is an operator U assigning to each $n \in \omega$ and each nowhere dense closed set D an open set $U(n,D)$ such that

(1) $D \subseteq U(n,D)$,
(2) if $E \subseteq D$, then $U(n,E) \subseteq U(n,D)$ and
(3) if $(D_i)_{i\in\omega}$ is a decreasing sequence of nowhere dense closed sets with empty intersection, then
$$\bigcap_{n\in\omega} U(n,D_n) = \emptyset.$$

X is nMCP if, in addition, U satisfies

(3′) if $(D_i)_{i\in\omega}$ is a decreasing sequence of nowhere dense closed sets with empty intersection, then
$$\bigcap_{n\in\omega} \overline{U(n,D_n)} = \emptyset.$$

It turns out that nMCM is equivalent to MCM, but that the situation for nMCP is not clear.

Let X be a space and, for each $x \in X$ and $n \in \omega$ let $g(n,x)$ be an open set containing x. We say that X is a β-space, or wN-space, or q-space (see [11,12,18]), if there exists a function g assigning to each $x \in X$ and $n \in \omega$, an open set $g(n,x)$ satisfying

(β) if $x \in g(n,y_n)$ for all n, then the sequence (y_n) has a cluster point;

(wN) if $g(n,x) \cap g(n,y_n) \neq \emptyset$ for all n, then the sequence (y_n) has a cluster point;
(q) if $y_n \in g(n,x)$ for all n, then the sequence (y_n) has a cluster point.

Theorem 10.

(1) *X is nMCM if and only if it is MCM if and only if it is a β-space.*
(2) *Suppose that X is a q-space. X is nMCP if and only if it is MCP if and only it is a wN space. In particular every first countable or locally (countably) compact nMCP space is MCP.*

Proof. For (1) recall that β-spaces are precisely MCM spaces [7]. Clearly every MCM space is nMCM, so it suffices to show that every nMCM space is a β-space. To see this define

$$g(n,x) = \begin{cases} \{x\} & \text{if } x \text{ is isolated,} \\ U(n,\{x\}) & \text{if } x \text{ is not isolated,} \end{cases}$$

so that for each $n \in \omega$, $g(n,x)$ is open and contains x. Suppose that $(y_n)_{n\in\omega}$ is a sequence of distinct points without a cluster point. There are two cases: either y_n is isolated for infinitely many $n \in \omega$, or there is some $k \in \omega$ such that y_j is not isolated for all $j \geqslant k$. In the first case, for some n and m, $g(n,y_n) \cap g(m,y_m) = \emptyset$ and so $\bigcap_{n\in\omega} g(n,y_n) = \emptyset$. In the second case, define $E_j = \{y_n \colon n \geqslant j\}$ if $j \geqslant k$ and $E_j = \{y_n \colon n \geqslant k\}$ otherwise. Then each E_j is closed and contains no isolated points. Clearly $(E_j)_{j\in\omega}$ is a decreasing sequence of closed nowhere dense sets with empty intersection and, by monotonicity of U, $g(n,y_n) \subseteq U(n,E_n)$ for each $n \in \omega$. Hence $\bigcap_{n\in\omega} g(n,y_n) = \emptyset$. It follows that X is a β-space.

The proof of (2) is similar. Suppose that X is a q-space. It was shown in [7] that X is MCP iff X is a wN-space. So clearly if X is a wN-space it is nMCP. Suppose that U is an nMCP operator for X such that $U(n+1,D) \subseteq U(n,D)$. Define g as in (1) and for each $x \in X$ and $n \in \omega$, choose $h(n,x)$ satisfying condition (q) above. Let $G(n,x) = g(n,x) \cap h(n,x)$. We claim that G satisfies the conditions for (wN).

Clearly $G(n,x)$ is an open set containing x for each x and n. So assume that $x_n \in G(n,x) \cap G(n,y_n)$ for each n. If y_n is isolated, then $y_n \in h(n,x)$. Hence, if infinitely many y_n are isolated, then by (q), they cluster. So suppose that there is some $k \in \omega$ such that y_j is not isolated for all $j \geqslant k$. Since X is a q-space, the sequence $(x_n)_{n\in\omega}$ has a cluster point z. Assume for a contradiction that the sequence $(y_n)_{n\in\omega}$ has no cluster points. Define E_j as in (1). Then $(E_j)_{n\in\omega}$ is a decreasing sequence of closed nowhere dense sets in X with empty intersection. By monotonicity of U, $x_n \in G(n,y_n) \subseteq U(n,E_n)$ for each n. Moreover $U(m,E_m) \subseteq U(n,E_n)$ for all $m \geqslant n$ and so, for each n, $x_m \in U(n,E_n)$ for all $m \geqslant n$. It follows that $z \in \overline{U(n,E_n)}$ for each n, but this is a contradiction as $\bigcap_{n\in\omega} \overline{U(n,E_n)} = \emptyset$. □

Similar arguments show that many results about MCP spaces hold for nMCP spaces, for example: every nMCP, Moore space is metrizable; first countable nMCP spaces are collectionwise Hausdorff; if an nMCP space is not collectionwise Hausdorff, then there is a measurable cardinal. However, we have been unable to determine whether nMCP and MCP coincide.

Finally let us note that Dowker [3] proved the following theorem.

Theorem 11. *The following properties of a Hausdorff space X are equivalent*:

(1) *X is countably paracompact and normal.*
(2) *X is countably metacompact and normal.*
(3) *If $g : X \to \mathbb{R}$ is lower semicontinuous, $h : X \to \mathbb{R}$ is upper semicontinuous, and $h < g$, then there exists a continuous function $f : X \to \mathbb{R}$ such that $h < f < g$.*
(4) *$X \times [0,1]$ is normal.*

The monotone versions of these conditions are not equivalent [7,8]. A space is monotonically normal and MCM iff it is monotonically normal and MCP. On the other hand, a space is stratifiable iff whenever $g : X \to \mathbb{R}$ is lower semicontinuous and $h : X \to \mathbb{R}$ is upper semicontinuous and $h < g$, then there exists a continuous function $f(g,h)$ such that $h < f(g,h) < g$ such that $f(g,h) \leqslant f(g',h')$ whenever $g \leqslant g'$ and $h \leqslant h'$.

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