Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 342 (2008) 1440–1454
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

An improved local well-posedness result for the one-dimensional Zakharov system

Hartmut Pecher

Fachbereich Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gaußstr. 20, D-42097 Wuppertal, Germany

Received 4 September 2007
Available online 24 January 2008
Submitted by M. Nakao

Abstract

The 1D Cauchy problem for the Zakharov system is shown to be locally well-posed for low regularity Schrödinger data $u_0 \in \widehat{H^{k,p}}$ and wave data $(n_0, n_1) \in \widehat{H^{l,p}} \times \widehat{H^{l-1,p}}$ under certain assumptions on the parameters k, l and $1 < p \leqslant 2$, where $\|u_0\|_{\widehat{H^{k,p}}} := \|\langle\xi\rangle^k \widehat{u_0}\|_{L^{p'}}$, generalizing the results for $p = 2$ by Ginibre, Tsutsumi and Velo. Especially we are able to improve the results from the scaling point of view, and also allow suitable $k < 0$, $l < -1/2$, i.e. data $u_0 \notin L^2$ and $(n_0, n_1) \notin H^{-1/2} \times H^{-3/2}$, which was excluded in the case $p = 2$.

Keywords: Zakharov system; Well-posedness; Fourier restriction norm method

1. Introduction and main results

Consider the $(1 + 1)$-dimensional Cauchy problem for the Zakharov system

$$iu_t + u_{xx} = nu, \tag{1}$$

$$n_{tt} - n_{xx} = \left(|u|^2\right)_{xx}, \tag{2}$$

$$u(0) = u_0, \quad n(0) = n_0, \quad n_t(0) = n_1, \tag{3}$$

where u is a complex-valued und n a real-valued function defined for $(x, t) \in \mathbf{R} \times \mathbf{R}^+$.

The Zakharov system was introduced in [16] to describe Langmuir turbulence in a plasma.

The Zakharov system (1)–(3) can be transformed into a first-order system in t as follows: With

$$n_\pm := n \pm iA^{-1/2} n_t, \quad \text{i.e.} \quad n = \frac{1}{2}(n_+ + n_-), \quad 2iA^{-1/2} n_t = n_+ - n_-, \quad A := -\partial_x^2,$$

we get

E-mail address: Hartmut.Pecher@math.uni-wuppertal.de.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.01.035

$$iu_t + u_{xx} = \frac{1}{2}(n_+ + n_-)u, \tag{4}$$

$$in_{\pm t} \mp A^{1/2} n_\pm = \pm A^{1/2}\bigl(|u|^2\bigr), \tag{5}$$

$$u(0) = u_0, \quad n_\pm(0) = n_0 \pm i A^{-1/2} n_1 =: n_{\pm 0}. \tag{6}$$

This problem was considered for data in L^2-based Sobolev spaces in detail in the last decade, especially low regularity local well-posedness results were given by Ginibre, Tsutsumi and Velo [10] for data $u_0 \in H^k$, $n_0 \in H^l$, $n_1 \in H^{l-1}$ under the following assumptions:

$$-\frac{1}{2} < k - l \leqslant 1, \qquad 2k \geqslant l + \frac{1}{2} \geqslant 0.$$

In this paper there were also given corresponding results in arbitrary space dimension. It was also shown that these results are sharp within the used method, namely the Fourier restriction norm method initiated by Bourgain and Klainerman–Machedon and further developed by Kenig, Ponce, Vega and others. It could also be shown by Colliander, Holmer and Tzirakis [6], that global well-posedness in the case $k = 0$, $l = -1/2$ holds true. Holmer [9] was able to show that the one-dimensional local well-posedness theory is sharp in the sense that the problem is locally ill-posed in some cases, where the assumptions on k, l in [10] are violated, more precisely: if $0 < k < 1$ and $2k > l + 1/2$, or, if $k \leqslant 0$ and $l > -1/2$, or, if $k = 0$ and $l < -3/2$. Moreover, the mapping data upon solution is not C^2, if $k \in \mathbf{R}$, $l < -1/2$. Ill-posedness for $k < 0$ and $l \leqslant -3/2$ was shown by Biagioni and Linares [1].

The minimal values $k = 0$, $l = -1/2$ are far from critical, if one compares them with those being critical for a scaling argument, namely $k = -1$ and $l = -3/2$. The heuristic scaling argument here is the following (for details we refer to [10]): Ignoring the term $A^{\frac{1}{2}} n_\pm$ in Eq. (5) the system (4)–(6) is invariant under the dilation

$$u(x,t) \to u_\mu(x,t) = \mu^{\frac{3}{2}} u\bigl(\mu x, \mu^2 t\bigr), \tag{7}$$

$$n_\pm(x,t) \to n_{\pm\mu}(x,t) = \mu^2 n_\pm\bigl(\mu x, \mu^2 t\bigr). \tag{8}$$

Because

$$\bigl\|u_\mu(x,0)\bigr\|_{\dot H^k} = \mu^{k+1} \|u_0\|_{\dot H^k} \tag{9}$$

and

$$\bigl\|n_{\pm\mu}(x,0)\bigr\|_{\dot H^l} = \mu^{l+\frac{3}{2}} \|n_{\pm 0}\|_{\dot H^l} \tag{10}$$

the system is critical for $k = -1$ and $l = -\frac{3}{2}$. If namely the lifespan of (u, n_+, n_-) were T the lifespan of $(u_\mu, n_{+\mu}, n_{-\mu})$ would be $T\mu^{-2}$. So, if $k < -1$ or $l < -\frac{3}{2}$, one would have both the norm of the data and the lifespan of the solution (u, n_+, n_-) going to zero as $\mu \to \infty$, which strongly indicates ill-posedness.

It is interesting to compare the situation with the corresponding problem for the cubic Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0, \quad u(0) = u_0, \tag{11}$$

which is known to be (globally) well-posed for data $u_0 \in H^s$, $s \geqslant 0$ [15] (cf. also [5]), and locally ill-posed for $s < 0$ [13], whereas scaling considerations suggest as the critical value $s = -1/2$. This problem is of special interest also for the Zakharov system, because the cubic Schrödinger equation is the formal limit for $c \to \infty$ of the Zakharov system modified by replacing $\partial_t^2 - \partial_x^2$ by $c^{-2}\partial_t^2 - \partial_x^2$. Now, for nonlinear Schrödinger equations it was suggested to leave the H^s-scale of the data by Cazenave, Vega and Vilela [4] and Vargas and Vega [14]. For the cubic Schrödinger equation local (and even global) well-posedness has been shown for data with infinite L^2-norm. A. Grünrock [8] was able to show in this case local well-posedness for data $u_0 \in \widehat{H^{s,r}}$, where

$$\|u_0\|_{\widehat{H^{s,r}}} := \bigl\|\langle\xi\rangle^s \widehat{u_0}\bigr\|_{L^{r'}_\xi}, \quad 1/r + 1/r' = 1,$$

if $s \geqslant 0$ and $1 < r < \infty$. Moreover, he could show global well-posedness for $2 \geqslant r \geqslant 5/3$, $u_0 \in \widehat{H^{0,r}}$, and also local ill-posedness for the cubic Schrödinger equation in $\widehat{H^{s,r}}$ for any $1 < r < \infty$ and $-1/r' < s < 0$. The well-posedness results were proven by a modified Fourier restriction norm method (for $p \neq 2$), which was developed by A. Grünrock in [7], where these ideas were applied to the modified KdV equation.

The aim of the present paper is to prove local well-posedness results for the Zakharov system with data $u_0 \in \widehat{H^{k,p}}$, $n_0 \in \widehat{H^{l,p}}$, $n_1 \in \widehat{H^{l-1,p}}$ under suitable assumptions on k, l, p, which allow to weaken the assumptions on the data from the scaling point of view, thus improving the L^2-based results in this sense, and also allow to get results for certain data $u_0 \notin L^2$ and $(n_0, n_1) \notin H^{-1/2} \times H^{-3/2}$. Details are given in Sections 2 and 3. Especially we can show that local well-posedness holds for data $(u_0, n_0, n_1) \in \widehat{H^{k,p}} \times \widehat{H^{l,p}} \times \widehat{H^{l-1,p}}$ for suitable $k < 0$, $l < -1/2$ and $1 < p < 2$ in contrast to the above-mentioned ill-posedness results of Holmer [9] for the Zakharov system, and also in contrast to Grünrock's ill-posedness results for the cubic Schrödinger equation, so that the limit of the c-dependent Zakharov system as $c \to \infty$ must be singular. We are also able to choose $k = 0$ and $l > -1/2$, a choice which was not possible in the L^2-case (cf. [9] again).

We prove our results by a modification of the Fourier restriction norm method, originally due to J. Bourgain [2,3], and derive the crucial estimates for the nonlinearities using a variant of the Schwarz method introduced by Kenig, Ponce and Vega [11,12] adapted to the L^p-theory. In principle these estimates are proven along the lines of [10].

We recall the modified Fourier restriction norm method in the following. For details we refer to the paper of A. Grünrock (cf. [7, Chapter 2]). Our solution spaces are the Banach spaces

$$X_r^{l,b} := \{ f \in \mathcal{S}'(\mathbf{R}^2):\ \|f\|_{X_r^{l,b}} < \infty \},$$

where $l, b \in \mathbf{R}$, $1 < r < \infty$, $1/r + 1/r' = 1$ and

$$\|f\|_{X_r^{l,b}} := \left(\int d\xi\, d\tau\, \langle \xi \rangle^{lr'} \langle \tau + \phi(\xi) \rangle^{br'} \big| \hat{f}(\xi, \tau) \big|^{r'} \right)^{1/r'},$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is a given smooth function of polynomial growth. The dual space of $X_r^{l,b}$ is $X_{r'}^{-l,-b}$, and the Schwartz space is dense in $X_p^{l,r}$. The embedding $X_r^{l,b} \subset C^0(\mathbf{R}, \widehat{H^{l,r}})$ is true for $b > 1/r$. We have

$$\|f\|_{X_r^{l,b}} = \left(\int d\xi\, d\tau\, \langle \xi \rangle^{lr'} \langle \tau \rangle^{br'} \big| \mathcal{F}\big(e^{-it\phi(-i\partial_x)} f\big)(\xi, \tau) \big|^{r'} \right)^{1/r'}$$

and

$$\big\| \psi e^{it\phi(-i\partial_x)} u_0 \big\|_{X_r^{l,b}} \leqslant c_\psi \|u_0\|_{\widehat{H^{l,r}}}$$

for any $\psi \in C_0^\infty(\mathbf{R}_t)$.

If v is a solution of the inhomogeneous problem

$$i v_t - \phi(-i\partial_x) v = F, \quad v(0) = 0,$$

and $\psi \in C_0^\infty(\mathbf{R}_t)$ with $\operatorname{supp} \psi \subset (-2, 2)$, $\psi \equiv 1$ on $[-1, 1]$, $\psi(t) = \psi(-t)$, $\psi(t) \geqslant 0$, $\psi_\delta(t) := \psi(\frac{t}{\delta})$, $0 < \delta \leqslant 1$, we have for $1 < r < \infty$, $b' + 1 \geqslant b \geqslant 0 \geqslant b' > -1/r'$,

$$\|\psi_\delta v\|_{X_r^{l,b}} \leqslant c \delta^{1+b'-b} \|F\|_{X_r^{l,b'}}.$$

For the reduced wave part $\phi(\xi) = \pm|\xi|$ we use the notation $X_{\pm,r}^{l,b}$ instead of $X_r^{l,b}$, whereas for the Schrödinger part $\phi(\xi) = \xi^2$ we simply use $X_r^{l,b}$. We also use the localized spaces

$$X_r^{l,b}(0, T) := \{ f = \tilde{f}|_{[0,T] \times \mathbf{R}}:\ \tilde{f} \in X_r^{l,b} \},$$

where

$$\|f\|_{X_r^{l,b}(0,T)} := \inf\{ \|\tilde{f}\|_{X_r^{l,b}}:\ f = \tilde{f}|_{[0,T] \times \mathbf{R}} \}.$$

Especially we use [7, Theorem 2.3], which we repeat for convenience.

Theorem 1.1. *Consider the Cauchy problem*

$$u_t - i\phi(-i\partial_x) u = N(u), \quad u(0) = u_0 \in \widehat{H^{s,r}}, \tag{12}$$

where N is a nonlinear function of u and its spatial derivatives. Assume for given $s \in \mathbf{R}$, $1 < r < \infty$, $\alpha \geqslant 1$ there exist $b > 1/r$, $b - 1 < b' \leqslant 0$ such that the estimates

$$\|N(u)\|_{X_r^{s,b'}} \leqslant c\|u\|_{X_r^{s,b}}^{\alpha}$$

and

$$\|N(u) - N(v)\|_{X_r^{s,b'}} \leqslant c\big(\|u\|_{X_r^{s,b}}^{\alpha-1} + \|v\|_{X_r^{s,b}}^{\alpha-1}\big)\|u - v\|_{X_r^{s,b}}$$

are valid. Then there exist $T = T(\|u_0\|_{\widehat{H^{s,r}}}) > 0$ and a unique solution $u \in X_r^{s,b}[0,T]$ of (12). *This solution belongs to $C^0([0,T],\widehat{H^{s,r}})$, and the mapping $u_0 \mapsto u$, $\widehat{H^{s,r}} \to X_r^{s,b}(0,T_0)$ is locally Lipschitz continuous for any $T_0 < T$.*

The main result of this paper is the following

Theorem 1.2. *Let $1 < p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \geqslant b, b_1 > \frac{1}{p}$.*

- *In the case $k \geqslant 0$ assume*
$$l \geqslant -\frac{1}{p}, \qquad k - l < 2(1-b_1), \qquad l \leqslant 2k - \frac{1}{p'}, \qquad l + 1 - k < \frac{1}{p} + 2(1-b), \qquad l + 1 - k \leqslant 2b_1.$$
- *In the case $k < 0$ assume*
$$k \geqslant -\frac{1}{p}, \qquad l \geqslant -\frac{1}{p}, \qquad l + k > \frac{1}{p} - 2b_1, \qquad l + k > \frac{1}{p} - 2b, \qquad l + k > -\frac{1}{p} - 2(1-b_1),$$
$$k - l < 2(1-b_1), \qquad 2k > \frac{1}{p} - b_1, \qquad 2k \geqslant l + \frac{1}{p'}, \qquad 2k > -(1-b).$$

Let $u_0 \in \widehat{H^{k,p}}$, $n_{\pm 0} \in \widehat{H^{l,p}}$. Then the Cauchy problem (4)–(6) *is locally well-posed, i.e. there exists a unique local solution $u \in X_p^{k,b_1}(0,T)$, $n_\pm \in X_{\pm,p}^{l,b}(0,T)$. This solution satisfies $u \in C^0([0,T],\widehat{H^{k,p}})$, $n_\pm \in C^0([0,T],\widehat{H^{l,p}})$, and the mapping data upon solution is locally Lipschitz continuous.*

The estimates for the nonlinearities are given in Section 3 and the short proof of this theorem as a consequence of these estimates in Section 4.

Remark. The assumption $n_{\pm 0} \in \widehat{H^{l,p}}$ requires n_0, $A^{-1/2}n_1 \in \widehat{H^{l,p}}$. This last assumption on n_1 can also be replaced by the condition $n_1 \in \widehat{H^{l-1,p}}$. One way to see this is to modify the transformation of the original Zakharov system into the first-order system in t as follows: replace the wave equation by $n_{tt} - n_{xx} + n = (|u|^2)_{xx} + n$ and define $n_\pm := n \pm i\tilde{A}^{-1/2}n_t$, where $\tilde{A} := -\partial_x^2 + 1$. This leads to the modified reduced wave equation:

$$in_{\pm t} \mp \tilde{A}^{1/2}n_\pm = \pm A\tilde{A}^{-1/2}\big(|u|^2\big) \mp (1/2)\tilde{A}^{-1/2}(n_+ + n_-).$$

Now it is easy to see that this modified nonlinear term can be estimated exactly in the same way as the original term $A^{1/2}(|u|^2)$, and also the additional linear term is harmless. This remark was already used by [10].

Thus we have

Theorem 1.3. *Let k, l, b, b_1, p fulfill the assumptions of Theorem* 1.2. *Let $u_0 \in \widehat{H^{k,p}}$, $n_0 \in \widehat{H^{l,p}}$, $n_1 \in \widehat{H^{l-1,p}}$. Then the Cauchy problem* (1)–(3) *is locally well-posed, i.e. there exists a unique solution*

$$u \in X_p^{k,b_1}(0,T), \qquad n \in X_{+,p}^{l,b}(0,T) + X_{-,p}^{l,b}(0,T), \qquad n_t \in X_{+,p}^{l-1,b}(0,T) + X_{-,p}^{l-1,b}(0,T).$$

This solution satisfies

$$u \in C^0\big([0,T],\widehat{H^{k,p}}\big), \qquad n \in C^0\big([0,T],\widehat{H^{l,p}}\big), \qquad n_t \in C^0\big([0,T],\widehat{H^{l-1,p}}\big),$$

and the mapping data upon solution is locally Lipschitz continuous.

We use the notation $\langle\lambda\rangle := (1+\lambda^2)^{1/2}$, and $a\pm$ to denote a number slightly larger (respectively smaller) than a.

2. Comparison with earlier results

It is interesting to compare our results with those of [10] for the case $p=2$. The lowest admissible choice in this case was $k=0$, $l=-1/2$, $p=2$. This is contained in our results, too.

- A choice, which improves this result from the scaling point of view for the Schrödinger part is $k=0$, $p=1+\epsilon$, $-\frac{2}{p'}<l\leqslant-\frac{1}{p'}$ (with $b=b_1=\frac{1}{p}+$) and $\epsilon>0$ small. It is easily checked that this choice is admissible due to Theorem 1.2.
  $\widehat{H^{k,p}}$ scales like H^σ, where $\sigma=k-\frac{1}{p}+\frac{1}{2}$, here: $\sigma=\frac{1}{2}-\frac{1}{1+\epsilon}\to-\frac{1}{2}$ $(\epsilon\to 0)$, $\widehat{H^{l,p}}$ scales like H^λ, where $\lambda=l-\frac{1}{p}+\frac{1}{2}$, here: $\lambda\to-\frac{1}{2}$ $(\epsilon\to 0)$.
  That $\widehat{H^{k,p}}$ scales like H^σ here just means that (cf. (7) and (9)):
  $$\big\||\xi|^k\widehat{u_\mu}(\xi,0)\big\|_{L^{p'}_\xi}=:\big\|u_\mu(x,0)\big\|_{\widehat{\dot H^{k,p}}}=\mu^{k-\frac{1}{p}+\frac{3}{2}}\|u_0\|_{\widehat{\dot H^{k,p}}}$$
  and
  $$\big\|u_\mu(x,0)\big\|_{\dot H^\sigma}=\mu^{\sigma+1}\|u_0\|_{\dot H^\sigma}$$
  and the exponents of μ here coincide.
- Another admissible choice improving the result from the scaling point of view for the wave part is $k=0$, $l=-\frac{1}{p}$ (with $b=b_1=\frac{1}{p}+$) and $2\geqslant p>\frac{3}{2}$. The conditions of Theorem 1.2 are fulfilled:
  1. $k-l<2(1-b_1)\Leftrightarrow\frac{1}{p}<2(1-\frac{1}{p})\Leftrightarrow p>\frac{3}{2}$,
  2. $l\leqslant 2k-\frac{1}{p'}\Leftrightarrow p\leqslant 2$,
  3. $l+1-k<\frac{1}{p}+2(1-b)\Leftrightarrow-\frac{1}{p}+1<\frac{1}{p}+2(1-\frac{1}{p})$, which is fulfilled, and
  4. $l+1-k\leqslant 2b_1\Leftrightarrow-\frac{1}{p}+1\leqslant\frac{2}{p}\Leftrightarrow 1\leqslant\frac{3}{p}$.

  $\widehat{H^{k,p}}$ scales like H^σ with $\sigma=-\frac{1}{p}+\frac{1}{2}\to-\frac{1}{6}$ $(p\to\frac{3}{2})$, $\widehat{H^{l,p}}$ scales like H^λ with $\lambda=-\frac{2}{p}+\frac{1}{2}\to-\frac{5}{6}$ $(p\to\frac{3}{2})$.

It is also interesting to remark that it is possible to choose $k<0$ and $l<-\frac{1}{2}$ (with a suitable $1<p<2$), and nevertheless achieve local well-posedness for the Zakharov system (see details below). In this situation Holmer [9] proved in the L^2-case that the mapping data upon solution is not C^2, so that a contraction mapping method as in our case cannot be applied. Moreover, the cubic nonlinear Schrödinger equation (11) is known to be ill-posed for suitable data $u_0\in\widehat{H^{k,p}}$ for any $-\frac{1}{p'}<k<0$ and $p>1$ (cf. [8]). This equation, as already remarked in the introduction, is the formal limit as $c\to\infty$ of a sequence of velocity-dependent Zakharov systems (replacing $\partial_x^2-\partial_t^2$ by $c^{-2}\partial_x^2-\partial_t^2$). So this limit must be singular in some sense.

In order to determine the minimal k, which fulfills all the assumptions in Theorem 1.2 we argue as follows:

1. The conditions $2k>\frac{1}{p}-b_1$ and $k<l+2(1-b_1)$ require $\frac{1}{2p}-\frac{1}{2}b_1<l+2-2b_1\Leftrightarrow b_1<\frac{2}{3}(l+2)-\frac{1}{3p}$.
2. The conditions $2k\geqslant l+\frac{1}{p'}$ and $k<l+2(1-b_1)$ require $\frac{l}{2}+\frac{1}{2p'}<l+2-2b_1\Leftrightarrow b_1<\frac{l}{4}+\frac{3}{4}+\frac{1}{4p}$.

Thus b_1 has to be chosen such that $\frac{1}{p}<b_1<\min(\frac{2}{3}(l+2)-\frac{1}{3p},\frac{l}{4}+\frac{3}{4}+\frac{1}{4p})$, so that the condition $2k>\frac{1}{p}-b_1$ can only be fulfilled, if

$$2k>\frac{1}{p}-\frac{2}{3}(l+2)+\frac{1}{3p}=\frac{4}{3p}-\frac{2}{3}(l+2) \tag{13}$$

and

$$2k>\frac{1}{p}-\frac{l}{4}-\frac{3}{4}-\frac{1}{4p}=\frac{3}{4p}-\frac{1}{4}(l+3). \tag{14}$$

Moreover we need

$$2k \geqslant l + \frac{1}{p'} = l + 1 - \frac{1}{p}. \tag{15}$$

The lower bound for $2k$ in (13) and (15) is minimized, if

$$\frac{4}{3p} - \frac{2}{3}(l+2) = l + 1 - \frac{1}{p} \quad \Leftrightarrow \quad \frac{1}{p} = \frac{5}{7}l + 1. \tag{16}$$

One easily checks that under this assumption all 3 lower bounds for $2k$ coincide. Thus we end up with (from (13)):

$$2k > \frac{4}{3}\left(\frac{5}{7}l + 1\right) - \frac{2}{3}(l+2) = \frac{2}{7}l \quad \Leftrightarrow \quad k > \frac{l}{7}.$$

The minimal and optimal choice for l here is $l = -\frac{1}{p}$ (because $l \geqslant -\frac{1}{p}$), which means by (16): $p = \frac{12}{7}$, and thus $k > -\frac{1}{12}$ (from $k > \frac{l}{7}$), and by (16):

$$\frac{5}{7}l = \frac{1}{p} - 1 = -\frac{5}{12} \quad \Leftrightarrow \quad l = -\frac{7}{12}.$$

Moreover, we should choose $b_1 < \frac{2}{3}(l+2) - \frac{1}{3p} = \frac{3}{4}$.

It is now completely elementary to see that the choice $k = -\frac{1}{12} + \epsilon$, $l = -\frac{7}{12}$, $b = b_1 = \frac{3}{4} - \epsilon$, $p = \frac{12}{7}$ ($\epsilon > 0$ small) meets all the assumptions of Theorem 1.2.

In this situation we have $\widehat{H^{k,p}}$ scales like H^σ with $\sigma = -\frac{1}{6} + \epsilon$, and $\widehat{H^{l,p}}$ scales like H^λ with $\lambda = -\frac{2}{3}$.

This is an improvement from the scaling point of view for both the Schrödinger and the wave part, compared to the L^2-result of [10], where $\sigma = 0$ and $\lambda = -\frac{1}{2}$.

3. Nonlinear estimates

In order to estimate the nonlinearities we use the following simple application of Hölder's inequality.

Lemma 3.1. *For $1/p + 1/p' = 1$, $1 < p < \infty$, the following estimate holds*:

$$\left| \iint \hat{v}(\zeta)\widehat{v_1}(\zeta_1)\widehat{v_2}(\zeta_2) K(\zeta_1,\zeta_2)\, d\zeta_1\, d\zeta_2 \right| \leqslant \sup_{\zeta_1}\left(\int \left|K(\zeta_1,\zeta_2)\right|^p d\zeta_2 \right)^{1/p} \|\widehat{v_1}\|_{L^p} \|\hat{v}\|_{L^{p'}} \|\widehat{v_2}\|_{L^{p'}},$$

where $\zeta := \zeta_1 - \zeta_2$.

Proof.

$$\begin{aligned}
&\left| \iint \hat{v}(\zeta)\widehat{v_1}(\zeta_1)\widehat{v_2}(\zeta_2) K(\zeta_1,\zeta_2)\, d\zeta_1\, d\zeta_2 \right| \\
&\quad \leqslant \|\widehat{v_1}\|_{L^p} \left(\int \left| \int \hat{v}(\zeta_1-\zeta_2)\widehat{v_2}(\zeta_2) K(\zeta_1,\zeta_2)\, d\zeta_2 \right|^{p'} d\zeta_1 \right)^{1/p'} \\
&\quad \leqslant \|\widehat{v_1}\|_{L^p} \left\{ \int \left[\left(\int \left|\hat{v}(\zeta_1-\zeta_2)\widehat{v_2}(\zeta_2)\right|^{p'} d\zeta_2 \right) \left(\int \left|K(\zeta_1,\zeta_2)\right|^p d\zeta_2 \right)^{p'/p} \right] d\zeta_1 \right\}^{1/p'} \\
&\quad \leqslant \|\widehat{v_1}\|_{L^p} \left(\sup_{\zeta_1} \int \left|K(\zeta_1,\zeta_2)\right|^p d\zeta_2 \right)^{1/p} \left(\iint \left|\hat{v}(\zeta_1-\zeta_2)\widehat{v_2}(\zeta_2)\right|^{p'} d\zeta_2\, d\zeta_1 \right)^{1/p'} \\
&\quad \leqslant \sup_{\zeta_1}\left(\int \left|K(\zeta_1,\zeta_2)\right|^p d\zeta_2 \right)^{1/p} \|\widehat{v_1}\|_{L^p} \|\hat{v}\|_{L^{p'}} \|\widehat{v_2}\|_{L^{p'}}. \qquad \square
\end{aligned}$$

Remark. Similarly one can prove

$$\left| \iint \hat{v}(\zeta)\widehat{v_1}(\zeta_1)\widehat{v_2}(\zeta_2) K(\zeta_1,\zeta_2)\, d\zeta_1\, d\zeta_2 \right| \leqslant \sup_{\zeta}\left(\int \left|K(\zeta+\zeta_2,\zeta_2)\right|^p d\zeta_2 \right)^{1/p} \|\hat{v}\|_{L^p} \|\widehat{v_1}\|_{L^{p'}} \|\widehat{v_2}\|_{L^{p'}}.$$

Our first aim is to estimate the nonlinearity $f = n_\pm u$ in $X_p^{k,-c_1}$ for given $n_\pm \in X_{\pm,p}^{l,b}$ and $u \in X_p^{k,b_1}$. We estimate $\hat{f}(\xi_1', \tau_1) = (\widehat{n_\pm} * \hat{u})(\xi_1', \tau_1)$ in terms of $\widehat{n_\pm}(\xi, \tau)$ and $\hat{u}(\xi_2', \tau_2)$, where $\xi = \xi_1' - \xi_2'$, $\tau = \tau_1 - \tau_2$. We also introduce the variables $\sigma_1 = \tau_1 + \xi_1'^2$, $\sigma_2 = \tau_2 + \xi_2'^2$, $\sigma = \tau \pm |\xi|$, so that

$$z := \xi_1'^2 - \xi_2'^2 \mp |\xi| = \sigma_1 - \sigma_2 - \sigma. \tag{17}$$

Define $\widehat{v_2} = \langle \xi_2' \rangle^k \langle \sigma_2 \rangle^{b_1} \hat{u}$ and $\hat{v} = \langle \xi \rangle^l \langle \sigma \rangle^b \widehat{n_\pm}$, so that $\|u\|_{X_p^{k,b_1}} = \|\widehat{v_2}\|_{L^{p'}}$ and $\|n_\pm\|_{X_{\pm,p}^{l,b}} = \|\hat{v}\|_{L^{p'}}$. In order to estimate f in $X_p^{k,-c_1}$ we take its scalar product with a function in $X_{p'}^{-k,c_1}$ with Fourier transform $\langle \xi_1' \rangle^k \langle \sigma_1 \rangle^{-c_1} \widehat{v_1}$ with $\widehat{v_1} \in L^p$.

In the sequel we want to show an estimate of the form

$$|S| \leqslant c \|\hat{v}\|_{L^{p'}} \|\widehat{v_1}\|_{L^p} \|\widehat{v_2}\|_{L^{p'}},$$

where

$$S := \int \frac{|\hat{v}\widehat{v_1}\widehat{v_2}| \langle \xi_1' \rangle^k}{\langle \sigma \rangle^b \langle \sigma_1 \rangle^{c_1} \langle \sigma_2 \rangle^{b_1} \langle \xi_2' \rangle^k \langle \xi \rangle^l} \, d\xi_1' \, d\xi_2' \, d\tau_1 \, d\tau_2.$$

This directly gives the desired estimate

$$\|n_\pm u\|_{X_p^{k,-c_1}} \leqslant c \|n_\pm\|_{X_{\pm,p}^{l,b}} \|u\|_{X_p^{k,b_1}}. \tag{18}$$

Proposition 3.1. *The estimate* (18) *holds under the following assumptions*:

$$k \geqslant 0, \qquad l \geqslant -1/p, \qquad k - l \leqslant 2c_1, \qquad k - l \leqslant 2/p,$$

where $c_1 \geqslant 0$, $b > 1/p$, $b_1 > 1/p$, $1 < p \leqslant 2$.

Remark. We simplify (17) as follows. If (17) holds with the minus sign and if $\xi_1' \geqslant \xi_2'$ (respectively $\xi_1' \leqslant \xi_2'$), we have

$$z = \xi_1'^2 - \xi_2'^2 - |\xi_1' - \xi_2'| = (\xi_1' \mp 1/2)^2 - (\xi_2' \mp 1/2)^2 = \xi_1^2 - \xi_2^2$$

where $\xi_i = \xi_i' \mp 1/2$. Thus the region $\xi_1' \geqslant \xi_2'$ (respectively $\xi_1' \leqslant \xi_2'$) of S is majorized by

$$\overline{S} = c \int \frac{|\hat{v}(\xi, \tau)\widehat{v_1}(\xi_1 \pm 1/2, \tau_1)\widehat{v_2}(\xi_2 \pm 1/2, \tau_2)| \langle \xi_1 \rangle^k}{\langle \sigma \rangle^b \langle \sigma_1 \rangle^{c_1} \langle \sigma_2 \rangle^{b_1} \langle \xi_2 \rangle^k \langle \xi \rangle^l} \, d\xi_1 \, d\xi_2 \, d\tau_1 \, d\tau_2,$$

where now

$$\begin{aligned} &z = \xi_1^2 - \xi_2^2 = \sigma_1 - \sigma_2 - \sigma, \qquad \xi = \xi_1 - \xi_2, \qquad \tau = \tau_1 - \tau_2, \\ &\sigma_i = \tau_i + (\xi_i \pm 1/2)^2, \qquad \sigma = \tau \pm |\xi| = \tau \pm |\xi_1 - \xi_2|. \end{aligned} \tag{19}$$

Also, the plus sign in (17) can be treated similarly by again defining $\xi_i = \xi_i' \pm 1/2$. If one wants to estimate $\overline{S}$ by $c\|\hat{v}\|_{L^{p'}} \|\widehat{v_1}\|_{L^p} \|\widehat{v_2}\|_{L^{p'}}$, the variables ξ_i and $\xi_i \pm 1/2$ are completely equivalent, thus we do not distinguish between them.

Proof of Proposition 3.1. According to Lemma 3.1 we have to show

$$C^p := \sup_{\xi_1, \sigma_1} \langle \sigma_1 \rangle^{-c_1 p} \langle \xi_1 \rangle^{kp} \int \frac{d\xi_2 \, d\sigma_2}{\langle \sigma \rangle^{bp} \langle \sigma_2 \rangle^{b_1 p} \langle \xi \rangle^{lp} \langle \xi_2 \rangle^{kp}} < \infty.$$

Case 1. $|\xi_1| \leqslant 2|\xi_2|$ ($\Rightarrow |\xi| \leqslant 3|\xi_2|$).

If $|\xi_2| \leqslant 1$ we have $\langle \xi \rangle \sim 1$ and thus

$$C^p \leqslant c \sup_{\xi_1, \sigma_1} \int_{|\xi_2| \leqslant 1} d\xi_2 \int_0^\infty \langle \sigma_2 \rangle^{-b_1 p} \, d\sigma_2 < \infty,$$

because $b_1 > 1/p$. If $|\xi_2| \geqslant 1$ we get from (19) for $\xi_1, \sigma_1, \sigma_2$ fixed: $\frac{d\sigma}{d\xi_2} = 2\xi_2$, and thus

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int \langle \xi \rangle^{-lp} \langle \xi_2 \rangle^{-1} \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p} \, d\sigma \, d\sigma_2.$$

For $l \geqslant 0$ we immediately have

$$C^p \leqslant c \int \langle \sigma \rangle^{-bp} \, d\sigma \int \langle \sigma_2 \rangle^{-b_1 p} \, d\sigma_2 < \infty,$$

whereas for $l \leqslant 0$ we use our assumption $l \geqslant -1/p$ and get the same bound by using $\langle \xi \rangle^{-lp} \langle \xi_2 \rangle^{-1} \leqslant c \langle \xi_2 \rangle^{-lp-1} \leqslant c$.

Case 2. $|\xi_1| \geqslant 2|\xi_2|$ ($\Rightarrow |\xi| \sim |\xi_1|$).

From (19) we conclude $\xi_1^2 \leqslant c(|\sigma_1| + |\sigma_2| + |\sigma|)$ and distinguish three cases.

Case 2a. $|\sigma_1|$ dominant, i.e. $|\sigma_1| \geqslant |\sigma_2|, |\sigma|$ ($\Rightarrow \xi_1^2 \leqslant c|\sigma_1|$).

This implies, using our assumption $k - l \leqslant 2c_1$,

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \langle \xi_1 \rangle^{(k-l-2c_1)p} \int d\xi_2 \, d\sigma_2 \, \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p} \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2 \, d\sigma_2 \, \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p}.$$

If $|\xi_2| \leqslant 1$ this is immediately bounded by $c \int_{|\xi_2| \leqslant 1} d\xi_2 \int d\sigma_2 \, \langle \sigma_2 \rangle^{-b_1 p} < \infty$, whereas for $|\xi_2| \geqslant 1$ we use $\frac{d\sigma}{d\xi_2} = 2\xi_2 \sim 2\langle \xi_2 \rangle$ again and get the bound

$$c \int d\sigma \, \langle \sigma \rangle^{-bp} \int d\sigma_2 \, \langle \sigma_2 \rangle^{-b_1 p} < \infty,$$

using $b, b_1 > 1/p$.

Case 2b. $|\sigma_2|$ dominant ($\Rightarrow \xi_1^2 \leqslant c|\sigma_2|$).

We have

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \langle \xi_1 \rangle^{(k-l)p} \int d\xi_2 \, d\sigma_2 \, \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p} \langle \xi_2 \rangle^{-kp}.$$

The case $k \leqslant l$ is simple and leads to

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2 \, d\sigma_2 \, \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p},$$

which can be handled as in Case 2a.

The case $k > l$ is treated as follows:

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2 \, d\sigma_2 \, \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p + \frac{(k-l)p}{2}} \langle \xi_2 \rangle^{-kp}.$$

Substituting $y = \xi_2^2$, thus $d\xi_2 = \frac{dy}{2|y|^{1/2}}$, leads to

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int |y|^{-\frac{1}{2}} \langle y \rangle^{-\frac{kp}{2}} \int \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p + \frac{(k-l)p}{2}} \, d\sigma_2 \, dy.$$

From (19) we have $\langle \sigma \rangle = \langle \sigma_2 - (\sigma_1 - \xi_1^2 + y) \rangle$, and thus by [10, Lemma 4.2], using $b, b_1 > 1/p$,

$$\int d\sigma_2 \, \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p + \frac{(k-l)p}{2}} \leqslant c \big\langle \sigma_1 - \xi_1^2 + y \big\rangle^{-1 + \frac{(k-l)p}{2} -},$$

because $-b_1 p + \frac{(k-l)p}{2} < -1 + 1 = 0$ using our assumptions $b_1 > 1/p$ and $k - l \leqslant 2/p$. Thus

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int |y|^{-\frac{1}{2}} \langle y\rangle^{-\frac{kp}{2}} \langle \sigma_1 - \xi_1^2 + y\rangle^{-1+\frac{(k-l)p}{2}-} \, dy.$$

The supremum occurs for $\sigma_1 = \xi_1^2$ by [10, Lemma 4.1], so that

$$C^p \leqslant c \int |y|^{-\frac{1}{2}} \langle y\rangle^{-\frac{kp}{2}} \langle y\rangle^{-1+\frac{(k-l)p}{2}-} \, dy = \int |y|^{-\frac{1}{2}} \langle y\rangle^{-1-\frac{lp}{2}-} \, dy < \infty,$$

because $l \geqslant -1/p$.

Case 2c. $|\sigma|$ dominant.

This case can be treated like Case 2b, which completes the proof. □

It is also possible to prove (18) in certain cases where k is negative. This is done in the following

Proposition 3.2. *Estimate* (18) *holds under the following conditions*:

$$k \leqslant 0, \qquad l \geqslant -\frac{1}{p}, \qquad k \geqslant -\frac{1}{p},$$
$$k - l \leqslant 2c_1, \qquad k - l \leqslant \frac{2}{p}, \qquad l + k > \frac{1}{p} - 2b_1, \qquad l + k > \frac{1}{p} - 2b, \qquad l + k \geqslant -\frac{1}{p} - 2c_1,$$

where $c_1 \geqslant 0$ *and* $b_1, b > \frac{1}{p}$.

Proof.

Case 1. $|\xi_1| \sim |\xi_2|$.

This case can be treated exactly like Case 1 in the previous proposition, using $l \geqslant -1/p$.

Case 2. $|\xi_1| \ll |\xi_2|$ ($\Rightarrow |\xi| \sim |\xi_2|$).

Using the notation of the previous proposition we have

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \langle \sigma_1\rangle^{-c_1 p} \langle \xi_1\rangle^{kp} \int d\xi_2 \, d\sigma_2 \, \langle \xi_2\rangle^{-lp-kp} \langle \sigma\rangle^{-bp} \langle \sigma_2\rangle^{-b_1 p}.$$

From (19) we get $\xi_2^2 \leqslant c(|\sigma_1| + |\sigma_2| + |\sigma|)$.

Case 2a. $|\sigma_1|$ dominant ($\Rightarrow \xi_2^2 \leqslant c|\sigma_1|$).

Thus

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2 \, d\sigma_2 \, \langle \xi_2\rangle^{-lp-kp-2c_1 p} \langle \sigma\rangle^{-bp} \langle \sigma_2\rangle^{-b_1 p}.$$

If $|\xi_2| \leqslant 1$ we easily get the bound

$$C^p \leqslant c \int_{|\xi_2|\leqslant 1} d\xi_2 \int \langle \sigma_2\rangle^{-b_1 p} \, d\sigma_2 < \infty.$$

If $|\xi_2| \geqslant 1$ we use (19) and get for fixed $\xi_1, \sigma_1, \sigma_2$: $\frac{d\sigma}{d\xi_2} = 2|\xi_2| \sim 2\langle \xi_2\rangle$, so that, using the condition $l + k \geqslant -\frac{1}{p} - 2c_1$, we have $\langle \xi_2\rangle^{-(l+k+2c_1)p-1} \leqslant c$, and thus the bound

$$C^p \leqslant c \int d\sigma\, \langle\sigma\rangle^{-bp} \int d\sigma_2\, \langle\sigma_2\rangle^{-b_1 p} < \infty$$

by $b, b_1 > 1/p$.

Case 2b. $|\sigma_2|$ dominant ($\Rightarrow \xi_2^2 \leqslant c|\sigma_2|$).

Ignoring the factor $\langle\sigma_1\rangle^{-c_1 p}\langle\xi_1\rangle^{kp}$ we estimate

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-lp-kp+2-2b_1 p+} \langle\sigma\rangle^{-bp} \langle\sigma_2\rangle^{-1-}.$$

The case $|\xi_2| \leqslant 1$ is easy again, and for $|\xi_2| \geqslant 1$ we again use $\frac{d\sigma}{d\xi_2} = 2|\xi_2| \sim 2\langle\xi_2\rangle$ and $\langle\xi_2\rangle^{-lp-kp+2-2b_1 p-1+} \leqslant c$, using our assumption $l + k > \frac{1}{p} - 2b_1$, and thus

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int d\sigma\, \langle\sigma\rangle^{-bp} \int d\sigma_2\, \langle\sigma_2\rangle^{-1-} < \infty.$$

Case 2c. $|\sigma|$ dominant.

This case can be handled like Case 2b, using the assumption $l + k > \frac{1}{p} - 2b$.

Case 3. $|\xi_1| \gg |\xi_2|$ ($\Rightarrow |\xi| \sim |\xi_1|$).

We have

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \langle\sigma_1\rangle^{-c_1 p} \langle\xi_1\rangle^{(k-l)p} \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-kp} \langle\sigma\rangle^{-bp} \langle\sigma_2\rangle^{-b_1 p}.$$

Again we distinguish three cases.

Case 3a. $|\sigma_1|$ dominant ($\Rightarrow \xi_1^2 \leqslant c|\sigma_1|$).

By our assumption $k - l \leqslant 2c_1$ we get

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \langle\xi_1\rangle^{(k-l-2c_1)p} \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-kp} \langle\sigma\rangle^{-bp} \langle\sigma_2\rangle^{-b_1 p} \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-kp} \langle\sigma\rangle^{-bp} \langle\sigma_2\rangle^{-b_1 p},$$

which is easily handled for $|\xi_2| \leqslant 1$, whereas for $|\xi_2| \geqslant 1$, using again $\frac{d\sigma}{d\xi_2} = 2|\xi_2| \sim 2\langle\xi_2\rangle$ and our assumption $k \geqslant -1/p$, we arrive at

$$C^p \leqslant c \int \langle\sigma\rangle^{-bp}\, d\sigma \int \langle\sigma_2\rangle^{-b_1 p}\, d\sigma_2 < \infty.$$

Case 3b. $|\sigma_2|$ dominant ($\Rightarrow \xi_1^2 \leqslant c|\sigma_2|$).

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \langle\xi_1\rangle^{(k-l)p} \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-kp} \langle\sigma\rangle^{-bp} \langle\sigma_2\rangle^{-b_1 p}.$$

In the case $k \leqslant l$ we have

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-kp} \langle\sigma\rangle^{-bp} \langle\sigma_2\rangle^{-b_1 p},$$

which is simple to handle for $|\xi_2| \leqslant 1$, and for $|\xi_2| \geqslant 1$ we use $\frac{d\sigma}{d\xi_2} = 2|\xi_2| \sim 2\langle\xi_2\rangle$ and the assumption $k \geqslant -1/p$ and estimate

$$C^p \leqslant c \int d\sigma\, \langle\sigma\rangle^{-bp} \int d\sigma_2\, \langle\sigma_2\rangle^{-b_1 p} < \infty.$$

In the case $k \geqslant l$ we use $\xi_1^2 \leqslant c|\sigma_2|$ and the substitution $y = \xi_2^2$ and $d\xi_2 = \frac{dy}{2|y|^{1/2}}$ to get

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int d\xi_2\, d\sigma_2\, \langle \xi_2 \rangle^{-kp} \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p + \frac{(k-l)p}{2}}$$
$$\leqslant c \sup_{\xi_1,\sigma_1} \int |y|^{-\frac{1}{2}} \langle y \rangle^{-\frac{kp}{2}} \int \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{-b_1 p + \frac{(k-l)p}{2}}\, d\sigma_2\, dy.$$

Now $\langle \sigma \rangle = \langle \sigma_2 - (\sigma_1 - \xi_1^2 + y) \rangle$ by (19), and [10, Lemma 4.2] implies

$$\int \langle \sigma \rangle^{-bp} \langle \sigma_2 \rangle^{b_1 p + \frac{(k-l)p}{2}}\, d\sigma_2 \leqslant c \big\langle \sigma_1 - \xi_1^2 + y \big\rangle^{-1 + \frac{(k-l)p}{2} -},$$

where we used the assumption $k - l \leqslant 2/p$. Thus by [10, Lemma 4.3], using $k \geqslant -1/p$ and $l \geqslant -1/p$,

$$C^p \leqslant c \sup_{\xi_1,\sigma_1} \int |y|^{-\frac{1}{2}} \langle y \rangle^{-\frac{kp}{2}} \big\langle \sigma_1 - \xi_1^2 + y \big\rangle^{-1 + \frac{(k-l)p}{2} -}\, dy \leqslant c \int |y|^{-\frac{1}{2}} \langle y \rangle^{-\frac{kp}{2}} \langle y \rangle^{-1 + \frac{(k-l)p}{2} -}\, dy$$
$$= c \int |y|^{-\frac{1}{2}} \langle y \rangle^{-1 - \frac{lp}{2} -}\, dy < \infty.$$

Case 3c. $|\sigma|$ dominant.

This case is treated like Case 3b. □

Next, we want to estimate the nonlinearity $(|u|^2)_x$, namely

$$|W| \leqslant c \|\hat{v}\|_{L^p} \|\widehat{v_1}\|_{L^{p'}} \|\widehat{v_2}\|_{L^{p'}},$$

where

$$W := \int \frac{|\hat{v}\widehat{v_1}\widehat{v_2}| \langle \xi \rangle^l |\xi|}{\langle \sigma \rangle^c \langle \sigma_1 \rangle^{b_1} \langle \sigma_2 \rangle^{b_1} \langle \xi_1' \rangle^k \langle \xi_2' \rangle^k}\, d\xi_1'\, d\xi_2'\, d\tau_1\, d\tau_2.$$

This implies the desired estimate

$$\big\| \big(|u|^2\big)_x \big\|_{X^{l,-c}_{\pm,p}} \leqslant c \|u\|^2_{X^{k,b_1}_p}. \tag{20}$$

Proposition 3.3. *Estimate* (20) *holds under the following conditions*:

$$k \geqslant 0, \qquad l \leqslant 2k - \frac{1}{p'}, \qquad l + 1 - k \leqslant \frac{1}{p} + 2c, \qquad l + 1 - k \leqslant 2b_1,$$

where $c \geqslant 0$, $b_1 > 1/p$, $1 < p \leqslant 2$.

Proof. According to the remark after Lemma 3.1 we have to show

$$C^p := \sup_{\xi,\sigma} \langle \sigma \rangle^{-cp} \langle \xi \rangle^{lp} |\xi|^p \int d\xi_2\, d\sigma_2\, \langle \xi_1 \rangle^{-kp} \langle \xi_2 \rangle^{-kp} \langle \sigma_1 \rangle^{-b_1 p} \langle \sigma_2 \rangle^{-b_1 p} < \infty. \tag{21}$$

Case 1. $|\xi_1| \sim |\xi_2|$ ($\Rightarrow |\xi| \leqslant c|\xi_1|, c|\xi_2|$).

Applying the remark after Proposition 3.1 we have

$$\xi_1^2 - \xi_2^2 = (\xi + \xi_2)^2 - \xi_2^2 = \sigma_1 - \sigma_2 - \sigma.$$

Thus, for fixed ξ, σ, σ_2, we have $\frac{d\sigma_1}{d\xi_2} = 2\xi$, so that

$$C^p \leqslant c \sup_{\xi,\sigma} \langle \xi \rangle^{lp} |\xi|^{p-1} \langle \xi \rangle^{-2kp} \int \langle \sigma_1 \rangle^{-b_1 p}\, d\sigma_1 \int \langle \sigma_2 \rangle^{-b_1 p}\, d\sigma_2.$$

This is easily seen to be finite under the assumption $l - 2k + 1 \leqslant 1/p \Leftrightarrow l \leqslant 2k - \frac{1}{p'}$.

Case 2. $|\xi_1| \gg |\xi_2|$ ($\Rightarrow |\xi| \sim |\xi_1|$) (and analogously $|\xi_2| \gg |\xi_1|$).

Case 2a. $|\sigma|$ dominant ($\Rightarrow \xi^2 \sim \xi_1^2 \leqslant c|\sigma|$).

Using the relation $\frac{d\sigma_1}{d\xi_2} = 2\xi$ again and ignoring the term $\langle\xi_2\rangle^{-kp}$ we arrive at

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{(-2c+l-k)p} |\xi|^p \int d\xi_2\, d\sigma_2\, \langle\sigma_1\rangle^{-b_1 p} \langle\sigma_2\rangle^{-b_1 p}$$
$$\leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{(-2c+l-k)p} |\xi|^{p-1} \int \langle\sigma_1\rangle^{-b_1 p}\, d\sigma_1 \int \langle\sigma_2\rangle^{-b_1 p}\, d\sigma_2,$$

which can be seen to be finite under the assumption $l + 1 - k \leqslant \frac{1}{p} + 2c$.

Case 2b. $|\sigma_1|$ dominant ($\Rightarrow \xi^2 \sim \xi_1^2 \leqslant c|\sigma_1|$) (and similarly $|\sigma_2|$ dominant).

We have

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{(l-k)p} |\xi|^p \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-kp} \langle\sigma_1\rangle^{-b_1 p} \langle\sigma_2\rangle^{-b_1 p}.$$

Case 2bα. In the case $l + 1 - k \geqslant 1/p$ we introduce the variable $z := \xi_1^2 - \xi_2^2 = (\xi + \xi_2)^2 - \xi_2^2$ and get for fixed ξ: $\frac{dz}{d\xi_2} = 2\xi$. We also have $z = \xi^2 + 2\xi\xi_2 \Leftrightarrow \xi_2 = \frac{z-\xi^2}{2\xi}$ and $z \sim \xi_1^2 \sim \xi^2$, so that we get

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{(l-k)p} \langle\xi\rangle^{p-1} \int_{0 \leqslant z \leqslant c\xi^2} dz\, d\sigma_2\, \langle\sigma_1\rangle^{-b_1 p} \langle\sigma_2\rangle^{-b_1 p} \left\langle \frac{z-\xi^2}{2\xi} \right\rangle^{-kp}$$
$$\leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{-1} \int_{0 \leqslant z \leqslant c\xi^2} dz\, d\sigma_2\, \langle\sigma_1\rangle^{\frac{(l-k+1)p}{2} - b_1 p} \langle\sigma_2\rangle^{-b_1 p} \left\langle \frac{z-\xi^2}{2\xi} \right\rangle^{-kp}.$$

Now we have by [10, Lemma 4.2]

$$\int d\sigma_2\, \langle\sigma_1\rangle^{\frac{(l-k+1)p}{2} - b_1 p} \langle\sigma_2\rangle^{-b_1 p} \leqslant c \langle\sigma + z\rangle^{\frac{(l-k+1)p}{2} - b_1 p},$$

where we used the assumption $l - k + 1 \leqslant 2b_1$ as well as (19), namely $\sigma_1 - \sigma_2 = \sigma + z$, so that we arrive at

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{-1} \int_0^{c\xi^2} dz\, \langle\sigma + z\rangle^{\frac{(l-k+1)p}{2} - b_1 p} \left\langle \frac{z-\xi^2}{2\xi} \right\rangle^{-kp}.$$

With $y := z - \xi^2$ we have

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{-1} \int_{-c\xi^2}^{c\xi^2} \langle\sigma + \xi^2 + y\rangle^{\frac{(l-k+1)p}{2} - b_1 p} \left\langle \frac{y}{2\xi} \right\rangle^{-kp} dy = c \sup_{\xi} \langle\xi\rangle^{-1} \int_{-c\xi^2}^{c\xi^2} \langle y\rangle^{\frac{(l-k+1)p}{2} - b_1 p} \left\langle \frac{y}{2\xi} \right\rangle^{-kp} dy,$$

where we used [10, Lemma 4.3]. The case $|\xi| \leqslant 1$ is easily handled. If $|\xi| \geqslant 1$ we get

(a) in the region $|y| \leqslant |\xi|$ we have $\langle \frac{y}{2\xi} \rangle^{-kp} \sim 1$ and $\langle y\rangle^{\frac{(l-k+1)p}{2} - b_1 p} \leqslant 1$ by our assumption $l - k + 1 \leqslant 2b_1$, so that the integral is bounded by $c|\xi|$, thus $C^p < \infty$.

(b) In the region $|\xi| \leqslant |y| \leqslant c\xi^2$ we have $\langle \frac{y}{2\xi} \rangle \sim |\frac{y}{2\xi}| \sim \frac{\langle y \rangle}{\langle \xi \rangle}$, so that

$$C^p \leqslant c \sup_\xi \langle \xi \rangle^{-1} \langle \xi \rangle^{kp} \int_{|\xi|}^{c\xi^2} \langle y \rangle^{\frac{(l-k+1)p}{2} - b_1 p - kp} \, dy.$$

If $\frac{(l-k+1)p}{2} - b_1 p - kp < -1$ we have

$$C^p \leqslant c \sup_\xi \langle \xi \rangle^{-1+kp+\frac{(l-k+1)p}{2} - b_1 p - kp + 1} = c \sup_\xi \langle \xi \rangle^{\frac{(l-k+1)p}{2} - b_1 p},$$

which is finite under the assumption $l - k + 1 \leqslant 2b_1$.

If $\frac{(l-k+1)p}{2} - b_1 p - kp \geqslant -1$ we have

$$C^p \leqslant c \sup_\xi \langle \xi \rangle^{-1+kp+(l-k+1)p-2b_1p-2kp+2+} = c \sup_\xi \langle \xi \rangle^{(l-2k+1)p-2b_1p+1+} < \infty,$$

because $(l-2k+1)p - 2b_1 p + 1+ \leqslant 0 \Leftrightarrow l - 2k < 2b_1 - 1 - \frac{1}{p}$, which is fulfilled under the assumption $l - 2k \leqslant \frac{1}{p} - 1 = -\frac{1}{p'}$ for $b_1 > \frac{1}{p}$.

Case 2b$\boldsymbol{\beta}$. In the case $l + 1 - k \leqslant \frac{1}{p}$ we directly get by $\frac{d\sigma_1}{d\xi_2} = 2\xi$ (for σ, ξ, σ_2 fixed), ignoring the term $\langle \xi_2 \rangle^{-kp}$ in the integral

$$C^p \leqslant c \sup_{\xi,\sigma} \langle \xi \rangle^{(l-k)p} |\xi|^{p-1} \int \langle \sigma_1 \rangle^{-b_1 p} \, d\sigma_1 \int \langle \sigma_2 \rangle^{-b_1 p} \, d\sigma_2 < \infty. \qquad \square$$

The case $k < 0$ is considered in the following

Proposition 3.4. *Estimate* (20) *holds under the following conditions*:

$$k \leqslant 0, \qquad l \leqslant 2k - \frac{1}{p'}, \qquad 2k \geqslant -c, \qquad 2k > \frac{1}{p} - b_1,$$

where $\frac{1}{p'} > c \geqslant 0$, $b_1 > \frac{1}{p}$, $1 < p \leqslant 2$.

Proof. We again have to show (21) as in the previous proof.

Case 1. $|\xi_1| \sim |\xi_2|$, and ξ_1, ξ_2 have different signs.

In this case we have $|\xi| = |\xi_1 - \xi_2| \sim 2|\xi_1| \sim 2|\xi_2|$, and thus

$$C^p \leqslant c \sup_{\xi,\sigma} \langle \xi \rangle^{lp} |\xi|^p \int d\xi_2 \, d\sigma_2 \, \langle \sigma_1 \rangle^{-b_1 p} \langle \sigma_2 \rangle^{-b_1 p} |\xi|^{-2kp}$$
$$\leqslant c \sup_{\xi,\sigma} \langle \xi \rangle^{(l-2k)p} |\xi|^{p-1} \int \langle \sigma_1 \rangle^{-b_1 p} \, d\sigma_1 \int \langle \sigma_2 \rangle^{-b_1 p} \, d\sigma_2,$$

using $\frac{d\sigma_1}{d\xi_2} = 2\xi$ again. Under the assumption $l - 2k + 1 \leqslant \frac{1}{p} \Leftrightarrow 2k \geqslant l + \frac{1}{p'}$ this is finite.

Case 2. $|\xi_1| \sim |\xi_2|$, and ξ_1, ξ_2 have equal signs.

This implies $|\xi_1 + \xi_2| \sim 2|\xi_1| \sim 2|\xi_2|$, and thus by (19)

$$\xi(\xi_1 + \xi_2) = (\xi_1 - \xi_2)(\xi_1 + \xi_2) = \sigma_1 - \sigma_2 - \sigma$$

and

$$|\xi||\xi_2| \leqslant c\big(|\sigma_1| + |\sigma_2| + |\sigma|\big).$$

Case 2a. $|\sigma_1|$ dominant ($\Rightarrow |\xi\xi_2| \leqslant c|\sigma_1|$) ($|\sigma_2|$ dominant can be handled in the same way).

This also implies $|\xi|\langle\xi_2\rangle \leqslant c\langle\sigma_1\rangle$, which is evident for $|\xi_2| \geqslant 1$, whereas $|\xi_2| \leqslant 1$ implies $|\xi| = |\xi_1 - \xi_2| \leqslant c|\xi_2|$ (using $|\xi_1| \sim |\xi_2|$), so that $|\xi|\langle\xi_2\rangle \leqslant c \leqslant c\langle\sigma_1\rangle$. Thus

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{lp}|\xi|^p \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-2kp}\langle\sigma_1\rangle^{-1-}\langle\sigma_2\rangle^{-b_1p}\langle\xi_2\rangle^{1-b_1p+}|\xi|^{1-b_1p+}.$$

Under the assumption $2k > \frac{1}{p} - b_1$ we have by $|\xi| \leqslant c|\xi_2|$ and $\frac{d\sigma_1}{d\xi_2} = 2\xi$,

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{lp-2kp+1-b_1p+}|\xi|^{p-b_1p+} \int d\sigma_1\, \langle\sigma_1\rangle^{-1-} \int d\sigma_2\, \langle\sigma_2\rangle^{-b_1p}.$$

Assuming $b_1 \leqslant 1$ without loss of generality, this is finite, provided $lp - 2kp + 1 - b_1p + p - b_1p < 0 \Leftrightarrow l - 2k + 1 < -\frac{1}{p} + 2b_1$, which is fulfilled under our assumption $2k \geqslant l + \frac{1}{p'} \Leftrightarrow l - 2k + 1 \leqslant \frac{1}{p}$, because $b_1 > \frac{1}{p}$.

Case 2b. $|\sigma|$ dominant ($\Rightarrow |\xi|\langle\xi_2\rangle \leqslant c\langle\sigma\rangle$).

We have

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{lp}|\xi|^{p-cp} \int d\xi_2\, d\sigma_2\, \langle\xi_2\rangle^{-2kp-cp}\langle\sigma_1\rangle^{-b_1p}\langle\sigma_2\rangle^{-b_1p}.$$

Using $\langle\xi_2\rangle^{-2kp-cp} \leqslant c\langle\xi\rangle^{-2kp-cp}$ (by the assumption $2k \geqslant -c$) and $\frac{d\sigma_1}{d\xi_2} = 2\xi$, we get

$$C^p \leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{lp-2kp-cp}|\xi|^{p-cp-1} \int d\sigma_1\, \langle\sigma_1\rangle^{-b_1p} \int d\sigma_2\, \langle\sigma_2\rangle^{-b_1p}.$$

The assumption $c \leqslant \frac{1}{p'}$ implies $p - cp - 1 \geqslant 0$. Moreover we have $lp - 2kp - cp + p - cp - 1 \leqslant 0 \Leftrightarrow l - 2k + 1 \leqslant \frac{1}{p} + 2c$, which is fulfilled under the assumption $l - 2k + 1 \leqslant \frac{1}{p} \Leftrightarrow 2k \geqslant l + \frac{1}{p'}$, so that C^p is finite.

Case 3. $|\xi_1| \gg |\xi_2|$ ($\Rightarrow |\xi| \sim |\xi_1|$ and $|\xi_2| \ll |\xi|$) (and similarly $|\xi_2| \gg |\xi_1|$).

We have by $d\xi_2 = \frac{d\sigma_1}{2\xi}$,

$$\begin{aligned} C^p &\leqslant c \sup_{\xi,\sigma} \int d\xi_2\, d\sigma_2\, \langle\xi\rangle^{(l-k)p}|\xi|^p\langle\xi_2\rangle^{-kp}\langle\sigma_1\rangle^{-b_1p}\langle\sigma_2\rangle^{-b_1p} \\ &\leqslant c \sup_{\xi,\sigma} \langle\xi\rangle^{(l-k)p}|\xi|^p\langle\xi\rangle^{-kp}|\xi|^{-1} \int d\sigma_1\, \langle\sigma_1\rangle^{-b_1p} \int d\sigma_2\, \langle\sigma_2\rangle^{-b_1p}, \end{aligned}$$

which is finite, provided $l - 2k + 1 \leqslant \frac{1}{p} \Leftrightarrow 2k \geqslant l + \frac{1}{p'}$. □

4. Proof of Theorem 1.2

Proof. We construct a solution of the system of integral equations which belongs to our Cauchy problem by the contraction mapping principle. This can be achieved by using our Propositions 3.1–3.4, which give the necessary estimates for the nonlinearities, if one chooses $c_1 = 1 - b_1-$ and $c = 1 - b-$. In this case the assumptions on the parameters in these propositions reduce to the assumptions in the theorem. We may apply Theorem 1.1 to our system, because its generalization from the case of a single equation to a system is evident. □

References

[1] H.A. Biagioni, F. Linares, Ill-posedness for the Zakharov system with generalized nonlinearity, Proc. Amer. Math. Soc. 131 (2003) 3113–3121.
[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3 (1993) 107–156.

[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, Geom. Funct. Anal. 3 (1993) 209–262.
[4] T. Cazenave, L. Vega, M.C. Vilela, A note on the nonlinear Schrödinger equation in weak L^p spaces, Commun. Contemp. Math. 3 (2001) 153–162.
[5] T. Cazenave, F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Anal. 14 (1990) 807–836.
[6] J. Colliander, J. Holmer, N. Tzirakis, Low regularity global well-posedness for the Zakharov and Klein–Gordon–Schrödinger systems, arXiv: math.AP/0603595, Trans. Amer. Math. Soc., in press.
[7] A. Grünrock, An improved local well-posedness result for the modified KdV equation, Int. Math. Res. Not. 2004 (61) (2004) 3287–3308.
[8] A. Grünrock, Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, Int. Math. Res. Not. 2005 (41) (2005) 2525–2558.
[9] J. Holmer, Local ill-posedness of the 1D Zakharov system, Electron. J. Differential Equations 2007 (24) (2007) 1–22.
[10] J. Ginibre, Y. Tsutsumi, G. Velo, On the Cauchy problem for the Zakharov system, J. Funct. Anal. 151 (1997) 384–436.
[11] C.E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996) 573–603.
[12] C.E. Kenig, G. Ponce, L. Vega, Quadratic forms for the 1D semilinear Schrödinger equation, Trans. Amer. Math. Soc. 348 (1996) 3323–3353.
[13] C.E. Kenig, G. Ponce, L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Math. J. 106 (2001) 617–633.
[14] A. Vargas, L. Vega, Global wellposedness for 1D non-linear Schrödinger equation for data with infinite L^2 norm, J. Math. Pures Appl. 80 (2001) 1029–1044.
[15] Y. Tsutsumi, L^2-solutions for nonlinear Schrödinger equations and nonlinear groups, Funkcial. Ekvac. 30 (1987) 115–125.
[16] V.E. Zakharov, Collapse of Langmuir waves, Soviet Phys. JETP 35 (1972) 908–914.