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Enhanced group analysis and conservation laws of variable coefficient reaction–diffusion equations with power nonlinearities

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Abstract

A class of variable coefficient $(1+1)$-dimensional nonlinear reaction–diffusion equations of the general form $f(x)u_t = (g(x)u^n u_x)_x + h(x)u^m$ is investigated. Different kinds of equivalence groups are constructed including ones with transformations which are nonlocal with respect to arbitrary elements. For the class under consideration the complete group classification is performed with respect to convenient equivalence groups (*generalized extended* and *conditional* ones) and with respect to the set of all local transformations. Usage of different equivalences and coefficient gauges plays the major role for simple and clear formulation of the final results. The corresponding set of admissible transformations is described exhaustively. Then, using the most direct method, we classify local conservation laws. Some exact solutions are constructed by the classical Lie method.

Keywords: Nonlinear diffusion equations; Equivalence transformations; Lie symmetries; Conservation laws

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1. Introduction

In this paper we study nonlinear partial differential equations (PDEs) of the general class

$$f(x)u_t = \big(g(x)u^n u_x\big)_x + h(x)u^m \tag{1}$$

with Lie symmetry point of view. Here $f = f(x)$, $g = g(x)$ and $h = h(x)$ are arbitrary smooth functions of their variables, $f(x)g(x) \neq 0$, n and m are arbitrary constants. The linear case is excluded from consideration as well-investigated. We additionally assume the diffusion coefficient to be nonlinear, i.e. $n \neq 0$. The case $n = 0$ is quite singular and will be investigated separately.

Various simple transport models of electron temperature in a confined plasma are reducible to equations of the form (1) [19]. Here u represents the temperature, $f(x)$ the density and $g(x)$ the density-dependent part of the thermal diffusion. The term $h(x)u^m$ corresponds to the heat source. Furthermore a number of special cases of the class (1) has been used to model successfully problems in mathematical physics, chemistry and biology [8,27,28,34,42].

While there is no existing general theory for solving nonlinear PDEs, many special cases have yielded to appropriate changes of variables. Point transformations are the ones which are mostly used. These are transformations in the space of the dependent and the independent variables of a PDE. Probably the most useful point transformations of PDEs are those which form a continuous Lie group of transformations, which leave the equation invariant. Symmetries of this PDE are then revealed, perhaps enabling new solutions to be found directly or via similarity reductions. The classical method of finding Lie symmetries is first to find infinitesimal transformations, with the benefit of linearization, and then to extend these to groups of finite transformations. This method is easy to apply and well established in the last few years [4,12,15,30]. A large number of diffusion-type equations has been studied using Lie group analysis. See for example, [10,16,38].

A goal in the present work is to classify the Lie symmetries of the general class (1). The problem of classification of Lie symmetries is dated back in the late fifties, when Ovsiannikov [31] determined all the forms of the well-known nonlinear diffusion equation

$$u_t = \big[f(u)u_x\big]_x$$

that admit such symmetries. Since this latter work a number of articles on the classification of Lie symmetries of diffusion type equations appeared in the literature. See for example, [3,7,9,10,16,25,29,38]. The Lie symmetries of (1) when $f = g = h = 1$ can be found in [9,10] (see also [14, Chapter 10]). In fact, Dorodnitsyn [9,10] carried out the classification for the nonlinear diffusion equation with a source,

$$u_t = \big[k(u)u_x\big]_x + q(u). \tag{2}$$

Recently Popovych and Ivanova [38] presented the classification of Lie symmetries for the general class of diffusion–convection equations

$$f(x)u_t = \big[g(x)A(u)u_x\big]_x + B(u)u_x$$

and later Ivanova and Sophocleous [16] derived the Lie symmetries for the class

$$f(x)u_t = \big[g(x)A(u)u_x\big]_x + h(x)B(u)u_x. \tag{3}$$

Furthermore in the present work, using the most direct method [39], we carry out two classifications of local conservation laws up to equivalence relations. Conservations laws of (1) when $f = g = h = 1$ can be found in [11].

The presence of three functions in Eq. (1) makes our task very difficult. However knowledge of the equivalence transformations for class (1) enable us to consider an equivalent, but simpler form. For this reason, in the next section we construct different kinds of equivalence groups for the class under consideration. Extensive investigation and usage of equivalence transformations have the two-fold effect for further group analysis of class (1): reduction of the number of cases to be studied and simplification of form of both equations and basis elements of their Lie algebras in these cases. Moreover, presentation of classification results in a closed form becomes possible only after application of generalizations of usual point equivalence group.

The group classification of the equations from class (1) is carried out in Section 3. It is shown that usage of the *generalized extended* equivalence group and *conditional* equivalence groups plays a crucial role in solving this problem. Importance of true choice of gauge for arbitrary elements with respect to equivalence transformations is discussed and illustrated by examples. Let us note that similar investigation on generalizations of equivalence group and different possibilities in choice of gauges was first fulfilled in [17,18] for class (3).

In Section 4 we look for the additional equivalence transformations between obtained cases of symmetry extension, which are inequivalent with respect to transformations from the adduced equivalence groups. As a result, the problem of group classification in class (1) with respect to the set of all local transformations is solved in passing. Connection between additional equivalence transformations and conditional equivalence groups is demonstrated.

Results of Section 5 are key for this paper. The set of *form-preserving transformations* [22] (called also *admissible* [37] ones) of class (1) is described exhaustively. Roughly speaking, any point transformation which links a pair of equations from the class under consideration is called an admissible transformation for this class. See [37] for rigorous definitions and problem statements. At the best of our knowledge, first the set of admissible transformations was described in non-trivial case, namely for a class of generalized Burgers equations, by Kingston and Sophocleous in [21]. Although the same problem has been already solved for a number of different classes of PDEs [6,22,23,37,40], these results are not well known.

The set of admissible transformations of class (1) has an interesting structure. The class (1) can be presented as a union of normalized subclasses. Some of the subclasses admit non-trivial extensions of the equivalence group, which will be conditional equivalence groups for the whole class (1). Moreover, there are subclasses intersecting each other. In such cases interaction of the corresponding conditional equivalence groups gives admissible transformations which cannot be interpreted in group terms. By-products of investigation of admissible transformations are, in particular, exhaustive description of the additional equivalence transformations between symmetry extension cases and derivation of all condition resulting in non-trivial conditional equivalence groups.

After presenting necessary notions and tools (Section 6), in Section 7 we classify local conservations laws of the equations from class (1).

Studying Eqs. (1) in the framework of Lie group analysis is completed in Section 8 with construction of exact solutions for some equations from this class. We use results of fulfilled group classification and apply the classical Lie method. The main idea of the section is to show simplification introduced in finding exact solutions with equivalence transformations.

2. Equivalence transformations and choice of investigated class

In this section we derive the equivalence transformations of class (1). These transformations enable us to reduce the class under consideration to a simpler form. We also present the equivalence transformations of the reduced class and a special subclass.

Note 1. The value of the arbitrary element m is undefined if $h=0$. In this case, it is most convenient for one to assume m equal to $n+1$. Sometimes the value $m=1$ is also acceptable.

The usual equivalence group $G^{\sim}$ of class (1) is formed by the nondegenerate point transformations in the space of (t,x,u,f,g,h,n,m), which are projectible on the space of (t,x,u), i.e. they have the form

$$(\tilde t,\tilde x,\tilde u)=\big(T^t,T^x,T^u\big)(t,x,u),$$
$$(\tilde f,\tilde g,\tilde h,\tilde n,\tilde m)=\big(T^f,T^g,T^h,T^n,T^m\big)(t,x,u,f,g,h,n,m),$$

and transform any equation from class (1) for the function $u=u(t,x)$ with the arbitrary elements (f,g,h,n,m) to an equation from the same class for the function $\tilde u=\tilde u(\tilde t,\tilde x)$ with the new arbitrary elements $(\tilde f,\tilde g,\tilde h,\tilde n,\tilde m)$.

Theorem 1. *$G^{\sim}$ consists of the transformations*

$$\tilde t=\delta_1 t+\delta_2,\qquad \tilde x=\varphi(x),\qquad \tilde u=\delta_3 u,$$
$$\tilde f=\frac{\delta_0\delta_1}{\delta_3\varphi_x}f,\qquad \tilde g=\frac{\delta_0\varphi_x}{\delta_3^{n+1}}g,\qquad \tilde h=\frac{\delta_0}{\delta_3^m\varphi_x}h,\qquad \tilde n=n,\qquad \tilde m=m,$$

where δ_j $(j=\overline{0,3})$ are arbitrary constants, $\delta_0\delta_1\delta_3\neq 0$, φ is an arbitrary smooth function of x, $\varphi_x\neq 0$.

It appears that class (1) admits other equivalence transformations which do not belong to $G^{\sim}$ and form, together with usual equivalence transformations, a *generalized extended equivalence group*. Restrictions on transformations can be weakened in two directions. We admit the transformations of the variables t, x and u can depend on arbitrary elements (the prefix "generalized" [26]), and this dependence are not point necessarily and have to become point with respect to (t,x,u) after fixing values of arbitrary elements. The explicit form of the new arbitrary elements $(\tilde f,\tilde g,\tilde h,\tilde n,\tilde m)$ is determined via (t,x,u,f,g,h,n,m) in some non-fixed (possibly, nonlocal) way (the prefix "extended"). We construct the complete (in this sense) generalized extended equivalence group $\hat G^{\sim}$ of class (1), using the direct method [22].

Theorem 2. *The generalized extended equivalence group $\hat G^{\sim}$ of class* (1) *is formed by the transformations*

$$\tilde t=\delta_1 t+\delta_2,\qquad \tilde x=\varphi(x),\qquad \tilde u=\psi(x)u,$$
$$\tilde f=\frac{\delta_0\delta_1}{\varphi_x\psi^{n+2}}f,\qquad \tilde g=\frac{\delta_0\varphi_x}{\psi^{2n+2}}g,\qquad \tilde h=\frac{\delta_0}{\varphi_x\psi^{m+n+1}}h,\qquad \tilde n=n,\qquad \tilde m=m,$$

where δ_j $(j=0,1,2)$ are arbitrary constants, $\delta_0\delta_1\neq 0$, φ is an arbitrary smooth function of x, $\varphi_x\neq 0$. The function $\psi(x)$ is determined by the formula

$$\psi(x)=\begin{cases}(1-(n+1)F(x))^{-\frac{1}{n+1}}, & n\neq -1,\\ e^{F(x)}, & n=-1,\end{cases}\qquad \text{where } F(x)=\delta_3\int\frac{dx}{g(x)}+\delta_4.$$

Note 2. The above representation of the function $\psi(x)$ guarantees that this family of transformations is continuously parameterized with the parameter n.

From Theorem 1 we deduce that the transformation

$$\tilde t = t, \qquad \tilde x = \int \frac{dx}{g(x)}, \qquad \tilde u = u \tag{4}$$

maps equation (1) to

$$\tilde f(\tilde x)\tilde u_{\tilde t} = \big(\tilde u^n \tilde u_{\tilde x}\big)_{\tilde x} + \tilde h(\tilde x)\tilde u^m,$$

where $\tilde f(\tilde x) = g(x)f(x)$, $\tilde g(\tilde x) = 1$ and $\tilde h(\tilde x) = g(x)h(x)$. (Likewise any equation of form (1) can be reduced to the same form with $\tilde f(\tilde x) = 1$.) That is why, without loss of generality, we can restrict ourselves to investigation of the equations of the general form

$$f(x)u_t = \big(u^n u_x\big)_x + h(x)u^m. \tag{5}$$

All results on symmetries, solutions and conservation laws of class (5) can be extended to class (1) with transformations (4).

It appears that we can deduce the equivalence group for class (5) from Theorems 1 and 2 by setting $\tilde g = g = 1$. The results are summarized in the following theorem.

Theorem 3. *The generalized equivalence group $G_1^\sim$ of the class* (5), *where $n \neq -1$, consists of the transformations*

$$\tilde t = \delta_1 t + \delta_2, \qquad \tilde x = \frac{\delta_3 x + \delta_4}{\delta_5 x + \delta_6} =: \varphi(x), \qquad \tilde u = \delta_7 \varphi_x^{\frac{1}{2n+2}} u,$$

$$\tilde f = \delta_1 \delta_7^n \varphi_x^{-\frac{3n+4}{2n+2}} f, \qquad \tilde h = \delta_7^{-m+n+1} \varphi_x^{-\frac{m+3n+3}{2n+2}} h, \qquad \tilde n = n, \qquad \tilde m = m,$$

where δ_j $(j = \overline{1,7})$ are arbitrary constants, $\delta_1\delta_7 \neq 0$ and $\delta_3\delta_6 - \delta_4\delta_5 = \pm 1$.

For $n = -1$ transformations from group $G_1^\sim$ take the form

$$\tilde t = \delta_1 t + \delta_2, \qquad \tilde x = \delta_3 x + \delta_4, \qquad \tilde u = \delta_5 e^{\delta_6 x} u,$$

$$\tilde f = \frac{\delta_1}{\delta_3^2 \delta_5 e^{\delta_6 x}} f, \qquad \tilde h = \frac{1}{\delta_3^2 \delta_5^m e^{m\delta_6 x}} h, \qquad \tilde n = n, \qquad \tilde m = m,$$

where δ_j $(j = \overline{1,6})$ are arbitrary constants, $\delta_1\delta_3\delta_5 \neq 0$.

These equivalence transformations can be employed to simplify the forms of $f(x)$ and $h(x)$ in the study of Lie symmetries of (5) which takes place in the next section.

Note 3. Theorem 3 can be reformulated in the following way: The usual equivalence group $G_1^\sim$ of class (5) for values $n \neq -1$ is formed by the transformations

$$\tilde t = \delta_1 t + \delta_2, \qquad \tilde x = \frac{1}{\varepsilon(n+1)\delta_3(1 - (n+1)(\delta_3 x + \delta_4))} - \frac{1}{\varepsilon(n+1)\delta_3} =: \varphi(x),$$

$$\tilde u = \big(1 - (n+1)(\delta_3 x + \delta_4)\big)^{-\frac{1}{n+1}} u = \varepsilon^{\frac{1}{2n+2}} \varphi_x^{\frac{1}{2n+2}} u =: \psi(x) u,$$

$$\tilde f = \delta_1 \varepsilon^{\frac{n}{2n+2}} \varphi_x^{-\frac{3n+4}{2n+2}} f, \qquad \tilde h = \varepsilon^{\frac{-m+n+1}{2n+2}} \varphi_x^{-\frac{m+3n+3}{2n+2}} h, \qquad \tilde n = n, \qquad \tilde m = m,$$

where δ_j $(j = \overline{1,4})$ and ε are arbitrary constants, $\varepsilon\delta_1\delta_3 \neq 0$.

This representation of the transformations from $G_1^\sim$ is more clumsily than in the theorem but it shows that the family of the above transformations is continuously parameterized with the parameter n, including the value $n=-1$. In fact,

$$\lim_{n\to -1}\varphi(x)=\frac{x}{\varepsilon}+\frac{\delta_4}{\varepsilon\delta_3},\qquad \lim_{n\to -1}\psi(x)=e^{\delta_3 x+\delta_4}.$$

Since the parameters n and m are invariants of all the above equivalence transformations, class (1) (or class (5)) can be presented as the union of disjoint subclasses where each from the subclasses corresponds to fixed values of n and m. This representation allows us to give other interpretation of the above results. For example, the generalized equivalence group $G_1^\sim$ from Theorem 3, more exactly its projection to the complementary set of variables and arbitrary elements, can be considered as a family of usual equivalence groups of the subclasses parameterized with n and m.

The question occurs: could one obtain wider equivalence groups for some of the subclasses, a priori assuming the parameters n and m satisfy a condition? The answer is positive in the case of $m=n+1$. Namely, the following statement is true.

Theorem 4. *The class of equations*

$$f(x)u_t=\big(g(x)u^n u_x\big)_x+h(x)u^{n+1}\tag{6}$$

admits the equivalence group $G^\sim_{m=n+1}$ *consisting of the transformations*:

$$\tilde t=\delta_1 t+\delta_2,\qquad \tilde x=\varphi(x),\qquad \tilde u=\psi(x)u,$$
$$\tilde f=\frac{\delta_0\delta_1}{\psi^{n+2}\varphi_x}f,\quad \tilde g=\frac{\delta_0\varphi_x}{\psi^{2n+2}}g,\quad \tilde h=\delta_0\frac{h-\psi^{n+1}(\psi^{-(n+2)}\psi_x g)_x}{\psi^{2n+2}\varphi_x}\quad (\tilde n=n),$$

where φ *and* ψ *are arbitrary functions of* x, δ_j $(j=0,1,2)$ *are arbitrary constants,* $\delta_0\delta_1\varphi_x\psi\neq 0$.

It should be emphasized that $G^\sim_{m=n+1}$ is the usual equivalence group of class (6) even through n is assumed as an arbitrary element. Moreover, it is wider than the generalized extended equivalence group $\hat G^\sim$ of whole class (1). We will also called $G^\sim_{m=n+1}$ as the *conditional equivalence group* of class (1) under the condition $m=n+1$ on the arbitrary elements. It is a non-trivial conditional equivalence group of class (1) in the sense that it is not a subgroup of $\hat G^\sim$. Let us note that the notion of conditional equivalence group was first used in classification of systems of two nonlinear Laplace equations [36] (see also [37]).

Gauging g with the condition $g=1$, we impose the restrictions $\delta_0\varphi_x=\psi^{2n+2}$ on parameters of $G^\sim_{m=n+1}$ and obtain the equivalence group $G^\sim_{1,m=n+1}$ of the class of equations

$$f(x)u_t=\big(u^n u_x\big)_x+h(x)u^{n+1}.\tag{7}$$

Corollary 1. *The equivalence group* $G^\sim_{1,m=n+1}$ *of class* (7) *is formed by the transformations*

$$\tilde t=\delta_1 t+\delta_2,\qquad \tilde x=\varphi(x),\qquad \tilde u=\psi(x)u,$$
$$\tilde f=\frac{\delta_0^2\delta_1}{\psi^{3n+4}}f,\quad \tilde h=\delta_0^2\frac{h-\psi^{n+1}[\psi^{-(n+2)}\psi_x]_x}{\psi^{4n+4}}\quad (\tilde n=n),$$

where φ *and* ψ *are arbitrary functions satisfying the condition* $\delta_0\varphi_x=\psi^{2n+2}$, δ_j $(j=0,1,2)$ *are arbitrary constants,* $\delta_0\delta_1\psi\neq 0$.

Similarly to $G_1^{\sim}$, $G_{1,m=n+1}^{\sim}$ is the generalized equivalence group for the whole class (7) since transformations of variables explicitly depend on the arbitrary element n. And $G_{1,m=n+1}^{\sim}$ becomes the usual equivalence group under restriction to any subclass with a fixed value of n.

Simultaneous usage of both the generalized equivalence group $G_1^{\sim}$ and the conditional equivalence group $G_{1,m=n+1}^{\sim}$ via gauging arbitrary elements leads to crucial simplification of solving the group classification problem and presentation of the obtained results. See Note 6 and Example 3 for some explanations.

There exists another condition on arbitrary elements, which gives a non-trivial conditional equivalence group. All such conditions and the corresponding conditional equivalence groups are systematically investigated as components of classification of admissible transformations in Section 5.

Note 4. Due to physical sense of Eq. (1), the function u should satisfy the condition $u \geqslant 0$. In this case we have to demand for the multipliers of u to be positive in all transformations. If we avoid positiveness of u then we have to use the modular of u as base of powers which are not determined for negative values of base. The same statement is true for similar expressions in transformations and other places. The necessary changes in formulas are obvious.

3. Lie symmetries

In the section we present a complete classification of Lie symmetries for class (5). We search for operators of the form $\Gamma = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$, which generate one-parameter groups of point symmetry transformations of equations from class (5). These operators satisfy the necessary and sufficient criterion of infinitesimal invariance, i.e. action of the rth prolongation $\Gamma^{(r)}$ of Γ to the (rth order) DE results in identically zero, modulo the DE under consideration. Here we require that

$$\Gamma^{(2)}\big\{f(x)u_t - u^n u_{xx} - nu^{n-1}u_x^2 - h(x)u^m\big\} = 0 \tag{8}$$

identically, modulo equation (5).

After elimination of u_t due to (5), Eq. (8) becomes an identity in six variables, t, x, u, u_x, u_{xx} and u_{tx}. In fact, Eq. (8) is a multivariable polynomial in the variables u_x, u_{xx} and u_{tx}. The coefficients of the different powers of these variables must be zero, giving the determining equations on the coefficients τ, ξ and η. Since Eq. (5) has a specific form (it is a quasi-linear evolution equation, the right-hand side of (5) is a polynomial in the pure derivatives of u with respect to x, etc.), the forms of the coefficients can be simplified. That is, $\tau = \tau(t)$, $\xi = \xi(t,x)$ [22] and, moreover, $\eta = \zeta(t,x)u$ and $\xi = \xi(x)$ (see Section 5). To derive the latter condition, we partially split with respect to u. Finally we obtain the *classifying* equations which include both the residuary uncertainties in coefficients of the operator and the arbitrary elements of the class under consideration:

$$f_x\xi = f(n\zeta + \tau_t - 2\xi_x), \qquad 2(n+1)\zeta_x = \xi_{xx},$$
$$uf^2\zeta_t - u^{n+1}f\zeta_{xx} + u^m\big(hf_x\xi - fh_x\xi + (1-m)fh\zeta - fh\tau_t\big) = 0.$$

The third equation is a polynomial in u and should be split with respect to u for all values of the parameters n and m, which correspond to the nonlinear case $n \neq 0$. Splitting gives essentially different results in three exclusive cases: 1. $m \neq 1, n+1$; 2. $m = 1$; 3. $m = n+1$. The obtained equations enable us to derive the forms of $\tau(t)$, $\xi(x)$, $\zeta(t,x)$, $f(x)$ and $h(x)$ depending on values of n and m and consequently the desired Lie symmetries will be constructed.

Table 1
Results of group classification of class (1)

	n	$f(x)$	$h(x)$	Basis of $A^{\max}$
General case				
1	$\forall$	$\forall$	$\forall$	∂_t
2	$\forall$	$f_1(x)$	$h_1(x)$	$\partial_t, (d+2b-pn)t\partial_t+((n+1)ax^2+bx+c)\partial x+(ax+p)u\partial_u$
3	$\forall$	1	ε	$\partial_t, \partial_x, 2(1-m)t\partial_t+(1+n-m)x\partial_x+2u\partial_u$
$m=1$, $h\neq 0$, $(h/f)_x=0$				
4	$\forall$	$\forall$	εf	$\partial_t, e^{-\varepsilon nt}(\partial_t+\varepsilon u\partial_u)$
5	$\forall$	$f_1(x)$	εf	$\partial_t, e^{-\varepsilon nt}(\partial_t+\varepsilon u\partial_u), n((n+1)ax^2+bx+c)\partial x+(nax+2b+d)u\partial_u$
6	$\neq -\frac{4}{3}$	1	ε	$\partial_t, \partial_x, e^{-\varepsilon nt}(\partial_t+\varepsilon u\partial_u), nx\partial x+2u\partial_u$
7	$-\frac{4}{3}$	1	ε	$\partial_t, \partial_x, e^{\frac{4}{3}\varepsilon t}(\partial_t+\varepsilon u\partial_u), -\frac{4}{3}x\partial x+2u\partial_u, -\frac{1}{3}x^2\partial_x+xu\partial_u$
$m=n+1$ or $h=0$				
8	$\forall$	$\forall$	$\forall$	$\partial_t, nt\partial_t-u\partial_u$
9	$\neq -\frac{4}{3}$	1	αx^{-2}	$\partial_t, nt\partial_t-u\partial_u, 2t\partial_t+x\partial_x$
10	$\neq -\frac{4}{3}$	1	ε	$\partial_t, nt\partial_t-u\partial_u, \partial_x$
11	$\neq -\frac{4}{3}$	1	0	$\partial_t, \partial_x, nt\partial_t-u\partial_u, 2t\partial_t+x\partial_x$
12	$-\frac{4}{3}$	e^x	α	$\partial_t, t\partial_t+\frac{3}{4}u\partial_u, \partial_x-\frac{3}{4}u\partial_u$
13	$-\frac{4}{3}$	1	0	$\partial_t, \partial_x, \frac{4}{3}t\partial t+u\partial_u, 2t\partial_t+x\partial_x, -\frac{1}{3}x^2\partial_x+xu\partial_u$

Here α is arbitrary constant, $\alpha\neq 0$ in case 9, $\varepsilon=\pm 1$,

$$f_1(x)=\exp\left[\int\frac{-(3n+4)ax+d}{(n+1)ax^2+bx+c}\,dx\right],$$
$$h_1(x)=\exp\left[\int\frac{-(3(n+1)+m)ax+(1+m-n)p-2b}{(n+1)ax^2+bx+c}\,dx\right],$$

and it can be assumed up to equivalence with respect to $G_1^{\sim}$ that the parameter tuple (a,b,c,d) takes only the following non-equivalent values:

$$(\varepsilon,0,1,0)\quad\text{if}\quad n=-1\quad\text{or}\quad(1,0,1,d')\quad\text{if}\quad n\neq -1,\quad(0,1,0,d'),\quad(0,0,1,1),$$

where d' is arbitrary constant. In all the cases we put $g(x)=1$.

In case 8 the parameter-functions f and h can be additionally gauged with equivalence transformations from $G^{\sim}_{1,m=n+1}$. For example, we can put $f=1$ if $n\neq -4/3$ and $f=e^x$ otherwise.

In Table 1 we list tuples of parameter-functions $f(x)$ and $h(x)$, the constant parameters m and n and bases of the invariance algebras in all possible inequivalent cases of Lie symmetry extension. The operators from Table 1 form bases of the maximal Lie invariance algebras iff the corresponding values of the parameters are inequivalent to ones with more abundant Lie invariance algebras. In cases 2 and 5 we do not try to use equivalence transformations as much as possible since otherwise a number of similar simplified cases would be derived, see remarks after the table.

Note 5. It should be emphasized that we adduce only the cases of extensions of maximal Lie invariance algebra, which are inequivalent with respect to $G_1^{\sim}$ if $m\neq n+1$ and with respect to $G^{\sim}_{1,m=n+1}$ if $m=n+1$ or $h=0$. During solving the group classification problem, we derive some cases of extensions, which are transformed with equivalence transformations to representatives from the table in a non-trivial way. Below we give some examples of such transformations.

Example 1. The 'variable coefficient' equation

$$|x|^{-\frac{3n+4}{n+1}}u_t = \left(u^n u_x\right)_x + \varepsilon|x|^{-\frac{3n+3+m}{n+1}}u^m, \quad n \neq -1,\ m \neq n+1, \tag{9}$$

is transformed to the 'constant coefficient' equation

$$\tilde u_{\tilde t} = \left(\tilde u^n \tilde u_{\tilde x}\right)_{\tilde x} + \varepsilon \tilde u^m \tag{10}$$

(case 3 of Table 1 if $m \neq 1$ and case 6 if $m = 1$) by the transformation

$$\tilde t = t, \qquad \tilde x = \frac{1}{x} =: \varphi(x), \qquad \tilde u = |x|^{-\frac{1}{n+1}}u = |\varphi_x|^{\frac{1}{2n+2}}u$$

which belongs to the equivalence group $G_1^\sim$ of the class (5). Hence, using the maximal Lie invariance algebra of case 3, we can derive the basis elements of the maximal Lie invariance algebra of (9):

$$\partial_t, \quad (n+1)x^2\partial_x + xu\partial_u, \quad 2(1-m)t\partial_t - (n+1-m)x\partial_x + \frac{n+1+m}{n+1}u\partial_u$$

if $m \neq 1$ and

$$\partial_t, \quad e^{-\varepsilon nt}(\partial_t + \lambda u\partial_u), \quad (n+1)x^2\partial_x + xu\partial_u, \quad n(n+1)x\partial x - (n+2)u\partial_u$$

if $m = 1$. In the latter case we additionally assume $n \neq -4/3$ for the algebra to be really maximal.

Example 2. In an analogous way, the 'variable coefficient' equation

$$e^x u_t = \left(\frac{u_x}{u}\right)_x + \varepsilon e^x u \tag{11}$$

is reduced by the transformation $\tilde t = t$, $\tilde x = x$, $\tilde u = e^x u$ from the equivalence group $G_1^\sim$ to Eq. (10) with $n = -1$ and $m = 1$ (case 6 with $n = -1$). The basis elements of the maximal Lie invariance algebra of (11) have the form

$$\partial_t, \quad e^{\varepsilon t}(\partial_t + \varepsilon u\partial_u), \quad \partial_x - u\partial_u, \quad x\partial_x - (x+2)u\partial_u.$$

Note 6. Right choice of a gauge of the parameter tuple (f, g, h) is another crucial point of our investigations. Really, the major choice has been made from the very outset of classification when the parameter-function g was put equal to 1. It is the gauge that leads to maximal simplification of both the whole solving and the final results. Although all gauges are equivalent from an abstract point of view, only the gauge $g = 1$ allows one to exhaustively solve the problem of group classification of class (1) with reasonable quantity of calculations. Even after the other simplest gauge $f = 1$ chosen, calculations become too cumbersome and sophisticated.

The expression of transformations from the conditional equivalence group $G^\sim_{m=n+1}$, i.e. the equivalence group of the subclass (6) separated from class (1) with the condition $m = n+1$, contains one more arbitrary function of x in comparison with the equivalence group of the whole class. This fact makes an additional gauge of the parameter tuple (f, g, h) is possible and necessary in case of $m = n+1$. Different ways of the additional gauge with the conditional equivalence group $G^\sim_{m=n+1}$ (or $G^\sim_{1,m=n+1}$) are discussed in the next example.

Example 3. Let $m = n+1$ and the gauge $g = 1$ be already fixed. The kernel $A^{\ker}_{1,m=n+1}$ of maximal Lie invariance algebras of the subclass (7) is generated by the operators ∂_t and $nt\partial_t - u\partial_u$ (case 8 of Table 1).

It seems on the face of it that the optimal choice for the additional gauge is $h=0$. This gauge results to essential simplification of both the general form of initial equations and the determining equations. Indeed, the system of determining equations are then reduced to the following one:

$$\xi f_x = (n\zeta - 2\xi_x + \tau_t)f, \qquad \xi_{xx} = 2(n+1)\zeta_x, \qquad \zeta_{xx} = 0, \qquad \zeta_t = 0.$$

Hence, $\zeta = C_1x + C_2$, $\xi = C_1(n+1)x^2 + C_3x + C_4$, $\tau = C_5t + C_6$. The main difficulty arises after taking into account the first equation. It implies that extension of $A^{\mathrm{ker}}_{m=n+1}$ is possible iff

$$\big(\alpha x^2 + \beta x + \gamma\big)f_x = \delta f,$$

where α, β, γ and δ are arbitrary constants, $(\alpha,\beta,\gamma)\neq(0,0,0)$. Moreover, the equivalence group of the gauged class '$g=1$, $h=0$' is formed by the transformations from $G^{\sim}_{m=n+1}$ which additionally satisfy the conditions $\delta_0\varphi_x = \psi^{2n+2}$, $(\psi^{-(n+1)}\psi_x)_x = 0$. Although classification may be carried out in this way, complexity of both the expression of f in some extension cases and equivalence transformations to be applied results in necessity of careful and sophisticated investigations of reducibility between different cases.

Another way is to begin with gauging the parameter-function f. The obtained gauge

$$f = 1 \quad \text{if} \quad n \neq -\frac{4}{3} \quad \text{and} \quad f = e^x \quad \text{or} \quad (f,h) = (1,0) \quad \text{otherwise}$$

seems to be more complicated than $h=0$. But it is the gauge that straight leads to exhaustive classification, and formulation of results having a simplest form (cases 8–13 of Table 1).

Note 7. We tested different ways of classification. Implemented 'experiments' result in the following conclusion. Application of equivalence transformations is more effective on earlier stages of classification. An expedient way is to gauge arbitrary elements as much as possible on preparatory classification stage and additionally to gauge arbitrary elements under classification every time when sufficient conditions to do it arises. The worst choice is to refuse to use gauges completely.

After analyzing the classification adduced in Table 1, we can formulate the following theorem which is similar to a one from [38].

Theorem 5. *If an equation of form* (1) *is invariant with respect to a Lie algebra of dimension not less than four then it can be reduced by means of point transformations to a one with constant values of the parameters f, g and h. If $m\neq 1$, $n+1$, the similar statement is also true for three-dimensional Lie invariance algebras.*

4. Additional equivalence transformations

Let a class of differential equations be classified with respect to its equivalence group $G^{\sim}$ and projection of $G^{\sim}$ on the space of the equation variables be narrower than the whole pseudogroup of point transformations in this space. Then there can exist point transformations between $G^{\sim}$-inequivalent cases of symmetry extension, which are called *additional equivalence transformations*. Knowledge of them is important since it simplifies further application of group classification results.

The first non-trivial and interesting example of such transformations was given in [24] (see also [14, Chapter 10]). Under classification of ‘constant coefficient’ nonlinear reaction–diffusion equations (2) in [9,10], equations of the form

$$u_t = \left(u^{-4/3}u_x\right)_x + bu^{-1/3},$$

where b is an arbitrary constant, arose as cases of symmetry extension, which are inequivalent with respect to the corresponding equivalence group for different values of b. It was shown in [24] that the transformation

$$t' = t, \qquad x' = \begin{cases} e^{-2\sqrt{b/3}x}, & b > 0, \\ \tan(\sqrt{-b/3}x), & b < 0, \end{cases} \qquad u' = \begin{cases} e^{3\sqrt{b/3}x}u, & b > 0, \\ \cos^3(\sqrt{-b/3}x)u, & b < 0, \end{cases}$$

maps any such equation with the equation

$$u'_{t'} = \left(u'^{-4/3}u'_{x'}\right)_{x'}$$

which is of the same form with $b' = 0$. Therefore, it is an additional equivalence transformation in class (2). It is an additional equivalence transformation in classes (1) and (5) and, at the same time, a member of the equivalence group $G^{\sim}_{1,m=n+1}$ of class (7). That is why the case $b \neq 0$ does not arise under our classification. Let us note that application of generalized, extended and conditional equivalence groups simultaneously with the usual one results in implicit but effective usage of additional equivalences.

Another additional equivalence transformation in class (2) [14, Chapter 10]

$$t' = \frac{1}{\varepsilon n}e^{\varepsilon nt}, \qquad x' = x, \qquad u' = e^{-\varepsilon t}u \tag{12}$$

links the equations

$$u_t = \left(u^n u_x\right)_x + \varepsilon u \quad \text{and} \quad u'_{t'} = \left(u'^n u'_{x'}\right)_{x'}. \tag{13}$$

This transformation belongs to no equivalence group found in Section 2. It can be obviously extended to a wider subclass of (1). Namely, it connects the equations

$$f(x)u_t = \left(g(x)u^n u_x\right)_x + \varepsilon f(x)u \quad \text{and} \quad f(x')u'_{t'} = \left(g(x')u'^n u'_{x'}\right)_{x'}$$

and, therefore, reduces cases 4–7 of Table 1 to the set of cases ‘$m = n + 1$ or $h = 0$,’ i.e. it is an additional equivalence transformation in class (1). Any other additional equivalence transformations is a composition of transformations of form (12) and transformations from $G^{\sim}_{1,m=n+1}$. (It is proved in the next section in the framework of admissible transformations.) For example, if $n = -\frac{4}{3}$ Eqs. (13) are also connected by the mapping

$$t' = \frac{1}{\varepsilon n}e^{\varepsilon nt}, \qquad x' = \frac{\delta}{x}, \qquad u' = e^{-\varepsilon t}x^3 u, \qquad \delta = \pm 1.$$

Seeming scarcity of additional equivalence transformations in class (1), e.g. in comparison with the class $f(x)u_t = [g(x)A(u)u_x]_x + h(x)B(u)u_x$ admitting a large number of additional equivalence transformations [16,38], is connected with usage of conditional equivalence groups under classification. As it is demonstrated in the next section, the transformations of form (12) can be also included in the framework of conditional equivalence.

As a result, the following statement is derived.

Theorem 6. *Up to point transformations, a complete list of extensions of the maximal Lie invariance group of equations from class* (1) *is exhausted by cases* 1–3 *and* 8–13 *of Table* 1.

5. Classification of form-preserving (admissible) transformations

Due to special structure of equations from class (1), the following problem can be solved completely. To describe all point transformations each of which connects a pair of equations from class (1). Such transformations are called *form-preserving* [22] or *admissible* [37] *transformations*. See [37] for stronger definitions. They can be naturally interpreted in terms of the category theory [41]. Let us note that there exists an infinitesimal equivalent of this notion [6].

Since class (1) can be gauged with transformations from its usual equivalence group $G^{\sim}$, it is enough for one to solve the similar problem in class (5). To do it, we consider a pair of equations from the class under consideration, i.e. (5) and

$$\tilde f(\tilde x)\tilde u_{\tilde t}=\left(\tilde u^{\tilde n}\tilde u_{\tilde x}\right)_{\tilde x}+\tilde h(\tilde x)\tilde u^{\tilde m} \tag{14}$$

and assume that these equations are connected via a point transformation of the general form

$$\tilde t=T(t,x,u),\qquad \tilde x=X(t,x,u),\qquad \tilde u=U(t,x,u).$$

We have to derive determining equations in functions T, X and U and to solve them depending on values of arbitrary elements in (5) and (14).

After substitution of expressions for the tilde-variables into (14), we obtain an equation in the tildeless variables. It should be an identity on the manifold determined by (5) in the second-order jet space over the variables $(t,x\,|\,u)$. Splitting of this identity with respect to the unconstrained variables in a true way results at first in the equations

$$T_u=T_x=X_u=U_{uu}=0$$

that agrees with results on more general classes of evolution equations [22,38,41]. We split further, taking into account the above equations and also splitting with respect to u partially. As a results, the following relations are deduced:

$$\begin{aligned}
&\tilde n=n,\qquad \tilde m=m\quad\text{or}\quad (m,\tilde m)\in\big\{(1,n+1),(n+1,1)\big\},\\
&T=T(t),\qquad X=X(x),\qquad U=V(t,x)u,\qquad T_tX_xV\neq 0,\\
&2(n+1)X_xV_x=X_{xx}V,\\
&V^n=\frac{\tilde f}{f}\frac{{X_x}^2}{T_t},\\
&\frac{\tilde f}{f}\frac{V}{T_t}hu^m+\tilde f\frac{V_t}{T_t}u=\frac{1}{X_x}\left(V^n\frac{V_x}{X_x}\right)_x u^{n+1}+\tilde hV^{\tilde m}u^{\tilde m}.
\end{aligned} \tag{15}$$

Splitting of Eq. (15) with respect to u and subsequent integration of the determining equations depends on values of m and $\tilde m$ appreciably. Consider different cases separately.

If $\tilde m=m\neq 1$, Eq. (15) implies $V_t=0$ hence $T_{tt}=0$, i.e. $V=V(x)$ and $T=\delta_1t+\delta_2$. The other conditions are

$$\begin{aligned}
&\tilde m=m\neq 1,\ n+1:\quad \frac{\tilde f}{f}=\delta_1\frac{V^n}{X_x},\quad \frac{\tilde h}{h}=\frac{V^{n+1-m}}{X_x^2},\\
&n=-1:\quad X_{xx}=0,\quad (\ln V)_{xx}=0,\\
&n\neq -1:\quad \big(|X_x|^{-1/2}\big)_{xx}=0,\quad V^{2n+2}=\delta_0X_x;\\
&\tilde m=m=n+1:\quad \frac{\tilde f}{f}=\delta_1\frac{V^n}{X_x},\quad \tilde h=\frac{1}{X_x^2}h-\frac{1}{X_xV^{n+1}}\left(V^n\frac{V_x}{X_x}\right)_x,\quad V^{2n+2}=\delta_0X_x.
\end{aligned}$$

Therefore, in the case $m \neq 1, n+1$ (the case $m = n+1$) any admissible transformation is determined by a transformation from the equivalence group $G_1^{\sim}$ (the conditional equivalence group $G^{\sim}_{1,m=n+1}$).

If $\tilde m = m = 1$, the function V is conveniently presented as $V = |T_t|^{-1/n}\psi(x)$. The value $n = -1$ is again singular:

$$\begin{aligned}
&n = -1: \quad X_{xx} = 0, \quad (\ln\psi)_{xx} = 0 \quad \Longrightarrow \quad X = \delta_3 x + \delta_4, \quad \psi = \delta_5 e^{\delta_6 x},\\
&n \neq -1: \quad \left(|X_x|^{-1/2}\right)_{xx} = 0, \quad 2(n+1)\frac{\psi_x}{\psi} = \frac{X_{xx}}{X_x}\\
&\qquad\qquad \Longrightarrow \quad X = \frac{\delta_3 x + \delta_4}{\delta_5 x + \delta_6}, \quad \psi = \delta_7 |X_x|^{\frac{1}{2n+2}},
\end{aligned}$$

where $\delta_3\delta_6 - \delta_4\delta_5$ has a fixed value which can be assumed equal to ± 1. The arbitrary elements f and h are transformed by the formulas

$$\frac{\tilde f}{f} = \frac{\psi^n}{X_x^2}\operatorname{sign} T_t, \qquad \frac{\tilde h}{\tilde f} = \frac{1}{T_t}\frac{h}{f} + \frac{1}{n}\left(\frac{1}{T_t}\right)_t, \quad \text{hence} \quad \left(\frac{\tilde h}{\tilde f}\right)_x = \frac{1}{T_t}\left(\frac{h}{f}\right)_x.$$

In view of the latter condition, $(\tilde h/\tilde f)_x \neq 0$ and $T_{tt} = 0$ if $(h/f)_x \neq 0$. Therefore, any admissible transformation in this case corresponds to a transformation from the equivalence group $G_1^{\sim}$.

Different situation is in case of $(h/f)_x = 0$. Then $(\tilde h/\tilde f)_x = 0$, i.e. the condition $(h/f)_x = 0$ is invariant with respect to $G_1^{\sim}$. Let the constants h/f and $\tilde h/\tilde f$ be denoted by α and $\tilde\alpha$, respectively. Then we get the following equation for the function $T(t)$:

$$\left(\frac{1}{T_t}\right)_t = -n\alpha\frac{1}{T_t} + n\tilde\alpha.$$

We integrate these equations and present the general solution in such form that continuous dependence of it on the parameters α and $\tilde\alpha$ is obvious:

$$\begin{aligned}
&\alpha\tilde\alpha \neq 0: \quad \frac{e^{n\tilde\alpha T} - 1}{n\tilde\alpha} = \delta_1\frac{e^{n\alpha t} - 1}{n\alpha} + \delta_2, \qquad \alpha = 0,\ \tilde\alpha \neq 0: \quad \frac{e^{n\tilde\alpha T} - 1}{n\tilde\alpha} = \delta_1 t + \delta_2,\\
&\alpha \neq 0,\ \tilde\alpha = 0: \quad T = \delta_1\frac{e^{n\alpha t} - 1}{n\alpha} + \delta_2, \qquad \alpha = \tilde\alpha = 0: \quad T = \delta_1 t + \delta_2.
\end{aligned}$$

The above-described transformations form the equivalence group $G^{\sim}_{1,m=1,(h/f)_x=0}$ of the subclass of class (1), which is separated with the conditions $g = 1$, $m = 1$, $(h/f)_x = 0$. It is a generalized equivalence group even through n is fixed since it contains transformations with respect to t, which depends on (constant) ratio of the arbitrary elements h and f. Unlike $G^{\sim}_{1,m=n+1}$, we do not use $G^{\sim}_{1,m=1,(h/f)_x=0}$ under group classification of class (5) since application of this conditional equivalence group does not have crucial influence on classification and the corresponding conditions on arbitrary elements are less obvious. At the same time, transformations from $G^{\sim}_{1,m=1,(h/f)_x=0}$ play the role of additional equivalence transformations after completing the classification (see the previous section).

If $h\tilde h \neq 0$, $m = 1$ and $\tilde m = n+1$ then the determining equations yield

$$\frac{h}{f} = \frac{1}{n}\frac{T_{tt}}{T_t}, \quad \text{i.e.} \quad \left(\frac{h}{f}\right)_x = 0, \qquad \tilde h = -\frac{1}{X_x V^{n+1}}\left(V^n\frac{V_x}{X_x}\right)_x.$$

Hence both Eqs. (5) and (14) are mapped to the subclass '$h = 0$' of the class under consideration. These mappings are realized by the transformations from the corresponding conditional equivalence groups. We can assume that their images coincides. Therefore, the admissible transformation is composition of a transformation $\mathcal{T}_1$ of (5) to the equation $f(x)u_t = (u^n u_x)_x$ and a transformation $\mathcal{T}_2$ of the equation $f(x)u_t = (u^n u_x)_x$ to (14). The transformation $\mathcal{T}_1$ belongs to $G^{\sim}_{1,m=1,(h/f)_x=0}$ and the transformation $\mathcal{T}_2$ belongs to $G^{\sim}_{1,m=n+1}$, and we can put $X = x$, $\psi = 1$ in $\mathcal{T}_1$.

The case $h\tilde{h} \neq 0$, $m = n + 1$ and $\tilde{m} = 1$ is considered in similar way.

All the possible cases are exhausted.

Let us summarize investigation of the admissible transformations in class (5) in the following statement.

Theorem 7. *Let the equations $f(x)u_t = (u^n u_x)_x + h(x)u^m$ and $\tilde{f}(x)u_t = (u^{\tilde{n}} u_x)_x + \tilde{h}(x)u^{\tilde{m}}$ be connected via a point transformation in the variables t, x and u. Then*

$$\tilde{n} = n \quad \text{and either} \quad \tilde{m} = m \quad \text{or} \quad (m, \tilde{m}) = (1, n+1) \quad \text{or} \quad (m, \tilde{m}) = (n+1, 1).$$

The transformation is determined by a transformation from

(a) $G^{\sim}_1$ *if either $m \neq 1, n+1$ or $m = 1$, $(h/f)_x \neq 0$;*
(b) $G^{\sim}_{1,m=n+1}$ *if $m = \tilde{m} = n+1$;*
(c) $G^{\sim}_{1,m=1,(h/f)_x=0}$ *if $m = \tilde{m} = 1$, $(h/f)_x = 0$; then also $(\tilde{h}/\tilde{f})_x = 0$.*

If $m = 1$ and $\tilde{m} = n+1$ then $(h/f)_x = 0$ and the transformation is composition of two transformations from $G^{\sim}_{1,m=1,(h/f)_x=0}$ and $G^{\sim}_{1,m=n+1}$ with the intermediate equation having $h = 0$.

The case $m = n+1$ and $\tilde{m} = 1$ is similar.

Corollary 2. *Class* (5) *with $n \neq 0$ can be presented as the union of normalized subclasses separated by the conditions*

$$h \neq 0, \quad m \neq 1,\ n+1; \qquad m = 1, \quad (h/f)_x \neq 0;$$
$$m = 1, \quad (h/f)_x = 0; \qquad m = n+1.$$

Only two latter subclasses have a non-empty intersection, and the intersection being the normalized subclass '$h = 0$.'

Roughly speaking, the class of differential equations is called normalized if any admissible transformation in this class belongs to its equivalence group. See [37] for strong definitions.

6. Basic definitions and statements on conservation laws

In this section we give basic definitions and statements on conservation laws, following the well-known textbook of Olver [30] in general outlines. Then we formulate the notion of equivalence of conservation laws with respect to equivalence groups, which was first introduced in [39]. This notion is a base for modification of the direct method of construction of conservation laws, which is applied in Section 7 for exhaustive classification of local conservation laws of Eqs. (1).

Let $\mathcal{L}$ be a system $L(x, u_{(\rho)}) = 0$ of l differential equations $L^1 = 0, \ldots, L^l = 0$ for m unknown functions $u = (u^1, \ldots, u^m)$ of n independent variables $x = (x_1, \ldots, x_n)$. Here $u_{(\rho)}$ denotes the set of all the derivatives of the functions u with respect to x of order no greater than ρ, including u as the derivatives of the zero order. Let $\mathcal{L}_{(k)}$ denote the set of all algebraically independent differential consequences that have, as differential equations, orders no greater than k. We identify $\mathcal{L}_{(k)}$ with the manifold determined by $\mathcal{L}_{(k)}$ in the jet space $J^{(k)}$.

Definition 1. A *conserved vector* of the system $\mathcal{L}$ is an n-tuple $F = (F^1(x, u_{(r)}), \dots, F^n(x, u_{(r)}))$ for which the divergence $\operatorname{Div} F := D_i F^i$ vanishes for all solutions of $\mathcal{L}$ (i.e. $\operatorname{Div} F|_{\mathcal{L}} = 0$).

In Definition 1 and below $D_i = D_{x_i}$ denotes the operator of total differentiation with respect to the variable x_i, i.e. $D_i = \partial_{x_i} + u^a_{\alpha,i}\partial_{u^a_\alpha}$, where u^a_α and $u^a_{\alpha,i}$ stand for the variables in jet spaces, which correspond to derivatives $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$ and $\partial u^a_\alpha/\partial x_i$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We use the summation convention for repeated indices and assume any function as its zero-order derivative. The notation $V|_{\mathcal{L}}$ means that values of V are considered only on solutions of the system $\mathcal{L}$.

Definition 2. A conserved vector F is called *trivial* if $F^i = \hat{F}^i + \check{F}^i$, $i = \overline{1,n}$, where $\hat{F}^i$ and $\check{F}^i$ are, likewise F^i, functions of x and derivatives of u (i.e. differential functions), $\hat{F}^i$ vanish on the solutions of $\mathcal{L}$ and the n-tuple $\check{F} = (\check{F}^1, \dots, \check{F}^n)$ is a null divergence (i.e. its divergence vanishes identically).

The triviality concerning the vanishing conserved vectors on solutions of the system can be easily eliminated by confining on the manifold of the system, taking into account all its necessary differential consequences. A characterization of all null divergences is given by the following lemma (see e.g. [30]).

Lemma 1. *The n-tuple $F = (F^1, \dots, F^n)$, $n \geqslant 2$, is a null divergence* ($\operatorname{Div} F \equiv 0$) *iff there exist smooth functions v^{ij} $(i, j = \overline{1,n})$ of x and derivatives of u, such that $v^{ij} = -v^{ji}$ and $F^i = D_j v^{ij}$.*

The functions v^{ij} are called *potentials* corresponding to the null divergence F. If $n = 1$ any null divergence is constant.

Definition 3. Two conserved vectors F and F' are called *equivalent* if the vector-function $F' - F$ is a trivial conserved vector.

The above definitions of triviality and equivalence of conserved vectors are natural in view of the usual ‘empiric’ definition of conservation laws of a system of differential equations as divergences of its conserved vectors, i.e. divergence expressions which vanish for all solutions of this system. For example, equivalent conserved vectors correspond to the same conservation law. It allows us to formulate the definition of conservation law in a rigorous style (see e.g. [43]). Namely, for any system $\mathcal{L}$ of differential equations the set $\mathrm{CV}(\mathcal{L})$ of conserved vectors of its conservation laws is a linear space, and the subset $\mathrm{CV}_0(\mathcal{L})$ of trivial conserved vectors is a linear subspace in $\mathrm{CV}(\mathcal{L})$. The factor space $\mathrm{CL}(\mathcal{L}) = \mathrm{CV}(\mathcal{L})/\mathrm{CV}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\mathrm{CV}(\mathcal{L})$ with respect to the equivalence relation adduced in Definition 3.

Definition 4. The elements of $\mathrm{CL}(\mathcal{L})$ are called *conservation laws* of the system $\mathcal{L}$, and the whole factor space $\mathrm{CL}(\mathcal{L})$ is called as *the space of conservation laws* of $\mathcal{L}$.

That is why we assume description of the set of conservation laws as finding $\mathrm{CL}(\mathcal{L})$ that is equivalent to construction of either a basis if $\dim \mathrm{CL}(\mathcal{L}) < \infty$ or a system of generatrices in the infinite-dimensional case. The elements of $\mathrm{CV}(\mathcal{L})$ which belong to the same equivalence class giving a conservation law $\mathcal{F}$ are considered all as conserved vectors of this conservation law, and we will additionally identify elements from $\mathrm{CL}(\mathcal{L})$ with their representatives in $\mathrm{CV}(\mathcal{L})$. For $F \in \mathrm{CV}(\mathcal{L})$ and $\mathcal{F} \in \mathrm{CL}(\mathcal{L})$ the notation $F \in \mathcal{F}$ will denote that F is a conserved vector corresponding to the conservation law $\mathcal{F}$. In contrast to the order r_F of a conserved vector F as the maximal order of derivatives explicitly appearing in F, the *order of the conservation law* $\mathcal{F}$ is called $\min\{r_F \mid F \in \mathcal{F}\}$. Under linear dependence of conservation laws we understand linear dependence of them as elements of $\mathrm{CL}(\mathcal{L})$. Therefore, in the framework of "representative" approach conservation laws of a system $\mathcal{L}$ are considered *linearly dependent* if there exists linear combination of their representatives, which is a trivial conserved vector.

Let the system $\mathcal{L}$ be totally nondegenerate [30]. Then application of the Hadamard lemma to the definition of conserved vector and integrating by parts imply that divergence of any conserved vector of $\mathcal{L}$ can be always presented, up to the equivalence relation of conserved vectors, as a linear combination of left side of independent equations from $\mathcal{L}$ with coefficients λ^μ being functions on a suitable jet space $J^{(k)}$:

$$\mathrm{Div}\, F = \lambda^\mu L^\mu. \tag{16}$$

Here the order k is determined by $\mathcal{L}$ and the allowable order of conservation laws, $\mu = \overline{1, l}$.

Definition 5. Formula (16) and the l-tuple $\lambda = (\lambda^1, \dots, \lambda^l)$ are called the *characteristic form* and the *characteristic* of the conservation law $\mathrm{Div}\, F = 0$ correspondingly.

The characteristic λ is *trivial* if it vanishes for all solutions of $\mathcal{L}$. Since $\mathcal{L}$ is nondegenerate, the characteristics λ and $\tilde{\lambda}$ satisfy (16) for the same F and, therefore, are called *equivalent* iff $\lambda - \tilde{\lambda}$ is a trivial characteristic. Similarly to conserved vectors, the set $\mathrm{Ch}(\mathcal{L})$ of characteristics corresponding to conservation laws of the system $\mathcal{L}$ is a linear space, and the subset $\mathrm{Ch}_0(\mathcal{L})$ of trivial characteristics is a linear subspace in $\mathrm{Ch}(\mathcal{L})$. The factor space $\mathrm{Ch}_{\mathrm{f}}(\mathcal{L}) = \mathrm{Ch}(\mathcal{L}) / \mathrm{Ch}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\mathrm{Ch}(\mathcal{L})$ with respect to the above characteristic equivalence relation.

We can essentially simplify and order classification of conservation laws, taking into account additionally symmetry transformations of a system or equivalence transformations of a whole class of systems. Such problem is similar to one of group classification of differential equations.

Proposition 1. *Any point transformation g maps a class of equations in the conserved form into itself. More exactly, the transformation g: $\tilde{x} = x_g(x, u)$, $\tilde{u} = u_g(x, u)$ prolonged to the jet space $J^{(r)}$ transforms the equation $D_i F^i = 0$ to the equation $D_i F^i_g = 0$. The transformed conserved vector F_g is determined by the formula*

$$F^i_g(\tilde{x}, \tilde{u}_{(r)}) = \frac{D_{x_j} \tilde{x}_i}{|D_x \tilde{x}|} F^j(x, u_{(r)}), \quad \text{i.e.} \quad F_g(\tilde{x}, \tilde{u}_{(r)}) = \frac{1}{|D_x \tilde{x}|} (D_x \tilde{x}) F(x, u_{(r)}) \tag{17}$$

in the matrix notions. Here $|D_x \tilde{x}|$ is the determinant of the matrix $D_x \tilde{x} = (D_{x_j} \tilde{x}_i)$.

Note 8. In the case of one dependent variable ($m = 1$) g can be a contact transformation: $\tilde{x} = x_g(x, u_{(1)})$, $\tilde{u}_{(1)} = u_{g(1)}(x, u_{(1)})$. Similar notes are also true for the statements below.

Definition 6. Let G be a symmetry group of the system $\mathcal{L}$. Two conservation laws with the conserved vectors F and F' are called *G-equivalent* if there exists a transformation $g \in G$ such that the conserved vectors F_g and F' are equivalent in the sense of Definition 3.

Any transformation $g \in G$ induces a linear one-to-one mapping g_* in $\mathrm{CV}(\mathcal{L})$, transforms trivial conserved vectors only to trivial ones (i.e. $\mathrm{CV}_0(\mathcal{L})$ is invariant with respect to g_*) and therefore induces a linear one-to-one mapping g_f in $\mathrm{CL}(\mathcal{L})$. It is obvious that g_f preserves linear (in)dependence of elements in $\mathrm{CL}(\mathcal{L})$ and maps a basis (a set of generatrices) of $\mathrm{CL}(\mathcal{L})$ in a basis (a set of generatrices) of the same space. In such way we can consider the G-equivalence relation of conservation laws as well-determined on $\mathrm{CL}(\mathcal{L})$ and use it to classify conservation laws.

Proposition 2. *If the system $\mathcal{L}$ admits a one-parameter group of transformations then the infinitesimal generator $X = \xi^i \partial_i + \eta^a \partial_{u^a}$ of this group can be used for construction of new conservation laws from known ones. Namely, differentiating equation* (17) *with respect to the parameter ε and taking the value $\varepsilon = 0$, we obtain the new conserved vector*

$$\widetilde{F}^i = -X_{(r)} F^i + \left(D_j \xi^i\right) F^j - \left(D_j \xi^j\right) F^i. \tag{18}$$

Here $X_{(r)}$ denotes the rth prolongation [30,32] *of the operator X.*

Note 9. Formula (18) can be directly extended to generalized symmetry operators (see, for example, [5]). A similar statement for generalized symmetry operators in evolutionary form ($\xi^i = 0$) was known earlier [13,30]. It was used in [20] to introduce a notion of basis of conservation laws as a set which generates a whole set of conservation laws with action of generalized symmetry operators and operation of linear combination.

Proposition 3. *Any point transformation g between systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces a linear one-to-one mapping g_* from* $\mathrm{CV}(\mathcal{L})$ *into* $\mathrm{CV}(\tilde{\mathcal{L}})$, *which maps* $\mathrm{CV}_0(\mathcal{L})$ *into* $\mathrm{CV}_0(\tilde{\mathcal{L}})$ *and generates a linear one-to-one mapping g_f from* $\mathrm{CL}(\mathcal{L})$ *into* $\mathrm{CL}(\tilde{\mathcal{L}})$.

Corollary 3. *Any point transformation g between systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces a linear one-to-one mapping $\hat{g}_\mathrm{f}$ from* $\mathrm{Ch_f}(\mathcal{L})$ *into* $\mathrm{Ch_f}(\tilde{\mathcal{L}})$.

It is possible to obtain an explicit formula for correspondence between characteristics of $\mathcal{L}$ and $\tilde{\mathcal{L}}$. Let $\tilde{\mathcal{L}}^\mu = \Lambda^{\mu\nu} \mathcal{L}^\nu$, where $\Lambda^{\mu\nu} = \Lambda^{\mu\nu\alpha} D^\alpha$, $\Lambda^{\mu\nu\alpha}$ are differential functions, $\alpha = (\alpha_1, \dots, \alpha_n)$ runs the multi-indices set ($\alpha_i \in \mathbb{N} \cup \{0\}$), $\mu, \nu = \overline{1, l}$. Then

$$\lambda^\mu = \Lambda^{\nu\mu *}\left(|D_x \tilde{x}| \tilde{\lambda}^\nu\right).$$

Here $\Lambda^{\nu\mu *} = (-D)^\alpha \cdot \Lambda^{\mu\nu\alpha}$ is the adjoint to the operator $\Lambda^{\nu\mu}$. For a number of cases, e.g. if $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are single partial differential equations ($l = 1$), the operators $\Lambda^{\mu\nu}$ are simply differential functions (i.e. $\Lambda^{\mu\nu\alpha} = 0$ for $|\alpha| > 0$) and, therefore, $\Lambda^{\nu\mu *} = \Lambda^{\mu\nu}$.

Consider the class $\mathcal{L}|_\mathcal{S}$ of systems $\mathcal{L}_\theta$: $L(x, u_{(\rho)}, \theta(x, u_{(\rho)})) = 0$ parameterized with the parameter-functions $\theta = \theta(x, u_{(\rho)})$. Here L is a tuple of fixed functions of x, $u_{(\rho)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(x, u_{(\rho)}) = (\theta^1(x, u_{(\rho)}), \dots, \theta^k(x, u_{(\rho)}))$ running the set $\mathcal{S}$ of solutions of the system $S(x, u_{(\rho)}, \theta_{(q)}(x, u_{(\rho)})) = 0$. This system consists of differential equations on θ, where x and $u_{(\rho)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order no greater than q. In what follows we call the functions θ arbitrary elements. Denote the point transformations group preserving the form of the systems from $\mathcal{L}|_{\mathcal{S}}$ as $G^{\sim} = G^{\sim}(L, S)$.

Consider the set $P = P(L, S)$ of all pairs each of which consists of a system $\mathcal{L}_\theta$ from $\mathcal{L}|_{\mathcal{S}}$ and a conservation law $\mathcal{F}$ of this system. In view of Proposition 3, action of transformations from $G^{\sim}$ on $\mathcal{L}|_{\mathcal{S}}$ and $\{\mathrm{CV}(\mathcal{L}_\theta) \mid \theta \in \mathcal{S}\}$ together with the pure equivalence relation of conserved vectors naturally generates an equivalence relation on P.

Definition 7. Let $\theta, \theta' \in \mathcal{S}$, $\mathcal{F} \in \mathrm{CL}(\mathcal{L}_\theta)$, $\mathcal{F}' \in \mathrm{CL}(\mathcal{L}_{\theta'})$, $F \in \mathcal{F}$, $F' \in \mathcal{F}'$. The pairs $(\mathcal{L}_\theta, \mathcal{F})$ and $(\mathcal{L}_{\theta'}, \mathcal{F}')$ are called *$G^{\sim}$-equivalent* if there exists a transformation $g \in G^{\sim}$ which transforms the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\theta'}$ and such that the conserved vectors F_g and F' are equivalent in the sense of Definition 3.

Classification of conservation laws with respect to $G^{\sim}$ will be understood as classification in P with respect to the above equivalence relation. This problem can be investigated in the way that is similar to group classification in classes of systems of differential equations, especially it is formulated in terms of characteristics. Namely, we construct firstly the conservation laws that are defined for all values of the arbitrary elements. (The corresponding conserved vectors may depend on the arbitrary elements.) Then we classify, with respect to the equivalence group, arbitrary elements for each of that the system admits additional conservation laws.

In an analogues way we also can introduce equivalence relations on P, which are generated by either generalizations of usual equivalence groups or all admissible point or contact transformations (called also form-preserving ones [22]) in pairs of equations from $\mathcal{L}|_{\mathcal{S}}$.

Note 10. It can be easy shown that all the above equivalences are indeed equivalence relations, i.e. they have the usual reflexive, symmetric and transitive properties.

7. Conservation laws

We search (local) conservation laws of equations from class (1), applying the modification of the most direct method, which was proposed in [39] and preliminaries of which are described in the previous section. It is based on using the definition of conservation laws, the notion of equivalence of conservation laws with respect to a transformation group and classification up to the equivalence group of a class of differential equations. In view of results of the previous section, it is sufficient for exhaustive investigation if we classify conservation laws of equations only from class (5). Let us note that other kinds of the direct methods, which are based e.g. on the characteristic form of conservation laws [1,2] could be also applied.

There are two independent variables t and x in equations under consideration, which play a part of the time and space variables correspondingly. Therefore, the general form of constructed conservation laws will be

$$D_t F(t, x, u_{(r)}) + D_x G(t, x, u_{(r)}) = 0, \tag{19}$$

where D_t and D_x are the operators of total differentiation with respect to t and x. The components F and G of the conserved vector (F, G) are called the *density* and the *flux* of the conservation law. Two conserved vectors (F, G) and (F', G') are equivalent if there exist such functions $\hat{F}$, $\hat{G}$ and H of t, x and derivatives of u that $\hat{F}$ and $\hat{G}$ vanish for all solutions of $\mathcal{L}$ and $F' = F + \hat{F} + D_x H$, $G' = G + \hat{G} - D_t H$.

The following lemma on order of local conservation laws for more general class of second-order evolution equations, which covers class (5), is used below.

Lemma 2. [18] *Any local conservation law of any second-order* $(1+1)$*-dimensional quasi-linear evolution equation has the first order and, moreover, there exists its conserved vector with the density depending at most on* t, x, *and* u *and the flux depending at most on* t, x, u *and* u_x.

Theorem 8. *A complete list of Eqs.* (5) *having non-trivial conservation laws is exhausted by the following ones*:

(1) $m = n+1$.

$$n \neq -1: \quad \left(\varphi^i fu, -\varphi^i u^n u_x + \varphi^i_x \frac{u^{n+1}}{n+1}\right), \quad \varphi^i, \quad i = 1, 2,$$
$$n = -1: \quad \left(xfu, -xu^{-1}u_x + \ln u\right), \quad x; \qquad \left(fu, -u^{-1}u_x\right), \quad 1.$$

(2) $m = 1$, $h = \mu f$.

$$n \neq -1: \quad \left(xe^{-\mu t} fu, e^{-\mu t}\left(-xu^n u_x + \frac{u^{n+1}}{n+1}\right)\right), \quad xe^{-\mu t};$$
$$\left(e^{-\mu t} fu, -e^{-\mu t}u^n u_x\right), \quad e^{-\mu t},$$
$$n = -1: \quad \left(xe^{-\mu t} fu, e^{-\mu t}\left(-xu^{-1}u_x + \ln u\right)\right), \quad xe^{-\mu t};$$
$$\left(e^{-\mu t} fu, -e^{-\mu t}u^{-1}u_x\right), \quad e^{-\mu t}.$$

Here β_1 *and* β_2 *are arbitrary constants. The functions* $\varphi^i = \varphi^i(x)$, $i = 1, 2$, *form a fundamental set of solutions of the second-order linear ordinary differential equation* $\varphi_{xx} + (n+1)h\varphi = 0$. (*Together with constraints on the parameter-functions* f *and* h *we also adduce conserved vectors and characteristics of the basis elements of the corresponding space of conservation laws.*)

Proof. In view of Lemma 2, we can assume at once that $F = F(t,x,u)$ and $G = G(t,x,u,u_x)$. Let us substitute the expression for u_t deduced from (5) into (19) and decompose the obtained equation with respect to u_{xx}. The coefficient of u_{xx} gives the equation $u^n f^{-1} F_u + G_{u_x} = 0$, therefore $G = -u^n f^{-1} F_u u_x + \hat{G}(t,x,u)$. Taking into account the latter expression for G and splitting the rest of Eq. (19) with respect to the powers of u_x, we obtain the system of PDEs on the functions F and $\hat{G}$ of the form

$$F_{uu} = 0, \qquad -u^n\left(\frac{F_u}{f}\right)_x + \hat{G}_u = 0, \qquad F_t + \frac{h}{f}u^m F_u + \hat{G}_x = 0. \tag{20}$$

Solving first two equations of (20) yields $F = \Phi(t,x)fu + F^0(t,x)$, $\hat{G} = \Phi_x \int u^n + G^0(t,x)$. The case $n = -1$ is special with respect to integration of the function u^n. So

$$G = \begin{cases} -\Phi u^n u_x + \Phi_x \frac{u^{n+1}}{n+1} + G^0(t,x), & n \neq -1, \\ -\Phi u^{-1} u_x + \Phi_x \ln u + G^0(t,x), & n = -1. \end{cases}$$

(It is convenient to separate f as a multiplier in the coefficient of u in the expression of F.) In further consideration the major role is played by a differential consequence of system (20) that can be written as

$$f\Phi_t + \Phi_{xx}u^n + mh\Phi u^{m-1} = 0. \tag{21}$$

Indeed, it is the unique classifying condition for this problem. In all the classification cases we obtain the equation $F^0_t + G^0_x = 0$. Therefore, we can assume $F^0 = G^0 = 0$ up to conserved vector equivalence and have to suppose for existence of non-trivial conserved vectors that $\Phi \neq 0$.

It follows from (21) that Eq. (5) possesses non-trivial conserved vectors only for the special values of the parameters n and m, namely, if

$$(n,m) \in \{(0,0),\ (0,1),\ (n',n'+1),\ (n',1),\ n' \in \mathbb{R}\}.$$

We exclude from consideration the cases (0, 0) and (0, 1) since they correspond to the linear case of Eq. (5).

(1) $m = n + 1$. Equation (21) is split to two equations $\Phi_t = 0$ and $\Phi_{xx} + (n+1)h\Phi = 0$, i.e. Φ depends only on x and runs the set of solutions of the second equation which is homogeneous linear second-order ordinary differential one. $\Phi = C_1 x + C_2$ if $n = -1$.

(2) $m = 1$. Split of (21) results in the equations $\Phi_{xx} = 0$ and $f\Phi_t + h\Phi = 0$. The equation $\Phi_{xx} = 0$ gives $\Phi = \Phi^1(t)x + \Phi^2(t)$. Substitution Φ in this form to the second equation yields $\Phi^1_t f x + \Phi^2_t f + \Phi^1 h x + \Phi^2 h = 0$. It follows from the latter equation in view of the condition $\Phi \neq 0$ that $h = \mu f$ and $\Phi = C_3 e^{-\mu t} x + C_4 e^{-\mu t}$. □

Corollary 4. *For any* (*nonlinear*) *equation from class* (1)*, the dimension of the space of local conservation laws is equal to either* 0 *or* 2*. In the second case the equation can be reduced by a point transformation to an equation from the same class, where* $g = 1$ *and* $h = 0$*. Then a basis of the corresponding space of characteristics is formed by the identically unity* ($\varphi^1 = 1$) *and the function* $\varphi^2 = x$.

8. Similarity solutions

The Lie symmetry operators found as a result of solving the group classification problem can be applied to construction of exact solutions of the corresponding equations. The method of reductions with respect to subalgebras of Lie invariance algebras is well known and quite algorithmic to use in most cases; we refer to the standard textbooks on the subject [30,32]. Exhaustive group analysis, including Lie reductions, of nonlinear constant–coefficient reaction–diffusion equations with nonlinearities of general form was carried out in [10] (see also [14]). A number of exact solutions of such equations are tabulated (see e.g. [35]). Then the problem on Lie reductions and Lie exact solutions of equations from class (1), which have four- or five-dimensional Lie invariance algebra (or even three-dimensional Lie invariance algebra in case $m \neq n + 1$) can be assumed as solved in view of Theorem 5.

The Lie reduction algorithm is more effective for equations having nice symmetry properties. Together with the above facts, it gives solid argumentation for us to choose the cases from Table 1 with at least one non-constant parameter-function of x and a three-dimensional Lie invariance algebra. Namely, these are cases 9 and 12. (Let us remember that case 5 can be transformed by an additional equivalence transformation to the case with the same value of f and $\varepsilon = 0$ and then reduced by equivalence transformations to one of cases 9–12 depending on the values of n and f.)

We have shown that equations

$$u_t = (u^n u_x)_x + \alpha x^{-2} u^{n+1} \quad \text{and} \quad e^x u_t = \left(u^{-\frac{4}{3}} u_x\right)_x + \alpha u^{-\frac{1}{3}}$$

(Table 1, case 9 and case 12) admit the three-dimensional Lie invariance algebras generated by the operators

$$X_1 = \partial_t, \qquad X_2 = t\partial_t - \frac{u}{n}\partial_u, \qquad X_3 = x\partial_x + \frac{2u}{n}\partial_u$$

and

$$X_1 = \partial_t, \qquad X_2 = t\partial_t + \frac{3}{4}u\partial_u, \qquad X_3 = \partial_x - \frac{3}{4}u\partial_u$$

correspondingly. Due to strong simplification of these cases with equivalence transformations, the operators are simplified to simple linear combinations of translation and scale operators and, therefore, are handy for further usage. The both tuples of operators satisfy the commutation relations

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = 0.$$

It means that the algebras are isomorphic to the algebra $\mathfrak{g}_2 \oplus \mathfrak{g}_1$ being the direct sum of the two-dimensional non-Abelian Lie algebra $\mathfrak{g}_2$ and the one-dimensional Lie algebra $\mathfrak{g}_1$. An optimal set of subalgebras of $\mathfrak{g}_2 \oplus \mathfrak{g}_1$ can be easily constructed with application of a standard technique [30,32]. Another way is to take the set from [33]. (In this paper optimal sets of subalgebras are listed for all three- and four-dimensional algebras.) The used optimal set consists of

one-dimensional subalgebras: $\langle X_2 - \mu X_3\rangle$, $\langle X_3\rangle$, $\langle X_1 \pm X_3\rangle$, $\langle X_1\rangle$;

two-dimensional subalgebras: $\langle X_1, X_3 - \nu X_2\rangle$, $\langle X_1, X_2\rangle$,

where μ and ν are arbitrary constants.

Lie reduction to algebraic equations can be made only with the first two-dimensional subalgebra; the second one does not satisfy the transversality condition [30]. The corresponding ansatzes and reduced algebraic equations have the form

$$9. \quad u = Cx^{\sigma}, \quad \text{where } \sigma = \frac{\nu+2}{n}; \quad C^{n+1}\big((n+1)\sigma^2 - \sigma + \alpha\big) = 0;$$

$$12. \quad u = Ce^{\sigma x}, \quad \text{where } \sigma = -\frac{3}{4}(\nu+1); \quad C^{-\frac{1}{3}}\big(\sigma^2 - 3\alpha\big) = 0.$$

Here C is an unknown constant to be found. The reduced equations are compatible and have non-trivial (non-zero) solutions only for some values of σ and, moreover, become identities for these values of σ. As a result, the following stationary solutions are constructed:

$$9. \quad u = Cx^{\sigma}, \quad \text{where } (n+1)\sigma^2 - \sigma + \alpha = 0;$$

$$12. \quad u = Ce^{\sigma x}, \quad \text{where } \sigma^2 = 3\alpha.$$

Here C is an arbitrary constant. These solutions can be also obtained with step-by-step reductions with respect to one-dimensional subalgebras.

The ansatzes and reduced equations corresponding to the one-dimensional subalgebras from the optimal system are collected in Table 2.

Some reduced equations are integrated completely:

$$9.2. \quad \varphi = \left(C - \left(\alpha n + 2 + \frac{4}{n}\right)\omega\right)^{-\frac{1}{n}};$$

$$9.4. \quad \varphi = \begin{cases} C_1\omega^{\frac{1}{2(n+1)}}(\ln\omega + C_2)^{\frac{1}{n+1}}, & \text{if } \alpha' = 0,\\ (C_1\omega^{\varkappa_1} + C_2\omega^{\varkappa_2})^{\frac{1}{n+1}}, & \text{if } \alpha' > 0,\\ \omega^{\frac{1}{2(n+1)}}(C_1\sin(\sigma\ln\omega) + C_2\cos(\sigma\ln\omega))^{\frac{1}{n+1}}, & \text{if } \alpha' < 0, \end{cases}$$

where $\alpha' = 1 - 4\alpha(n+1)$, $\varkappa_{1,2} = \frac{1\pm\sqrt{\alpha'}}{2}$, $\sigma = \frac{\sqrt{-\alpha'}}{2}$;

$$12.2. \quad \varphi = \left(C + \left(\frac{4}{3}\alpha - \frac{1}{4}\right)\omega\right)^{\frac{3}{4}};$$

Table 2
Similarity reductions of cases 9 and 12

N	X	ω	$u=$	Reduced ODE
9.1	$X_2-\mu X_3$	xt^{μ}	$t^{-\frac{1+2\mu}{n}}\varphi(\omega)$	$(\varphi^n\varphi_\omega)_\omega-\mu\omega\varphi_\omega+\frac{1+2\mu}{n}\varphi+\alpha\omega^{-2}\varphi^{n+1}=0$
9.2	X_3	t	$x^{\frac{2}{n}}\varphi(\omega)$	$n^2\varphi_\omega-(\alpha n^2+2n+4)\varphi^{n+1}=0$
9.3	$X_3\pm X_1$	$xe^{\mp t}$	$e^{\pm\frac{2t}{n}}\varphi(\omega)$	$(\varphi^n\varphi_\omega)_\omega\pm\omega\varphi_\omega\mp\frac{2}{n}\varphi+\alpha\omega^{-2}\varphi^{n+1}=0$
9.4	X_1	x	$\varphi(\omega)$	$(\varphi^n\varphi_\omega)_\omega+\alpha\omega^{-2}\varphi^{n+1}=0$
12.1	$X_2-\mu X_3$	$x+\mu\ln t$	$t^{\frac{3}{4}(\mu+1)}\varphi(\omega)$	$(\varphi^{-\frac{4}{3}}\varphi_\omega)_\omega-\mu e^{\omega}\varphi_\omega-\frac{3}{4}(\mu+1)e^{\omega}\varphi+\alpha\varphi^{-\frac{1}{3}}=0$
12.2	X_3	t	$e^{-\frac{3}{4}x}\varphi(\omega)$	$16\varphi_\omega+(3-16\alpha)\varphi^{-\frac{1}{3}}=0$
12.3	$X_3\pm X_1$	$x\mp t$	$e^{\mp\frac{3}{4}t}\varphi(\omega)$	$(\varphi^{-\frac{4}{3}}\varphi_\omega)_\omega\pm e^{\omega}\varphi_\omega\pm\frac{3}{4}e^{\omega}\varphi+\alpha\varphi^{-\frac{1}{3}}=0$
12.4	X_1	x	$\varphi(\omega)$	$(\varphi^{-\frac{4}{3}}\varphi_\omega)_\omega+\alpha\varphi^{-\frac{1}{3}}=0$

$$12.4.\quad \varphi=\begin{cases} C_1(\omega+C_2)^{-3}, & \text{if } \alpha=0,\\ (C_1e^{\varkappa\omega}+C_2e^{-\varkappa\omega})^{-3}, & \text{if } \alpha>0,\\ (C_1\sin(\sigma\omega)+C_2\cos(\sigma\omega))^{-3}, & \text{if } \alpha<0,\end{cases}$$

where $\varkappa=\sqrt{\alpha/3}$, $\sigma=\sqrt{-\alpha/3}$.

The above solutions of reduced equations results after substitution to the corresponding ansatzes to exact solutions of the initial equations.

Let us note that the obtained solutions can by transformed by equivalence or admissible transformations to exact solutions of much more complicated reaction–diffusion equations than the investigated ones.

9. Conclusion

In this paper different properties and objects concerning variable coefficient $(1+1)$-dimensional nonlinear reaction–diffusion equations (1) are investigated in the framework of classical group analysis: equivalence transformations, Lie symmetries, conservation laws and exact solutions.

The cornerstone of presented investigation is application of different generalizations of the usual equivalence group, including generalized, extended and conditional equivalence groups and admissible transformations, in an optimal way via gauging arbitrary elements. It is the tool that makes principal simplifications in solving the group classification problem, construction of conservation laws and finding exact solutions. The set of admissible transformations of class (1) has an interesting structure which is exhaustively described by Theorem 7.

It seems natural for one to consider other classes of PDEs in the same way. The direct extension is investigation of variable coefficient $(1+1)$-dimensional nonlinear reaction–diffusion equations for different (e.g. non-power) nonlinearities. Thus, we will complete studying the case $n=0$ which is omitted in this paper.

The adduced results form a basis for advanced analysis of class (1) with 'modern' symmetry methods. In particular, classification of conservation laws creates the necessary prerequisites for studying potential symmetries [4]. It is undoubted that generalized equivalences will continue to play a singular role in these investigations.

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