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Gorenstein homological dimensions and Auslander categories ☆

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Abstract

In this paper, we study Gorenstein projective and flat modules over a Noetherian ring R. For an R-module M, we show that Gorenstein projective dimension of M is finite if and only if Gorenstein flat dimension of M is finite provided the Krull dimension of R is finite. Moreover, in the case that R is local, we prove that Gorenstein projective dimension of an R-module M is finite if and only if $\hat{R} \otimes_R M$ belongs to the Auslander category of $\hat{R}$.

Keywords: Gorenstein flat dimension; Gorenstein projective dimension; Cotorsion module; Precover; Preenvelope

1. Introduction

Throughout this paper, R will denote a commutative ring with non-zero identity and $\hat{R}$ will denote the completion of a local ring $(R, \mathfrak{m})$. When discussing the completion of a local ring $(R, \mathfrak{m})$, we will mean the $\mathfrak{m}$-adic completion.

Auslander and Bridger [2] introduced the G-dimension, $\text{G-dim}_R M$, for every finitely generated R-module M (see also [1]). They proved the inequality $\text{G-dim}_R M \leqslant \text{pd}_R M$, with $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M$ is finite. The G-dimension has strong parallels to the pro-

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jective dimension. For instance, for a local Noetherian ring $(R, \mathfrak{m})$, the following conditions are equivalent:

(i) R is Gorenstein.
(ii) $\text{G-dim}_R\, R/\mathfrak{m} < \infty$.
(iii) All finitely generated R-modules have finite G-dimension.

This characterization of Gorenstein rings is parallel to Auslander–Buchsbaum–Serre characterization of regular rings. G-dimension also differs from projective dimension in that it is defined only for finitely generated modules. Enochs and Jenda defined in [8] Gorenstein projective modules (i.e. modules of G-dimension 0) whether the modules are finitely generated or not. Also, they defined a homological dimension, namely the Gorenstein projective dimension, $\operatorname{Gpd}_R(-)$, for arbitrary (non-finitely generated) modules. It is known that for finitely generated modules, the Gorenstein projective dimension agrees with the G-dimension. Along the same lines, Gorenstein flat and Gorenstein injective modules were introduced in [8,9].

Let R be a Cohen–Macaulay local ring admitting a dualizing module D. Foxby [11] defined $\mathcal{G}_0(R)$ to be the class of R-modules M such that $\operatorname{Tor}_i^R(D, M) = \operatorname{Ext}_R^i(D, D \otimes_R M) = 0$ for all $i \geqslant 1$ and such that the natural map $M \to \operatorname{Hom}_R(D, D \otimes_R M)$ is an isomorphism. In [10] Enochs, Jenda and Xu characterize Gorenstein projective and flat dimensions in terms of $\mathcal{G}_0(R)$.

Let R be a Noetherian ring with dualizing complex $\mathbf{D}$. The Auslander category $A(R)$ with respect to $\mathbf{D}$ is defined in [3, 3.1]. In [5], it is shown that the modules in $A(R)$ are precisely those of finite Gorenstein projective dimension (Gorenstein flat dimension), see [5, Theorem 4.1]. This may be viewed as an extension of the results of [10]. Note that, by [3, Proposition 3.4], if R is a Cohen–Macaulay local ring with a dualizing module, then an R-module M (viewed as a complex) is in $A(R)$ if and only if $M \in \mathcal{G}_0(R)$.

The main aim of this paper is to extend the characterization of finiteness of Gorenstein projective and flat dimensions in [5] to arbitrary local Noetherian rings.

Let R be a local Noetherian ring, possibly without a dualizing complex, and let $\mathbf{D}$ denote the dualizing complex of $\hat{R}$. We define $A'(R)$ to be the class of R-modules M such that $\hat{R} \otimes_R M \in A(\hat{R})$. In Sections 2 and 3, we characterize Gorenstein projective and flat modules in terms of the class $A'(R)$. To be more precise, we show the following results.

Theorem A. *Let R be a local Noetherian ring and M an R-module.*

(i) *(See Theorem 2.5.) M is Gorenstein flat if and only if M belongs to $A'(R)$ and $\operatorname{Tor}_i^R(E, M) = 0$ for all injective R-modules E and all $i > 0$.*
(ii) *(See Corollary 3.3.) M is Gorenstein projective if and only if M belongs to $A'(R)$ and $\operatorname{Ext}_R^i(M, P) = 0$ for all projective R-modules P and all $i > 0$.*

Also, by using the class $A'(R)$, we characterize modules of finite Gorenstein projective and flat dimensions. Namely, we prove the following result.

Theorem B. (See Proposition 3.1, Theorem 3.4, and Corollary 3.5.) *Let R be a Noetherian ring of finite Krull dimension d and M an R-module.*

(i) $\operatorname{Gfd}_R M \leqslant \operatorname{Gpd}_R M$ *(here $\operatorname{Gfd}_R M$ denotes the Gorenstein flat dimension of M).*
(ii) *If $\operatorname{Gpd}_R M < \infty$ or $\operatorname{Gfd}_R M < \infty$, then $\operatorname{Max}\{\operatorname{Gfd}_R M, \operatorname{Gpd}_R M\} \leqslant d$.*
(iii) *If R is local, then $\operatorname{Gpd}_R M < \infty$ if and only if $M \in A'(R)$.*

Setup and notation. If M is any R-module, we use $\mathrm{pd}_R\, M$, $\mathrm{fd}_R\, M$ and $\mathrm{id}_R\, M$ to denote the usual projective, flat and injective dimension of M, respectively. Furthermore, we write $\mathrm{Gpd}_R\, M$ and $\mathrm{Gfd}_R\, M$ for the Gorenstein projective and Gorenstein flat dimension of M, respectively. Let $\mathcal{X}$ be any class of R-modules and let M be an R-module. An $\mathcal{X}$-precover of M is an R-homomorphism $\varphi : X \to M$, where $X \in \mathcal{X}$ and such that the sequence,

$$\mathrm{Hom}_R(X', X) \xrightarrow{\mathrm{Hom}_R(X',\varphi)} \mathrm{Hom}_R(X', M) \to 0$$

is exact for every $X' \in \mathcal{X}$. If, moreover, $\varphi f = \varphi$ for $f \in \mathrm{Hom}_R(X, X)$ implies f is an automorphism of X, then φ is called an $\mathcal{X}$-cover of M. Also, an $\mathcal{X}$-preenvelope and $\mathcal{X}$-envelope of M are defined "dually." By $\overline{P(R)}$ and $\overline{F(R)}$, we denote the classes of all R-modules with finite projective and flat dimension, respectively.

We shall use the following facts without comment. If R is Noetherian of finite Krull dimension, then $\overline{P(R)} = \overline{F(R)}$ (see [14, Theorem 4.2.8]). Also, if R is Noetherian then for any $M \in \overline{P(R)}$, we have $\mathrm{pd}_R(M) \leqslant \dim R$, where $\dim R$ denotes the Krull dimension of R (see [13, Theorem 3.2.6]).

2. Gorenstein flat dimension

Let R be a local Noetherian ring and let $\mathbf{D}$ denote the dualizing complex of $\hat{R}$. Let $\mathcal{D}_b(\hat{R})$ denote the full subcategory of $\mathcal{D}(\hat{R})$ (the derived category of $\hat{R}$-modules) consisting of complexes X with $H_n(X) = 0$ for $|n| \gg 0$, see [3]. Let $A(\hat{R})$ denote the full subcategory of $\mathcal{D}_b(\hat{R})$, consisting of those complexes X for which $\mathbf{D} \otimes^{\mathbf{L}}_{\hat{R}} X \in \mathcal{D}_b(\hat{R})$ and the canonical morphism

$$\gamma_X : X \longrightarrow \mathbf{R}\mathrm{Hom}_{\hat{R}}\big(\mathbf{D}, \mathbf{D} \otimes^{\mathbf{L}}_{\hat{R}} X\big),$$

is an isomorphism. Now, we define $A'(R)$ to be the class of all R-modules M such that $\hat{R} \otimes_R M \in A(\hat{R})$.

Lemma 2.1. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of modules over a local Noetherian ring R. Then if any two of M', M, M'' are in $A'(R)$, so is the third.*

Proof. The exact sequence $0 \to M' \to M \to M'' \to 0$ yields the exact sequence $0 \to \hat{R} \otimes_R M' \to \hat{R} \otimes_R M \to \hat{R} \otimes_R M'' \to 0$. Now, the conclusion follows from [4, Lemma 3.1.13]. □

Proposition 2.2. *Let R be a local Noetherian ring and M an R-module. If $\mathrm{Gfd}_R\, M < \infty$, then $M \in A'(R)$.*

Proof. By [12, Proposition 3.10], we have $\mathrm{Gfd}_{\hat{R}}(\hat{R} \otimes_R M) < \infty$. Using [5, Theorem 4.1], we conclude that $\hat{R} \otimes_R M$ belongs to $A(\hat{R})$. So, the assertion follows by the definition. □

In the proof of the following lemma we use the method of the proof of [10, Lemma 3.1].

Lemma 2.3. *Let R be a Noetherian ring, K be a cotorsion R-module of finite flat dimension and M be an R-module. If $\mathrm{Tor}_i^R(E, M) = 0$ for all $i > 0$ and all injective R-modules E, then $\mathrm{Ext}^i_R(M, K) = 0$ for all $i > 0$.*

Proof. We prove by induction on $\mathrm{fd}_R K = n$. First, let K be flat and cotorsion. Then K is a summand of a module of the form $\mathrm{Hom}_R(E, E')$ where E and E' are injective [7, Lemma 2.3]. It is enough to show that $\mathrm{Ext}^i_R(M, \mathrm{Hom}_R(E, E')) = 0$ for all $i > 0$. We have the following isomorphisms

$$\mathrm{Ext}^i_R\big(M, \mathrm{Hom}_R(E, E')\big) \cong \mathrm{Hom}_R\big(\mathrm{Tor}^R_i(E, M), E'\big)$$

for all $i \geqslant 0$. Thus $\mathrm{Ext}^i_R(M, \mathrm{Hom}_R(E, E')) = 0$ for all $i > 0$. Now, suppose, inductively, that $n > 0$. Let $F \to K$ be a flat cover of K with kernel L. Then $F \to K$ is surjective. So, we have the exact sequence

$$\mathrm{Ext}^i_R(M, F) \longrightarrow \mathrm{Ext}^i_R(M, K) \longrightarrow \mathrm{Ext}^{i+1}_R(M, L),$$

for all $i > 0$. Also, L is cotorsion by [7, Lemma 2.2] and $\mathrm{fd}_R L = n - 1$. Since K and L are cotorsion, then so is F. Hence, by inductive hypothesis, $\mathrm{Ext}^i_R(M, K) = 0$, for all $i > 0$. □

Lemma 2.4. *Let R be a Noetherian ring and M an R-module.*

(i) *If R is a local ring and $M \in A'(R)$, then there exists a monomorphism $M \to L$ with* $\mathrm{fd}_R L < \infty$.
(ii) *Assume $\psi : M \to L$ is a monomorphism such that* $\mathrm{fd}_R L < \infty$ *and that* $\mathrm{Tor}^R_i(E, M) = 0$ *for all injective R-modules E and all* $i > 0$. *Then M possesses a monic $\overline{F(R)}$-preenvelope $M \to F$, in which F is flat.*
(iii) *Let R-homomorphism $f : M \to L'$ be an $\overline{P(R)}$-preenvelope. Assume $\varphi : M \to L$ is a monomorphism such that* $\mathrm{pd}_R L < \infty$ *and that* $\mathrm{Ext}^i_R(M, N) = 0$ *for all projective R-modules N and all* $i > 0$. *Then there exists a monic $\overline{P(R)}$-preenvelope $M \to P$, in which P is projective.*

Proof. (i) Since M belongs to $A'(R)$, $\mathrm{Gfd}_{\hat{R}}(M \otimes_R \hat{R})$ is finite by the definition and [5, Theorem 4.1]. Therefore, by [5, Lemma 2.19], we have an exact sequence of $\hat{R}$-modules and $\hat{R}$-homomorphisms $0 \to M \otimes_R \hat{R} \to L$, where flat dimension of L is finite as an $\hat{R}$-module. So, we obtain an exact sequence $0 \to M \to L$ of R-modules and R-homomorphisms. The flat dimension of L is finite as an R-module, because every flat $\hat{R}$-module is also flat as an R-module.

(ii) Using [6, Proposition 5.1], there exists a flat preenvelope $f : M \to F$. First, we show that f is $\overline{F(R)}$-preenvelope. To this end, let $\psi' : M \to L'$ be an R-homomorphism such that $\mathrm{fd}_R L' < \infty$ and let $0 \to K \to F' \xrightarrow{\pi} L' \to 0$ be an exact sequence such that $\pi : F' \to L'$ is a flat cover. Then K is of finite flat dimension and also by [7, Lemma 2.2], it is cotorsion. Lemma 2.3 implies that $\mathrm{Ext}^i_R(M, K) = 0$ for all $i > 0$. So, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, K) \longrightarrow \mathrm{Hom}_R(M, F') \longrightarrow \mathrm{Hom}_R(M, L') \longrightarrow \mathrm{Ext}^1_R(M, K) = 0.$$

Therefore, there exists an R-homomorphism $h : M \to F'$ such that $\pi h = \psi'$. Since $f : M \to F$ is flat preenvelope, there exists an R-homomorphism $g : F \to F'$ such that $h = gf$. Hence, there exists the R-homomorphism $\pi g : F \to L'$ such that $\pi g f = \psi'$. Thus f is $\overline{F(R)}$-preenvelope.

Since f is an $\overline{F(R)}$-preenvelope, there is an R-homomorphism $\theta : F \to L$ such that $\theta f = \psi$. So f is monic, because ψ is monic.

(iii) Since $\varphi : M \to L$ is monic, it turns out that $f : M \to L'$ is also monic. Now, let $0 \to K \to P \xrightarrow{\pi} L' \to 0$ be an exact sequence such that P is projective R-module. It is easy to see that $K \in \overline{P(R)}$. On the other hand, by hypothesis and induction on projective dimension, $\mathrm{Ext}^i_R(M, Q) = 0$ for all $i > 0$ and for all $Q \in \overline{P(R)}$. Therefore, $\mathrm{Ext}^1_R(M, K) = 0$. Hence $f : M \to L'$ has a lifting $M \to P$ which is monic and still an $\overline{P(R)}$-preenvelope. □

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, M an R-module, and n a non-negative integer. Then the following conditions are equivalent*:

(i) $\mathrm{Gfd}_R M \leqslant n$.
(ii) $M \in A'(R)$ *and* $\mathrm{Tor}^R_i(E, M) = 0$ *for all injective R-modules E and all $i > n$.*

Proof. (i) ⇒ (ii). By Proposition 2.2, M belongs to $A'(R)$. Also, [12, Theorem 3.14], implies the last assertion in (ii).

(ii) ⇒ (i). There is an exact sequence

$$0 \longrightarrow C \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where the modules $P_0, \ldots, P_{n-1}$ are projective. It is enough to show that C is Gorenstein flat. For this, by [12, Theorem 3.6], it is enough to show that C admits a right flat resolution

$$\mathbf{X} = 0 \longrightarrow C \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$$

such that $\mathrm{Hom}_R(\mathbf{X}, Y)$ is exact for all flat R-modules Y (i.e. C admits a co-proper right flat resolution) and $\mathrm{Tor}^R_i(E, C) = 0$ for all injective R-modules E and all $i > 0$.

We have

$$\mathrm{Tor}^R_{i-n}(N, C) \cong \mathrm{Tor}^R_i(N, M),$$

for all R-modules N and for all $i > n$. Therefore $\mathrm{Tor}^R_i(E, C) = 0$ for all injective R-modules E and all $i > 0$.

By Lemma 2.1 and Proposition 2.2, the module C belongs to $A'(R)$. So, Lemma 2.4(i) implies that there exists an exact sequence $0 \to C \to L$ of R-modules and R-homomorphisms such that $\mathrm{fd}_R L < \infty$. Using Lemma 2.4(ii), there exists a monomorphism $f : C \to K$ which is a flat preenvelope. We obtain the short exact sequence $0 \to C \xrightarrow{f} K \to B \to 0$. For every flat R-module F' we have the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(B, F') \longrightarrow \mathrm{Hom}_R(K, F') \longrightarrow \mathrm{Hom}_R(C, F') \longrightarrow 0,$$

because f is a flat preenvelope. Let E be an injective R-module and $E_R(R/\mathfrak{m})$ be the injective envelope of $R/\mathfrak{m}$. Since $\mathrm{Hom}_R(E, E_R(R/\mathfrak{m}))$ is a flat R-module and $E_R(R/\mathfrak{m})$ is an injective cogenerator, we conclude that

$$0 \longrightarrow C \otimes_R E \longrightarrow K \otimes_R E \longrightarrow B \otimes_R E \longrightarrow 0$$

is an exact sequence. So, $\mathrm{Tor}^R_i(E, B) = 0$ for all $i > 0$ and all injective R-modules E, because K is a flat R-module and $\mathrm{Tor}^R_i(E, C) = 0$ for all $i > 0$. Also, by Lemma 2.1 and Proposition 2.2, we obtain $B \in A'(R)$. Then proceeding in this manner, we get the desired co-proper right flat resolution of C. $\square$

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $M \in A'(R)$. Then* $\operatorname{Gfd}_R M = \operatorname{Gfd}_{\hat{R}}(\hat{R} \otimes_R M)$. *In particular, if $M \in A'(R)$ then* $\operatorname{Gfd}_R M < \infty$.

Proof. By [12, Proposition 3.10], $\operatorname{Gfd}_{\hat{R}}(\hat{R} \otimes_R M) \leqslant \operatorname{Gfd}_R M$. We show that $\operatorname{Gfd}_R M \leqslant \operatorname{Gfd}_{\hat{R}}(\hat{R} \otimes_R M)$. As M belongs to $A'(R)$, we get that $\hat{R} \otimes_R M$ belongs to $A(\hat{R})$. So, by [5, Theorem 4.1], $\operatorname{Gfd}_{\hat{R}}(\hat{R} \otimes_R M)$ is finite. Set $\operatorname{Gfd}_{\hat{R}}(\hat{R} \otimes_R M) = t$ and let

$$0 \longrightarrow C \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

be an exact sequence of R-modules and R-homomorphisms such that P_i's are projective. We prove that C is Gorenstein flat and so the assertions hold.

We obtain the exact sequence

$$0 \longrightarrow C \otimes_R \hat{R} \longrightarrow P_{t-1} \otimes_R \hat{R} \longrightarrow \cdots \longrightarrow P_1 \otimes_R \hat{R} \longrightarrow P_0 \otimes_R \hat{R} \longrightarrow M \otimes_R \hat{R} \longrightarrow 0.$$

By [12, Theorem 3.14], $C \otimes_R \hat{R}$ is a Gorenstein flat $\hat{R}$-module. By Lemma 2.1 and Proposition 2.2 the module C belongs to $A'(R)$. In view of Theorem 2.5, it is enough to show that $\operatorname{Tor}_i^R(E, C) = 0$ for all injective R-modules E and all $i > 0$.

Let E be an injective R-module and let $\operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$ denote by $(-)^\vee$, where $E_R(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. From the natural monomorphism $E \to (E^\vee)^\vee$, we conclude that E is a direct summand of $(E^\vee)^\vee$. So, it is enough to show that $\operatorname{Tor}_i^R((E^\vee)^\vee, C) = 0$ for all $i > 0$. By the next result, $\operatorname{id}_{\hat{R}} (E^\vee)^\vee$ is finite. It therefore follows from [12, Theorem 3.14] that $\operatorname{Tor}_i^{\hat{R}}(C \otimes_R \hat{R}, (E^\vee)^\vee) = 0$ for all $i > 0$. Suppose $\mathbf{F}_\bullet \to C$ is a flat resolution of C. For every $i > 0$, we have

$$\begin{aligned} \operatorname{Tor}_i^R\big(C, \big(E^\vee\big)^\vee\big) &\cong H_i\big(\mathbf{F}_\bullet \otimes_R \big(E^\vee\big)^\vee\big) \\ &\cong H_i\big((\mathbf{F}_\bullet \otimes_R \hat{R}) \otimes_{\hat{R}} \big(E^\vee\big)^\vee\big) \\ &\cong \operatorname{Tor}_i^{\hat{R}}\big(C \otimes_R \hat{R}, \big(E^\vee\big)^\vee\big). \end{aligned}$$

The last isomorphism comes from the fact that $F_\bullet \otimes_R \hat{R}$ is a flat resolution of $C \otimes_R \hat{R}$, considered as an $\hat{R}$-module. Thus, $\operatorname{Tor}_i^R(C, (E^\vee)^\vee) = 0$ for all $i > 0$. $\square$

Lemma 2.7. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and let K be an R-module such that $\operatorname{id}_R(K)$ is finite. Let $\operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$ denote by $(-)^\vee$, where $E_R(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. The R-module $(K^\vee)^\vee$ considered with the $\hat{R}$-module structure coming from $E_R(R/\mathfrak{m})$, that is, $(\hat{r} f)(x) = \hat{r}(f(x))$, for all $\hat{r} \in \hat{R}$, $f \in \operatorname{Hom}_R(K^\vee, E_R(R/\mathfrak{m}))$ and $x \in K^\vee$. Then $\operatorname{id}_{\hat{R}} (K^\vee)^\vee$ is finite.*

Proof. We deduce that $\operatorname{fd}_R(K^\vee)$ is finite. It is easy to see that $\operatorname{fd}_{\hat{R}}(K^\vee \otimes_R \hat{R})$ is finite. By the adjoint isomorphism, we have the following isomorphism

$$\operatorname{Hom}_{\hat{R}}\big(K^\vee \otimes_R \hat{R}, E_R(R/\mathfrak{m})\big) \cong \operatorname{Hom}_R\big(K^\vee, E_R(R/\mathfrak{m})\big),$$

as an $\hat{R}$-modules. This ends the proof, because the injective dimension of $\mathrm{Hom}_{\hat{R}}(K^{\vee} \otimes_R \hat{R}, E_R(R/\mathfrak{m}))$ is finite as an $\hat{R}$-module. $\square$

3. Gorenstein projective dimension

In this section over a Noetherian ring R with finite Krull dimension, we show that Gorenstein projective dimension of an R-module is finite if and only if its Gorenstein flat dimension is finite.

Proposition 3.1. *Let R be a Noetherian ring of finite Krull dimension and M be an R-module. Then $\mathrm{Gfd}_R M \leqslant \mathrm{Gpd}_R M$.*

Proof. See [12, Remark 3.3 and Proposition 3.4]. $\square$

Theorem 3.2. *Let R be a Noetherian ring of finite Krull dimension, M an R-module, and n a non-negative integer. Then the following conditions are equivalent*:

(i) $\mathrm{Gpd}_R M \leqslant n$.
(ii) $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{Ext}^i_R(M, P) = 0$ *for all projective R-modules P and all $i > n$.*

Proof. Assume that $\mathrm{Gpd}_R M \leqslant n$. Then $\mathrm{Gfd}_R M < \infty$, by Proposition 3.1. Also, [12, Theorem 2.20], implies that $\mathrm{Ext}^i_R(M, P) = 0$ for all projective R-modules P and all $i > n$.

Next, we show that (ii) $\Rightarrow$ (i). There is an exact sequence

$$0 \longrightarrow C \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where the modules $P_0, \ldots, P_{n-1}$ are projective. It is enough to show that C is Gorenstein projective. For this, by [12, Proposition 2.3], it is enough to show that C admits a right projective resolution

$$\mathbf{X} = 0 \longrightarrow C \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$$

such that $\mathrm{Hom}_R(\mathbf{X}, Y)$ is exact for every projective R-module Y (i.e. M admits a co-proper right projective resolution) and $\mathrm{Ext}^i_R(C, P) = 0$ for all projective R-modules P and all $i > 0$.

It is easy to deduce from the assumption that $\mathrm{Ext}^i_R(C, P) = 0$ for all projective R-modules P and all $i > 0$.

It follows from [12, Theorem 3.15] that $\mathrm{Gfd}_R C < \infty$. Since $\mathrm{Gfd}_R C < \infty$, it follows from [5, Lemma 2.19] that there exists a monomorphism $C \to L$ with $\mathrm{fd}_R L < \infty$.

Let $i > 0$. By assumption and induction on projective dimension, $\mathrm{Ext}^i_R(C, Q) = 0$ for all $Q \in \overline{P(R)}$. On the other hand, we have

$$\mathrm{Ext}^i_R\big(C, \mathrm{Hom}_R(E, E')\big) \cong \mathrm{Hom}_R\big(\mathrm{Tor}^R_i(C, E), E'\big)$$

for all injective R-modules E and E'. Therefore, $\mathrm{Tor}^R_i(E, C) = 0$ for all injective R-modules E. Note that, for each non-zero R-module N, there exists an injective R-module E' such that $\mathrm{Hom}_R(N, E') \neq 0$ and $\mathrm{Hom}_R(E, E')$ is flat for every injective R-modules E and E'.

Now, using parts (ii) and (iii) of Lemma 2.4, there exists a monic $\overline{P(R)}$-preenvelope $\psi : C \to Q$, in which Q is projective. We consider the exact sequence

$$0 \longrightarrow C \xrightarrow{\psi} Q \longrightarrow B \longrightarrow 0,$$

where $B = \operatorname{Coker} \psi$. Let P be a projective R-module. Applying the functor $\operatorname{Hom}_R(-, P)$ to the above exact sequence, we see that $\operatorname{Ext}_R^i(B, P) = 0$ for all $i > 0$, because $\psi : C \to Q$ is a projective preenvelope. Also, $\operatorname{Gfd}_R B < \infty$, by [12, Theorem 3.15]. Then proceeding in this manner, we get the desired co-proper right projective resolution for M. □

We can deduce from Proposition 2.2, Corollary 2.6 and Theorem 3.2 the following result.

Corollary 3.3. *Let R be a local Noetherian ring, M an R-module, and n a non-negative integer. Then the following conditions are equivalent*:

(i) $\operatorname{Gpd}_R(M) \leqslant n$.
(ii) $M \in A'(R)$ *and* $\operatorname{Ext}_R^i(M, P) = 0$ *for all projective R-modules P and all $i > n$.*

Theorem 3.4. *Let R be a Noetherian ring of finite Krull dimension d and M be an R-module. Then the following conditions are equivalent*:

(i) $\operatorname{Gfd}_R M < \infty$.
(ii) $\operatorname{Gpd}_R M < \infty$.

Moreover, if one of the above conditions holds, then $\operatorname{Gpd}_R M \leqslant d$.

Proof. (i) ⇒ (ii). We prove the claim by induction on $\operatorname{Gfd}_R M$. First, let M be a Gorenstein flat R-module. Let F be a flat R-module. Consider the minimal pure injective resolution

$$0 \longrightarrow F \longrightarrow PE^0(F) \longrightarrow PE^1(F) \longrightarrow \cdots$$

(see [14, pp. 39 and 92]). Note that, by [14, Lemma 3.1.6], $PE^n(F)$ is flat for all $n \geqslant 0$ and also, by [14, Corollary 4.2.7], $PE^n(F) = 0$ for all $n > d$. It is well known that every pure injective module is cotorsion. So, by [12, Proposition 3.22], $\operatorname{Ext}_R^i(M, PE^j(F)) = 0$ for all $j \geqslant 0$ and all $i \geqslant 1$. Therefore, $\operatorname{Ext}_R^{d+i}(M, F) \cong \operatorname{Ext}_R^i(M, PE^d(F))$ for all $i \geqslant 1$, and so $\operatorname{Ext}_R^{d+i}(M, F) = 0$ for all $i \geqslant 1$. In view of the above fact, $\operatorname{Gpd}_R M < \infty$ follows from Theorem 3.2.

Now, let $\operatorname{Gfd}_R M = t > 0$ and let $0 \to K \to P \to M \to 0$ be an exact sequence such that P is projective. By [12, Proposition 3.12], $\operatorname{Gfd}_R K = t - 1$. Hence, induction hypothesis implies that $\operatorname{Gpd}_R M < \infty$.

(ii) ⇒ (i). This follows from Proposition 3.1.

Now, if either $\operatorname{Gpd}_R M < \infty$ or equivalently $\operatorname{Gfd}_R M < \infty$, then, by [5, Lemma 2.17], $\operatorname{Gpd}_R M = \operatorname{pd}_R H$, where H is an R-module. This completes the proof. □

Now, we are ready to deduce the main result of this section by using Proposition 2.2, Corollary 2.6 and Theorem 3.4.

Corollary 3.5. *Let R be a local Noetherian ring of Krull dimension d and M an R-module. Then the following conditions are equivalent*:

(i) $\mathrm{Gfd}_R\, M < \infty$.
(ii) $\mathrm{Gpd}_R\, M < \infty$.
(iii) $M \in A'(R)$.

Moreover, if one of the above conditions holds, then $\mathrm{Gpd}_R\, M \leqslant d$.

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