A Fuzzy Algebraic System Based on the Theory of Falling Shadows

Xue-hai Yuan

Department of Mathematics, Liaoning Normal University, Dalian, 116029, People’s Republic of China

and

E. Stanley Lee

Department of Industrial Engineering, Kansas State University, Manhattan, Kansas 66506

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Based on the concept of a falling shadow, a theoretical approach of the fuzzy algebraic system is established. A fuzzy subalgebraic system of an algebraic system is considered as a falling shadow of the “cloud of the subalgebraic systems.” When an algebraic system is a ring, fuzzy subrings (fuzzy ideals, fuzzy prime ideals) of a ring are seen as falling shadows of “the cloud of subrings” (“cloud of ideals,” “cloud of prime ideals”). We shall show that fuzzy algebraic systems defined in $t$-norms are consequences of our definition under certain conditions.

1. INTRODUCTION

Following the definition of a fuzzy subgroup of a group first suggested by Rosenfeld [1] and Anthony and Sherwood [2], various similar definitions such as fuzzy subring, fuzzy sublattice, and fuzzy subalgebraic system have been suggested and used in the literature. Because these definitions are primarily based on $t$-norms, they are intuitive generalizations of class theory and do not have any theoretical foundation.

In this paper, a theoretical basis is established by defining the fuzzy subalgebraic system of an algebraic system based on the theory of a falling shadow.
shadow which was first formulated by Wang [13]. The main characteristic of this approach is that a fuzzy subalgebraic system is considered as the falling shadow of the “cloud of the subalgebraic system.” In other words, a fuzzy subgroup of a group is considered as the falling shadow of the “cloud of the subgroup,” a fuzzy subring of a ring is considered as the falling shadow of the “cloud of the subring,” and a fuzzy idea (prime idea) of a ring is considered as the falling shadow of the “cloud of the idea” (prime idea). We shall show that fuzzy algebraic systems defined based on \( \tau \)-norms are consequences of our definition under certain conditions.

2. PRELIMINARIES

Definition 2.1 [5]. Let \( S \) be an algebraic system, \( \sigma \) be any \( n \)-element operation on \( S \), and \( A \) be a fuzzy subset of \( S \) on \( [0,1] \). If

\[
A(\sigma(x_1, \ldots, x_n)) \geq T_n(A(x_1), A(x_2), \ldots, A(x_n)),
\]

\[
\forall x_i \in S \ (1 \leq i \leq n)
\]

then \( A \) is called a \( T \)-fuzzy subalgebraic system of \( S \), where \( T \) is a \( \tau \)-norm and for \( \lambda_1, \ldots, \lambda_n \in [0,1] \),

\[
T_n(\lambda_1, \ldots, \lambda_n) = \begin{cases} 
\lambda_1, & \text{if } n = 1 \\
T(\lambda_1, \lambda_2), & \text{if } n = 2 \\
T(\lambda_1, T_{n-1}(\lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)), & \text{if } n > 2.
\end{cases}
\]

Example 2.1. When \( S = G \) is a group, if

\[
A(xy) \geq T(A(x), A(y)), \quad A(x^{-1}) \geq A(x)
\]

then \( A \) is called a \( (T-) \)-fuzzy subgroup of \( G \) [2].

Example 2.2. When \( S = R \) is a ring, if

\[
A(x + y) \geq T(A(x), A(y)), \quad A(-x) \geq A(x), \\
A(xy) \geq T(A(x), A(y))
\]

then \( A \) is called a \( (T-) \)-fuzzy ring of \( R \).

Example 2.3. When \( T(x, y) = \min(x, y) \), for \( x, y \in [0,1] \), if

\[
A(\sigma(x_1, \ldots, x_n)) \geq \min\{A(x_1), A(x_2), \ldots, A(x_n)\}
\]

then \( A \) is called a \( \Lambda \)-fuzzy subalgebraic system of \( S \). In this case, for any \( \lambda \in [0,1] \), \( A_\lambda = \{x \in S \mid A(x) \geq \lambda\} \) is a subalgebraic system of \( S \).
DEFINITION 2.2 [3]. Let $R_1$ and $R_2$ be rings, and let $A_1$ and $A_2$ be fuzzy subrings of $R_1$ and $R_2$, respectively. $A_1$ and $A_2$ are isomorphic if and only if there is an isomorphism $\phi$: $R_1 \to R_2$ such that $A_1 = A_2 \circ \phi$.

3. THE THEORY OF FALLING SHADOWS

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [7] pointed out the equivalence of a fuzzy set and a class of random sets. At about the same time, Wang and Sanchez [8] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. A statistical experiment was carried out to determine the membership function of the fuzzy concept “young” by three distinct groups of students. The resulting membership functions of “young” were almost identical for the three groups. This result signifies that the stability of the membership function of fuzzy concepts does exist in the theory of fuzzy sets. The mathematical structure of the theory of falling shadows [13] is thus formulated. The following is a brief summary of this theory.

Given a universe of discourse $X$, $P(X)$ denotes the power set of $X$. For each $x \in X$, let

$$\hat{x} = \{A \mid x \in A \text{ and } A \subseteq X\}.$$ 

For each $A \in \mathcal{P}(X)$, let

$$\hat{A} = \{\hat{x} \mid x \in A\}.$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be hyper-measurable on $X$ if it is a $\sigma$-field in $\mathcal{P}(X)$ and $X \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and the hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on $X$, a random set on $X$ is defined to be a mapping $\xi$: $\Omega \to \mathcal{P}(X)$ that is $\mathcal{A} \cap \mathcal{B}$ measurable, that is,

$$\forall C \in \mathcal{B}, \xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \subseteq C\} \in \mathcal{A}.$$ 

Suppose that $\xi$ is a random set on $X$. Let

$$A(x) = P(\omega \mid x \in \xi(\omega)) \quad \text{for each } x \in X.$$ 

Fuzzy set $A$ is called a falling shadow of the random set $\xi$ and $\xi$ is called a cloud on $A$.

For example, $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{B}, m)$, where $\mathcal{A}$ is a Borel field on $[0,1]$ and $m$ the usual Lebesgue measure. Let $A$ be a fuzzy subset on $X$ and $A_{\lambda} = \{x \in X \mid A(x) \geq \lambda\}$ be a $\lambda$-cut of $A$. Then

$$\xi: [0,1] \to \mathcal{P}(X) \lambda \to A_{\lambda}$$

is a random set and $\xi$ is a cloud of $A$. We shall call $\xi$ defined above as the cut-cloud of $A$ [7].
4. FUZZY SUBALGEBRAIC SYSTEMS BASED ON THE THEORY OF FALLING SHADOWS

**Definition 4.1.** Let $S$ be an algebraic system, let $(\Omega, \mathcal{A}, P)$ be a probability space, and let

$$\xi: \Omega \to \mathcal{P}(S)$$

be a random set. If $\xi(\omega)$ is a subalgebraic system of $S$ for any $\omega \in \Omega$, then the falling shadow $H$ of the random set $\xi$, i.e.,

$$H(x) = P(\omega \mid x \in \xi(\omega)),$$

is called a II-fuzzy subalgebraic system of $S$.

Partially, when $S = G$ is a group, then $\xi(\omega)$ is a subgroup of $G$ for any $\omega \in \Omega$ and consequently $H$ is called a II-fuzzy subgroup of $G$.

When $S = R$ is a ring, then $\xi(\omega)$ is a subring (idea, prime idea) for any $\omega \in \Omega$ and consequently $H$ is called a II-fuzzy subring (idea, prime idea, respectively) of ring $R$.

**Theorem 1.** Let $H$ be a $L$-fuzzy subalgebraic system of $S$. Then $H$ is also a II-fuzzy subalgebraic system of $S$.

**Proof.** Since $H_\lambda$ is a subalgebraic system of $S$ for each $\lambda \in [0, 1]$, let $\xi: [0, 1] \to \mathcal{P}(S)$ be a random set and $\xi(\lambda) = H_\lambda$, i.e., $\xi$ is a cut-cloud of $H$. Then $H$ is a II-fuzzy subalgebraic system of $S$.

**Corollary 1.** If $H$ is a $L$-fuzzy subgroup of group $G$, then $H$ is also a II-fuzzy subgroup of group $G$.

**Corollary 2.** If $H$ is a $L$-fuzzy subring of ring $R$, then $H$ is also a II-fuzzy subgroup of ring $R$.

**Theorem 4.2.** Let $H$ be a II-fuzzy subalgebraic system of algebraic system $S$, let $\sigma$ be a 1-element operation on $S$, and let $\tau$ be a 2-element operation on $S$. Then for any $x, y \in S$, we have (1) $H(\sigma(x)) \supseteq H(x)$, (2) $H(\tau(x, y)) \supseteq T^\sigma(H(x), H(y))$, where $T^\sigma(\lambda, \mu) = \max(\lambda + \mu - 1, 0)$ for any $\lambda, \mu \in [0, 1]$.  

**Proof.** Let $H$ be a falling shadow of random set $\xi$, then $\xi(\omega)$ is a subalgebraic system of $S$ for any $\omega \in \Omega$. Then $\{\omega \mid \sigma(x) \in \xi(\omega)\} \supseteq \{\omega \mid x \in \xi(\omega)\}$ and $P(\omega \mid \sigma(x) \in \xi(\omega)) \supseteq P(\omega \mid x \in \xi(\omega))$, i.e., $H(\sigma(x)) \supseteq H(x)$.

Since

$$\{\omega \mid \tau(x, y) \in \xi(\omega)\} \supseteq \{\omega \mid x \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\},$$
then
\[ H(\tau(x, y)) = P(\omega | \tau(x, y) \in \xi(\omega)) \]
\[ \geq P(\{ \omega | x \in \xi(\omega) \} \cap \{ \omega | y \in \xi(\omega) \}) \]
\[ \geq P(\omega | x \in \xi(\omega)) + P(\omega | y \in \xi(\omega)) \]
\[ - P(\omega | x \in \xi(\omega) \text{ or } y \in \xi(\omega)) \]
\[ \geq H(x) + H(y) - 1. \]

Hence \( H(\tau(x, y)) \geq \max\{H(x) + H(y) - 1, 0\} \).

**Corollary 1.** If \( H \) is a \( II \)-fuzzy subgroup of a group \( G \), then (1) \( H(x^{-1}) \geq H(x) \), (2) \( H(xy) \geq T^m(H(x), H(y)) \).

**Corollary 2.** If \( H \) is a \( II \)-fuzzy subring of a ring \( R \), then (1) \( H(-x) \geq H(x) \), (2) \( H(x + y) \geq T^m(H(x), H(y)) \), (3) \( H(xy) \geq T^m(H(x), H(y)) \).

**Example 4.1.** Let \( H \) be a subalgebraic system of algebraic system \( S \) and \( (\Omega, \mathcal{A}, P) \) be a probability space. Let
\[ F(S) = \{ f | f: \Omega \to S \text{ is a mapping} \}. \]
For any \( n \)-element operation \( \sigma \) on \( S \), let
\[ \sigma(f_1, \ldots, f_n)(\omega) = \sigma(f_1(\omega), \ldots, f_n(\omega)) \]
then \( \sigma(f_1, \ldots, f_n) \in F(S) \) and consequently \( F(S) \) is also an algebraic system.
For \( f \in F(S) \), let
\[ S_f = \{ \omega | \omega \in \Omega \text{ and } f(\omega) \in H \} \in \mathcal{A} \]
and
\[ \xi: \Omega \to \mathcal{P}(F(S)) \]
\[ \omega \to S_\omega = \{ f | f \in F(S) \text{ and } f(\omega) \in H \}. \]
Clearly, \( S_\omega \) is a subalgebraic system of \( F(S) \).
Since \( \xi^{-1}(f) = \{ \omega | f \in \xi(\omega) \} = \{ \omega | f \in S_\omega \} = \{ \omega | f(\omega) \in H \} = S_f \in \mathcal{A} \), then \( \xi \) is a random set on \( F(S) \). Let
\[ A(f) = P(\omega | f(\omega) \in H) \]
then \( A \) is a \( II \)-fuzzy subalgebraic system of \( F(S) \). A \( II \)-fuzzy subalgebraic system \( H \) obtained in this manner is called a \( II \)-fuzzy subalgebraic system generated by \( H \).
Partially, when \( S = G \) is a group and \( H \) is a subgroup of \( G \), then \( A \) obtained as above is called a \( \mu \)-fuzzy subgroup generated by \( H \).

When \( S = R \) is a ring and \( H \) is a subring (idea, prime idea, respectively) then, \( A \) obtained as above is called a \( \mu \)-fuzzy subring (idea, prime idea, respectively) generated by \( H \).

5. \( \mu \)-FUZZY SUBRING (IDEA, PRIME IDEA) OF A RING

Let \( S = R \) be a ring, \((\Omega,\mathcal{A},P)\) be a probability space, \( \xi: \Omega \to \mathcal{P}(R) \) be a random set, and \( H \) be a falling shadow of \( \xi \). For \( x \in R \), let \( I_x^\xi = \{\omega | x \in \xi(\omega)\} \), then \( I_x^\xi \in \mathcal{A} \).

**Theorem 5.1.** Let \( H \) be a \( \mu \)-fuzzy subring of ring \( R \), then

1. If \( I_x^\xi \subseteq I_y^\xi \) or \( I_x^\xi \supseteq I_y^\xi \), then \( H(x+y) \geq \min\{H(x), H(y)\} \) and \( H(xy) \geq \min\{H(x), H(y)\} \).

2. If \( I_x^\xi \) and \( I_y^\xi \) are independent random events, then \( H(x+y) \geq H(x)H(y) \) and \( H(xy) \geq H(x)H(y) \).

We know that the proof is direct from \( I_{x+y}^\xi \supseteq I_x^\xi \cap I_y^\xi \) and \( I_{xy}^\xi \supseteq I_x^\xi \cap I_y^\xi \).

**Theorem 5.2.** Let \( H \) be a \( \mu \)-fuzzy idea of ring \( R \), then

1. \( H(xy) \geq \max\{H(x), H(y)\} \).

2. If \( I_x^\xi \) and \( I_y^\xi \) are independent random events, then \( H(xy) \geq H(x) + H(y) - H(x)H(y) \).

3. If \( I_x^\xi \) and \( I_y^\xi \) are disjoint events, then \( H(xy) \geq \min\{H(x) + H(y), 1\} \).

**Proof.** Since \( I_{xy}^\xi \supseteq I_x^\xi \cup I_y^\xi \), it follows that

1. \( H(xy) = P(I_{xy}^\xi) \geq P(I_x^\xi \cup I_y^\xi) = P(I_x^\xi) + P(I_y^\xi) - P(I_x^\xi \cap I_y^\xi) \)

2. \( H(xy) \geq P(I_x^\xi \cup I_y^\xi) = P(I_x^\xi) + P(I_y^\xi) - P(I_x^\xi \cap I_y^\xi) = H(x) + H(y) - H(x)H(y) \).

This is clear.
**Theorem 5.3.** Let $H$ be a II-fuzzy prime idea of ring $R$, then

1. If $I^\xi_x \supseteq I^\xi_y$ or $I^\xi_y \subseteq I^\xi_x$, then $H(xy) = \max(H(x), H(y))$.
2. If $I^\xi_x$ and $I^\xi_y$ are independent random events, then $H(xy) = H(x) + H(y) - H(x)H(y)$.
3. If $I^\xi_x$ and $I^\xi_y$ are disjoint random events, then $H(xy) = \min(H(x) + H(y), 1)$.

**Proof.** Since $H$ is a II-fuzzy prime idea of $R$, so $\xi(\omega)$ is a prime idea of $R$ for each $\omega \in \Omega$. Hence $xy \in \xi(\omega) \iff x \in \xi(\omega)$ or $y \in \xi(\omega)$, and it follows that $I^\xi_{xy} = I^\xi_x \cup I^\xi_y$. From Theorem 5.2, we know that (1), (2), and (3) are true.

**Remark.** The formulas in Theorems 5.1–5.3 are precisely those fuzzy algebraic systems defined by the $t$-norm and $s$-norm of Zadeh, Lukasiewicz, and the probability formula.

**Theorem 5.4.** Let $R$ be a ring of integers and $H$ be a II-fuzzy prime idea of $R$. Then $H(xy) = H(x) + H(y) - H((x, y))$, where the $(x, y)$ represent the greatest common divisor of $x$ and $y$.

**Proof.** Since $(x, y)$ is the greatest common divisor of $x$ and $y$, so there are integers $a, b \in R$ such that $ax + by = (x, y)$. It follows that $I^\xi_{xy} = I^\xi_{(x, y)}$ and

$$H(xy) = P\left(I^\xi_{xy}\right) = P\left(I^\xi_x\right) + P\left(I^\xi_y\right) - P\left(I^\xi_x \cap I^\xi_y\right)$$

$$= H(x) + H(y) - H((x, y)).$$

**Corollary 1.** If $(x, y) = 1$ and $\xi(\omega) \neq R$ for each $\omega \in \Omega$, then $H(xy) = \min(H(x) + H(y), 1)$.

**Corollary 2.** If $x$ is a divisor of $y$ or $y$ is a divisor of $x$, then $H(xy) = \max(H(x), H(y))$.

**Example.** Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $P(\omega) = 1/6$ for each $\omega \in \Omega$. Let $R$ be a ring of integers and $\xi$ be a random set satisfying $\xi(1) = \xi(2) = 249$, $\xi(3) = \xi(6) = 3$, $\xi(4) = 7$, $\xi(5) = 5$, where $(i) = \{ai|a \in R\}$. Then $\xi(\omega)$ is a prime idea for each $\omega \in \Omega$ and $I^\xi_{14} = \{1, 2, 4\}$, $I^\xi_{6} = \{1, 2, 3, 6\}$, $I^\xi_{14} \cap I^\xi_{6} = \{1, 2\}$.

Let $H$ be falling shadows of $\xi$. Then

$$H(14) = P(I^\xi_{14}) = \frac{1}{2}, \quad H(6) = P(I^\xi_{6}) = \frac{2}{3}, \quad H(2) = P(I^\xi_{2}) = \frac{1}{3}$$

and

$$H(84) = H(14) + H(6) - H(2) = \frac{5}{6}.$$
We notice that $I^x$ and $I^y$ are independent random events and $I^x$ and $I^y$ are disjoint events.

**Theorem 5.5.** Every II-fuzzy subring of a ring $R$ is isomorphic to a II-fuzzy subring generated by a subring $\mathcal{R}$ of some ring $\mathcal{R}$.

**Proof.** Let $A$ be a II-fuzzy subring of ring $R$ and $A$ be falling shadows of the random set $\xi$ on $R$. Let $R_\omega = R$ for each $\omega \in \Omega$, $\mathcal{R} = \prod_{\omega \in \Omega} R_\omega$, $\mathcal{R} = \prod_{\omega \in \Omega} \xi(\omega)$. For $x \in R$, let mapping $f_x$ satisfy

$$f_x : \Omega \rightarrow \mathcal{R},$$

$$\omega \rightarrow f_x^\omega$$

where

$$f_x^\omega(\omega') = \begin{cases} 0, & \text{if } \omega' \neq \omega \\ x, & \text{if } \omega' = \omega. \end{cases}$$

Let $F(\mathcal{R}) = \{ f_x | x \in R \}$. Then

$$(f_x + f_y)(\omega) = f_x(\omega) + f_y(\omega) = f_x^\omega + f_y^\omega = f_{x+y}(\omega)$$

$$(-f_x)(\omega) = -f_x(\omega) = -f_x^\omega = f_{-x}(\omega)$$

$$(f_x \cdot f_y)(\omega) = f_x(\omega) \cdot f_y(\omega) = f_x^\omega \cdot f_y^\omega = f_{xy}(\omega).$$

Hence $f_x + f_y = f_{x+y}$, $-f_x = f_{-x}$, $f_x \cdot f_y = f_{xy}$ and consequently $F(\mathcal{R})$ can make a ring. Let

$$\phi : R \rightarrow F(\mathcal{R})$$

$$x \rightarrow f_x.$$

Then $\phi$ is a mapping and

$$\phi(x + y) = f_{x+y} = f_x + f_y = \phi(x) + \phi(y)$$

$$\phi(-x) = f_{-x} = -f_x = -\phi(x)$$

$$\phi(xy) = f_{xy} = f_x \cdot f_y = \phi(x) \cdot \phi(y).$$

Hence $\phi$ is a homomorphism of rings. Clearly $\phi$ is an isomorphism. Let

$$\eta : \Omega \rightarrow \mathcal{P}(F(\mathcal{R}))$$

$$\omega \rightarrow \eta(\omega) = \{ f_x^\omega | x \in \mathcal{R} \}.$$

Then $\eta(\omega)$ is a subring of $F(\mathcal{R})$ for each $\omega \in \Omega$ and

$$\eta^{-1}(f_x) = \{ \omega | f_x \in \eta(\omega) \} = \{ \omega | f_x^\omega \in \mathcal{R} \} = \{ \omega | x \in \xi(\omega) \} \in \mathcal{R}. $$
It follows that $\eta$ is a random set on $F(\mathcal{R})$ and falling shadows $B$ of $\eta$ satisfy

$$B(\phi(x)) = B(f_x) = P(\eta^{-1}(f_x)) = P(\omega | x \in \xi(\omega)) = A(x).$$

Hence $A$ and $B$ are isomorphic.

**Corollary 1.** Let $A$ be a II-fuzzy subgroup of group $G$. Then $A$ is isomorphic to a II-fuzzy subgroup generated by a subgroup $\mathcal{F}$ of some group $\mathcal{G}$.

**Corollary 2.** Every II-fuzzy idea (prime idea, respectively) of a ring $R$ is isomorphic to a II-fuzzy idea (prime idea, respectively) generated by an idea (prime idea, respectively) of some ring $\mathcal{R}$.

**References**