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Stochastic comparisons of multivariate mixture models

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ABSTRACT

In this paper we consider sufficient conditions in order to stochastically compare random vectors of multivariate mixture models. In particular we consider stochastic and convex orders, the likelihood ratio order, and the hazard rate and mean residual life dynamic orders. Applications to proportional hazard models and mixture models in risk theory are also given.

1. Introduction

Stochastic comparisons of random vectors are of interest in the context of risk theory and reliability and some other fields. In this paper we consider specific results about stochastic comparisons of random vectors $(T_1, T_2, \ldots, T_n)$ where the joint distribution function is given by

$$F(t_1, \ldots, t_n) = \int_{\mathbb{R}^m} \prod_{i=1}^{n} F_i(t_i|\theta_1, \ldots, \theta_m) \mathrm{d}\Pi(\theta_1, \ldots, \theta_m), \tag{1.1}$$

where Π is an m-dimensional probability distribution and, for any vector $(\theta_1, \ldots, \theta_m)$ in the support of Π, $F_i(\cdot|\theta_1, \ldots, \theta_m)$ is a one-dimensional distribution function. For example, this model can be used in the following situations:

- *Applications in finance*: In risk management, the default risk of an obligor can be assumed to depend on some random factors, such as macroeconomics variables. Given a realization of the factors, defaults of a firm are assumed to be independent. Two particular models of interest in this context are the case where the conditional defaults follow a Bernoulli model or a Poisson model. An extension of these static mixture models, arises when the default times of n firms $T_1, \ldots, T_n$ are independent given the observation of some m-dimensional random economic factors $\boldsymbol{\Theta}$. If $F_i(\cdot|\boldsymbol{\theta})$ denotes the conditional distribution of $T_i(\boldsymbol{\theta}) \equiv [T_i|\boldsymbol{\Theta} = \boldsymbol{\theta}]$, and Π denotes the distribution of $\boldsymbol{\Theta}$, then the joint distribution of $(T_1, \ldots, T_n)$ is given by (1.1).

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- *Applications in reliability*: Let us consider that $T_1, \ldots, T_n$ are the random lifetimes of n components, which are working in an m-dimensional random environment $\mathbf{\Theta}$. Given $\mathbf{\Theta} = \boldsymbol{\theta}$, that is, given specific values of the environment, the components are independent. Again the joint distribution of $T_1, \ldots, T_n$ is given, with the notation of the previous example, by (1.1).

If $F_i(\cdot|\boldsymbol{\theta})$ is absolutely continuous, for each $\boldsymbol{\theta}$ in the support of $\mathbf{\Theta}$, with a density function $f_i(\cdot|\boldsymbol{\theta})$, then the distribution of $(T_1, \ldots, T_n)$ is absolutely continuous, and the joint density function is given by

$$f(t_1, \ldots, t_n) = \int_{\mathbb{R}^m} \prod_{i=1}^{n} f_i(t_i|\theta_1, \ldots, \theta_m) \mathrm{d}\Pi(\theta_1, \ldots, \theta_m). \tag{1.2}$$

This model is known as *multivariate mixture model*, and can be applied not only in risk theory and reliability, but also in some other different contexts and situations, for example it can be used to model *heterogeneity* (see [1]) and if $F_i(\cdot|\theta_1, \ldots, \theta_m) = F(\cdot|\theta_i)$, where F denotes a distribution function, then (1.1) describes special *frailty models*. If the components are not only independent, but also equally distributed, that is $F_i(\cdot|\boldsymbol{\theta}) = F(\cdot|\boldsymbol{\theta})$, for all $i = 1, \ldots, n$, then the random vector $(T_1, \ldots, T_n)$ is exchangeable. The role of this model in reliability, in the exchangeable case, can be seen in [2,3].

For this model several authors have provided conditions to obtain dependence properties of the random vector $(T_1, \ldots, T_n)$, as can be seen in [4–8]. See also [9]. For a recent review of this topic see [10]. An analysis of stochastic dependence for such models in the context of financial applications and reliability and survival analysis is presented in the recent papers by Denuit and Frostig [11] and Frostig and Denuit [12] (in this respect see also Section 4).

The purpose of this paper is, given two random vectors as above, $(S_1, \ldots, S_n, \mathbf{\Theta}_1)$ and $(T_1, \ldots, T_n, \mathbf{\Theta}_2)$, to provide conditions to obtain stochastic comparisons of $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$. In particular we shall present a detailed discussion about stochastic orders of, the so-called, dynamic type. These stochastic orders are particularly interesting from a conceptual and mathematical point view. They (see e.g. [13]) are defined in terms of the notion of history, which describes the state of a random vector of lifetimes, of a set of components, at a given time t; this approach leads to the notions of multivariate hazard rate and multivariate mean residual life orders. Since several notions of positive dependence can be defined in terms of stochastic orderings, we will also obtain some results in the spirit of Shaked and Spizzichino [5] as corollaries of our achievements. In particular we obtain sufficient conditions for the HIF notion, which was left as an open problem in that paper.

The paper is organized as follows. In Section 2 we recall the definitions and main properties of the stochastic orders and dependence properties considered in this paper. The main results will be given in Section 3 and applications to proportional hazard models and mixture models in credit risk will be given in Section 4. Along the paper for any event A the notation $[X|A]$ stands for any random variable whose distribution is the conditional distribution of X given A. By $=_{\mathrm{st}}$ we denote equality in law.

2. Previous notions and results on stochastic orders and related concepts

In this section we recall the definitions of some stochastic orders that will be considered along the paper. Also some facts for positive dependence notions are considered. For the definitions and properties of stochastic orders, the reader can refer to [13,14].

Given two random vectors $\mathbf{X}$ and $\mathbf{Y}$ we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the usual multivariate stochastic order, denoted by $\mathbf{X} \leq_{\mathrm{st}} \mathbf{Y}$, if

$$E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})], \tag{2.3}$$

for all increasing functions $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, for which the previous expectations exist. In the univariate case, given two random variables X and Y, with survival functions $\overline{F}$ and $\overline{G}$, then (2.3) is equivalent to

$$\overline{F}(t) \leq \overline{G}(t), \quad \text{for all } t \in \mathbb{R}.$$

Given two random variables X and Y we say that X is less than Y in the convex [increasing convex] order, denoted by $X \leq_{\mathrm{cx[icx]}} Y$, if

$$E[\phi(X)] \leq E[\phi(Y)],$$

for all convex [increasing convex] functions ϕ, for which previous expectations exist. In the multivariate case there are several possibilities to extend this concept, depending on the kind of convexity that we consider.

Given two random vectors $\mathbf{X}$ and $\mathbf{Y}$ we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate convex [increasing convex] order, denoted by $\mathbf{X} \leq_{\mathrm{cx[icx]}} \mathbf{Y}$, if

$$E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})], \tag{2.4}$$

for all convex [increasing convex] functions $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, for which the previous expectations exist.

If (2.4) holds for all componentwise convex [increasing componentwise convex] functions ϕ, then we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the componentwise convex [increasing componentwise convex] order, denoted by $\mathbf{X} \leq_{\mathrm{ccx[iccx]}} \mathbf{Y}$. Some other appropriate classes of functions defined on $\mathbb{R}^n$ can be considered to extend convex orders to the multivariate case, by means of a difference operator. Let Δ_i^ϵ be the ith difference operator defined for a function $\phi : \mathbb{R}^n \to \mathbb{R}$ as

$$\Delta_i^\epsilon \phi(\mathbf{x}) = \phi(\mathbf{x} + \epsilon \mathbf{1}_i) - \phi(\mathbf{x})$$

where $\mathbf{1}_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$. A function ϕ is said to be directionally convex if $\Delta_i^\epsilon \Delta_j^\delta \phi(\mathbf{x}) \geq 0$ for all $1 \leq i \leq j \leq n$ and $\epsilon, \delta \geq 0$. We observe that directionally convex functions are also known as ultramodular functions (see [15]). A function ϕ is said to be supermodular if $\Delta_i^\epsilon \Delta_j^\delta \phi(\mathbf{x}) \geq 0$ for all $1 \leq i < j \leq n$ and $\epsilon, \delta \geq 0$. If ϕ is twice differentiable then, it is directionally convex if $\partial^2 \phi / \partial x_i \partial x_j \geq 0$ for every $1 \leq i \leq j \leq n$, and it is supermodular if $\partial^2 \phi / \partial x_i \partial x_j \geq 0$ for every $1 \leq i < j \leq n$. Clearly a function ϕ is directionally convex if it is supermodular and it is componentwise convex.

When we consider directionally convex [increasing directionally convex] functions in (2.4) then we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the directionally convex [increasing directionally convex] order, denoted by $\mathbf{X} \leq_{\text{dir-cx[idir-cx]}} \mathbf{Y}$. If we consider supermodular [increasing supermodular] functions in (2.4) then we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the supermodular [increasing supermodular] order, denoted by $\mathbf{X} \leq_{\text{sm[ism]}} \mathbf{Y}$.

The supermodular order is a well known tool to compare dependence structures of random vectors whereas the directionally convex order not only compares the dependence structure but also the variability of the marginals.

Now we consider some notions in the absolutely continuous case. Some remarks for the discrete case will be given along the section.

Given two random variables X and Y, with densities f and g, respectively, we say that X is less than Y in the likelihood ratio order, denoted by $X \leq_{\text{lr}} Y$, if

$$f(s)g(t) \geq f(t)g(s) \quad \text{for all } s \leq t.$$

In the multivariate case, given two random vectors $\mathbf{X}$ and $\mathbf{Y}$, with joint densities f and g, respectively, we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate likelihood ratio order, denoted by $\mathbf{X} \leq_{\text{lr}} \mathbf{Y}$, if

$$f(\mathbf{x} \wedge \mathbf{y})g(\mathbf{x} \vee \mathbf{y}) \geq f(\mathbf{x})g(\mathbf{y}), \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

where $(\mathbf{x} \wedge \mathbf{y}) = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$ and $(\mathbf{x} \vee \mathbf{y}) = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$.

The likelihood ratio order is related to the MTP_2 dependence notion, as we will recall next.

Given a random vector $\mathbf{X}$ with density f, we say that $\mathbf{X}$ is MTP_2 (multivariate totally positive of order 2) if

$$f(\mathbf{x} \wedge \mathbf{y})f(\mathbf{x} \vee \mathbf{y}) \geq f(\mathbf{x})f(\mathbf{y}), \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \tag{2.5}$$

that is, if $\mathbf{X} \leq_{\text{lr}} \mathbf{X}$.

Recall that any function $f : \mathbb{R}^n \to \mathbb{R}$, which satisfies (2.5) is said to be MTP_2.

In the discrete case the definition of multivariate likelihood ratio order can be given by replacing the joint density by the joint probability function.

Next we consider some orders of interest in reliability, where the random variables denote the random lifetimes of some units or systems. These orders are motivated from a time-dynamic point of view and for the definitions and properties the reader can refer to [16–19,13].

Given two random variables X and Y, with survival functions $\overline{F}$ and $\overline{G}$, respectively, we say that X is less than Y in the hazard rate order, denoted by $X \leq_{\text{hr}} Y$, if

$$\overline{F}(t)\overline{G}(s) \leq \overline{F}(s)\overline{G}(t) \quad \text{for all } s \leq t.$$

In the absolutely continuous case, given two non-negative random variables X and Y with hazard rates r and s respectively, then $X \leq_{\text{hr}} Y$ if $r(t) \geq s(t)$ for all $t \geq 0$.

In the multivariate case it is possible to provide several extensions. We consider the time-dynamic definition of the multivariate hazard rate order introduced by Shaked and Shanthikumar [16]. For some other extensions, from a mathematical point of view, see [20].

Let us consider a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ where the X_i's can be considered as the lifetimes of n units. For $t \geq 0$ let h_t denote the list of units which have failed and their failure times. More explicitly, a history h_t will denote

$$h_t = \{\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{\bar{I}} > t\mathbf{e}\},$$

where $I = \{i_1, \ldots, i_k\}$ is a subset of $\{1, \ldots, n\}$, $\bar{I}$ is its complement with respect to $\{1, \ldots, n\}$, $\mathbf{X}_I$ will denote the vector formed by the components of $\mathbf{X}$ with index in I and $0 < x_{i_j} < t$ for all $j = 1, \ldots, k$ and $\mathbf{e}$ denotes vectors of 1's, where the dimension can be determined from the context.

Now we proceed to give the definition of the multivariate hazard rate order. Given the history h_t, as above, let $j \in \bar{I}$, its multivariate conditional hazard rate, at time t, is defined as follows:

$$\eta_j(t|h_t) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathrm{P}[t < X_j \leq t + \Delta t | h_t]. \tag{2.6}$$

Clearly $\eta_j(t|h_t)$ is the "probability" of instant failure of component j, given the history h_t.

Now let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with hazard rate functions $\eta_{\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot}(\cdot|\cdot)$, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate hazard rate order, denoted by $\mathbf{X} \le_{\text{hr}} \mathbf{Y}$, if, for every $t \ge 0$,

$$\eta_i(t|h_t) \ge \lambda_i(t|h'_t)$$

where

$$h_t = \{\mathbf{X}_{I\cup J} = \mathbf{x}_{I\cup J}, \mathbf{X}_{\overline{I\cup J}} > t\mathbf{e}\} \tag{2.7}$$

and

$$h'_t = \{\mathbf{Y}_I = \mathbf{y}_I, \mathbf{Y}_{\bar{I}} > t\mathbf{e}\}, \tag{2.8}$$

whenever $I \cap J = \emptyset$, $\mathbf{0} \le \mathbf{x}_I \le \mathbf{y}_I \le t\mathbf{e}$, and $\mathbf{0} \le \mathbf{x}_J \le t\mathbf{e}$, where $i \in \overline{I \cup J}$.

Given two histories as above, we say that h_t is more severe than h'_t.

The multivariate hazard rate order, as the multivariate likelihood ratio order, is not necessarily reflexive. In fact if a random vector $\mathbf{X}$ satisfies $\mathbf{X} \le_{\text{hr}} \mathbf{X}$, then it is said to have the HIF property (hazard increasing upon failure, see [19]), and it can be considered as a positive dependence property. Also the HIF notion can be considered as a mathematical formalization of the *default contagion* notion in risk theory. Loosely speaking, the default contagion notion means that the conditional probability of default for a non-defaulted firm increases given the information that some other firms have defaulted. In particular, concerning the HIF notion, we have that if the information becomes the worst, that is, the number of defaulted firms is larger and the default times are earlier, then the probability of default for a non-defaulted firm increases.

We want to point out that the definition of the multivariate hazard rate order in the discrete case does not follow just considering discrete hazard rates in (2.6). In this case some additional considerations have to be taken into account (see [21,22]).

Another stochastic order of interest, from a time-dynamic point of view, is the mean residual order.

Given two random variables X and Y, with mean residual lives

$$m(t) = E[X - t|X > t],$$

and

$$l(t) = E[Y - t|Y > t],$$

respectively, we say that X is less than Y in the mean residual life order, denoted by $X \le_{\text{mrl}} Y$, if, for every $t \ge 0$,

$$m(t) \le l(t).$$

In the multivariate case, given an n-dimensional random vector $\mathbf{X}$, and a history $h_t = \{\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{\bar{I}} > t\mathbf{e}\}$, then for the component $j \in \bar{I}$, its multivariate conditional mean residual function, at time t, is defined as follows:

$$m_j(t|h_t) = E[X_j - t|h_t].$$

In this case $m_j(t|h_t)$ is the expected residual life of component j, given the history h_t.

In a similar way to the multivariate hazard rate order, Shaked and Shanthikumar [17] define the multivariate mean residual life order. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with multivariate conditional mean residual life functions $m_{\cdot}(\cdot|\cdot)$ and $l_{\cdot}(\cdot|\cdot)$, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate mean residual life order, denoted by $\mathbf{X} \le_{\text{mrl}} \mathbf{Y}$, if, for every $t \ge 0$,

$$m_i(t|h_t) \le l_i(t|h'_t)$$

where h_t and h'_t are given in (2.7) and (2.8), respectively.

Again the multivariate mean residual life order is not necessarily reflexive. If a random vector $\mathbf{X}$ satisfies $\mathbf{X} \le_{\text{mrl}} \mathbf{X}$, then it is said to have the MRL-DF property (mean residual life decreasing upon failure, see [17]), and it is considered as a positive dependence property. Again the MRL-DF notion can be interpreted in the context of default contagion. In this case we have that the mean time to default of a non-defaulted firm decreases, when the number of defaulted firms increases and the default times are earlier.

To conclude we include the following relationships among the previous orders.

Univariate case:

$$\begin{array}{ccccc} \le_{\text{lr}} \Rightarrow & \le_{\text{hr}} \Rightarrow & \le_{\text{st}} \\ & \Downarrow & \Downarrow \\ & \le_{\text{mrl}} \Rightarrow & \le_{\text{icx}}. \end{array}$$

Multivariate case:

$$\begin{array}{ccc} \le_{\text{lr}} \Rightarrow & \le_{\text{hr}} \Rightarrow & \le_{\text{st}} \\ & \Downarrow & \Downarrow \\ & \le_{\text{mrl}} & \le_{\text{icx}} \end{array}$$

and

$$\le_{\text{cx[icx]}} \Leftarrow \le_{\text{ccx[iccx]}} \Rightarrow \le_{\text{dir-cx[idir-cx]}} \Leftarrow \le_{\text{sm[ism]}}. \tag{2.9}$$

3. Stochastic comparisons of conditionally independent random vectors

We are now ready to start with our treatment of the general case in which we have two random vectors $(S_1,\ldots,S_n,\boldsymbol{\Theta}_1)$ and $(T_1,\ldots,T_n,\boldsymbol{\Theta}_2)$. Generally, unless stated otherwise, we shall assume that $S_1,\ldots,S_n$ and $T_1,\ldots,T_n$ are independent random variables given $\boldsymbol{\Theta}_1=\boldsymbol{\theta}$ and $\boldsymbol{\Theta}_2=\boldsymbol{\theta}$ for any value of $\boldsymbol{\theta}$, respectively, and $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$ are m-dimensional random vectors. As first we describe sufficient conditions for the multivariate stochastic order of $(S_1,\ldots,S_n)$ and $(T_1,\ldots,T_n)$, in the usual sense.

Theorem 3.1. *If, for all i:* $1,\ldots,n$,

(i) $[S_i|\boldsymbol{\Theta}_1=\boldsymbol{\theta}]$ *or* $[T_i|\boldsymbol{\Theta}_2=\boldsymbol{\theta}]$, *or both, are increasing in the stochastic order in* $\boldsymbol{\theta}$,
(ii) $[S_i|\boldsymbol{\Theta}_1=\boldsymbol{\theta}]\le_{\mathrm{st}}[T_i|\boldsymbol{\Theta}_2=\boldsymbol{\theta}]$, *for all* $\boldsymbol{\theta}$,
and
(iii) $\boldsymbol{\Theta}_1\le_{\mathrm{st}}\boldsymbol{\Theta}_2$,

then

$$(S_1,\ldots,S_n)\le_{\mathrm{st}}(T_1,\ldots,T_n).$$

Proof. Let $\phi:\mathbb{R}^n\mapsto\mathbb{R}$ be an increasing function. We are going to prove condition (2.3).

Let us suppose that (i) holds for $S_i(\boldsymbol{\theta})\equiv[S_i|\boldsymbol{\Theta}_1=\boldsymbol{\theta}]$ (the proof in the other case is similar). Then it is easy to see that

$$E[\phi(S_1(\boldsymbol{\theta}),\ldots,S_n(\boldsymbol{\theta}))]\quad\text{is increasing in }\boldsymbol{\theta}.\tag{3.10}$$

Also condition (ii) is equivalent to (see Theorem 6.B.16(b) in [13]),

$$(S_1(\boldsymbol{\theta}),\ldots,S_n(\boldsymbol{\theta}))\le_{\mathrm{st}}(T_1(\boldsymbol{\theta}),\ldots,T_n(\boldsymbol{\theta})),\qquad\text{for all }\boldsymbol{\theta}.\tag{3.11}$$

Let us denote by Π_1 and Π_2 the joint distribution function of $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$, respectively. We have the following chain of inequalities (we assume that the conditions of Fubini's theorem hold)

$$\begin{aligned}E[\phi(S_1,\ldots,S_n)]&=\int_{\mathbb{R}^m}E[\phi(S_1(\boldsymbol{\theta}),\ldots,S_n(\boldsymbol{\theta}))]\mathrm{d}\Pi_1(\boldsymbol{\theta})\\&\ge\int_{\mathbb{R}^m}E[\phi(S_1(\boldsymbol{\theta}),\ldots,S_n(\boldsymbol{\theta}))]\mathrm{d}\Pi_2(\boldsymbol{\theta})\\&\ge\int_{\mathbb{R}^m}E[\phi(T_1(\boldsymbol{\theta}),\ldots,T_n(\boldsymbol{\theta}))]\mathrm{d}\Pi_2(\boldsymbol{\theta})=E[\phi(T_1,\ldots,T_n)],\end{aligned}$$

where the first inequality follows from (iii) and (3.10), and the second inequality from (3.11). Therefore $(S_1,\ldots,S_n)\le_{\mathrm{st}}(T_1,\ldots,T_n)$. □

Remark 3.2. Condition (i) is a well known property. Given a family $\{X(\boldsymbol{\theta}),\boldsymbol{\theta}\in\chi\subseteqq\mathbb{R}^m\}$ of random variables, if $X(\boldsymbol{\theta})$ is increasing in the stochastic order in $\boldsymbol{\theta}$, then $\{X(\boldsymbol{\theta}),\boldsymbol{\theta}\in\chi\subseteqq\mathbb{R}^m\}$ is said to be stochastically increasing (SI). The SI notion can be seen also, in the literature, as a condition of positive dependence. If (X,Θ) is a bivariate vector, X is SI in Θ if $[X|\Theta=\theta]$ is SI (see [23,13]).

The previous theorem is a particular case of the following one.

Theorem 3.3. *Let* $(S_1,\ldots,S_n,\boldsymbol{\Theta}_1)$ *and* $(T_1,\ldots,T_n,\boldsymbol{\Theta}_2)$ *be random vectors, where* $\boldsymbol{\Theta}_1$ *and* $\boldsymbol{\Theta}_2$ *are m-dimensional random vectors, and the components* $S_1,\ldots,S_n$ *and* $T_1,\ldots,T_n$ *are not necessarily conditionally independent random variables given* $\boldsymbol{\Theta}_1=\boldsymbol{\theta}$ *and* $\boldsymbol{\Theta}_2=\boldsymbol{\theta}$ *for all* $\boldsymbol{\theta}$, *respectively. If*

(i) $[S_1,\ldots,S_n|\boldsymbol{\Theta}_1=\boldsymbol{\theta}]$ *or* $[T_1,\ldots,T_n|\boldsymbol{\Theta}_2=\boldsymbol{\theta}]$, *or both, are increasing in the stochastic order in* $\boldsymbol{\theta}$,
(ii) $[S_1,\ldots,S_n|\boldsymbol{\Theta}_1=\boldsymbol{\theta}]\le_{\mathrm{st}}[T_1,\ldots,T_n|\boldsymbol{\Theta}_2=\boldsymbol{\theta}]$, *for all* $\boldsymbol{\theta}$,
and
(iii) $\boldsymbol{\Theta}_1\le_{\mathrm{st}}\boldsymbol{\Theta}_2$,

then

$$(S_1,\ldots,S_n)\le_{\mathrm{st}}(T_1,\ldots,T_n).$$

Proof. The proof follows similar steps to that of Theorem 3.1. □

In the conditionally independent case, (i) and (ii) in Theorem 3.1 are equivalent to (i) and (ii) in Theorem 3.3, respectively.

Next we consider the stochastic comparison for some convex and related orders. We first consider a result for the ism order in the same situation as that in Theorem 3.1, and we assume also that $[S_i|\boldsymbol{\Theta}_1=\boldsymbol{\theta}]=_{\mathrm{st}}[T_i|\boldsymbol{\Theta}_2=\boldsymbol{\theta}]$, for all $\boldsymbol{\theta}$. We will denote by $\overline{F}_i(\cdot|\boldsymbol{\theta})$ the common survival function of $[S_i|\boldsymbol{\Theta}_1=\boldsymbol{\theta}]$ and $[T_i|\boldsymbol{\Theta}_2=\boldsymbol{\theta}]$.

Theorem 3.4. *If, for all i:* $1, \ldots, n$,

(i) $[S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta}]$ *is increasing or decreasing [increasing], at the same time, in the stochastic order in* $\boldsymbol{\theta}$,
(ii) $\overline{F}_i(\cdot|\boldsymbol{\theta})$ *is componentwise convex in* θ
and
(iii) $\mathbf{\Theta}_1 \leq_{\text{ccx[iccx]}} \mathbf{\Theta}_2$,

then

$$(S_1, \ldots, S_n) \leq_{\text{ism}} (T_1, \ldots, T_n).$$

Proof. Let ϕ be an increasing supermodular function, we assume that ϕ is twice differentiable, then following [24,12]

$$\frac{\partial^2}{\partial\theta_i\partial\theta_j}E[\phi(S_1, S_2, \ldots, S_n)|\Theta_1 = \theta] = \sum_{l<k}\int \frac{\partial^2}{\partial x_l \partial x_k}\phi(x_1, \ldots, x_n)\frac{\partial}{\partial\theta_i}\overline{F}_l(x_l|\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\overline{F}_k(x_k|\boldsymbol{\theta})\prod_{h\neq l,k}\mathrm{d}F_h(x_h|\boldsymbol{\theta})$$
$$+\sum_{l=1}^{n}\int\frac{\partial}{\partial x_l}\phi(x_1, \ldots, x_n)\frac{\partial^2}{\partial\theta_i\partial\theta_j}\overline{F}_l(x_l|\boldsymbol{\theta})\prod_{h\neq l}\mathrm{d}F_h(x_h|\boldsymbol{\theta}).$$

Therefore, under (i) and (ii), $E[\phi(S_1, S_2, \ldots, S_n)|\Theta_1 = \theta]$ is componentwise convex [increasing componentwise convex] in $\boldsymbol{\theta}$, and the result follows as in the previous result. □

Now we consider a different situation. Let $(S_1, \ldots, S_n, \mathbf{\Theta}_1)$, $(T_1, \ldots, T_n, \mathbf{\Theta}_2)$ be random vectors, where $\mathbf{\Theta}_1 = (\Theta_{1,1}, \ldots, \Theta_{1,n})$ and $\mathbf{\Theta}_2 = (\Theta_{2,1}, \ldots, \Theta_{2,n})$ are n-dimensional random vectors, and the conditional distributions of S_i $[T_i]$ given $\mathbf{\Theta}_1 = \boldsymbol{\theta}$ $[\mathbf{\Theta}_2 = \boldsymbol{\theta}]$ just depends on $\Theta_{1,i} = \theta_i$ $[\Theta_{2,i} = \theta_i]$. In this case we will fix the notation $S_i(\theta_i) \equiv S_i(\boldsymbol{\theta})$ $[T_i(\theta_i) \equiv T_i(\boldsymbol{\theta})]$

Now we can state the following theorem.

Theorem 3.5. *If, for all i:* $1, \ldots, n$,

(i) $S_i(\theta) \leq_{\text{cx[icx]}} T_i(\theta)$, *for all* θ,
(ii) $E[\phi(S_i(\theta))]$ *or* $E[\phi(T_i(\theta))]$, *or both, are convex [increasing convex] in* θ *for all convex [increasing convex] functions* ϕ,
and
(iii) $\mathbf{\Theta}_1 \leq_{\text{ccx[iccx]}} \mathbf{\Theta}_2$

then

$$(S_1, \ldots, S_n) \leq_{\text{ccx[iccx]}} (T_1, \ldots, T_n).$$

Proof. Following [24] it is not difficult to show that if (ii) holds for $S_i(\theta)$, $i = 1, 2, \ldots, n$, then $E[\phi(S_1(\theta_1), \ldots, S_n(\theta_n))]$ is componentwise convex [increasing componentwise convex] in $\boldsymbol{\theta}$ (the proof when (i) holds for $T_i(\theta)$, $i = 1, 2, \ldots, n$, is similar). Now (i) is equivalent to $(S_1(\theta_1), \ldots, S_n(\theta_n)) \leq_{\text{ccx[iccx]}} (T_1(\theta_1), \ldots, T_n(\theta_n))$, and the proof follows similar steps to that of Theorem 3.1. □

Related results can be found in [12]. In particular we observe that Proposition 3.5 in [12] can be obtained from Theorems 3.4 and 3.5.

Remark 3.6. Recall that given a family $\{X(\theta), \theta \in \chi \subseteqq \mathbb{R}\}$ of random variables, if $X(\theta)$ is SI and $E[\phi(X(\theta))]$ is increasing convex in $\theta \in \chi$, for all increasing convex functions ϕ, then $\{X(\theta), \theta \in \chi \subseteqq \mathbb{R}\}$ is said to be stochastically increasing and convex (SICX) (see [13]). Therefore condition (ii) (in the increasing convex case) is weaker than the SICX notion.

From (2.9) we get the following result.

Corollary 3.7. *Under the same conditions as in the previous theorem, then*

$$(S_1, \ldots, S_n) \leq_{\text{dir-cx,cx[idir-cx,icx]}} (T_1, \ldots, T_n).$$

In the same situation for $(S_1, \ldots, S_n, \mathbf{\Theta}_1)$, $(T_1, \ldots, T_n, \mathbf{\Theta}_2)$ as that in the previous result, we consider now the multivariate likelihood ratio order.

Theorem 3.8. *If, for all i:* $1, \ldots, n$,

(a) $S_i(\theta) =_{\text{st}} T_i(\theta)$, *for all* θ,
(b) $S_i(\theta) \leq_{\text{lr}} S_i(\theta')$ *for all* $\theta \leq \theta'$,
and
(c) $\mathbf{\Theta}_1 \leq_{\text{lr}} \mathbf{\Theta}_2$

then

$$(S_1, \ldots, S_n) \le_{\mathrm{lr}} (T_1, \ldots, T_n).$$

Proof. Let us denote by $f_i(t_i|\theta)$ the density function of $S_i(\theta)$. It is not difficult to show that condition (ii) implies that

$$f(t_1, \ldots, t_n|\theta_1, \ldots, \theta_n) = \prod_{i=1}^{n} f_i(t_i|\theta_i), \quad \text{is } \mathrm{MTP}_2 \text{ in } (t_1, \ldots, t_n, \theta_1, \ldots, \theta_n).$$

Now from Theorem 2.4 by Karlin and Rinott [25], then $(S_1, \ldots, S_n) \le_{\mathrm{lr}} (T_1, \ldots, T_n)$. □

When $\mathbf{\Theta}_1 =_{\mathrm{st}} \mathbf{\Theta}_2$, then we have the following result obtained by Shaked and Spizzichino [5].

Corollary 3.9. *If, for all i:* $1, \ldots, n$,

(i) $S_i(\theta) \le_{\mathrm{lr}} S_i(\theta')$ *for all* $\theta \le \theta'$,
and
(ii) $\mathbf{\Theta}$ *is* MTP_2,

then

$$(S_1, \ldots, S_n) \quad \textit{is } \mathrm{MTP}_2.$$

Following the results given for the likelihood ratio order, it is natural to conjecture the following result.

Conjecture 3.10. *Let* $(S_1, S_2, \ldots, S_n, \mathbf{\Theta}_1)$, $(T_1, T_2, \ldots, T_n, \mathbf{\Theta}_2)$ *be random vectors as in Theorem 3.8, if for all i:* $1, \ldots, n$,

(i) $S_i(\theta) =_{\mathrm{st}} T_i(\theta)$, *for all* θ,
(ii) $S_i(\theta) \le_{\mathrm{hr}} S_i(\theta')$ *for all* $\theta \le \theta'$,
and
(iii) $\mathbf{\Theta}_1 \le_{\mathrm{lr}} \mathbf{\Theta}_2$

then

$$(S_1, \ldots, S_n) \le_{\mathrm{hr}} (T_1, \ldots, T_n).$$

A consequence of this result would be the following: Let $(S_1, S_2, \ldots, S_n, \mathbf{\Theta})$ be a random vector where $(S_1, S_2, \ldots, S_n)$ is conditionally independent given $\mathbf{\Theta} = \boldsymbol{\theta}$. If $S_i(\theta) \le_{\mathrm{hr}} S_i(\theta')$ for all $\theta \le \theta'$ and i: $1, \ldots, n$ then

$$(S_1, \ldots, S_n) \le_{\mathrm{hr}} (S_1, \ldots, S_n) (\Leftrightarrow (S_1, \ldots, S_n) \in \mathrm{HIF}).$$

However this result is not true, for a counterexample see Example 3.5 in [5].

Anyway, it is possible to give some conditions under which we can obtain the dynamical hazard rate order.

Let us consider two random vectors $(S_1, \ldots, S_n, \mathbf{\Theta}_1)$ and $(T_1, \ldots, T_n, \mathbf{\Theta}_2)$, where $\mathbf{\Theta}_1$ and $\mathbf{\Theta}_2$ are m-dimensional random vectors, and let us denote by $s_t^i(\boldsymbol{\theta})$ and $r_t^i(\boldsymbol{\theta})$ the hazard rates of $[(S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta}]$ and $[T_i|\mathbf{\Theta}_2 = \boldsymbol{\theta})]$, respectively. Let us denote by η and λ the dynamic multivariate hazard rates of $\mathbf{S} = (S_1, \ldots, S_n)$ and $\mathbf{T} = (T_1, \ldots, T_n)$, respectively. Finally $\mathbf{S}$ and $\mathbf{T}$ are conditionally independent given $\mathbf{\Theta}_1 = \boldsymbol{\theta}$ and $\mathbf{\Theta}_2 = \boldsymbol{\theta}$, respectively. In this case, following arguments similar to those in the proof of Proposition 2.39 in [3], we have the following equalities

$$\eta_j(t|h_t) = \int_{\mathbb{R}^m} s_t^j(\boldsymbol{\theta}) \pi_1(\boldsymbol{\theta}|h_t) \mathrm{d}\boldsymbol{\theta} \tag{3.12}$$

and

$$\lambda_j(t|h'_t) = \int_{\mathbb{R}^m} r_t^j(\boldsymbol{\theta}) \pi_2(\boldsymbol{\theta}|h'_t) \mathrm{d}\boldsymbol{\theta}, \tag{3.13}$$

where h_t and h'_t are histories for $\mathbf{S}$ and $\mathbf{T}$, respectively, and $\pi_i(\boldsymbol{\theta}|D)$ is the conditional density of $\mathbf{\Theta}_i$, for $i = 1, 2$ given a history D.

Now we state the following result.

Theorem 3.11. *If, for all i:* $1, \ldots, n$,

(i) $[S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta}]$ *(or* $[T_i|\mathbf{\Theta}_2 = \boldsymbol{\theta}]$*) is increasing [decreasing] in the hazard rate order in* $\boldsymbol{\theta}$,
(ii) $[S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta}] \le_{\mathrm{hr}} [T_i|\mathbf{\Theta}_2 = \boldsymbol{\theta}]$, *for all* $\boldsymbol{\theta}$,
and
(iii) $[\mathbf{\Theta}_1|h_t] \le_{\mathrm{st}} [\ge_{\mathrm{st}}] [\mathbf{\Theta}_2|h'_t]$, *for every two histories* h_t *and* h'_t, *for* $(S_1, \ldots, S_n)$ *and* $(T_1, \ldots, T_n)$ *respectively, where* h_t *is more severe than* h'_t,

then

$$(S_1, \dots, S_n) \le_{\rm hr} (T_1, \dots, T_n).$$

Proof. First we observe that condition (i) is equivalent to the condition $s_t^i(\boldsymbol{\theta})$ (or $r_t^i(\boldsymbol{\theta})$) is decreasing [increasing] in $\boldsymbol{\theta}$, for all $t > 0$. We will consider the case in which $s_t^i(\boldsymbol{\theta})$ is decreasing [increasing] in $\boldsymbol{\theta}$, for all $t > 0$, the other case follows under similar arguments.

Let us denote by η and λ the multivariate conditional hazard rates of $(S_1, \dots, S_n)$ and $(T_1, \dots, T_n)$, respectively. And finally, let us consider two histories h_t and h'_t as in (2.7) and (2.8). Let $j \in \overline{I \cup J}$, recalling (3.12) and (3.13), we have the following chain of inequalities

$$\begin{aligned}\eta_j(t|h_t) = \int_\chi s_t^j(\boldsymbol{\theta})\pi_1(\boldsymbol{\theta}|h_t)\mathrm{d}\boldsymbol{\theta} &= E[s_t^j(\boldsymbol{\Theta}_1|h_t)] \\ &\ge E[s_t^j(\boldsymbol{\Theta}_2|h'_t)] \\ &\ge E[r_t^j(\boldsymbol{\Theta}_2|h'_t)] = \lambda_j(t|h'_t),\end{aligned}$$

where the first inequality follows from (i) and (iii), and the second inequality follows from (ii). Therefore $(S_1, \dots, S_n) \le_{\rm hr} (T_1, \dots, T_n)$. □

An important consequence of this theorem is the following result.

Theorem 3.12. *If, for all i:* $1, \dots, n$,

(i) $[S_i|\boldsymbol{\Theta} = \boldsymbol{\theta}]$ *is increasing*[*decreasing*] *in the hazard rate order in* $\boldsymbol{\theta}$
and
(ii) $[\boldsymbol{\Theta}|h_t] \le_{\rm st} [\ge_{\rm st}] [\boldsymbol{\Theta}|h'_t]$, *for every two histories* h_t *and* h'_t, *for* $(S_1, \dots, S_n)$, *where* h_t *is more severe than* h'_t,

then

$$(S_1, \dots, S_n) \in HIF.$$

Therefore we provide conditions for the HIF property of random vectors with conditionally independent components, which was an open problem in [5].

Next we consider the mean residual life order. In the conditionally independent case, we provide first the following expression for the multivariate mean residual life.

Proposition 3.13. *Let* $(S_1, \dots, S_n)$ *be a random vector with the property of being conditionally independent given* $\boldsymbol{\Theta}$. *Denote by* $m_t^j(\boldsymbol{\theta})$ *the mean residual life of* $[S_j|\boldsymbol{\Theta} = \boldsymbol{\theta}]$, *for* $j: 1, \dots, n$, *and denote by* $m_\cdot(\cdot|\cdot)$ *the multivariate conditional mean residual life of* $(S_1, \dots, S_n)$, *then*

$$m_j(t|h_t) = \int_\chi m_t^j(\boldsymbol{\theta})\pi(\boldsymbol{\theta}|h_t)\mathrm{d}\boldsymbol{\theta}, \tag{3.14}$$

where $h_t = \{S_I = s_I, S_{\bar{I}} > t\}$ *and* $j \in \bar{I}$, *and* $\pi(\cdot|h_t)$ *is the conditional density function of* $\boldsymbol{\Theta}$ *given* h_t.

Proof. Let us consider the history $h_t = \{\mathbf{S}_I = \mathbf{s}_I, \mathbf{S}_{\bar{I}} > t\mathbf{e}\}$, where $I = \{i_1, \dots, i_k\}$ and $\bar{I} = \{j_1, \dots, j_{n-k}\}$. Let $j \in \bar{I}$, then

$$m_j(t|h_t) = E[S_j - t|h_t] = E\{E[S_j - t|\boldsymbol{\Theta}]|h_t\} = E[m_t^i(\boldsymbol{\Theta})|h_t],$$

where the last inequality follows from the conditional independence of $S_1, \dots, S_n$ given $\boldsymbol{\Theta}$. □

Now we give conditions for the mrl order multivariate mixture models.

Theorem 3.14. *If, for all i:* $1, \dots, n$,

(i) $[S_i|\boldsymbol{\Theta}_1 = \boldsymbol{\theta}]$(*or* $[T_i|\boldsymbol{\Theta}_2 = \boldsymbol{\theta}]$) *is increasing* [*decreasing*] *in the mean residual life order in* $\boldsymbol{\theta}$,
(ii) $[S_i|\boldsymbol{\Theta}_1 = \boldsymbol{\theta}] \le_{\rm mrl} [T_i|\boldsymbol{\Theta}_2 = \boldsymbol{\theta}]$, *for all* $\boldsymbol{\theta}$,
and
(iii) $[\boldsymbol{\Theta}_1|h_t] \le_{\rm st} [\ge_{\rm st}] [\boldsymbol{\Theta}_2|h'_t]$, *for every two histories* h_t *and* h'_t, *for* $(S_1, \dots, S_n)$ *and* $(T_1, \dots, T_n)$ *respectively, where* h_t *is more severe than* h'_t,

then

$$(S_1, \dots, S_n) \le_{\rm mrl} (T_1, \dots, T_n).$$

Proof. The proof is similar to that of Theorem 3.11. First we observe that condition (i) is equivalent to the condition $m_t^i(\boldsymbol{\theta})$ (or $l_t^i(\boldsymbol{\theta})$) is increasing [decreasing] in $\boldsymbol{\theta}$, for all $t > 0$. We will consider the case in which $m_t^i(\boldsymbol{\theta})$ is increasing [decreasing] in $\boldsymbol{\theta}$, for all $t > 0$, the other case follows under similar arguments.

Let us denote by m and l the multivariate conditional mean residual lives of $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$, respectively. And finally, let us consider two histories h_t and h'_t for $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$, such that h_t is more severe than h'_t. Recalling (3.14), we have the following chain of inequalities

$$
\begin{aligned}
m_i(t|h_t) = \int_{\chi} m_t^i(\boldsymbol{\theta})\pi_1(\boldsymbol{\theta}|h_t)\mathrm{d}\boldsymbol{\theta} &= E[m_t^i(\boldsymbol{\Theta}_1|h_t)] \\
&\le E[m_t^i(\boldsymbol{\Theta}_2|h'_t)] \\
&\le E[l_t^i(\boldsymbol{\Theta}_2|h'_t)] = l_i(t|h'_t),
\end{aligned}
$$

where the first inequality follows from (i) and (iii), and the second inequality follows from (ii). Therefore $(S_1, \ldots, S_n) \le_{\mathrm{mrl}} (T_1, \ldots, T_n)$. $\square$

Now it is possible to provide the following result.

Theorem 3.15. *If, for all i:* $1, \ldots, n$*,*

(i) $[S_i|\boldsymbol{\Theta} = \boldsymbol{\theta}]$ *is increasing*[*decreasing*] *in the mean residual life order in* $\boldsymbol{\theta}$
and
(ii) $[\boldsymbol{\Theta}|h_t] \le_{\mathrm{st}} [\ge_{\mathrm{st}}] [\boldsymbol{\Theta}|h'_t]$, *for every two histories h_t and h'_t, for* $(S_1, \ldots, S_n)$*, where h_t is more severe than h'_t,*

then

$$(S_1, \ldots, S_n) \in \text{MRL-DF}.$$

In view of previous theorems, it seems that condition (iii) in Theorems 3.11 and 3.14 is the most difficult to verify, so it would be interesting to describe some sufficient conditions for

$$(\boldsymbol{\Theta}_1|h_t) \le_{\mathrm{st}}[\ge_{\mathrm{st}}](\boldsymbol{\Theta}_2|h'_t)$$

where h_t is more severe than h'_t. Now we describe some results in such a direction. We will give conditions for likelihood ratio order of $(\boldsymbol{\Theta}_1|h_t)$ and $(\boldsymbol{\Theta}_2|h'_t)$, which in turns implies the stochastic order. The results are given for random vectors not necessarily with the property of being conditionally independent. We start with a result concerning the case when Θ_1 and Θ_2 are real random variables.

Theorem 3.16. *Let* $(S_1, \ldots, S_n, \Theta_1)$ *and* $(T_1, \ldots, T_n, \Theta_2)$*, be random vectors, where Θ_1 and Θ_2 are random variables. If*

(i) $\overline{G}(\mathbf{t}|\theta)/\overline{F}(\mathbf{t}|\theta)$ *is increasing*[*decreasing*] *in θ, for all $\mathbf{t}$, where $\overline{G}(\mathbf{t}|\theta)$ and $\overline{F}(\mathbf{t}|\theta)$ are the survival functions of* $(S_1, \ldots, S_n|\Theta_1 = \theta)$ *and* $(T_1, \ldots, T_n|\Theta_2 = \theta)$*,*
(ii) $\Theta_1 \le_{\mathrm{lr}}[\ge_{\mathrm{lr}}]\Theta_2$*,*
and
(iii) *for every* $K \subseteq \{1, \ldots, n\}$ *and* $\mathbf{x}_K \le \mathbf{y}_K$

$$\frac{g_{\mathbf{T}_K}(\mathbf{y}_K|\mathbf{T}_{\overline{K}} > t\mathbf{e}_K; \Theta_2 = \theta)}{f_{\mathbf{S}_K}(\mathbf{x}_K|\mathbf{S}_{\overline{K}} > t\mathbf{e}_K; \Theta_1 = \theta)} \quad \text{is increasing in } \theta,$$

where by $f_{\mathbf{S}_K}(\mathbf{x}_K|B)$ and $g_{\mathbf{T}_K}(\mathbf{y}_K|B)$ we denote the conditional density functions of $\mathbf{S}_K$ and $\mathbf{T}_K$, respectively, given some event B

then

$$[\Theta_1|h_t] \le_{\mathrm{lr}} [\ge_{\mathrm{lr}}] [\Theta_2|h'_t]$$

where h_t is more severe than h'_t.

Proof. Let

$$h'_t = \left\{\mathbf{T}_I = \mathbf{y}_I, \mathbf{T}_J > t\mathbf{e}_J, \mathbf{T}_{\overline{I \bigcup J}} > t\mathbf{e}_{\overline{I \bigcup J}}\right\}$$

and

$$h_t = \left\{\mathbf{S}_I = \mathbf{x}_I, \mathbf{S}_J = \mathbf{x}_J, \mathbf{S}_{\overline{I \bigcup J}} > t\mathbf{e}_{\overline{I \bigcup J}}\right\},$$

where $\mathbf{x}_I \le \mathbf{y}_I \le t\mathbf{e}_I$ and $\mathbf{x}_J \le t\mathbf{e}_J$.

If we denote by $\pi_i(\theta|A)$, the conditional density of Θ_i given A, for $i = 1, 2$, and by $f_{\mathbf{S}_I}(\mathbf{x}_I|B)$ and $g_{\mathbf{T}_I}(\mathbf{y}_I|B)$ we denote the conditional density functions of $\mathbf{S}_I$ and $\mathbf{T}_I$, respectively, given some event B, then we have

$$\pi_1(\theta|h_t) \propto \pi_1\left(\theta|\mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}\right) f_{\mathbf{S}_I}\left(\mathbf{x}_I|\mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_1 = \theta\right) f_{\mathbf{S}_J}\left(\mathbf{x}_J|\mathbf{S}_I = \mathbf{x}_I; \mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_1 = \theta\right)$$

and

$$\pi_2(\theta|h'_t) \propto \pi_2\left(\theta|\mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}\right) g_{\mathbf{T}_I}\left(\mathbf{y}_I|\mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_2 = \theta\right)$$
$$\times \int_t^\infty \cdots \int_t^\infty g_{\mathbf{T}_J}\left(\mathbf{y}_J|\mathbf{T}_I = \mathbf{y}_I; \mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_2 = \theta\right) \mathrm{d}\mathbf{y}_J.$$

Now we prove that $[\Theta_1|h_t] \le_{\mathrm{lr}} [\Theta_2|h'_t]$, the proof for $[\Theta_1|h_t] \ge_{\mathrm{lr}} [\Theta_2|h'_t]$ follows under similar arguments. The result will follow if we prove that

$$\frac{\pi_2\left(\theta|\mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}\right)}{\pi_1\left(\theta|\mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}\right)} \tag{3.15}$$

and

$$\frac{g_{\mathbf{T}_I}\left(\mathbf{y}_I|\mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_2 = \theta\right) \int_t^\infty \cdots \int_t^\infty g_{\mathbf{T}_J}\left(\mathbf{y}_J|\mathbf{T}_I = \mathbf{y}_I; \mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_2 = \theta\right) \mathrm{d}\mathbf{y}_J}{f_{\mathbf{S}_I}\left(\mathbf{x}_I|\mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_1 = \theta\right) f_{\mathbf{S}_J}\left(\mathbf{x}_J|\mathbf{S}_I = \mathbf{x}_I; \mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_1 = \theta\right)} \tag{3.16}$$

are increasing in θ.

As far as (3.15) is concerned, we have that

$$\frac{\pi_2\left(\theta|\mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}\right)}{\pi_1\left(\theta|\mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}\right)} \propto \frac{\overline{G}_{\overline{I\bigcup J}}\left(t\mathbf{e}_{\overline{I\bigcup J}}|\theta\right) \pi_2(\theta)}{\overline{F}_{\overline{I\bigcup J}}\left(t\mathbf{e}_{\overline{I\bigcup J}}|\theta\right) \pi_1(\theta)}$$

and therefore, by (i) and (ii), is increasing in θ.

Finally we observe that (3.16) can be written as

$$\int_t^\infty \cdots \int_t^\infty \frac{g_{\mathbf{T}_I}\left(\mathbf{y}_I|\mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_2 = \theta\right) g_{\mathbf{T}_J}\left(\mathbf{y}_J|\mathbf{T}_I = \mathbf{y}_I; \mathbf{T}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_2 = \theta\right)}{f_{\mathbf{S}_I}\left(\mathbf{x}_I|\mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_1 = \theta\right) f_{\mathbf{S}_J}\left(\mathbf{x}_J|\mathbf{S}_I = \mathbf{x}_I; \mathbf{S}_{\overline{I\bigcup J}} > t\mathbf{e}_{\overline{I\bigcup J}}; \Theta_1 = \theta\right)} \mathrm{d}\mathbf{y}_J$$

and this expression is increasing in θ by (iii). □

Remark 3.17. Now we extend the previous result to the multivariate case, where Θ_1 and Θ_2 are replaced by random vectors, of the same dimension, $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$. First we note (see [26] and [3, p. 109]) that in the multivariate case, given two random vectors $\mathbf{S}$ and $\mathbf{T}$ with joint densities $f_{\mathbf{S}}$ and $g_{\mathbf{T}}$, if $\mathbf{S}$ or $\mathbf{T}$, or both, are MTP$_2$ and $f_{\mathbf{S}}(\mathbf{u})/g_{\mathbf{T}}(\mathbf{u})$ is increasing in $\mathbf{u}$, then $\mathbf{S} \le_{\mathrm{lr}} \mathbf{T}$. Clearly if $\mathbf{S} \le_{\mathrm{lr}} \mathbf{T}$ then $f_{\mathbf{S}}(\mathbf{u})/g_{\mathbf{T}}(\mathbf{u})$ is increasing in $\mathbf{u}$.

Theorem 3.18. *Let $(S_1, \ldots, S_n, \boldsymbol{\Theta}_1)$ and $(T_1, \ldots, T_n, \boldsymbol{\Theta}_2)$, be random vectors, where $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$ are random vectors of the same dimension. If*

(i) *$\overline{G}(\mathbf{t}|\boldsymbol{\theta})/\overline{F}(\mathbf{t}|\boldsymbol{\theta})$ is increasing [decreasing] in $\boldsymbol{\theta}$, for all $\mathbf{t}$, where $\overline{G}(\mathbf{t}|\boldsymbol{\theta})$ and $\overline{F}(\mathbf{t}|\theta)$ are the survival functions of $[S_1, \ldots, S_n|\boldsymbol{\Theta}_1 = \boldsymbol{\theta}]$ and $[T_1, \ldots, T_n|\boldsymbol{\Theta}_2 = \boldsymbol{\theta}]$,*

(ii) *$\boldsymbol{\Theta}_1 \le_{\mathrm{lr}} [\ge_{\mathrm{lr}}] \boldsymbol{\Theta}_2$,*

(iii) *$\boldsymbol{\Theta}_1$ is MTP$_2$ and $f(\cdot|\boldsymbol{\theta})$ is MTP$_2$ in $\boldsymbol{\theta}$ or $\boldsymbol{\Theta}_2$ is MTP$_2$ and $g(\cdot|\boldsymbol{\theta})$ is MTP$_2$ in $\boldsymbol{\theta}$, or both, where $f(\cdot|\boldsymbol{\theta})$ and $g(\cdot|\boldsymbol{\theta})$ are the conditional densities of $\mathbf{S}$ and $\mathbf{T}$ given $\boldsymbol{\Theta}_1 = \boldsymbol{\theta}$ and $\boldsymbol{\Theta}_2 = \boldsymbol{\theta}$, respectively, and*

(iv) *for every $K \subseteq \{1, \ldots, n\}$ and $\mathbf{x}_K \le \mathbf{y}_K$*

$$\frac{g_{\mathbf{T}_K}(\mathbf{y}_K|\mathbf{T}_{\overline{K}} > t\mathbf{e}_K; \boldsymbol{\Theta}_2 = \boldsymbol{\theta})}{f_{\mathbf{S}_K}(\mathbf{x}_K|\mathbf{S}_{\overline{K}} > t\mathbf{e}_K; \boldsymbol{\Theta}_1 = \boldsymbol{\theta})} \quad \text{is increasing in } \boldsymbol{\theta},$$

then

$$[\boldsymbol{\Theta}_1|h_t] \le_{\mathrm{lr}} [\ge_{\mathrm{lr}}] [\boldsymbol{\Theta}_2|h'_t]$$

where h_t is more severe than h'_t.

Proof. As in the previous theorem we give the proof for $[\mathbf{\Theta}_1|h_t] \leq_{\text{lr}} [\mathbf{\Theta}_2|h'_t]$. Let h_t and h'_t be as in the previous theorem. Now under the same notation as in the previous theorem and from Remark 3.17, the result follows if we prove that

$$\pi_2(\boldsymbol{\theta}|h'_t)/\pi_1(\boldsymbol{\theta}|h_t) \quad \text{is increasing in } \boldsymbol{\theta} \tag{3.17}$$

and that

$$[\mathbf{\Theta}_1|h_]) \quad \text{or} \quad [\mathbf{\Theta}_2|h'_t], \quad \text{or both, are MTP}_2. \tag{3.18}$$

The proof of (3.17) follows, from (i), (ii) and (iv), under the same arguments of the proof of Theorem 3.16.

Now we prove (3.18). We give the proof for $[\mathbf{\Theta}_1|h_t]$, the proof for $[\mathbf{\Theta}_2|h'_t]$ is analogous. Given a history $h_t = \{\mathbf{S}_I = \mathbf{s}_I, \mathbf{S}_{\bar{I}} > t\mathbf{e}\}$, we observe that

$$\pi_1(\boldsymbol{\theta}|h_t) \propto \pi_1(\boldsymbol{\theta})\mathrm{P}[\mathbf{S}_{\bar{I}} > t\mathbf{e}|\mathbf{\Theta}_1 = \boldsymbol{\theta}] f_{\mathbf{S}_I}(\mathbf{S}_I|\mathbf{S}_{\bar{I}} > t\mathbf{e}, \mathbf{\Theta}_1 = \boldsymbol{\theta}),$$

and then (3.18) follows from (iii). □

It is important to note that condition (iii) in Theorems 3.11 and 3.14 just requires a stochastic comparison in the $\leq_{\text{st}}$ sense. In the two previous results we give conditions for the likelihood ratio order. Therefore it would be interesting to find other conditions, different from those of Theorems 3.16 and 3.18 for the usual stochastic order. This remains as an open problem. We want to point out that conditions (iii) and (iv) in Theorems 3.16 and 3.18, respectively, are implied by the condition $(S_1, \ldots, S_n, \mathbf{\Theta}_1) \leq_{\text{lr}} (T_1, \ldots, T_n, \mathbf{\Theta}_2)$.

To conclude this section we observe also that a combination of Theorems 3.12 and 3.18 can be used to provide the following result for the HIF property. This result can be compared with Corollary 3.9

Theorem 3.19. *Let $(S_1, \ldots, S_n, \mathbf{\Theta})$ be a random vector, where, for all values $\boldsymbol{\theta}$ in the support of $\mathbf{\Theta}$, $S_1, \ldots, S_n$ are conditionally independent given in $\mathbf{\Theta} = \boldsymbol{\theta}$. If, for all i: $1, \ldots, n$,*

(i) $S_i(\boldsymbol{\theta}) \leq_{\text{lr}} S_i(\boldsymbol{\theta}')$ *for all* $\boldsymbol{\theta} \leq \boldsymbol{\theta}'$,
(ii) $\mathbf{\Theta}$ *is* MTP$_2$
(iii) $f_i(\cdot|\boldsymbol{\theta})$ *is* MTP$_2$ *in* $\boldsymbol{\theta}$, *where* $f_i(\cdot|\boldsymbol{\theta})$ *is the conditional density of* $\mathbf{S}_i$ *given* $\mathbf{\Theta} = \boldsymbol{\theta}$,
and
(iv)

$$\frac{f_{\mathbf{S}_K}(\mathbf{y}_K|\mathbf{S}_{\bar{K}} > t\mathbf{e}_K; \mathbf{\Theta} = \boldsymbol{\theta})}{f_{\mathbf{S}_K}(\mathbf{x}_K|\mathbf{S}_{\bar{K}} > t\mathbf{e}_K; \mathbf{\Theta} = \boldsymbol{\theta})} \quad \textit{is increasing in } \boldsymbol{\theta},$$

then

$$(S_1, \ldots, S_n) \in \mathit{HIF}.$$

4. Applications

4.1. Mixture models in credit risk

An important application of multivariate mixture models is given in portfolio credit risk. In this context if we consider a loan portfolio with respect to n different obligors, the default risk of each obligor is assumed to depend on a set of economic factors, which are modelled stochastically. Given a realization of the factors, defaults of individual firms are assumed to be independent. Some important applications arise when the defaults are modelled via Bernoulli or Poisson random variables. In this case Belzunce and Semeraro [7] and Denuit and Frostig [11] have provided several results about dependence properties between the individual defaults. Next we provide some additional results about stochastic comparisons.

- **Bernoulli mixture models**
  Let us consider the case where the default probability of the ith firm, given some random economic factors $\mathbf{\Theta} = \boldsymbol{\theta}$, is given by $p_i(\boldsymbol{\theta})$. If we denote by $S_i(\boldsymbol{\theta})$ the indicator random variable of default of the ith firm, then $S_i(\boldsymbol{\theta})$ is a Bernoulli random variable with parameter $p_i(\boldsymbol{\theta}) = P[S_i(\boldsymbol{\theta}) = 1]$. If we consider the unconditional distribution of defaults of the n firms, $(S_1, S_2, \ldots, S_n)$, obtained by integrating over the distribution of the economic factors $\mathbf{\Theta}$, then $(S_1, S_2, \ldots, S_n)$ is said to follow a *Bernoulli mixture model* (see [23, p. 219]). Let us describe situations where some of the previous results can be applied.
  First we observe that given a function $\phi : \mathbb{R}^n \to \mathbb{R}$, we obtain that

$$E[\phi(S_i(\boldsymbol{\theta}))] = \phi(0) + (\phi(1) - \phi(0))p_i(\boldsymbol{\theta}). \tag{4.19}$$

Therefore if ϕ is increasing, then the behaviour of $E[\phi(S_i(\theta))]$, with respect to $\boldsymbol{\theta}$ depends only on $p_i(\boldsymbol{\theta})$. For example, if $p_i(\boldsymbol{\theta})$ is increasing in $\boldsymbol{\theta}$, then $E[\phi(S_i(\theta))]$ is increasing in $\boldsymbol{\theta}$, and therefore $S_i(\theta)$ is SI in $\boldsymbol{\theta}$ (see Remark 3.2. Let us consider another set of n Bernoulli random variables, $T_1(\boldsymbol{\theta}), T_2(\boldsymbol{\theta}), \ldots, T_n(\boldsymbol{\theta})$, where $P[T_i(\boldsymbol{\theta}) = 1] = q_i(\boldsymbol{\theta})$, that can be considered as the random defaults of another set of firms given a realization of the economic factors, $\boldsymbol{\Theta} = \boldsymbol{\theta}$. It is not difficult to see that, if $p_i(\boldsymbol{\theta}) \leq q_i(\boldsymbol{\theta})$ then $S_i(\boldsymbol{\theta}) \leq_{\text{st}} T_i(\boldsymbol{\theta})$. In fact it can be proved easily that $S_i(\boldsymbol{\theta}) \leq_{\text{lr}} T_i(\boldsymbol{\theta})$. Therefore as a consequence of Theorem 3.1 $(S_1, \ldots, S_n) \leq_{\text{st}} (T_1, \ldots, T_n)$. We can consider also the situation where the default probability of the ith firm can be computed under two different scenarios, $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$, as $P[\text{default of } i\text{th firm}|\boldsymbol{\Theta}_1 = \theta] = p_i(\boldsymbol{\theta})$ and $P[\text{default of } i\text{-th firm}|\boldsymbol{\Theta}_2 = \boldsymbol{\theta}] = q_i(\boldsymbol{\theta})$, respectively. Under previous conditions and assuming that $\boldsymbol{\Theta}_1 \leq_{\text{st}} \boldsymbol{\Theta}_2$, we have again, from Theorem 3.1, that $(S_1, \ldots, S_n) \leq_{\text{st}} (T_1, \ldots, T_n)$. Let us consider now the potential loss of the ith firm, given by e_i, where e_i is positive and deterministic. Then the portfolio loss under the two scenarios, are given by $L_1 = \sum_{i=1}^n e_i S_i$ and $L_2 = \sum_{i=1}^n e_i T_i$, and under previous considerations we obtain that $L_1 \leq_{\text{st}} L_2$.

Let us consider, for example, that the two scenarios are modelled by multivariate logit-normal distributions, that is, for $i = 1, 2$

$$\boldsymbol{\Theta}_i = \left(\frac{\exp\{Z_1^i\}}{1 + \exp\{Z_1^i\}}, \ldots, \frac{\exp\{Z_m^i\}}{1 + \exp\{Z_m^i\}} \right),$$

where $\mathbf{Z}^1 \sim N(\mu_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{Z}_2 \sim N(\mu_2, \boldsymbol{\Sigma}_2)$. Given that the multivariate stochastic order is preserved under increasing transformations, and from results for multivariate normal distributions (see [27]), if $\mu_{1i} \leq \mu_{2i}$ for all $1 \leq i \leq n$ and $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$, then $\boldsymbol{\Theta}_1 \leq_{\text{st}} \boldsymbol{\Theta}_2$.

In case that the probability of default, $p_i(\boldsymbol{\theta})$, depends only on the value θ_i, that is $p_i(\boldsymbol{\theta}) = p_i(\theta_i)$, we can consider applications for the lr and icx orders. For example, is not difficult to prove that, if $p_i(\theta_i)$ is increasing in θ_i, then $S_i(\theta_i)$ is increasing in the likelihood ratio order, and therefore we can apply Theorem 3.8. From (4.19), if we consider an increasing convex function ϕ, then, if $p_i(\theta_i)$ is increasing and convex, then we can apply Theorem 3.5.

- **Poisson mixture models**

  Another possibility in this context is to consider that "a company may potentially "default more than once" in the period of interest, albeit with a very low probability" (see [28]). In this case given n companies, the number of defaults for each company $(S_1, \ldots, S_n)$, is a random vector such that conditional on some random economic factors $\boldsymbol{\Theta} = \boldsymbol{\theta}$, the random vector $(S_1, \ldots, S_n|\boldsymbol{\Theta} = \boldsymbol{\theta})$ is a vector of independent Poisson distributed rvs with parameter $\lambda_i(\boldsymbol{\theta})$. Applications of this model can be found also in actuarial mathematics (see Section 10.2.4 in [28]).

  It is not difficult to see that if $\lambda_i(\boldsymbol{\theta})$ is increasing in $\boldsymbol{\theta}$ then $S_i(\boldsymbol{\theta})$ is increasing in the likelihood ratio order in $\boldsymbol{\theta}$, therefore is increasing in the stochastic order and therefore condition (i) in Theorem 3.1 is satisfied, and similar results to that considered for Bernoulli mixture models can be given. For example we can consider the case where the distribution of the parameters $(\lambda_1, \ldots, \lambda_n)$ is a multivariate lognormal distribution. This leads to a multivariate Poisson lognormal distribution for $(S_1, \ldots, S_n)$. Given that the multivariate lognormal distribution can be obtained from a multivariate normal distribution through an increasing transformation, and given that the usual stochastic order is preserved under increasing transformations, we can provide conditions on two scenarios distributed according to multivariate lognormal distributions, as in the Bernoulli case, to apply Theorem 3.1.

4.2. Proportional hazard models

Let us consider the case in which for all i: $1, \ldots, n$, $S_i(\boldsymbol{\theta})$ and $T_i(\boldsymbol{\theta})$ are non-negative random variables, with *proportional hazard rates* given by

$$p_i(\boldsymbol{\theta})r_i(t) \quad \text{and} \quad q_i(\boldsymbol{\theta})r_i(t),$$

where $p_i(\cdot)$ and $q_i(\cdot)$ are real functions and $r_i(\cdot)$ is a hazard rate. Then the distribution functions of $S_i(\boldsymbol{\theta})$ and $T_i(\boldsymbol{\theta})$ are given by

$$\overline{F}_i(t)^{p_i(\boldsymbol{\theta})} \quad \text{and} \quad \overline{F}_i(t)^{q_i(\boldsymbol{\theta})},$$

respectively, where $\overline{F}_i$ is the associated survival function to r_i, that is $\overline{F}_i(t) = \exp(-\int_0^t r_i(x)\mathrm{d}x)$, and the density functions of $S_i(\boldsymbol{\theta})$ and $T_i(\boldsymbol{\theta})$ are given by

$$p_i(\boldsymbol{\theta})r_i(t)\overline{F}_i(t)^{p_i(\boldsymbol{\theta})} \quad \text{and} \quad q_i(\boldsymbol{\theta})r_i(t)\overline{F}_i(t)^{q_i(\boldsymbol{\theta})},$$

respectively.

Applications of this model can be given in the context of lifetimes. For example, it can be used to describe risk models (see Chapters 7 and 10 in [29]). Dependence properties for this model have been provided by Shaked and Spizzichino [5] and Frostig and Denuit [12]. Other applications can be given in the context of *Bayesian minimal repair*.

Consider n independent units with survival functions $\overline{F}_i$, i : $1, \ldots, m$, and suppose that each one is imperfectly repaired upon failure. That is, upon failure of unit i, this is perfectly repaired with probability p_i or is minimally repaired with probability $1-p_i$. This model was proposed by Brown and Proschan [30] and it is known as *imperfect repair model*. Following [31], we can generalize this model assuming that p_i depends on some m-dimensional random environment $\boldsymbol{\Theta}$, that is, given

$\mathbf{\Theta} = \boldsymbol{\theta}$, then $p_i = p_i(\boldsymbol{\theta})$. If we denote by $\Pi(\boldsymbol{\theta})$ the joint distribution function of $\mathbf{\Theta}$, and $\mathbf{S} = (S_1, \ldots, S_n)$, denotes the waiting time of the first perfect repair of the n components, then the joint survival function of $\mathbf{S}$, is given by

$$\overline{F}(t_1, \ldots, t_m) = \int_{\mathbb{R}^m} \prod_{i=1}^{m} \overline{F}_i(t_i)^{p_i(\boldsymbol{\theta})} \mathrm{d}\Pi(\boldsymbol{\theta}),$$

For this model Belzunce and Semeraro [7] provide some dependence properties. Next we describe conditions on $p_i(\cdot)$ and $q_i(\cdot)$ under which we can apply some of the previous results to compare two random vectors $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$ as above, working on two random environments $\mathbf{\Theta}_1$ and $\mathbf{\Theta}_2$, respectively.

Let us consider that $p_i(\boldsymbol{\theta})$ (or $q_i(\boldsymbol{\theta})$) is decreasing in $\boldsymbol{\theta}$. Then it is not difficult to verify that $S_i(\boldsymbol{\theta}) \leq_{\mathrm{lr}} S_i(\boldsymbol{\theta}')$ for all $\boldsymbol{\theta} \leq \boldsymbol{\theta}'$, and therefore $S_i(\boldsymbol{\theta}) \leq_{\mathrm{st}} S_i(\boldsymbol{\theta}')$. If we assume also that $p_i(\boldsymbol{\theta}) \geq q_i(\boldsymbol{\theta})$, for all $\boldsymbol{\theta}$, then is not difficult to prove that $S_i(\boldsymbol{\theta}) \leq_{\mathrm{lr}} T_i(\boldsymbol{\theta})$, and therefore $S_i(\boldsymbol{\theta}) \leq_{\mathrm{st}} T_i(\boldsymbol{\theta})$. Let us assume that $\mathbf{\Theta}_1 \leq_{\mathrm{st}} \mathbf{\Theta}_2$, then from Theorem 3.1, $(S_1, \ldots, S_n) \leq_{\mathrm{st}} (T_1, \ldots, T_n)$.

Let us consider now that $\mathbf{\Theta}_1 = (\Theta_{1,1}, \ldots, \Theta_{1,n})$ and $\mathbf{\Theta}_2 = (\Theta_{2,1}, \ldots, \Theta_{2,n})$ are n-dimensional random vectors, and let us assume that $p_i(\theta_1, \ldots, \theta_n)$ and $q_i(\theta_1, \ldots, \theta_n)$ depend only on θ_i. Let us consider that $p_i(\theta)$ (or $q_i(\theta)$) is not only decreasing, but also concave in θ. Then from Theorem 8.C.1 and 8.C.5 in [13] it is possible to prove that $\{S_i(\theta), \theta \in \chi \subseteq \mathbb{R}\} \in$ SICX. Therefore from Remark 3.6 we have that $S_i(\theta)$ (or $T_i(\theta)$) satisfies condition (ii) in Theorem 3.5. Under the assumption $p_i(\theta) \geq q_i(\theta)$, for all θ, we have condition (i) in Theorem 3.5, therefore if $\mathbf{\Theta}_1 \leq_{\mathrm{iccx}} \mathbf{\Theta}_2$ then from Theorem 3.5, $(S_1, \ldots, S_n) \leq_{\mathrm{iccx}} (T_1, \ldots, T_n)$.

Let us consider as above that p_i only depends on θ_i, and that, for all i: $1, \ldots, n$, $p_i(\cdot) = q_i(\cdot)$. As mentioned before, if $p_i(\theta)$ is decreasing in θ, then $S_i(\theta) \leq_{\mathrm{lr}} S_i(\theta')$ for all $\theta \leq \theta'$. Therefore, from Theorem 3.8, if $\mathbf{\Theta}_1 \leq_{\mathrm{lr}} \mathbf{\Theta}_2$ then $(S_1, \ldots, S_n) \leq_{\mathrm{lr}} (T_1, \ldots, T_n)$.

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References

[1] E. Arjas, M. Bhattacharjee, Modelling heterogeneity in repeated time data: A hierarchical Bayesian approach, in: R. Soyer, T.A. Mazzuchi, N.D. Singpurwalla (Eds.), Mathematical Reliability: An Expository Perspective, Kluwer Academic Publishers, Boston, MA, 2004, pp. 71–86.
[2] D.V. Lindley, N.D. Singpurwalla, On exchangeable, causal and cascading failures, Statistical Science 17 (2002) 209–219.
[3] F. Spizzichino, Subjective Probability Models for Lifetimes, in: Monographs on Statistics and Applied Probability, vol. 91, Chapman & Hall/CRC, Boca Raton, FL, 2001.
[4] K. Jogdeo, On a probability bound of Marshall and Olkin, The Annals of Statistics 6 (1978) 232–234.
[5] M. Shaked, F. Spizzichino, Positive dependence properties of conditionally independent random lifetimes, Mathematics of Operations Research 23 (1998) 944–959.
[6] B.-E. Khaledhi, S.C. Kochar, Dependence properties of multivariate mixture distributions and their applications, Annals of the Institute of Statistical Mathematics 53 (2001) 620–630.
[7] F. Belzunce, P. Semeraro, Preservation of positive and negative orthant dependence concepts under mixtures and applications, Journal of Applied Probability 41 (2004) 961–974.
[8] T. Hu, J. Chen, C. Xie, Regression dependence in latent variable models, Probability in the Engineering and Informational Sciences 20 (2006) 363–379.
[9] M. Scarsini, F. Spizzichino, Simpson-type paradoxes, dependence and ageing, Journal of Applied Probability 36 (1999) 119–131.
[10] F. Spizzichino, Mixture models, multivariate: Aging and dependence, in: F. Ruggeri, R.S. Kenett, F.W. Faltin (Eds.), Encyclopedia of Statistics in Quality and Reliability, John Wiley & Sons Ltd., Chichester, UK, 2007, pp. 1112–1116.
[11] M. Denuit, E. Frostig, Comparison of dependence in factor models with application to credit risk portfolios, Probability in the Engineering and Informational Sciences 22 (2008) 151–160.
[12] E. Frostig, M. Denuit, Dependence in failure times due to environmental factors, Statistics & Probability Letters (2008) doi:10.1016/j.spl.2008.09.023.
[13] M. Shaked, J.G. Shanthikumar, Stochastic Orders, in: Springer Series in Statistics, Springer, New York, 2007.
[14] A. Müller, D. Stoyan, Comparison Methods for Stochastic Models and Risks, John Wiley & Sons, Ltd., Chichester, 2002.
[15] M. Marinacci, L. Montrucchio, Ultramodular functions, Mathematics of Operations Research 30 (2005) 311–332.
[16] M. Shaked, J.G. Shanthikumar, Multivariate hazard rates and stochastic ordering, Advances in Applied Probability 19 (1987) 123–137.
[17] M. Shaked, J.G. Shanthikumar, Dynamic multivariate mean residual life functions, Journal of Applied Probability 28 (1991a) 613–629.
[18] M. Shaked, J.G. Shanthikumar, Dynamic multivariate aging notions in reliability theory, Stochastic Process and their Applications 38 (1991b) 85–97.
[19] M. Shaked, J.G. Shanthikumar, Multivariate conditional hazard rate and mean residual life functions and their applications, in: Richard E. Barlow, Carlo A. Clarotti, Fabio Spizzichino (Eds.), Reliability and Decision Making, Chapman & Hall, London, 1993, pp. 137–155.
[20] T. Hu, B.-E. Khaledi, M. Shaked, Multivariate hazard rate orders, Journal of Multivariate Analysis 84 (2003) 173–189.
[21] M. Shaked, J.G. Shanthikumar, J.B. Valdez-Torres, Discrete probabilistic orderings in reliability theory, Statistica Sinica 4 (1994) 567–579.
[22] M. Shaked, J.G. Shanthikumar, J.B. Valdez-Torres, Discrete hazard rate functions, Computers and Operations Research 22 (1995) 391–402.
[23] H. Joe, Multivariate Models and Dependence Concepts, Chapman and Hall, London, 1997.
[24] M. Denuit, A. Müller, Smooth generators of integral stochastic orders, Annals of Applied Probability 12 (2002) 1174–1184.
[25] S. Karlin, Y. Rinott, Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions, Journal of Multivariate Analysis 10 (1980) 467–498.
[26] S.C. Kochar, On stochastic orderings between distributions and their sample spacings, Statistics & Probability Letters 42 (1999) 345–352.
[27] A. Müller, Stochastic ordering of multivariate normal distributions, Annals of the Institute of Statistical Mathematics 53 (2001) 567–575.
[28] A.J. McNeil, R. Frey, P. Embrechts, Quantitative Risk Management. Concepts, Techniques and Tools, in: Princeton Series in Finance, Princeton University Press, Princeton, NJ, 2005.
[29] P. Hougaard, Analysis of Multivariate Survival Data, Springer, New-York, 2000.
[30] M. Brown, F. Proschan, Imperfect repair, Journal of Applied Probability 20 (1983) 851–859.
[31] J.-H. Lim, K.-L. Lu, D.-H. Park, Bayesian imperfect repair model, Communications in Statistics-Theory and Methods 27 (1998) 965–984.