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The classical double copy for Taub–NUT spacetime

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ABSTRACT

The double copy is a much-studied relationship between gauge theory and gravity amplitudes. Recently, this was generalised to an infinite family of classical solutions to Einstein's equations, namely stationary Kerr–Schild geometries. In this paper, we extend this to the Taub–NUT solution in gravity, which has a double Kerr–Schild form. The single copy of this solution is a dyon, whose electric and magnetic charges are related to the mass and NUT charge in the gravity theory. Finally, we find hints that the classical double copy extends to curved background geometries.

1. Introduction

The study of scattering amplitudes in both gauge and gravity theories continues to be an active area of research, including the relationships between both types of theory. A relatively recent discovery is the *double copy* property linking gauge and gravity amplitudes, with or without supersymmetry [1–3]. This itself relies on a certain relationship (*BCJ duality*) being made manifest in the gauge theory, and is known to be true at tree-level [2,4–10], where it is equivalent to the KLT relations of [11]. Follow-up work has examined loop-level amplitudes [3,12–30], form-factors [31], or alternative theories [32]. All-order tests of the double copy are possible in certain kinematic limits [15,33–36], but a full proof at loop level – which relies on the existence of BCJ-dual amplitudes – has yet to be obtained (see Refs. [37–52] for related studies). It is not known, for example, if the copy is a genuinely non-perturbative property of both theories. One reason for this is that the perturbative construction of the double copy (which relies on replacing four-gluon vertices with pairs of three-gluon vertices) obscures a direct understanding at the level of the Lagrangian. Partial exceptions to this are the perturbatively constructed effective Lagrangian of Ref. [2] (see also Refs. [38,53]), and the self-dual sector of Refs. [20,54] where, in the latter, the Yang–Mills action can be made manifestly cubic [55], and a full interpretation of BCJ duality obtained. See Ref. [56] for a recent review.

The above discussion motivates an alternative way of examining the double copy, namely to look directly at solutions of the classical equations of motion in gauge and gravity theories, and to match these up according to a double copy prescription. If this matching can be argued to be the same as the BCJ double copy for amplitudes in a meaningful way, one gains a deeper understanding of the double copy, including its potentially nonperturbative role. A pioneering study in this regard was recently undertaken by some of the present authors in Ref. [57], which found an infinite class of classical solutions that double copy between Yang–Mills theory and gravity. On the gravity side, these correspond to stationary Kerr–Schild solutions, where the latter property leads to linearisation of the Einstein equations, such that the solution for the graviton field terminates at first-order in perturbation theory. The single copies of these solutions solve linearised Yang–Mills equations, and a number of examples were given in Ref. [57]. That the single copy procedure is indeed related to the BCJ double copy for amplitudes relies on performing the zeroth copy from the gauge theory to a biadjoint scalar theory. The latter has arisen in a number of recent studies [10,58–60], and its relevance in the present context is that the scalar field from the zeroth copy can be used to fix the double copy prescription between gauge theory and gravity. It was also shown in Ref. [57] that the self-dual sectors of gauge theory and gravity have a Kerr–Schild-like description.

A number of puzzles remain regarding the results of Ref. [57], not least the lack of a full understanding of the role that Kerr–Schild coordinates play. The aim in this paper is to generalise the results of Ref. [57], and to provide further evidence supporting the classical double copy. To this end, we consider the Taub–NUT solution [61,62] in General Relativity, which is known not to have a

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simple Kerr–Schild form. Nevertheless, we will obtain a single copy of this solution, which has a clear interpretation (in four spacetime dimensions) as a gauge-theory dyon. As for the simple Kerr–Schild solutions of Ref. [57], one may also take a zeroth copy to a biadjoint scalar theory, which can be used to fix the nature of the double copy, and to relate this to the double copy for amplitudes. The Schwarzschild black hole emerges as a special case.

Another generalisation is to consider Kerr–Schild geometries that have a non-trivial (i.e. non-Minkowski) background metric. We examine the Taub–NUT solution on (Anti-)de Sitter space, and find that the single copy also works in this case, namely that one may construct a gauge theory object that satisfies the curved space Maxwell equations. We find an extra term in the biadjoint field equation, which is consistent with the scalar being conformally coupled to the gravity background in four spacetime dimensions.

The structure of our paper is as follows. In Section 2, we review the results of Ref. [57]. In Section 3 we discuss the Taub–NUT solution and its gauge theory counterpart, and in Section 4 we look at the double copy in (Anti-)de Sitter space. We discuss our results and conclude in Section 5.

2. The Kerr–Schild double copy

A special family of solutions to Einstein's gravitational field equations comprises *Kerr–Schild* metrics (see e.g. [63] for a detailed review)

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \\ &= \bar{g}_{\mu\nu} + \kappa\, \phi\, k_\mu\, k_\nu, \end{aligned} \tag{1}$$

where $\kappa = \sqrt{16\pi G_N}$, G_N is Newton's constant, and $\bar{g}_{\mu\nu}$ is a background metric. Here ϕ is a scalar field, and k_μ is both null and geodetic with respect to the background. That is,

$$\bar{g}_{\mu\nu}\, k^\mu\, k^\nu = 0, \quad (k\cdot D)\, k_\mu = 0, \tag{2}$$

where D^μ is the covariant derivative in the metric $\bar{g}_{\mu\nu}$. The Kerr–Schild form is special in that the "graviton" explicitly decomposes into an outer product of the vector k_μ with itself. Furthermore, such solutions have the remarkable property that they linearise the Einstein equations. More specifically, the components of the Ricci tensor are[1]

$$R^\mu_\nu = \bar{R}^\mu_\nu - \kappa \left[h^\mu_\rho \bar{R}^\rho_\nu + \frac{1}{2} D_\rho \left(D_\nu h^{\mu\rho} + D^\mu h^\rho_\nu - D^\rho h^\mu_\nu \right) \right], \tag{3}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor associated with $\bar{g}_{\mu\nu}$. Reference [57] concentrated on a Minkowski background, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, and also stationary solutions ($\partial_0 \phi = \partial_0 k_\mu = 0$). It was then shown that the gauge field

$$A^a_\mu = c^a \phi\, k_\mu, \tag{4}$$

for constant c^a, solves the Yang–Mills equations, which due to the trivial colour dependence take a linearised form,

$$\partial^\mu F^a_{\mu\nu} = 0. \tag{5}$$

This gauge field of eq. (4) is then taken to be the single copy of the graviton of eq. (1), obtained by removing one of the Lorentz vectors k_μ from the graviton solution, and replacing coupling constant and charge factors. Carrying this one step further, one may remove the additional Lorentz factor, and define the biadjoint scalar field

$$\Phi^{aa'} = c^a \tilde{c}^{a'} \phi, \tag{6}$$

for constant colour charge vectors c^a and $\tilde{c}^{a'}$, where a and a' are associated to the Lie algebras of two distinct groups G and G'. This solves the equation of motion for the biadjoint scalar theory (which linearises),

$$\partial^2 \Phi^{aa'} - y f^{abc} \tilde{f}^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0, \tag{7}$$

and is then identified with the zeroth copy of the gauge theory solution. The zeroth copy is one way of fixing the single copy procedure of eq. (4), which would otherwise be ambiguous. A priori, one can choose to absorb an overall scalar function into the Kerr–Schild vector k_μ before taking the single copy. However, there is a unique choice that satisfies the biadjoint equation upon taking the zeroth copy. Furthermore, this has a physical interpretation as a scalar propagator integrated over the source charge [57]. This is the same procedure as the BCJ double copy for amplitudes, in which denominator factors (scalar propagators) are left untouched when the double copy is performed, but numerators are not.

3. The Taub–NUT solution

The Taub–NUT metric, first derived in Refs. [61,62], is a stationary, axisymmetric vacuum solution that is not asymptotically flat. It can be sourced by a pointlike object at the origin, which carries both electric charge and *NUT charge*. The latter is associated with the lack of spherical symmetry and asymptotic flatness, and is known to correspond to a magnetic monopole-like behaviour of the gravitational field at spatial infinity (see e.g. [64] for a review). We will see in this section that the Taub–NUT solution provides an interesting example of a Kerr–Schild double copy, which extends our previous results to metrics not of the form of eq. (1).

A general formulation of the Taub–NUT–Kerr–de Sitter metric has been given by Plebanski [65], and has later been shown to exhibit a *double Kerr–Schild form* in Ref. [66]. That is, one may write the metric in the form

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \\ &= \bar{g}_{\mu\nu} + \kappa \left(\phi\, k_\mu\, k_\nu + \psi\, l_\mu\, l_\nu \right), \end{aligned} \tag{8}$$

where $\bar{g}_{\mu\nu}$ is a de Sitter background metric, the vectors k_μ and l_μ satisfy the conditions

$$k^2 = l^2 = k\cdot l = 0, \quad (k\cdot D)k_\mu = 0, \quad (l\cdot D)l_\mu = 0, \tag{9}$$

and all contractions and covariant derivatives can be taken with respect to either the background or the full metric. This form is a clear generalisation of the standard Kerr–Schild form of eq. (1). However, it is not a straightforward extension: it is no longer true in general that the Einstein equations are linearised. One may show, in fact, that the mixed components of the Ricci tensor are

$$\begin{aligned} R^\mu_\nu = \bar{R}^\mu_\nu + \kappa &\left[-h^\mu_\rho \bar{R}^\rho_\nu + \frac{1}{2} D_\rho \left(D_\nu h^{\mu\rho} + D^\mu h^\rho_\nu - D^\rho h^\mu_\nu \right) \right] \\ &+ R^\mu_{\nu,\text{non-lin.}}, \end{aligned} \tag{10}$$

where the non-linear term is

$$\begin{aligned} R^\mu_{\nu,\text{non-lin.}} = -\frac{\kappa^2}{2} \Bigg[& \frac{1}{2} D^\mu h(k)^\rho_\delta D_\nu h(l)^\delta_\rho + h(l)^{\mu\delta} D_\rho D_\nu h(k)^\rho_\delta \\ & + D_\rho \Big(h(l)^{\rho\delta} D_\delta h(k)^\mu_\nu + 2h(l)^{\rho\delta} D_{(\nu} h(k)^{\mu)}_\delta \\ & - 2h(l)^{\mu\delta} D^{[\rho} h(k)_{\delta]\nu} \Big) \Bigg] + (k \leftrightarrow l), \end{aligned} \tag{11}$$

and we have defined the shorthand notation

$$h(k)_{\mu\nu} = \phi k_\mu k_\nu, \quad h(l)_{\mu\nu} = \psi l_\mu l_\nu. \tag{12}$$

[1] N.B. the Ricci tensor is only linearised for the index placement as chosen in eq. (3).

Remarkably, in Plebanski coordinates, as argued in Ref. [66], the full metric can be cast in a form such that eq. (11) vanishes, and the Ricci tensor indeed linearises. The explicit form of the background line element is

$$d\bar{s}^2 = -\frac{1}{q^2-p^2}\left[\bar{\Delta}_p(d\tilde{\tau}+q^2 d\tilde{\sigma})^2 + \bar{\Delta}_q(d\tilde{\tau}+p^2 d\tilde{\sigma})^2\right] - 2(d\tilde{\tau}+q^2 d\tilde{\sigma})dp - 2(d\tilde{\tau}+p^2 d\sigma)dq, \tag{13}$$

where

$$\bar{\Delta}_p = \gamma - \epsilon p^2 + \lambda p^4, \quad \bar{\Delta}_q = -\gamma + \epsilon q^2 - \lambda q^4. \tag{14}$$

Here ϵ is a constant, and γ is related to the angular momentum. Equation (13) is a solution to the Einstein equation with non-zero cosmological constant λ. The Kerr–Schild vectors are given in the $(\tilde{\tau}, \tilde{\sigma}, p, q)$ coordinate system (which has $(2, 2)$ signature) by

$$k_\mu = (1, q^2, 0, 0), \quad l_\mu = (1, p^2, 0, 0), \tag{15}$$

and the accompanying scalar functions by

$$\phi = \frac{2Np}{q^2-p^2}, \quad \psi = \frac{2mq}{q^2-p^2}. \tag{16}$$

The parameter m represents the mass of the solution, and N the NUT charge.

Let us now obtain and interpret the single copy of this solution, where we will focus for brevity on the case of vanishing angular momentum ($\gamma = 0$), which leads to a pointlike source. First, we note that the natural generalisation of the single copy procedure of eq. (4) in the double Kerr–Schild case is to construct a gauge field

$$A^a_\mu = c^a\left(\phi k_\mu + \psi l_\mu\right). \tag{17}$$

That is, the double copy of this solution proceeds term-by-term, analogously to how the BCJ double copy for amplitudes is applied separately to terms involving different scalar propagators. We have verified that the gauge field of eq. (17) satisfies the Yang–Mills equations (which linearise),

$$D^\mu F^a_{\mu\nu} = 0, \quad F^a_{\mu\nu} = D_\mu A^a_\nu - D_\nu A^a_\mu. \tag{18}$$

It is interesting already at this point that the Yang–Mills equations are satisfied even for a non-Minkowski background. We return to this in the following section.

For now, let us interpret the case $\lambda = 0$, for which $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. In taking the single copy, we will make the replacements

$$\frac{m\kappa}{2} \to (c_a T^a) g_s, \quad \frac{N\kappa}{2} \to (c_a T^a)\tilde{g}_s, \tag{19}$$

to be explained shortly. Having taken the single copy in Plebanski coordinates (where the Kerr–Schild form is manifest), we are free to transform to any other choice of coordinates. We will do this in two stages. First, following Refs. [65,66], one may transform to spheroidal coordinates according to

$$\tau = t + a\varphi, \quad \sigma = \frac{\varphi}{a}, \quad q = r, \quad p = a\cos\theta, \tag{20}$$

where the coordinates τ and σ are related to the counterparts used throughout this paper by

$$d\tilde{\tau} = d\tau + \frac{p^2 dp}{\Delta_p} - \frac{q^2 dq}{\Delta_q}, \quad d\tilde{\sigma} = d\sigma - \frac{dp}{\Delta_p} + \frac{dq}{\Delta_q}. \tag{21}$$

Next, one may take the parameter $a^2 \equiv \gamma$ (which is related to the angular momentum) to zero, so that the spheroidal radius becomes a spherical one. This coordinate transformation is subtle, in that the vector l^μ becomes singular as $a \to 0$. The prefactor ψ entering the gauge field, however, is $\mathcal{O}(a)$, such that gauge field A^a_μ itself is well-defined. In the spherical polar coordinate system (t, r, θ, ϕ), the field strength tensor then becomes

$$F = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu = -\frac{c_a T^a}{8\pi}\left(\frac{g_s}{r^2}\, dt \wedge dr + \tilde{g}_s \sin\theta\, d\theta \wedge d\phi\right), \tag{22}$$

where we have separated out the contributions from the constants g_s and $\tilde{g}_s$. The first term on the right-hand side of eq. (22) gives a pure electric field, corresponding to a Coulomb solution. Thus, the mass in the Taub–NUT metric single copies to a static colour charge, exactly as in the Schwarzschild case of Ref. [57]. This must in fact be the case, given that the Taub–NUT metric becomes the Schwarzschild metric as $N \to 0$. This explains our choice of factors in eq. (19).

The NUT charge contribution to the field strength tensor is a pure magnetic field, and we can interpret this in more detail by expressing eq. (22) as

$$F = -\frac{c_a T^a}{8\pi}\left(\frac{g_s}{r^2}\, dt \wedge dr + \star\frac{\tilde{g}_s}{r^2}\, dt \wedge dr\right). \tag{23}$$

Here, $\star$ denotes the Hodge dual of a 2-form, say $\omega_{\mu\nu}$,

$$\star\omega_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta}, \qquad \star^2 = -1. \tag{24}$$

Thus, the dual tensor for the NUT-charge term contains a pure electric field corresponding to a point charge of strength $\tilde{g}_s$. It follows that the magnetic field in the original field strength tensor corresponds to a magnetic monopole, where the NUT charge in the gravity theory single copies to the monopole charge in the gauge theory. This is perhaps to be expected, given that the NUT charge in the Taub–NUT metric is known to be associated with monopole-like behaviour [64], an analogy which has now been turned into an exact statement under the classical double copy. We have then chosen the constant $\tilde{g}_s$ in eq. (19) to obey the same normalisation as g_s in the (non-dual) field strength tensor.

Note that the transformation from the Plebanski coordinate system to the spherical coordinate system involves a change of signature (from $(2, 2)$ to $(1, 3)$), and thus a Wick rotation. In the Plebanski system itself, the two charges m and l appear on an equal footing, as is clear from eqs. (13)–(16). In other words, in this signature one cannot tell the difference in the gauge theory between an electric and a (dual) magnetic charge. For the (anti-) self-dual case, the gauge and gravity solutions can be interpreted as instantons (see also [64]).

As is well known, consistency of the monopole gauge field leads to the quantisation condition (in the present notation)

$$g_s \tilde{g}_s = \frac{n}{2}, \quad n \in \mathbb{Z}, \tag{25}$$

relating the electric and magnetic charges. This has an analogue in the gravity theory, as discussed in Refs. [67,68]. There, recovery of spherical symmetry demands a periodic time coordinate. This corresponds to quantisation of the energy of the dyon, or its mass in the non-relativistic approximation. There is then a quantisation condition relating the mass and NUT charge, which is the equivalent of eq. (25) from a double copy perspective.

As in the standard Kerr–Schild case, we may take the zeroth copy, which produces a biadjoint scalar field

$$\Phi^{aa'} = c^a \tilde{c}^{a'}\left(\phi + \psi\right). \tag{26}$$

Similarly to the results of Ref. [57], this is a solution of the linearised biadjoint eq. (7). In fact, both ϕ and ψ satisfy that equation separately. They have the interpretation of a scalar propagator integrated over the source charges, and are analogous to the scalar propagators that are not modified when double-copying scattering amplitudes. As has already been mentioned above, another property that links the generalised Kerr–Schild double copy to the corresponding story for amplitudes is that each Kerr–Schild term (involving a different scalar propagator) is copied individually, with no mixing between these terms on the gravity side.

The results of this section constitute an interesting generalisation of the Kerr–Schild double copies of Ref. [57], in that a double Kerr–Schild form is used. As remarked in Ref. [66], it is highly non-trivial that the particular double Kerr–Schild result for the Taub–NUT solution linearises the Einstein equations. One may also examine analogues of the Taub–NUT solution in higher dimensions. A family of higher-dimensional generalisations of the Plebanski metric was obtained in Ref. [69], and subsequently shown to have a multiple Kerr–Schild form [70], involving $n = \lfloor D/2 \rfloor$ linearly independent, mutually orthogonal null vector fields k_i^μ (here $\lfloor X \rfloor$ denotes the integer part of X):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i=1}^{n} \phi_i k_{i\mu} k_{i\nu}. \tag{27}$$

This form relies on a generalised set of Plebanski-like coordinates, in $(\lfloor D/2 \rfloor, \lfloor (D+1)/2 \rfloor)$ signature, and each function ϕ_i involves a parameter playing the role of a generalised NUT charge. Assuming that these metrics indeed linearise the Einstein equations,[2] one may construct a single copy gauge field term by term, as for the double Kerr–Schild example in $D = 4$:

$$A_\mu = \sum_{i=1}^{n} \phi_i k_{i\mu}. \tag{28}$$

Each term taken by itself satisfies the Maxwell equations using the argument of Ref. [57], given that it constitutes a time-independent single Kerr–Schild solution. That the complete multiple Kerr–Schild form satisfies the Maxwell equations then follows from linear independence of the generalised NUT charges. Note that, in the signature of the Plebanski-like metric, the NUT charges appear on an equal footing, as in the canonical Taub–NUT case. After analytic continuation to the physical $(1, D-1)$ signature, one parameter will play the role of an electric charge, obtained as a single copy of a mass parameter in the gravity theory.

As mentioned above, there have been previous observations that the Taub–NUT solution is analogous to a gauge theory dyon [64,68]. Such statements, however, are restricted to the weak gravity approximation, and are not embedded in a formal double copy relationship between Yang–Mills theory and gravity. Here, the use of the Kerr–Schild double copy makes the dyon–Taub–NUT relationship perturbatively exact, and also ties it to the double copy for scattering amplitudes.

4. Double copy in de Sitter space

In the previous section, we saw that the Kerr–Schild double copy can be extended to the case of a double Kerr–Schild solution, representing a dyon. Another possible generalisation is to consider the background metric $\bar{g}_{\mu\nu}$ to be non-Minkowski, and the Plebanski form of the Taub–NUT–Kerr–de Sitter metric provides just such an example.

We have, in fact, already seen above that the gauge field of eq. (17), obtained as a single copy of the Plebanski metric, solves the *(Anti-)de Sitter space* Maxwell equations of eq. (18). This already suggests that our interpretation of the Kerr–Schild double copy can indeed be generalised to curved backgrounds. In the Minkowski case, it was important to take the zeroth copy to a biadjoint scalar theory. This fixes the overall scalar function that is not squared when taking the double copy, and also helps to tie the classical double copy to the similar procedure for scattering amplitudes. One may then also examine the zeroth copy in the (Anti-) de Sitter background. For the general Taub–NUT–Kerr–de Sitter metric, one finds that the scalar field of eq. (26) satisfies

$$D^2 \Phi^{aa'} = -2\lambda \Phi^{aa'}. \tag{29}$$

This has an additional term on the RHS, and it is not immediately clear how to interpret this. However, it is intriguing to note that this term is in fact proportional to the Ricci curvature, such that eq. (29) is obtained from the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(D^\mu \Phi^{aa'})(D_\mu \Phi^{aa'}) - \frac{y}{6} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'} - \frac{\mathcal{R}}{12} \Phi^{aa'} \Phi^{aa'}, \tag{30}$$

corresponding to a non-minimal coupling of the biadjoint scalar to the gravity background. More than this, the coefficient of the extra term precisely coincides with a conformally coupled scalar in four spacetime dimensions. This perhaps can be explained from the fact that classical Yang–Mills theory is conformally invariant in four spacetime dimensions, and that the zeroth copy somehow preserves this invariance in the free scalar theory.[3] Whether or not this is the correct interpretation of this result, deserves further investigation.

It should be mentioned that it is possible to reinterpret the de Sitter double copy as a multiple Kerr–Schild double copy around Minkowski space. This is because the de Sitter metric itself can be written in the Kerr–Schild form [70]

$$g_{\mathrm{dS},\mu\nu} = \eta_{\mu\nu} + \lambda r^2 n_\mu n_\nu, \quad n_\mu = (1, \hat{e}_r). \tag{31}$$

The gauge field obtained via the Kerr–Schild single copy is

$$A_\mu = \rho r^2 n_\mu, \tag{32}$$

where we have replaced $\lambda \to \rho$. We can interpret the latter parameter by noting that the electrostatic potential in the Kerr–Schild gauge satisfies

$$\nabla^2 A_0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_0}{\partial r}\right) = 6\rho. \tag{33}$$

Thus, ρ plays the role of a uniform charge density, filling all space. This is exactly what one expects from the single copy of the cosmological constant, given that the latter is a uniform energy density. If one chooses the fiducial metric to be Minkowski rather than de Sitter space, the conformal coupling in the biadjoint scalar theory would be absent (due to the vanishing Ricci scalar), but one must then include the uniform charge density explicitly.

5. Discussion

In this paper, we have extended our investigation of the double copy for classical solutions, which commenced in Ref. [57]. In particular, we studied the Taub–NUT metric, which goes beyond the results of Ref. [57] in having a double Kerr–Schild form. The single

[2] The linearisation property is not explicitly stated in Ref. [70]. However, we have checked its validity up to $D = 7$.

[3] In higher dimensions, Yang–Mills theory is not conformally invariant, and the relevant coefficient of the $\mathcal{R}\Phi^{aa'}\Phi^{aa'}$ term does not coincide with the conformal coupling.

copy of this solution a dyon whose electric and magnetic charge copy to mass and NUT charge respectively in the gravity theory. A similar story exists in higher spacetime dimensions, for the generalised Plebanski metrics of Refs. [69,70].

We also examined the Taub–NUT solution in (Anti-)de Sitter space, and found that the single copy also works. This is a highly interesting result, given that this is the first example of a double copy involving a non-Minkowski background. The zeroth copy also works, provided one adds a conformal mass term to the theory, which vanishes in the Minkowski case. This interpretation of the curved space double copy is somewhat tentative, and more investigation is necessary (e.g. involving other background geometries, or scattering amplitudes on curved space).

Work on extending the classical double copy further is in progress.

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