

Constrained gauge fields from spontaneous Lorentz violation

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Received 19 October 2007; received in revised form 26 November 2007; accepted 7 December 2007

Available online 15 December 2007

Abstract

Spontaneous Lorentz violation realized through a nonlinear vector field constraint of the type $A_\mu A^\mu = M^2$ (M is the proposed scale for Lorentz violation) is shown to generate massless vector Goldstone bosons, gauging the starting global internal symmetries in arbitrary relativistically invariant theories. The gauge invariance appears in essence as a necessary condition for these bosons not to be superfluously restricted in degrees of freedom, apart from the constraint due to which the true vacuum in a theory is chosen by the Lorentz violation. In the Abelian symmetry case the only possible theory proves to be QED with a massless vector Goldstone boson naturally associated with the photon, while the non-Abelian symmetry case results in a conventional Yang–Mills theory. These theories, both Abelian and non-Abelian, look essentially nonlinear and contain particular Lorentz (and *CPT*) violating couplings when expressed in terms of the pure Goldstone vector modes. However, they do not lead to physical Lorentz violation due to the simultaneously generated gauge invariance.

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PACS: 11.15.-q; 11.30.Cp; 12.20.-m

Keywords: Origin of gauge symmetry; Spontaneous Lorentz invariance violation; Constrained gauge fields

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1. Introduction

One of the most interesting examples where quantum field theory might provide some guiding rules for the search for new physics could be that of the origin of internal symmetry patterns in particle physics owing to space–time properties at very small distances. In this connection, the relativistic or Lorentz invariance seems to play a special role with respect to the observed internal local symmetries. The old idea [1] that spontaneous Lorentz invariance violation (SLIV) may lead to an alternative theory of QED, with the photon as a massless vector Nambu–Goldstone boson, still remains extremely attractive in numerous theoretical contexts [2] (for some later developments, see the papers [3]). At the same time, Lorentz violation on its own has attracted considerable attention in recent years as an interesting phenomenological possibility appearing in various quantum field and string theories [4–9]. Actually, the SLIV idea is in accordance with superstring theory, particularly with the observation that the relativistic invariance could spontaneously be violated in superstrings [4].

The first models realizing the SLIV conjecture were based on the four fermion (current–current) interaction, where the gauge field appears as a fermion–antifermion pair composite state [1], in complete analogy with the massless composite scalar field in the original Nambu–Jona-Lasinio model [10]. Unfortunately, owing to the lack of a starting gauge invariance in such models and the composite nature of the Goldstone modes which appear, it is hard to explicitly demonstrate that these modes really form together a massless vector boson as a gauge field candidate. Actually, one must make a precise tuning of parameters, including a cancellation between terms of different orders in the $1/N$ expansion (where N is the number of fermion species involved), in order to achieve the massless photon case (see, for example, the last paper in [1]). Rather, there are in general three separate massless Goldstone modes, two of which may mimic the transverse photon polarizations, while the third one must be appropriately suppressed.

In this connection, a more instructive laboratory for SLIV consideration proves to be a simple class of QED type models [11–14] having from the outset a gauge invariant form. In these models the spontaneous Lorentz violation is realized through the nonlinear dynamical constraint $A_\mu A^\mu = n_\nu n^\nu M^2$ (where n_ν is a properly oriented unit Lorentz vector, $n_\nu n^\nu = \pm 1$, while M is the proposed SLIV scale) imposed on the starting vector field A_μ , in much the same way as it occurs for the corresponding scalar field in the nonlinear σ -model for pions [15]. Note that a correspondence with the nonlinear σ -model for pions may be somewhat suggestive, in view of the fact that pions are the only presently known Goldstones and their theory, chiral dynamics [15], is given by the nonlinearly realized chiral $SU(2) \times SU(2)$ symmetry rather than by an ordinary linear σ -model. The above constraint means in essence that the vector field A_μ develops some constant background value $\langle A_\mu(x) \rangle = n_\mu M$ and the Lorentz symmetry $SO(1, 3)$ formally breaks down to $SO(3)$ or $SO(1, 2)$ depending on the time-like ($n_\nu n^\nu > 0$) or space-like ($n_\nu n^\nu < 0$) nature of SLIV. This allows one to explicitly demonstrate that gauge theories, both Abelian and non-Abelian, can be interpreted as spontaneously broken theories [11–14], although the physical Lorentz invariance still remains intact.

However, the question naturally arises of whether a gauge symmetry is necessary to start with. If so, this would in some sense depreciate the latter approach as compared with those of the original composite models [1], where a gauge symmetry was hoped to be derived (while this has not yet been achieved). Remarkably, as we will see, it happens that one does not need to specially postulate the starting gauge invariance, when considering the nonlinear σ -model type spontaneous Lorentz violation in the framework of an arbitrary relativistically invariant La-

grangian for elementary vector and matter fields, which are proposed only to possess some global internal symmetry. In the present article we start by a priori only assuming a global symmetry but no gauge invariance, taking all the terms in the Lagrangian allowed by Lorentz invariance. With such a Lagrangian, the vector field A_μ typically develops a non-zero vacuum expectation value,

$$\langle A_\mu(x) \rangle = n_\mu M. \quad (1)$$

In the limit analogous to the approximation of the linear σ -model by the nonlinear σ -model, we get the nonlinear constraint¹

$$A^2 = n^2 M^2 \quad (A^2 \equiv A_\mu A^\mu, \quad n^2 \equiv n_\nu n^\nu). \quad (2)$$

In this paper we shall simply postulate that the existence of the constraint (2) is to be upheld by adjusting the parameters of the Lagrangian. We then show that the SLIV conjecture, which is related to the condensation of a generic vector field or vector field multiplet, happens by itself to be powerful enough to impose gauge invariance, provided that we allow the corresponding Lagrangian density to be adjusted to ensure self-consistency without losing too many degrees of freedom. Due to the Lorentz violation, this theory acquires on its own a gauge-type invariance, which gauges the starting global symmetry of the interacting vector and matter fields involved. In essence, the gauge invariance (with a proper gauge-fixing term) appears as a necessary condition for these vector fields not to be superfluously restricted in degrees of freedom. In fact the crucial equations (4) and (17) below express the relations needed to reduce the number of independent equations among the equations of motion and the constraint (2). But notice that we are not assuming gauge invariance to derive Eqs. (4) and (17); our philosophy is to derive gauge invariance not to put it in. Due to the constraint (2), the true vacuum in a theory is chosen by the Lorentz violation, SLIV. The self-consistency problem to which we adjusted the couplings in the Lagrangian might have been avoided by using a Lagrange multiplier associated with the constraint (2). However it is rather the philosophy of the present article to look for consistency of the equations of motion and the constraint, without introducing such a Lagrange multiplier.

In Section 2 we consider the global Abelian symmetry case, which eventually appears as ordinary QED taken in a nonlinear gauge. While such a model for QED was considered before on its own [11–14], we actually derive it now using the pure SLIV conjecture. Then in Section 3 we generalize our consideration to the global non-Abelian internal symmetry case and come to a conventional Yang–Mills theory with that symmetry automatically gauged. Specifically, we will see that in a theory with a symmetry group G having D generators not only the

¹ Actually, some way to appreciate a possible origin for the supplementary condition (2) might be by the inclusion of a “standard” quartic vector field potential $U(A_\mu) = -(m_A^2/2)A^2 + (\lambda_A/4)(A^2)^2$ in the vector field Lagrangian, as can be motivated to some extent [4] from superstring theory. This potential inevitably causes the spontaneous violation of Lorentz symmetry in a conventional way, much as an internal symmetry violation is caused in a linear σ model for pions [15]. As a result, one has a massive “Higgs” mode (with mass $\sqrt{2}m_A$) together with massless Goldstone modes associated with the photon. Furthermore, just as in the pion model, one can go from the linear model for the SLIV to the nonlinear one by taking the limit $\lambda_A \rightarrow \infty$, $m_A^2 \rightarrow \infty$ (while keeping the ratio m_A^2/λ_A to be finite). This immediately leads to the constraint (2) for the vector potential A_μ with $n^2 M^2 = m_A^2/\lambda_A$, as appears from the validity of its equation of motion. Another motivation for the nonlinear vector field constraint (2) might be an attempt to avoid an infinite self-energy for the electron in classical electrodynamics, as was originally suggested by Dirac [16] and extended later to various vector field theory cases [17].

pure Lorentz symmetry $SO(1, 3)$, but the larger accidental symmetry $SO(D, 3D)$ of the Lorentz violating vector field constraint also happens to be spontaneously broken. As a result, although the pure Lorentz violation still generates only one true Goldstone vector boson, the accompanying pseudo-Goldstone vector bosons related to the $SO(D, 3D)$ breaking also come into play properly completing the whole gauge field multiplet of the internal symmetry group taken. Remarkably, they appear to be strictly massless as well, being protected by the simultaneously generated non-Abelian gauge invariance. When expressed in terms of the pure Goldstone vector modes these theories, both Abelian and non-Abelian, look essentially nonlinear and contain Lorentz and CPT violating couplings. However, due to cancellations, they appear to be physically indistinguishable from the conventional QED and Yang–Mills theories. On the other hand, their generic, SLIV induced, gauge invariance could of course be broken by some high-order operators, stemming from very short gravity-influenced distances that would lead to the physical Lorentz violation. This and some other of our conclusions are discussed in Section 4.

2. Abelian theory

Suppose first that there is only one vector field A_μ and one complex matter field ψ , a charged fermion or scalar, in a theory given by a general Lorentz invariant Lagrangian $L(A, \psi)$ with the corresponding global $U(1)$ charge symmetry imposed. Before proceeding further, note first that, while a conventional variation principle requires the equation of motion

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu A_\mu)} = 0 \quad (3)$$

to be satisfied, the vector field A_μ , both massive and massless, still contains one superfluous component which is usually eliminated by imposing some supplementary condition. This is typically imposed by taking the 4-divergence of the Euler equation (3). Such a condition for the massive QED case (with the gauge invariant $F_{\mu\nu}F^{\mu\nu}$ form for the vector field kinetic term) is known to be the spin-1 or Lorentz condition $\partial_\mu A^\mu = 0$, while for the conventional massless QED many other conditions (gauges) may alternatively be taken.

Let us now subject the vector field $A_\mu(x)$ in a general Lagrangian $L(A_\mu, \psi)$ to the SLIV constraint (2), which presumably chooses the true vacuum in a theory. Once the SLIV constraint is imposed, any extra supplementary condition is no longer possible, since this would superfluously restrict the number of degrees of freedom for the vector field which is inadmissible. In fact a further reduction in the number of independent A_μ components would make it impossible to set the required initial conditions in the appropriate Cauchy problem and, in quantum theory, to choose self-consistent equal-time commutation relations² [18]. It is also well known [15] that there is no way to construct a massless field A_μ , which transforms properly as a 4-vector, as a linear combination of creation and annihilation operators for helicity ± 1 states.

Under this assumption of not getting too many constraints,³ we shall now derive gauge invariance. Since the 4-divergence of the vector field Euler equation (3) should be zero if the equations

² For example the need for more than two degrees of freedom is well known for a massive vector field and for quantum electrodynamics. In the massive vector field case there are three physical spin-1 states to be described by the A_μ , whereas for QED, apart from the two physical (transverse) photon spin states, one formally needs one more component in the A_μ (A_0 or A_3) as the Lagrange multiplier to get the Gauss law. So, in both cases only one component in the A_μ may be eliminated.

³ The fact that there is a threat of too many supplementary conditions (an inconsistency) is because we have chosen not to put a Lagrange multiplier term for the constraint (2) into Eq. (3). Had we explicitly introduced such a Lagrange

of motion are used, it means that this divergence must be expressible as a sum over the equations of motion multiplied by appropriate quantities. This implies that, without using the equations of motion but still using the constraint (2), we have an identity for the vector and matter (fermion field, for definiteness) fields of the following type:

$$\begin{aligned} & \partial_\mu \left(\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu A_\mu)} \right) \\ & \equiv \left(\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu A_\mu)} \right) (c) A_\mu + \left(\frac{\partial L}{\partial \psi} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu \psi)} \right) (it) \psi \\ & + \bar{\psi} (-it) \left(\frac{\partial L}{\partial \bar{\psi}} - \partial_\nu \frac{\partial L}{\partial (\partial_\nu \bar{\psi})} \right). \end{aligned} \tag{4}$$

Here the coefficients c and t of the Eulerians on the right-hand side (which vanish by themselves when the equations of motion are fulfilled) are some dimensionless constants whose particular values are conditioned by the starting Lagrangian $L(A_\mu, \psi)$ taken, for simplicity, with renormalisable coupling constants. This identity (4) implies the invariance of L under the vector and fermion field local transformations whose infinitesimal form is given by⁴

$$\delta A_\mu = \partial_\mu \omega + c \omega A_\mu, \quad \delta \psi = it \omega \psi \tag{5}$$

where $\omega(x)$ is an arbitrary function, only being restricted by the requirement to conform with the nonlinear constraint (2). Conversely, the identity (4) in its turn follows from the invariance of the Lagrangian L under the transformations (5). Both direct and converse assertions are in fact particular cases of Noether’s second theorem [19]. Apart from this invariance, one has now to confirm that the transformations (5) in fact form an Abelian symmetry group. Constructing the corresponding Lie bracket operation ($\delta_1 \delta_2 - \delta_2 \delta_1$) for two successive vector field variations we find that, while the fermion transformation in (5) is an ordinary Abelian local one with zero Lie bracket, for the vector field transformations there appears a non-zero result

$$(\delta_1 \delta_2 - \delta_2 \delta_1) A_\mu = c (\omega_1 \partial_\mu \omega_2 - \omega_2 \partial_\mu \omega_1) \tag{6}$$

unless the coefficient $c = 0$. Note also that for non-zero c the variation of A_μ given by (6) is an essentially arbitrary vector function. Such a freely varying A_μ is only consistent with a trivial Lagrangian (i.e. $L = \text{const}$). Thus, in order to have a non-trivial Lagrangian, it is necessary to have $c = 0$ and the theory then possesses an Abelian local symmetry.⁵

Thus we have shown how the choice of a true vacuum conditioned by the SLIV constraint (2) enforces the modification of the Lagrangian L , so as to convert the starting global $U(1)$ charge symmetry into a local one (5). Otherwise, the theory would superfluously restrict the number of degrees of freedom for the vector field and that would be inadmissible. This SLIV induced local Abelian symmetry (5) now allows the Lagrangian L to be determined in full. For a minimal theory with renormalisable coupling constants, it is in fact the conventional QED Lagrangian

multiplier term, $F(x)(A^2 - n^2 M^2)$, into the Lagrangian L , the equation of motion for the vector field A_μ would have changed, so that the 4-divergence of this equation would now determine the Lagrange multiplier function $F(x)$ rather than satisfy the identity (4) appearing below.

⁴ Actually, one can confirm this proposition by expanding the action with the transformed Lagrangian density $\int d^4x L(A', \psi')$ in terms of functional derivatives and then using the identity equation (4).

⁵ We will see below (Section 3) that non-zero c -type coefficients appear in the non-Abelian internal symmetry case, resulting eventually in a Yang–Mills gauge invariant theory.

which we eventually come to:

$$L(A_\mu, \psi) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma\partial - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi \quad (7)$$

with the SLIV constraint $A^2 = n^2M^2$ imposed on the vector field A_μ . In the derivation made, we were only allowed to use gauge transformations consistent with the constraint (2) which now plays the role of a gauge-fixing term for the resulting gauge invariant theory⁶ (7). Note that a quartic potential $U(A_\mu)$ of the type discussed in footnote 1 would give vanishing contributions on both sides of Eq. (4), when the nonlinear constraint (2) with the SLIV scale M^2 given in the footnote is imposed. Furthermore the contribution of such a potential to the Lagrangian (7) would then reduce to an inessential constant.

One can rewrite the Lagrangian $L(A_\mu, \psi)$ in terms of the physical photons now identified as being the SLIV generated vector Goldstone bosons. For this purpose let us take the following handy parameterization for the vector potential A_μ in the Lagrangian L :

$$A_\mu = a_\mu + \frac{n_\mu}{n^2}(n \cdot A) \quad (n \cdot A \equiv n_\nu A^\nu) \quad (8)$$

where a_μ is the pure Goldstonic mode satisfying

$$n \cdot a = 0 \quad (n \cdot a \equiv n_\nu a^\nu) \quad (9)$$

while the effective ‘‘Higgs’’ mode (or the A_μ component in the vacuum direction) is given by the scalar product $n \cdot A$. Substituting this parameterization (8) into the vector field constraint (2), one comes to the equation for $n \cdot A$:

$$n \cdot A = (M^2 - n^2 a^2)^{\frac{1}{2}} = M - \frac{n^2 a^2}{2M} + O(1/M^2) \quad (10)$$

where $a^2 = a_\mu a^\mu$ and taking, for definiteness, the positive sign for the square root and expanding it in powers of a^2/M^2 . Putting then the parameterization (8) with the SLIV constraint (10) into our basic gauge invariant Lagrangian (7), one comes to the truly Goldstonic model for QED. This model might seem unacceptable since it contains, among other terms, the inappropriately large Lorentz violating fermion bilinear $eM\bar{\psi}(\gamma \cdot n/n^2)\psi$, which appears when the expansion (10) is applied to the fermion current interaction term in the Lagrangian L (7). However, due to local invariance of the Lagrangian (7), this term can be gauged away by making an appropriate redefinition of the fermion field according to

$$\psi \rightarrow e^{ieM(x \cdot n/n^2)}\psi \quad (11)$$

through which the $eM\bar{\psi}(\gamma \cdot n/n^2)\psi$ term is exactly cancelled by an analogous term stemming from the fermion kinetic term. So, one eventually arrives at the essentially nonlinear SLIV Lagrangian for the Goldstonic a_μ field of the type (taken to first order in a^2/M^2)

⁶ As indicated in Refs. [11,16], the SLIV constraint equation for the corresponding finite gauge function $\omega(x)$, $(A_\mu + \partial_\mu\omega)(A^\mu + \partial^\mu\omega) = n^2M^2$, appears to be mathematically equivalent to the classical Hamilton–Jacobi equation of motion for a charged particle. Thus, this equation should have a solution for some class of gauge functions $\omega(x)$, inasmuch as there is a solution to the classical problem.

$$\begin{aligned}
 L(a_\mu, \psi) = & -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} \delta (n \cdot a)^2 - \frac{1}{4} f_{\mu\nu} h^{\mu\nu} \frac{n^2 a^2}{M} \\
 & + \bar{\psi} (i \gamma \partial + m) \psi - e a_\mu \bar{\psi} \gamma^\mu \psi + \frac{e n^2 a^2}{2M} \bar{\psi} (\gamma \cdot n) \psi.
 \end{aligned}
 \tag{12}$$

We have denoted its field strength tensor by $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, while $h_{\mu\nu} = n^\mu \partial^\nu - n^\nu \partial^\mu$ is a new SLIV oriented differential tensor acting on the infinite series in a^2 coming from the expansion of the effective ‘‘Higgs’’ mode (10), from which we have only included the first-order term $-n^2 a^2 / 2M$ throughout the Lagrangian $L(a_\mu, \psi)$. We have also explicitly introduced the orthogonality condition $n \cdot a = 0$ into the Lagrangian through the second term, which can be treated as the gauge fixing term (taking the limit $\delta \rightarrow \infty$). Furthermore we have retained the notation ψ for the redefined fermion field.

This nonlinear QED model was first studied on its own by Nambu long ago [11]. As one can see, the model contains the massless vector Goldstone boson modes (keeping the massive ‘‘Higgs’’ mode frozen), and in the limit $M \rightarrow \infty$ is indistinguishable from conventional QED taken in the general axial (temporal or pure axial) gauge. So, for this part of the Lagrangian $L(a_\mu, \psi)$ given by the zero-order terms in $1/M$, the spontaneous Lorentz violation simply corresponds to a non-covariant gauge choice in an otherwise gauge invariant (and Lorentz invariant) theory. Remarkably, also all the other (first and higher order in $1/M$) terms in $L(a_\mu, \psi)$ (12), though being by themselves Lorentz and *CPT* violating ones, appear not to cause physical SLIV effects due to strict cancellations in the physical processes involved. So, the nonlinear constraint (2) applied to the standard QED Lagrangian (7) appears in fact to be a possible gauge choice, while the *S*-matrix remains unaltered under such a gauge convention. This conclusion was first confirmed at the tree level [11] and recently extended to the one-loop approximation [13]. All the one-loop contributions to the photon–photon, photon–fermion and fermion–fermion interactions violating Lorentz invariance were shown to be exactly cancelled with each other, in the manner observed earlier for the simplest tree-order diagrams. This suggests that the vector field constraint $A^2 = n^2 M^2$, having been treated as a nonlinear gauge choice at the tree (classical) level, remains as just a gauge condition when quantum effects are taken into account as well.

To resume let us recall the steps made in the derivation above. We started with the most general Lorentz invariant Lagrangian $L(A_\mu, \psi)$, proposing only a global internal $U(1)$ symmetry for the charged matter fields involved. The requirement for the vector field equations of motion to be compatible with the true vacuum chosen by the SLIV (2) led us to the necessity for the identity (4) to be satisfied by the Lagrangian L . According to Noether’s second theorem [19], this identity implies the invariance of the Lagrangian L under the $U(1)$ charge gauge transformations of all the interacting fields. And, finally, this local symmetry allows us to completely establish the underlying theory, which appears to be standard QED (7) taken in the nonlinear gauge (2) or the nonlinear σ model-type QED in a general axial gauge—both preserving physical Lorentz invariance.

3. Non-Abelian theory

Now we extend our discussion to the non-Abelian global internal symmetry case for a general Lorentz invariant Lagrangian $\mathcal{L}(A_\mu, \psi)$ for the vector and matter fields involved. This symmetry is given by a general group G with D generators t^α ,

$$[t_\alpha, t_\beta] = i c_{\alpha\beta\gamma} t_\gamma, \quad \text{Tr}(t_\alpha t_\beta) = \delta_{\alpha\beta} \quad (\alpha, \beta, \gamma = 0, 1, \dots, D - 1),
 \tag{13}$$

where $c_{\alpha\beta\gamma}$ are the structure constants of G . The corresponding vector fields, which transform according to the adjoint representation of G , are given in the matrix form $A_\mu = A_\mu^\alpha t_\alpha$. The matter fields (fermions or scalars) are, for definiteness, taken in the fundamental representation column ψ^σ ($\sigma = 0, 1, \dots, d-1$) of G . Let us again, as in the above Abelian case, subject the vector field multiplet $A_\mu^\alpha(x)$ to a SLIV constraint of the form

$$\text{Tr}(A_\mu A^\mu) = n^2 M^2, \quad n^2 \equiv n_\mu^\alpha n^{\mu,\alpha} = \pm 1, \quad (14)$$

that presumably chooses the true vacuum in a theory. Here, as usual, we sum over repeated indices. This covariant constraint is not only the simplest one, but the only possible SLIV condition which could be written for the vector field multiplet A_μ^α and not be superfluously restricted (see discussion below).

Although we only propose the $SO(1, 3) \times G$ invariance of the Lagrangian $\mathcal{L}(A_\mu, \psi)$, the chosen SLIV constraint (14) in fact possesses a much higher accidental symmetry $SO(D, 3D)$ determined by the dimensionality D of the G adjoint representation to which the vector fields A_μ^α belong.⁷ This symmetry is indeed spontaneously broken at a scale M ,

$$\langle A_\mu^\alpha(x) \rangle = n_\mu^\alpha M \quad (15)$$

with the vacuum direction given now by the ‘unit’ rectangular matrix n_μ^α describing simultaneously both of the generalized SLIV cases, time-like ($SO(D, 3D) \rightarrow SO(D-1, 3D)$) or space-like ($SO(D, 3D) \rightarrow SO(D, 3D-1)$), respectively, depending on the sign of $n^2 \equiv n_\mu^\alpha n^{\mu,\alpha} = \pm 1$. This matrix has in fact only one non-zero element for both cases, subject to the appropriate $SO(D, 3D)$ rotation. They are, specifically, n_0^0 or n_3^0 provided that the vacuum expectation value (15) is developed along the $\alpha = 0$ direction in the internal space and along the $\mu = 0$ or $\mu = 3$ direction respectively in the ordinary four-dimensional one. As we shall soon see, in response to each of these two breakings, side by side with one true vector Goldstone boson corresponding to the spontaneous violation of the actual $SO(1, 3) \otimes G$ symmetry of the Lagrangian \mathcal{L} , $D-1$ vector pseudo-Goldstone bosons (PGB) related to a breaking of the accidental $SO(D, 3D)$ symmetry of the constraint (14) per se are also produced.⁸ Remarkably, in contrast to the familiar scalar PGB case [15], the vector PGBs remain strictly massless being protected by the simultaneously generated non-Abelian gauge invariance. Together with the above true vector Goldstone boson, they just complete the whole gauge field multiplet of the internal symmetry group G .

Let us now turn to the possible supplementary conditions which can be imposed on the vector fields in a general Lagrangian $\mathcal{L}(A_\mu, \psi)$, in order to finally establish its form. While generally D supplementary conditions may be imposed on the vector field multiplet A_μ^α , one of them in the case considered is in fact the SLIV constraint (14). One might think that the other conditions

⁷ Actually, in the same way as in the Abelian case (see footnote 1), such a SLIV constraint (14) might be related to the minimisation of some $SO(D, 3D)$ invariant vector field potential $\mathcal{U}(A_\mu) = -(m_A^2/2) \text{Tr}(A_\mu A^\mu) + (\lambda_A/4) [\text{Tr}(A_\mu A^\mu)]^2$ followed by taking the limit $m_A^2 \rightarrow \infty$, $\lambda_A \rightarrow \infty$ (while keeping the ratio m_A^2/λ_A finite). Notably, the inclusion into this potential of another possible, while less symmetrical, four-linear self-interaction term of the type $(\lambda'_A/4) \text{Tr}(A_\mu A^\mu A_\nu A^\nu)$ would lead, as one can easily confirm, to an unacceptably large number ($4D$) of vector field constraints at the potential minimum.

⁸ Note that in total there appear $4D-1$ pseudo-Goldstone modes, complying with the number of broken generators of $SO(D, 3D)$, both for time-like and space-like SLIV. From these $4D-1$ pseudo-Goldstone modes, $3D$ modes correspond to the D three component vector states as will be shown below, while the remaining $D-1$ modes are scalar states which will be excluded from the theory. In fact $D-r$ actual scalar Goldstone bosons (where r is the rank of the group G), arising from the spontaneous violation of G , are contained among these excluded scalar states.

would appear by taking 4-divergences of the equations of motion

$$\frac{\partial \mathcal{L}}{\partial A_\mu^\alpha} - \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu^\alpha)} = 0, \tag{16}$$

which are determined by a variation of the Lagrangian \mathcal{L} . The point is, however, that due to the G symmetry this operation would lead, on equal terms, to D independent conditions thus giving in total, together with the basic SLIV constraint (14), $D + 1$ constraints for the vector field multiplet A_μ^α which is inadmissible. Therefore, as in the above Abelian case, the 4-divergences of the Euler equations (16) should not produce supplementary conditions at all once the SLIV occurs. This means again that such 4-divergences should be arranged to vanish (though still keeping the global G symmetry) either identically or as a result of the equations of motion for vector and matter fields (fermion fields for definiteness) thus implying that, in the absence of these equations, there must hold a general identity of the type

$$\begin{aligned} & \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial A_\mu^\alpha} - \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu^\alpha)} \right) \\ & \equiv \left(\frac{\partial \mathcal{L}}{\partial A_\mu^\beta} - \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu^\beta)} \right) C_{\alpha\beta\gamma} A_\mu^\gamma + \left(\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu \psi)} \right) (iT_\alpha) \psi \\ & \quad + \bar{\psi} (-iT_\alpha) \left(\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu \bar{\psi})} \right). \end{aligned} \tag{17}$$

The coefficients $C_{\alpha\beta\gamma}$ and T_α of the Eulerians on the right-hand side of the identity (17) can readily be identified with the structure constants $c_{\alpha\beta\gamma}$ and generators t_α (13) of the group G . This follows because the right-hand side of the identity (17) must transform in the same way as the left-hand side, which transforms as the adjoint representation of G . Note that these coefficients consist of dimensionless constants corresponding to the starting ‘minimal’ Lagrangian $\mathcal{L}(A_\mu, \psi)$ which is taken, for simplicity, with renormalisable coupling constants. According to Noether’s second theorem [19], the identity (17) again means the invariance of \mathcal{L} under the vector and fermion field local transformations having the infinitesimal form

$$\delta A_\mu^\alpha = \partial_\mu \omega^\alpha + C_{\alpha\beta\gamma} \omega^\beta A_\mu^\gamma, \quad \delta \psi = iT_\alpha \omega^\alpha \psi \tag{18}$$

where $\omega^\alpha(x)$ are arbitrary functions only being restricted, again as in the above Abelian case, by the requirement to conform with the corresponding nonlinear constraint (14).

Note that the existence of the starting global G symmetry in the theory is important for our consideration, since without such a symmetry the basic identity (17) would be written with arbitrary coefficients $C_{\alpha\beta\gamma}$ and T_α . Then this basic identity may be required for only some particular vector field $A_\mu^{\alpha_0}$ rather than for the entire set A_μ^α . This would eventually lead to the previous pure Abelian theory case just for this $A_\mu^{\alpha_0}$ component leaving aside all the other ones. Just the existence of the starting global symmetry G ensures a non-Abelian group-theoretical solution for the local transformations (18) in the theory.

So, we have shown that in the non-Abelian internal symmetry case, as well as in the Abelian case, the imposition of the SLIV constraint (14) converts the starting global symmetry G into the local one G_{loc} . Otherwise, the theory would superfluously restrict the number of degrees of freedom for the vector field multiplet A_μ^α , which would certainly not be allowed. This SLIV induced local non-Abelian symmetry (18) now completely determines the Lagrangian \mathcal{L} , following the standard procedure (see, for example, [20]). For a minimal theory with renormalisable coupling

constants, this corresponds in fact to a conventional Yang–Mills type Lagrangian

$$\mathcal{L}(A_\mu, \psi) = -\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \bar{\psi}(i\gamma\partial - m)\psi + g\bar{\psi}A_\mu\gamma^\mu\psi \quad (19)$$

(where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and g stands for the universal coupling constant in the theory) with the SLIV constraint (14) imposed. These constrained gauge fields A_μ^α contain, as we directly confirm below, one true Goldstone and $D - 1$ pseudo-Goldstone vector bosons, corresponding to the spontaneous violation of the accidental $SO(D, 3D)$ symmetry of the constraint (14).

Actually, as in the above Abelian case, after the explicit use of the corresponding SLIV constraint (14), which is so far the only supplementary condition for the vector field multiplet A_μ^α , one can identify the pure Goldstone field modes a_μ^α as follows:

$$A_\mu^\alpha = a_\mu^\alpha + \frac{n_\mu^\alpha}{n^2}(n \cdot A), \quad n \cdot a \equiv n_\mu^\alpha a^{\mu,\alpha} = 0. \quad (20)$$

At the same time an effective ‘‘Higgs’’ mode (i.e., the A_μ^α component in the vacuum direction n_μ^α) is given by the product $n \cdot A \equiv n_\mu^\alpha A^{\mu,\alpha}$ determined by the SLIV constraint

$$n \cdot A = [M^2 - n^2 a^2]^{\frac{1}{2}} = M - \frac{n^2 a^2}{2M} + O(1/M^2) \quad (21)$$

where $a^2 = a_\nu^\alpha a^{\nu,\alpha}$. As earlier in the Abelian case, we take the positive sign for the square root and expand it in powers of a^2/M^2 . Note that, apart from the pure vector fields, the general Goldstonic modes a_μ^α contain $D - 1$ scalar fields, $a_0^{\alpha'}$ or $a_3^{\alpha'}$ ($\alpha' = 1, \dots, D - 1$), for the time-like ($n_\mu^\alpha = n_0^\alpha g_{\mu 0} \delta^{\alpha 0}$) or space-like ($n_\mu^\alpha = n_3^\alpha g_{\mu 3} \delta^{\alpha 0}$) SLIV, respectively. They can be eliminated from the theory if one imposes appropriate supplementary conditions on the a_μ^α fields which are still free of constraints. Using their overall orthogonality (20) to the physical vacuum direction n_μ^α , one can formulate these supplementary conditions in terms of a general axial gauge for the entire a_μ^α multiplet

$$n \cdot a^\alpha \equiv n_\mu a^{\mu,\alpha} = 0, \quad \alpha = 0, 1, \dots, D - 1. \quad (22)$$

Here n_μ is the unit Lorentz vector, analogous to that introduced in the Abelian case, which is now oriented in Minkowskian space–time so as to be parallel to the vacuum matrix⁹ n_μ^α . As a result, apart from the ‘‘Higgs’’ mode excluded earlier by the above orthogonality condition (20), all the other scalar fields are also eliminated, and only the pure vector fields, a_i^α ($i = 1, 2, 3$) or $a_{\mu'}^\alpha$ ($\mu' = 0, 1, 2$) for time-like or space-like SLIV, respectively, are left in the theory. Clearly, the components $a_i^{\alpha=0}$ and $a_{\mu'}^{\alpha=0}$ correspond to the Goldstone boson, for each type of SLIV, respectively, while all the others (for $\alpha = 1, \dots, D - 1$) are vector PGBs.

We now show that these Goldstonic vector fields, denoted generally as a_μ^α but with the supplementary conditions (22) understood, appear truly massless in the SLIV inspired gauge invariant Lagrangian \mathcal{L} (19) subject to the SLIV constraint (14). Actually, substituting the parameterization (20) with the SLIV constraint (21) into the Lagrangian (19), one is led to a highly nonlinear Yang–Mills theory in terms of the pure Goldstonic modes a_μ^α . However, as in the above Abelian

⁹ For such a choice the simple identity $n_\mu^\alpha \equiv \frac{n \cdot n^\alpha}{n^2} n_\mu$ holds, showing that the rectangular vacuum matrix n_μ^α has the factorized ‘‘two-vector’’ form.

case, one should first use the local invariance of the Lagrangian \mathcal{L} to gauge away the apparently large Lorentz violating terms, which appear in the theory in the form of fermion and vector field bilinears. As one can readily see, they stem from the expansion (21) when it is applied to the couplings $g\bar{\psi}A_\mu\gamma^\mu\psi$ and $-\frac{1}{4}g^2\text{Tr}([A_\mu, A_\nu]^2)$ respectively in the Lagrangian (19). Analogously to the Abelian case, we make the appropriate redefinitions of the fermion (ψ) and vector ($\mathbf{a}_\mu \equiv \mathbf{a}_\mu^\alpha t^\alpha$) field multiplets:

$$\psi \rightarrow U(\omega)\psi, \quad \mathbf{a}_\mu \rightarrow U(\omega)\mathbf{a}_\mu U(\omega)^\dagger, \quad U(\omega) = e^{igM(x \cdot \mathbf{n}^\alpha/n^2)t^\alpha}. \quad (23)$$

Since the phase of the transformation matrix $U(\omega)$ is linear in the space–time coordinate, the following equalities are evidently satisfied:

$$\partial_\mu U(\omega) = igM\mathbf{n}_\mu U(\omega) = igMU(\omega)\mathbf{n}_\mu, \quad \mathbf{n}_\mu \equiv \mathbf{n}_\mu^\alpha t^\alpha. \quad (24)$$

One can readily confirm that the above-mentioned Lorentz violating terms are thereby cancelled with the analogous bilinears stemming from their kinetic terms. So, the final Lagrangian for the Goldstonic Yang–Mills theory takes the form (to first order in (\mathbf{a}^2/M^2) ,

$$\begin{aligned} \mathcal{L}(\mathbf{a}_\mu^\alpha, \psi) = & -\frac{1}{4}\text{Tr}(\mathbf{f}_{\mu\nu}\mathbf{f}^{\mu\nu}) - \frac{1}{2}\delta(n \cdot \mathbf{a}^\alpha)^2 + \frac{1}{4}\text{Tr}(\mathbf{f}_{\mu\nu}\mathbf{h}^{\mu\nu})\frac{\mathbf{n}^2\mathbf{a}^2}{M} \\ & + \bar{\psi}(i\gamma\partial - m)\psi + g\bar{\psi}\mathbf{a}_\mu\gamma^\mu\psi - \frac{g\mathbf{n}^2\mathbf{a}^2}{2M}\bar{\psi}(\gamma \cdot \mathbf{n})\psi. \end{aligned} \quad (25)$$

Here the tensor $\mathbf{f}_{\mu\nu}$ is, as usual, $\mathbf{f}_{\mu\nu} = \partial_\mu\mathbf{a}_\nu - \partial_\nu\mathbf{a}_\mu - ig[\mathbf{a}_\mu, \mathbf{a}_\nu]$, while $\mathbf{h}_{\mu\nu}$ is a new SLIV oriented tensor of the type

$$\mathbf{h}_{\mu\nu} = \mathbf{n}_\mu\partial_\nu - \mathbf{n}_\nu\partial_\mu + ig([\mathbf{n}_\mu, \mathbf{a}_\nu] - [\mathbf{n}_\nu, \mathbf{a}_\mu])$$

acting on the infinite series in \mathbf{a}^2 coming from the expansion of the effective ‘‘Higgs’’ mode (21), from which we have only included the first-order term $-\mathbf{n}^2\mathbf{a}^2/2M$ throughout the Lagrangian $\mathcal{L}(\mathbf{a}_\mu^\alpha, \psi)$. We have explicitly introduced the (axial) gauge fixing term into the Lagrangian, corresponding to the supplementary conditions (22) imposed. We have also retained the original notations for the fermion and vector fields after the transformations (23).

The theory we here derived is in essence a generalization of the nonlinear QED model [11] for the non-Abelian case. As one can see, this theory contains the massless vector Goldstone and pseudo-Goldstone boson multiplet \mathbf{a}_μ^α gauging the starting global symmetry G and, in the limit $M \rightarrow \infty$, is indistinguishable from conventional Yang–Mills theory taken in a general axial gauge. So, for this part of the Lagrangian $\mathcal{L}(\mathbf{a}_\mu^\alpha, \psi)$ given by the zero-order terms in $1/M$, the spontaneous Lorentz violation again simply corresponds to a non-covariant gauge choice in an otherwise gauge invariant (and Lorentz invariant) theory. Furthermore one may expect that, as in the nonlinear QED model [11], all the first- and higher-order terms in $1/M$ in \mathcal{L} (25), though being by themselves Lorentz and CPT violating ones, do not cause physical SLIV effects due to the mutual cancellation of their contributions to the physical processes involved. Recent tree level calculations [14] related to the Lagrangian $\mathcal{L}(\mathbf{a}_\mu^\alpha, \psi)$ seem to confirm this proposition. Therefore, the SLIV constraint (14) applied to a starting general Lagrangian $\mathcal{L}(\mathbf{A}_\mu^\alpha, \psi)$, while generating the true Goldstonic vector field theory for the non-Abelian charge-carrying matter, is not likely to manifest itself in a physical Lorentz invariance violating way.

4. Conclusion

The spontaneous Lorentz violation realized through a nonlinear vector field constraint of the type $A^2 = M^2$ (M is the proposed scale for Lorentz violation) is shown to generate massless vector Goldstone bosons gauging the starting global internal symmetries involved, both in the Abelian and the non-Abelian symmetry case. The gauge invariance, as we have seen, directly follows from a general variation principle and Noether's second theorem [19], as a necessary condition for these bosons not to be superfluously restricted in degrees of freedom once the true vacuum in a theory is chosen by the SLIV constraint. It should be stressed that we can of course only achieve this derivation of gauge invariance by allowing all the coupling constants in the Lagrangian density to be determined from the requirement of avoiding any extra restriction imposed on the vector field(s) in addition to the SLIV constraint. Actually, this derivation excludes "wrong" couplings in the vector field Lagrangian, which would otherwise distort the final Lorentz symmetry broken phase with unphysical extra states including ghost-like ones. Note that this procedure might, in some sense, be inspired by string theory where the coupling constants are just vacuum expectation values of the dilaton and moduli fields [21]. So, the adjustment of coupling constants in the Lagrangian would mean, in essence, a certain choice for the vacuum configurations of these fields, which are thus correlated with the SLIV. Another important point for this gauge symmetry derivation is that we followed our philosophy of imposing the SLIV constraints, (2) and (14), respectively, without adding a Lagrange multiplier term, as one might have imagined should come with these constraints. Had we done so the equations of motion would have changed and the Lagrange multiplier might have picked up the inconsistency, which we required to be solved in the Abelian case by Eq. (4) and in the non-Abelian case by Eq. (17).

In the Abelian case a massless vector Goldstone boson appears, which is naturally associated with the photon. In the non-Abelian case it was shown that the pure Lorentz violation still generates just one genuine Goldstone vector boson. However the SLIV constraint (14) manifests a larger accidental $SO(D, 3D)$ symmetry, which is not shared by the Lagrangian \mathcal{L} . The spontaneous violation of this $SO(D, 3D)$ symmetry generates $D - 1$ pseudo-Goldstone vector bosons which, together with the genuine Goldstone vector boson, complete the whole gauge field multiplet of the internal symmetry group G . Remarkably, these vector bosons all appear to be strictly massless, as they are protected by the simultaneously generated non-Abelian gauge invariance. These theories, both Abelian and non-Abelian, though being essentially nonlinear, appear to be physically indistinguishable from the conventional QED and Yang–Mills theories due to their generic, SLIV enforced, gauge invariance. One could actually see that just this gauge invariance ensures that our theories do not have unreasonably large (proportional to the SLIV scale M) Lorentz violation in the fermion and vector field interaction terms. It appears also to ensure that all the physical Lorentz violating effects, even those suppressed by this SLIV scale, are non-observable.

In this connection, the only way for physical Lorentz violation then to appear would be if the above gauge invariance is somehow broken at very small distances. One could imagine how such a breaking might occur. Only gauge invariant theories provide, as we have learned, the needed number of degrees of freedom for the interacting vector fields once the SLIV occurs. Note that a superfluous restriction on a vector (or any other) field would make it impossible to set the required initial conditions in the appropriate Cauchy problem and, in quantum theory, to choose self-consistent equal-time commutation relations [18]. One could expect, however, that gravity could in general hinder the setting of the required initial conditions at extra-small distances. Eventually this would manifest itself in the violation of the above gauge invariance in a theory

through some high-order operators stemming from the gravity-influenced area, which could lead to physical Lorentz violation. We may return to this interesting possibility elsewhere.

Acknowledgements

We would like to thank Rabi Mohapatra for useful discussions and comments.

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