Totally bounded topological group topologies on the integers

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Abstract

We generalize an argument of W.W. Comfort, F.J. Trigos-Arrieta and T.S. Wu [Fund. Math. 143 (1993) 119–136] showing that if there is a non-trivial sequence converging to the identity in a locally compact Abelian group $G$, then $A := \{ \lambda \in \hat{G} : \lambda(x_n) \to 1 \}$ is a locally $\mu$-null subgroup of the character group $\hat{G}$ of $G$, where $\mu$ denotes Haar measure on $\hat{G}$.

Using a result of the same authors we show the existence of families $A$ and $B$ of dense subgroups of $\mathbb{T} \cong \hat{\mathbb{Z}}$ such that:

(i) $|A| = |B| = 2^\omega$;
(ii) each $A \in A$ and each $B \in B$ is algebraically isomorphic to the free Abelian group $\bigoplus^\omega \mathbb{Z}$;
(iii) the spaces $(\mathbb{Z}, \tau_A)$ ($A \in A$) are pairwise non-homeomorphic, and the spaces $(\mathbb{Z}, \tau_B)$ ($B \in B$) are pairwise non-homeomorphic (by $\tau_X$ we denote the weakest topology making all elements of $X$ continuous);
(iv) each group $(\mathbb{Z}, \tau_A)$ ($A \in A$) has a non-trivial convergent sequence; and
(v) every convergent sequence of $(\mathbb{Z}, \tau_B)$ ($B \in B$) is trivial.

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1. Introduction and motivation

Given an Abelian group $G$ and a non-trivial sequence $u_n$ in $G$, when is it possible to construct a Hausdorff group topology on $G$ that allows this sequence to converge? This question, especially for the integers, has been studied by many authors Graev [11, §8,
Theorem 6], Nienhuys [22], Zelenyuk and Protasov [28], Clark and Cates [3] and others. It is known that the answer is positive for any sequence (in the integers or the rationals) that satisfies appropriate “sparseness” conditions (e.g., while the sequence of squares in the integers \( \mathbb{Z} \) cannot converge to 0 in any Hausdorff group topology [3], any sequence \( u_n \) in \( \mathbb{Z} \) with \( \lim |u_{n+1}/u_n| = \infty \) allows such a topology (Trigos-Arrieta [27], also see Raczkowski and Trigos-Arrieta [25]), a complete description in the general case as well as many interesting examples can be found in [28]). The purpose of this paper is to provide some insights into this question in the realm of totally bounded group topologies.

All groups that we consider here are Abelian and all topologies are Hausdorff.

We consider a locally compact Abelian group \( G \) and a countably infinite subset \( S \) with no accumulation points. We show that the subgroups of continuous characters of \( G \) that induce on \( G \) a (necessarily totally bounded) group topology in which \( S \) forms a sequence converging to the identity, must be locally \( \mu \)-null with respect to any Haar measure \( \mu \) on \( \hat{G} \) (Definition 5). Although for discrete groups this means that such (sub)groups of characters are of measure 0, we give an example of when such a subgroup is of infinite measure (any locally \( \mu \)-null subset is either of measure 0 or of infinite measure, cf. Hewitt and Ross [12, 11.33]).

An argument in Berhanu et al. [2, Corollary 5.7], shows that on any Abelian group of cardinality \( \alpha \), one can induce \( 2^{2^{\omega}} \)-many distinct totally bounded, topological group topologies. This means, for example, that on the group of integers \( \mathbb{Z} \) there are \( 2^{\varepsilon} \)-many distinct such topologies. Of these, \( \varepsilon \)-many are metric (i.e., of weight \( \omega \) cf. [2, Theorem 4.3]). Sierpiński [26] has shown that countable, dense in itself, metrizable spaces are all homeomorphic to each other. Thus all metric topologies on \( \mathbb{Z} \) are all homeomorphic to each other as spaces.

On the other hand we have, on \( \mathbb{Z} \), \( 2^\varepsilon \)-many distinct totally bounded group topologies of weight \( \varepsilon \). On any countable set one can have only \( \varepsilon \)-many distinct permutations. Since a homeomorphism is among other things a permutation, this implies that we have on \( \mathbb{Z} \), \( 2^\varepsilon \)-many pairwise non-homeomorphic, totally bounded group topologies of weight \( \varepsilon \). The goal of this paper is to study these kinds of topologies.

Notation. We will adopt the notation from Comfort [5]. By \( \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{R} \) we denote the groups of integers, rationals and reals, respectively. In each case the operation is addition and the topology is the usual, unless otherwise specified.

For an Abelian group \( G \) we denote by \( G_d \) the group \( G \) equipped with the discrete topology. By \( wt(G) \) we denote the weight of the topological group \( G \), i.e., the minimal cardinality of a base for the topology on \( G \).

The brackets \( \langle \cdot, \cdot \rangle \) will serve a double purpose, since confusion is unlikely. By \( \langle G, \tau \rangle \) we denote the group \( G \) with topology \( \tau \), while for a subset \( S \subseteq G \), \( \langle S \rangle \) denotes the group generated by \( S \). For a real number \( x \), let \( [x] \) denote the integer part of \( x \), and \( \{x\} \) the fractional part of \( x \). We identify \( \mathbb{T} \) with the multiplicative subgroup \( \{e^{2\pi ix}: 0 \leq x < 1\} \) of the complex plane. We denote by \( \mathcal{H}(G) := \text{Hom}(G, \mathbb{T}) \), the group of all homomorphisms of \( G \) into \( \mathbb{T} \).
Definition 1. A character of a topological group $G$ is a continuous homomorphism $h : G \to \mathbb{T}$.

For a topological group $G$, we denote by $\hat{G}$ the group of all characters. We equip $\hat{G}$ with the compact-open topology, i.e., the topology whose basic open neighborhoods of the identity in $\hat{G}$ are of the form:

$$U(F, \varepsilon) := \{ \lambda \in \hat{G} : |\lambda(g) - 1| < \varepsilon, \ g \in F \},$$

where $F \subseteq G$ and $\varepsilon > 0$ is arbitrary.

It is proved by Pontryagin [23] and van Kampen [17] that the dual of a locally compact Abelian group (from now on LCA) is itself LCA. Moreover $G$ is discrete if and only if $\hat{G}$ is compact. In particular, the dual group of $\mathbb{Z}$ is $\mathbb{T}$ (see [12, 23.27 b]).

Definition 2. A subgroup $A$ of $\mathcal{H}(G)$ is said to separate points of $G$ if for every $g$ in $G \setminus \{0\}$ there is $h$ in $A$ such that $h(g) \neq 1$.

It is a theorem of Comfort and Ross [6] that every totally bounded topological group topology is the weakest topology on $G$ that makes all elements of a suitably chosen point-separating (equivalently, dense in $\hat{G}_d$) subgroup $A$ of $\mathcal{H}(G)$ continuous; and conversely, every such topology is totally bounded. We denote it as $\langle G, \tau_A \rangle$. In particular we have a one-to-one correspondence between dense subgroups of $\mathbb{T}$ and totally bounded topological group topologies of $\mathbb{Z}$. Our objective is to study what properties of dense subgroups $A$ of $\mathbb{T}$ influence what topological properties of $\langle \mathbb{Z}, \tau_A \rangle$.

For example, if $G$ is a countable Abelian group, by the above-mentioned Theorem of Sierpiński [26] we know that it is sufficient for point separating subgroups $A$ and $B$ to satisfy $|A| = |B| = \aleph_0$, for the spaces $\langle G, \tau_A \rangle$ and $\langle G, \tau_B \rangle$ to be homeomorphic. By [5, Section 4.2], the corresponding conclusion can fail when, for example, the point-separating subgroups are of cardinality $\mathfrak{c}$.

Now consider the usual Lebesgue measure on $\mathbb{T}$. In this paper we are able to make a very crude distinction, namely (a) we have constructed a Lebesgue measurable dense subgroup $A \subseteq \mathbb{T}$ of cardinality $\mathfrak{c}$ for which $\langle \mathbb{Z}, \tau_A \rangle$ has a non-trivial sequence converging to 0, hence has infinite compact sets; (b) we have shown that an algebraically isomorphic copy $B$ of such subgroup $A$ can be positioned in $\mathbb{T}$ in such a way as to be no longer Lebesgue measurable, hence by [7], inducing a totally bounded topological group topology, $\tau_B$ whose convergent sequences are all trivial. Thus we have an example of non-homeomorphic spaces $\langle \mathbb{Z}, \tau_A \rangle$ and $\langle \mathbb{Z}, \tau_B \rangle$, such that $A$ and $B$ are isomorphic as groups. Since a compact countable space is second countable (see Hodel [16, 7.2]) for countable groups to have all compact sets finite is equivalent to having no non-trivial converging sequences. The above fails for groups of higher cardinality, as there are examples of groups that have compact sets of cardinality $\mathfrak{c}$, without non-trivial converging sequences (see Eda et al. [9, Theorem 3.4]).

Additionally, we show that on $\mathbb{Z}$, there are $2^\mathfrak{c}$-many pairwise non-homeomorphic totally bounded group topologies in each of which the only convergent sequences are the trivial
ones. We also show that on \( \mathbb{Z} \) there are 2\(^c\)-many pairwise non-homeomorphic totally bounded group topologies all sharing the same convergent sequence.

2. Bohr topologies and convergent sequences

Using analytic apparatus we show in this section that any subgroup \( A \) of characters of any \textbf{LCA} group \( G \) that makes some fixed, but arbitrary sequence with no accumulation points in \( G \), convergent in \((G, \tau_A)\), is locally null with respect to Haar measure on \( \hat{G} \). This generalizes in particular the second conclusion in [7, Lemma 3.10].

For \( G \in \textbf{LCA} \), denote by \( \mu \) its normalized Haar measure (see [12, §15]). Let \( L_1(G) \) denote the collection of \( \mu \)-integrable continuous complex-valued functions on \( G \). For any topological space \( X \), let \( C_0(X) \) denote the set of all complex-valued continuous functions \( f \) on \( X \) such that for every \( \varepsilon > 0 \), there exists a compact set \( F \) of \( X \), depending on \( f \) and \( \varepsilon \), such that \( |f(x)| < \varepsilon \) for all \( x \in X \setminus F \).

**Definition 3.** For \( f \in L_1(G) \) define \( \widehat{f} : \hat{G} \to \mathbb{C} \) by

\[
\widehat{f}(\lambda) := \int_{\hat{G}} f(x) \lambda(x) \, d\mu.
\]

The function \( \widehat{f} \) is called the Fourier transform of \( f \).

We denote by \( A(\hat{G}) \) the space of all Fourier transforms of functions in \( L_1(G) \). The standard topology of \( \hat{G} \) is the compact-open topology. An alternative way to describe it is as the weak topology determined by \( A(\hat{G}) \), that is the weakest topology for which all the functions in \( A(\hat{G}) \) are continuous; see Loomis [21, p. 151], or Katznelson [18, p. 190]. Such a topology is referred to as the Gel'fand topology (cf. [12, C.23]). Using the above, we are now ready to state the generalized Riemann–Lebesgue Lemma (see Folland [10, Proposition 4.13] or Hewitt and Ross [13, Remark 28.42]):

**Lemma 4.** Let \( G \in \textbf{LCA} \). Then for all \( f \in L_1(G) \), we have \( \widehat{f} \in C_0(\hat{G}) \).

Following [12, Definition 11.26], we say:

**Definition 5.** Let \( X \) be a topological space and \( m \) a measure defined on it. If \( A \subseteq X \) and \( m(A) = 0 \) we say that \( A \) is a \( m \)-null set. If \( A \cap F \) is \( m \)-null for every \( F \subseteq X \), then \( A \) is said to be locally \( m \)-null.

Any locally \( m \)-null set is measurable (see [12, Theorem 11.30]). The proof of our next Theorem generalizes the construction of Kuipers and Niederreiter [19, Theorem 7.8].

**Theorem 6.** Let \( G \in \textbf{LCA} \) and let \( \rho \) denote a Haar measure on \( \hat{G} \). Let \( \{x_n\}_{n<\infty} \) be a faithfully indexed sequence in \( G \) with no accumulation points and define

\[
A := \{ \lambda \in \hat{G} : \lambda(x_n) \to 1 \}.
\]
Then $A$ is a locally $\rho$-null subgroup of $\hat{G}$.

**Proof.** We need to show that $\rho(A \cap F) = 0$ for all $F \subseteq \hat{G}$. Notice that the elements of $G$ can be seen as functions on $\hat{G}$. If $\lambda \in A$, then

$$
\lim_{n \to \infty} x_n(\lambda) = 1
$$

and $|x_n(\lambda)| \leq 1_{\hat{F}}(\lambda)$ for all $n < \omega$ and all $\lambda \in F$. By Lebesgue’s Theorem on Dominated Convergence, we have the equality

$$
\lim_{n \to \infty} \int_{\hat{A} \cap \hat{F}} x_n(\lambda) \, d\rho = \int_{\hat{A} \cap \hat{F}} \lim_{n \to \infty} x_n(\lambda) \, d\rho = \rho(A \cap F),
$$

as it is easy to see (i.e., first conclusion of [7, Lemma 3.10]) that $A$ is measurable. Clearly then $1_{\hat{A} \cap \hat{F}} \in L^1(\hat{G})$, hence from Lemma 4 and Pontryagin Duality we have $\hat{1}_{\hat{A} \cap \hat{F}} \in C_0(G)$, and since $(x_n)_{n < \omega}$ has no accumulation points

$$
\lim_{n \to \infty} \int_{\hat{A} \cap \hat{F}} x_n(\lambda) \, d\rho = \lim_{n \to \infty} \int_{\hat{G}} 1_{\hat{A} \cap \hat{F}}(\lambda) x_n(\lambda) \, d\rho = \lim_{n \to \infty} \hat{1}_{\hat{A} \cap \hat{F}}(x_n) = 0.
$$

In other words $\rho(A \cap F) = 0$. $\blacksquare$

The above proposition applies only to LCA groups that are non-compact, since in a compact space every infinite set has at least one accumulation point. As a corollary, we get the conclusion of [7, Lemma 3.10], referring to the measure of $A$.

**Corollary 7.** Let $G$ be a discrete Abelian group and let $\rho$ denote a Haar measure on $\hat{G}$. Let \{\{x_n\}_{n < \omega}\} be a faithfully indexed sequence in $G$, and define

$$
A := \{\lambda \in \hat{G}: \lambda(x_n) \to 1\}.
$$

Then $A$ is a $\rho$-measurable subgroup of $\hat{G}$ and $\rho(A) = 0$.

Moreover, if $G$ is a LCA group, then $\hat{G}$ is $\sigma$-compact if and only if $G$ is metric (see Hewitt and Stromberg [15, Theorem 3], or [5, 3.14]). Thus we have:

**Corollary 8.** If $G$ is LCA and metric and $A$ is defined as above, then $\rho(A) = 0$.

Below we will show that there is a LCA group $G$ with dense subgroups of $\hat{G}$ of infinite measure that induce on $G$ totally bounded topological group topologies with non-trivial convergent sequences. Recall that the group of real numbers $\mathbb{R}$ is algebraically isomorphic with $\bigoplus_{n < \omega} \mathbb{Q}$. $\mathbb{R}_d$ can be realized as $\Sigma_\alpha$ where $\alpha = (2, 3, \ldots)$ (see [12, (25.4)]), thus by [12, (23.21)] we have

$$
\mathbb{R}_d \cong (\Sigma_\alpha)^\mathbb{R}.
$$

**Example 9.** Let $B \subseteq \mathbb{R} \cong \mathbb{R}_d$ be a group that induces on $\mathbb{R}$ a topology in which the sequence $(n!)_{n < \omega}$ converges to 0 (see [24, Theorem 15]). Define $G := (\Sigma_\alpha)^\mathbb{R} \times \mathbb{R}$. Then $G$
is a LCA group in which the sequence \(((0, n!))_{n<\omega}\) has no accumulation points. We have \(\hat{G} \cong \mathbb{R}_d \times \mathbb{R}\). Let \(A := \mathbb{R}_d \times B\). Then \(A\) separates the points of \(G\), since \(\mathbb{R}_d\) separates the points of \((\Sigma_a)^c\) and \(B\) separates the points of \(\mathbb{R}\). Also in \((G, \tau_A)\) the sequence \(((0, n!))_{n<\omega}\) converges to \((0, 0)\). Hence by Theorem 6, \(A\) is \(\rho\)-measurable. It is in fact a locally \(\rho\)-null subgroup of \(\hat{G}\). But the measure of \(A\) is infinite since the measure of each set of the form \(\mathbb{R}_d \times \{b\}\) is infinite in \(\hat{G}\) (see [12, 11.33]).

3. Totally bounded group topologies on \(\mathbb{Z}\) with non-trivial convergent sequences

We know that whenever \(A \subseteq T\) with \(|A| = \aleph_0\), then \((\mathbb{Z}, \tau_A)\) is a metric group. As such, it necessarily contains convergent sequences. On the other hand, in \((\mathbb{Z}, \tau_T)\), often referred to as \(\mathbb{Z}^n\), every convergent sequence is trivial (see Leptin [20]). In [7] it is shown that \(\mathbb{Z}\) may admit a totally bounded topological group topology \(\tau\), in which every convergent sequence is trivial, yet this topology is not the topology inherited from the Bohr compactification of \(\mathbb{Z}_d\). Note that the character group \(A\) of \((\mathbb{Z}, \tau)\) is (dense and) uncountable.

Thus it is natural to ask whether there exists a subgroup \(A \subseteq T\) with \(|A| > \aleph_0\) such that \((\mathbb{Z}, \tau_A)\) has non-trivial convergent sequences. The positive answer follows from:

**Theorem 10.** Let \((u_n)_{n \in \mathbb{N}}\) be any sequence in \(\mathbb{Z}\) such that for all \(n \in \mathbb{N}\)

\[
u_{n+1}/u_n \geq n + 1.
\]

Then there exists a totally bounded, topological group topology on \(\mathbb{Z}\) of weight \(c\) in which \(u_n \to 0\).

**Proof.** With no loss of generality assume that \(u_n > 0\) for all \(n \in \mathbb{N}\). We will construct a dense subgroup \(A\) of \(\mathbb{Z}\) of cardinality \(c\) that induces on \(\mathbb{Z}\) a (necessarily totally bounded) topological group topology \(\tau_A\) in which \(u_n \to 0\).

Let \(C := \{\xi \in \{0, 1\}^\mathbb{N}: \xi_1 = 0\}\). For each \(\xi \in C\) let us define by induction a sequence of intervals \((I_n(\xi))_{n \in \mathbb{N}}\) in \(\mathbb{R}\) as follows:

1. \(I_1(\xi) := [0, 2]\).
2. Suppose the interval \(I_n(\xi) := [p_n(\xi)u_n, p_n(\xi) + 2/n]\) has been defined.
3. Let \(q_n(\xi)\) be the unique integer lying in the interval

\[
[p_n(\xi)u_{n+1}/u_n, p_n(\xi)u_{n+1}/u_{n+1} + 1).
\]

Set \(J_{n+1}(\xi) := [q_n(\xi) + \xi_n, q_n(\xi) + \xi_n + 2/(n + 1)]\).

Now define the compact intervals \(J_n\) (obtained by shrinking the closures of the intervals \(I_n\)):

\[
J_n(\xi) := [p_n(\xi)/u_n, (p_n(\xi) + 2/n)/u_n] - J_{n+1}(\xi).
\]

Then we have \(J_{n+1}(\xi) \subseteq J_n(\xi)\) because:

\[
\frac{p_n(\xi)}{u_n} \leq \frac{q_n(\xi) + \xi_n}{u_n + 1} \leq \frac{p_n(\xi) + 2/(n + 1)}{u_n + 1} \leq \frac{p_n(\xi) + 2/n}{u_n}.
\]

\[\text{(***)}\]
The middle of the inequality is obvious. To see the first one, notice that by the choice of $q_n(\xi)$ we have

$$\frac{p_n(\xi)u_{n+1}}{u_n} \leq q_n(\xi).$$

Then it follows that

$$\frac{p_n(\xi)}{u_n} \leq \frac{q_n(\xi)}{u_{n+1}} + \frac{1}{u_n} \leq q_n(\xi) + \frac{1}{u_n}.$$

To see the last inequality in (**), by the choice of $q_n(\xi)$ we have

$$q_n(\xi) < \frac{p_n(\xi)u_{n+1}}{u_n} + 1,$$

thus

$$\frac{q_n(\xi)}{u_{n+1}} < \frac{p_n(\xi)}{u_n} + \frac{1}{u_{n+1}}.$$

hence

$$\frac{q_n(\xi)}{u_{n+1}} + \frac{2}{u_n} \leq \frac{p_n(\xi)}{u_n} + \frac{2 + 2/(n + 1)}{u_{n+1}}.$$

The rest follows from the choice of the sequence $u_n$, for since we have

$$\frac{u_{n+1}}{u_n} \geq n + 1 \iff \frac{1}{u_{n+1}} \leq \frac{1}{u_n(n + 1)}.$$

a quick check shows that

$$\frac{2n + 4}{(n + 1)^2} \leq \frac{2}{n}.$$

Then the rest of the inequality follows, for

$$\frac{q_n(\xi)}{u_{n+1}} + \frac{2}{u_n} \leq \frac{p_n(\xi)}{u_n} + \frac{2 + 2/(n + 1)}{u_{n+1}} \leq \frac{p_n(\xi) + 2/n}{u_n(n+1)}.$$

Thus (***) holds. The nested sequence $(J_n(\xi))_{n \in \mathbb{N}}$ of compact intervals has non-empty intersection. Let $y_\xi \in \bigcap_{n \in \mathbb{N}} J_n(\xi)$. We claim that the numbers $y_\xi (\xi \in C)$ are distinct. To see this let $\xi \neq \varepsilon$, and say that $k$ is the first integer so that $\xi_k \neq \varepsilon_k$. Then, since $J_k(\xi) \cap J_k(\varepsilon) = \emptyset$, we have $J_k(\xi) \cap J_k(\varepsilon) = \emptyset$, and hence $y_{\xi} \neq y_{\varepsilon}$.

Since for all $n \in \mathbb{N}$ we have $\gamma_{\xi}u_n \in I_n(\xi)$, and $p_n(\xi)$ is an integer, then if $n \geq N$ we have $\{\gamma_{\xi}u_n\} \in [0, 2/N]$. Define $\lambda_{\xi} : \mathbb{Z} \to \mathbb{T}$ by $\lambda_{\xi}(x) = e^{2\pi i y_{\xi} x}$. Thus for all $\xi \in C$, $\lambda_{\xi}(u_n) \to 1$. Now define

$$A := \{\lambda_{\xi} : \xi \in C\}.$$

Then $A \subseteq \mathbb{T}$ and by Corollary 7, every $\lambda \in A$ satisfies $\lambda(u_n) \to 1$. Since $|A| = c$, the topology $\tau_A$ is as required. □

By Theorem 6, the subgroup $A$ of $\mathbb{T}$ constructed above is always measurable, hence by the Steinhaus–Weil Theorem (cf. Hewitt and Stromberg [14, 10.43]) it is of measure 0.

**Corollary 11.** There exists a totally bounded, topological group topology $\tau$ on $\mathbb{Z}$ of weight $c$ in which the sequence $(n!)_{n \in \mathbb{N}}$ converges to 0.
It was brought to our attention by the referee that if \( A \) is a dense subgroup of \( \mathbb{T} = \mathbb{Z} \) in which \( n! \to 0 \) in \( (\mathbb{Z}, \tau_A) \), then \( a^n! \to 1 \) for every \( a \in A \). The subgroup of all such elements \( a \in \mathbb{T} \) is denoted by \( \mathbb{T}! \) in Armacost [1, Definition 3.1]. This yields another proof of the corollary by considering \( \tau = \tau_\mathbb{T}! \) and noticing that \( |\mathbb{T}!| = \epsilon \) (see [8, Theorem 4.4.2, p. 150], where \( \mathbb{T}! \) is proved to consist precisely of those \( \sum_{i=2}^{\infty} x_i/i! \), in which \( \lim_{i \to \infty} x_i/i = 0 \); clearly this is satisfied for \( x_i = 0 \) or 1).

**Corollary 12.** Given any faithfully indexed sequence \( (u_n)_{n \in \mathbb{N}} \) in \( \mathbb{Z} \) there is a subsequence \( (u_{n_k})_{k \in \mathbb{N}} \) and a totally bounded topological group topology \( \tau \) on \( \mathbb{Z} \) such that:

(a) \( \text{wt}(\langle \mathbb{Z}, \tau \rangle) = \epsilon \), and

(b) \( (u_{n_k})_{k \in \mathbb{N}} \) converges to 0 in \( \langle \mathbb{Z}, \tau \rangle \).

**Proof.** Take any infinite sequence in \( \mathbb{Z} \). With no loss of generality assume it has infinitely many positive terms, and extract an infinite subsequence \( (u_n)_{n \in \mathbb{N}} \) so that for all \( n \in \mathbb{N} \)

\[
\frac{u_{n+1}}{u_n} \geq n + 1.
\]

Now apply Theorem 10. \( \square \)

It is interesting to understand whether the condition \( u_{n+1}/u_n \geq n + 1 \) in Theorem 10 is really necessary. Let us see that one cannot prove the theorem for \( u_n = p^n \), where \( p \) is a prime. The following is due to the referee:

**Theorem 13.** Every totally bounded, topological group topology on \( \mathbb{Z} \) in which the sequence \( p^n \to 0 \) is metrizable.

**Proof.** Suppose that \( \tau = \tau_A \) for some subgroup \( A \) of \( \mathbb{T} = \mathbb{Z} \) and \( p^n \to 0 \) in \( (\mathbb{Z}, \tau) \). Clearly, \( \chi(p^n) \to 1 \) in \( \mathbb{T} \) for every \( \chi \in A \), i.e., identifying \( A \) with a subgroup of \( \mathbb{T} \), for every \( a \in A \) one must have \( a^{p^n} \to 1 \) in \( \mathbb{T} \). But then \( a \) is \( p \)-torsion [1, Lemma 2.6], so that \( A \leq \mathbb{Z}(p^\infty) \). Since \( \mathbb{Z}(p^\infty) \) has no proper infinite subgroups, we get \( A = \mathbb{Z}(p^\infty) \). Hence \( \tau \) is the \( p \)-adic topology. In particular, \( \tau \) is metrizable. \( \square \)

**Question 1.** Suppose \( A \subset \mathbb{T} \) with \( A \) measurable, i.e., of measure 0. Must it follow that \( \langle \mathbb{Z}, \tau_A \rangle \) has non-trivial convergent sequences?

By Corollary 7, in case \( A = \mathbb{T} \) or when \( A \) is non-measurable we know that in \( \langle \mathbb{Z}, \tau_A \rangle \) every convergent sequence is trivial.

**Lemma 14.** If \( A \) is a subgroup of \( \mathbb{T} \) with \( |A| = \epsilon \), then \( A \) contains an isomorphic copy of \( \bigoplus_{a < \epsilon} \mathbb{Z}_a \).

**Proof.** Because \( |A| = \epsilon \) and \( \mathbb{T} \) contains only \( \aleph_0 \)-many torsion elements, \( A \) contains \( \epsilon \)-many independent non-torsion elements (see [12, A13]). The group generated by these elements is as required. \( \square \)
Corollary 15. There are $2^c$-many distinct totally bounded group topologies on $\mathbb{Z}$ in each of which there is a non-trivial sequence converging to 0.

Proof. Consider the group $A$ that generates the topology given in Corollary 11. It has cardinality $c$. By above lemmas, $A$ contains a copy of the free Abelian group $\bigoplus_{\alpha < c} \mathbb{Z}_\alpha$, which has $2^c$-many distinct subgroups. Since the topology induced by each of these subgroups is weaker than the one induced by $A$, as described in Corollary 11, it must have a convergent sequence. 

From the above corollaries follows in fact that there are $2^c$-many pairwise non-homeomorphic totally bounded group topologies on $\mathbb{Z}$ in each of which some fixed sequence is convergent (witness the sequence $(n!)_{n \in \mathbb{N}}$).

4. Totally bounded group topologies on $\mathbb{Z}$ in which every convergent sequence is trivial

In this section we will show that there exist $2^c$-many distinct totally bounded group topologies on $\mathbb{Z}$ in each of which every convergent sequence is trivial. A slight variation of the construction of [7, Theorem 3.12], yields the following:

Lemma 16. There exists a non-measurable subgroup $M$ of $\mathbb{T}$ such that algebraically

$$M = \bigoplus_{\alpha < c} \mathbb{Z}_\alpha.$$

Proof. Let $\{F_\xi : \xi < c\}$ be an enumeration of all uncountable, closed subsets of $\mathbb{T}$. There are only $c$-many such sets since the topology of $\mathbb{T}$ can be generated by a countable collection of intervals (those with rational end-points). Since every $F_\xi$ satisfies $|F_\xi| = c$ (see Cantor [4]) and $\mathbb{T}$ contains only $\aleph_0$-many torsion elements, one can choose non-torsion elements $p_0, q_0 \in F_0$ with $\langle\{q_0\}\rangle \cap \langle\{p_0\}\rangle = \{1\}$. Recursively, if $\xi < c$ and $p_\eta, q_\eta$ have been chosen for all $\eta < \xi$, choose non-torsion elements $p_\xi, q_\xi \in F_\xi$ with

$$\langle\{q_\xi\}\rangle \cap \langle\{p_\xi, p_\eta : \eta < \xi\}\cup\{q_\eta : \eta < \xi\}\rangle = \{1\},$$

$$\langle\{q_\xi\}\rangle \cap \langle\{p_\eta : \eta < \xi\}\cup\{q_\eta : \eta < \xi\}\rangle = \{1\}.$$

The group $M := \langle\{p_\xi : \xi < c\}\rangle$ is isomorphic with $\bigoplus_{\alpha < c} \mathbb{Z}_\alpha$ (see [12, A.18]), thus $(M_d)^\sim \simeq \mathbb{T}^c$ and $M$ is not measurable because:

(a) $\lambda(M) > 0$ is impossible, since the inequality implies that $M$ is open in $\mathbb{T}$ (by the Steinhaus–Weil Theorem (cf. [14, 10.43])) and then $M = \mathbb{T}$;

(b) $\lambda(M) = 0$ is impossible, because then $\mathbb{T} \setminus M$ is measurable and $\lambda(\mathbb{T} \setminus M) > 0$ thus implying, by regularity of $\lambda$, that there is an uncountable compact subset $F = F_\xi$ of $\mathbb{T} \setminus M$, but then $p_\xi \in F \subseteq (\mathbb{T} \setminus M) \cap M = \emptyset$. 

\[\square\]
By Corollary 7, the topology induced by a non-measurable subgroup of \( T \) produces only trivial convergent sequences.

**Proposition 17.** There are \( 2^c \)-many pairwise non-homeomorphic totally bounded group topologies on \( \mathbb{Z} \) in each of which every convergent sequence is trivial.

**Proof.** Consider the set
\[
\{ p_\xi, q_\xi : \xi < c \}
\]
constructed in the proof of Lemma 16. Notice that by choice of the elements this set is independent. Define \( A := \{ q_\xi : \xi < c \} \). Let \( \{ A_\xi \}_{\xi < 2^c} \) be an enumeration of all proper subsets of \( A \), and consider
\[
M_\xi := \{ p_\xi \}_{\xi < c} \cup A_\xi \}
\]
Then, by the Steinhaus–Weil Theorem (cf. [14, 10.43])
\[
\{ M_\xi \}_{\xi < 2^c}
\]
is a family of \( 2^c \)-many distinct non-measurable subgroups of \( T \), each of the form \( \bigoplus_{\alpha < c} \mathbb{Z}_\alpha \). Since \( M \subseteq M_\xi \), each \( M_\xi \) induces on \( \mathbb{Z} \) a required topology. This completes our proof. \( \blacksquare \)

The referee has contributed the result below which strengthens a fact from a previous version of this paper, the proof simplifies the old one.

**Proposition 18.** Every subgroup \( A \) of \( T \) with \( |A| = c \) is isomorphic to a non-measurable subgroup \( B \) of \( T \).

**Proof.** Let \( A_1 = \langle S \rangle + t(A) \), where \( t(A) \) is the torsion part of \( A \) and \( S \) is a maximal linearly independent subset of \( A \). Let \( M \) be as defined in Lemma 16. There exists an isomorphism \( i : \langle S \rangle \to M \) that can be extended to a monomorphism \( i_1 : A_1 \to M + t(T) \), since the sum \( \langle S \rangle + t(A) \) defining \( A_1 \) is direct and the same holds true for \( M + t(T) \). Since \( T \) is divisible \( i_1 \) can be extended to a homomorphism \( i_2 : A \to T \). By the choice of \( S \) every non-trivial subgroup of \( A \) non-trivially meets \( A_1 \). In particular, \( \ker i_2 = \{0\} \). Thus \( B = i_2(A) \cong A \) and obviously contains \( M \). To see that \( B \) is a proper subgroup of \( T \), we recall a property of the subgroup \( M \) mentioned implicitly in the proof of Proposition 17. Namely, the quotient \( T/M \) is non-torsion (as it contains an isomorphic copy of the free group \( \langle \{ q_\xi : \xi < c \} \rangle \)). On the other hand, the quotient \( A/\langle S \rangle \) is torsion by the choice of \( S \). Therefore, \( i_2(\langle S \rangle) = M \) implies that also \( B/M \) is torsion. This proves that \( B \neq T \). \( \blacksquare \)

**Remark.** Isomorphic subgroups of \( T \) can induce non-homeomorphic group topologies on \( \mathbb{Z} \). For example, let \( A_0 \) be a subgroup of the group \( A \), as constructed in the proof of Corollary 11, of the form \( \bigoplus_{\alpha < c} \mathbb{Z}_\alpha \) (Lemma 14). By Theorem 6, \( A_0 \) is measurable, as \( n! \to 0 \) in \( \tau_{A_0} \). On the other hand if \( B \) is a non-measurable isomorphic copy of \( A_0 \), as in Proposition 18, in \( \tau_B \) there are no non-trivial convergent sequences.
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References