PATHS IN r-PARTITE SELF-COMPLEMENTARY GRAPHS

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This paper aims at finding best possible paths in r-partite self-complementary (r-p.s.c.) graphs $G(r)$. It is shown that, every connected bi-p.s.c. graphs $G(2)$ of order $p$, with a bi-partite complementing permutation (bi-p.c.p) $\sigma$ having mixed cycles, has a (p-3)-path and this result is best possible. Further, if the graph induced on each cycle of bi-p.c.p. of $G(2)$ is connected then $G(2)$ has a hamiltonian path. Lastly the fact that every r-p.s.c. graph with an r-partite complementing permutation (r-p.c.p.) $\sigma$ which permutes the partitions and for which each cycle of $\sigma$ has non-empty intersection with at least four partitions of $G(r)$ has a hamiltonian path, is established. The graph obtained from $G(r)$ by adding a vertex $u$ constituting $(r+1)$-st partition of $G(r)$, which is the fixed point of $\sigma^u = (u)\sigma$ also has a hamiltonian path. The last two results generalize the result that every self-complementary graph has a hamiltonian path.

Introduction

The word "graph" will mean a finite, undirected graph without loops and multiple edges. For the notation and terminology not defined here we refer to Harary [4].

An "r-partite graph" $G(r)$ is a graph whose vertex set $V = V(G(r))$ can be partitioned into $r \geq 1$ non-empty subsets, also called partitions, so that no edge has both ends in any one subset. Let $A_1, \ldots, A_r$ constitute an $r$-partition of $V$ with $|A_i| = n_i$, $n_i \geq 1$ ($i = 1, \ldots, r$).

An $r$-partite graph $G(r)$ is said to be "complete r-partite" if each vertex is joined to every other vertex that is not in the same subset. Such a graph is denoted by $K_{n_1, \ldots, n_r}$. Clearly, $K_{n_1, \ldots, n_r}$ has $\sum_{i=1}^r n_i$ vertices and $\sum_{i=1}^r n_i(n_i - 1)$ edges.

Bipartition of a connected graph, if exists, is unique. But, in general, r-partition of a graph need not be unique. Henceforth, if $G(r)$ is given to be an $r$-partite graph, we assume that an $r$-partition of $G(r)$ is prescribed.

The "r-partite complement" $\bar{G}(r)$ of an $r$-partite graph $G(r)$ is again an $r$-partite graph with vertex set $V(G(r))$, satisfying the following conditions:

(i) for $u, v \in A_i$, $1 \leq i \leq r$, $u \neq v \notin E(\bar{G}(r))$,
(ii) for $u \in A_i$, $v \in A_j$, $1 \leq i \neq j \leq r$, $(u, v) \in E(\bar{G}(r))$ iff $(u, v) \notin E(G(r))$.

An r-partite graph $G(r)$ is said to be "r-partite self-complementary" (r-p.s.c.) if there exists an $r$-partition of $V(G(r))$ with respect to which $G(r)$ and $\bar{G}(r)$ are isomorphic.

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The concepts \( r \)-partite complement and \( r \)-p.s.c. graph are first defined and studied in Hebbare [5].

Remark. The class of classical self-complementary (s.c.) graphs, first studied by Ringel [7], and Sachs [8], is included in the class of \( r \)-p.s.c. graphs, with \( r \geq 1 \) and \( n_1 = \cdots = n_r = 1 \). We refer to a survey article by Bhaskara Rao [1] and the references given in there, for most of the existing literature on s.c. graphs.

Let \( G(r) \) be an \( r \)-p.s.c. graph with the vertex set \( V(G(r)) = \{1, 2, \ldots, p\} \). Then the isomorphism between \( G(r) \) and \( \bar{G}(r) \) can be represented as a permutation \( \sigma \) in the set \( V(G(r)) \). We then write, \( \sigma(G(r)) = \bar{G}(r) \), and call \( \sigma \) an "\( r \)-partite complementing permutation" (r-p.c.p) for \( G(r) \). We assume that all permutations are expressed as the product of disjoint cycles. Further, we do not distinguish the symbol of the permutation and vertices of the graph. Now, let \( \sigma = \sigma_1 \cdots \sigma_\lambda \) be the disjoint cycle representation of \( \sigma \). A cycle, \( \sigma_i \) \((i = 1, \ldots, \lambda)\) of \( \sigma \) is said to be "pure" if \( \sigma_i \subseteq A_j \) for some \( j \in \{1, 2, \ldots, r\} \) and "mixed" otherwise. In other words, a mixed cycle of \( \sigma \) contains vertices from at least two partitions of \( G(r) \).

Let \( \mathcal{C}(G(r)), \mathcal{C}_p(G(r)) \), and \( \mathcal{C}_m(G(r)) \) denote the set of all \( r \)-p.c.p., \( r \)-p.c.p. each of whose cycles is pure, and \( r \)-p.c.p. each of whose cycles is mixed, of \( G(r) \). We simply write \( \mathcal{C}, \mathcal{C}_p \) and \( \mathcal{C}_m \) for the above sets when there is no confusion. We list here some observations and theorems from Gangopadhyay and Hebbare [3] which will be useful in what follows.

Observation 1. Let \( G(r) \) be an \( r \)-p.s.c. graph and \( \sigma \in \mathcal{C} \). Then for any two vertices \( u \) and \( v \) belonging to different partitions of \( G(r) \), \((u, v) \in E \) if and only if \((\sigma(u), \sigma(v)) \notin E \) where \((u, v) \) denotes an edge of \( G(r) \).

Observation 2. For all \( r \)-p.s.c. graph \( G(r) \), \( \sum_{i=1}^{r} n_i \) must be even. In particular, when \( r = 2 \), \( n_1 \) or \( n_2 \) must be even and when \( r = 3 \), at least two of \( n_1, n_2 \) and \( n_3 \) must be even.

Observation 3. Let \( \{\sigma_{i_1}, \ldots, \sigma_{i_{\mu}}\} \) be a subset of the set of cycles of \( \sigma \) where \( 1 \leq \mu \leq \lambda \) such that the union of \( \sigma_{i_1}, \ldots, \sigma_{i_{\mu}} \) has non-empty intersection with \( k \) partitions \((1 \leq k \leq r)\) of \( G(r) \) and with no other. Then the graph induced on the vertices of \( \sigma_{i_1}, \ldots, \sigma_{i_{\mu}} \) is a \( k \)-p.s.c. graph with a \( k \)-p.c.p. being \( \sigma' = \sigma_{i_1} \cdots \sigma_{i_{\mu}} \).

An \( r \)-p.c.p. \( \sigma \) of an \( r \)-p.s.c. graph \( G(r) \) is said to be periodic if \( \sigma \) maps each \( A_i \), into some \( A_i \). The class of all periodic \( r \)-p.c.p.'s of \( G(r) \) is denoted by \( \mathcal{C}_p(G(r)) \).

Theorem 1.1. Let \( G(r) \) be an \( r \)-p.s.c. graph and let \( \sigma \in \mathcal{C}_p \). Then \( \sigma^2 \in \text{Aut} G(r) \), where \( \text{Aut} G(r) \) denotes the group of all automorphisms of \( G(r) \).

In particular, if \( \sigma \) is a bi-p.c.p. of a connected bi-p.s.c. graph \( G(2) \), then \( \sigma^2 \in \text{Aut} G(2) \), and if \( \sigma \) is a p-p.c.p. of a p-p.s.c. (i.e., s.c.) graph \( G(p) \), then \( \sigma^2 \in \text{Aut} G(p) \).
Paths in r-partite self-complementary graphs

Let \( \sigma \in \mathcal{C} \) and \( \sigma = \sigma_1 \cdots \sigma_\alpha \). A mixed cycle \( \sigma_1 \) of \( \sigma \), with \( |\sigma_1| = k\alpha \), is said to be a "\((k, \alpha)\)-cycle" if \( \sigma_1 \) has exactly \( \alpha \geq 1 \) vertices from each of the \( k \geq 2 \) partitions, say, \( A_1, \ldots, A_k \) of \( G(r) \) in the following order:

\[
\sigma_1 = (u_{11} \cdots u_{k1} u_{12} \cdots u_{k2} \cdots u_{1\alpha} \cdots u_{k\alpha})
\]

where

\[
u_{lm} \in A_l \ (l = 1, \ldots, k; \ m = 1, \ldots, \alpha).
\]

**Theorem 1.2.** Let \( G(r) \) be an \( r \)-p.s.c. graph and let \( \sigma \in \mathcal{C}^* \). Let \( \sigma_1 \) be a mixed cycle of \( \sigma \) having non-empty intersection with \( k \) of the partitions of \( G(r) \) and with no other. Then \( |\sigma_1| = k\alpha \), for some \( k \geq 2, \alpha \geq 1 \) and \( \sigma_1 \) is a \((k, \alpha)\)-cycle. Further, \( k \equiv 0 \pmod{4} \) when \( \alpha \) is odd.

**Theorem 1.3.** Let \( G(r) \) be an \( r \)-p.s.c. graph and let \( \sigma \in \mathcal{C}^* \). Let \( \sigma_1 \) be a \((k, \alpha_1)\)-cycle of \( \sigma \) having non-empty intersection with \( A_1, \ldots, A_k \) in the same order. Then the following hold:

(a) Any other cycle \( \sigma_2 \) of \( \sigma \) having non-empty intersection with any of the partitions \( A_1, \ldots, A_k \) is again a \((k, \alpha_2)\)-cycle, for some \( \alpha_2 \geq 1 \) and \( \sigma_2 \subseteq \bigcup_{j=1}^{k} A_j \).

(b) The order of the partitions of \( \sigma_2 \) is same as that of \( \sigma_1 \) upto a cyclic permutation.

As a consequence of Theorems 1.2 and 1.3 it follows that cycles of any connected bi-p.c.p. of a bi-p.s.c. graph are either all pure or all mixed.

**Theorem (Rédei [6]).** Let \( C \) be a set of \( n \) elements with a relation \(<\) such that, for all \( a \) and \( b \) (\( a \neq b \)) in \( C \), either \( a < b \) or \( b < a \). Then the elements of \( C \) may be arranged in a sequence \( a_1, a_2, \ldots, a_n \).

Note that, Rédei's theorem is equivalent to saying that every finite tournament has a hamiltonian path.

S.c. graphs by their very nature enjoy nice properties such as that every s.c. graph \( G \) has a hamiltonian path, a fact proved by Clapham [2]; if \( p \geq 8 \), for every integer \( l, 3 \leq l \leq p-2 \), \( G \) has an \( l \)-cycle and furthermore, if \( G \) is hamiltonian then \( G \) is pancyclic. Hence, the class of s.c. graphs can be classified into three classes according as the circumference being \( p-2 \), \( p-1 \) and \( p \). Further, each of the above three classes of s.c. graphs is characterized in terms of degree sequences, in particular, the case \( p \) characterizes the class of hamiltonian s.c. graphs. All these facts are proved by Bhaskara Rao and we refer to [1] for the relevant references.

The class of \( r \)-p.s.c. graphs is a natural generalization of s.c. graphs in the class of simple (without loops and multiple edges) graphs. In particular, we have the feeling that most of the results in s.c. graphs may be generalized or extended to \( r \)-p.s.c. graphs, especially to \( r \)-p.s.c. graphs with an \( r \)-p.c.p. \( \in \mathcal{C}^* \) consisting of only \((k, \alpha)\)-cycles.
Structural properties of r-p.c.p. of r-p.s.c. graphs are considered in Gangopadhyay and Hebbe [3] wherein, besides the results stated above, a generalization of Ringel and Sachs' Theorem (See [1]) for s.c. graphs to r-p.s.c. graphs is given.

This paper aims at determining the maximum length of a path that exists in any r-p.s.c. graph. It is shown that every connected bi-p.s.c. graph $G$ with $\mathcal{C}_m \neq \emptyset$ has a $(p-3)$-path, (i.e. a path consisting of exactly $p-3$ edges and $p-2$ vertices) and that this result is best possible and that $G$ has a hamiltonian path if the graph induced on each cycle of a $\sigma \in \mathcal{C}_m$ is connected. Lastly, the fact that for $r \geq 4$, every r-p.s.c. graph $G(r)$ with a $\sigma \in \mathcal{C}_m^*$ such that each cycle $\sigma_i$ of $\sigma$ has non-empty intersection with at least 4 partitions of $G(r)$, has a hamiltonian path, is also established.

The graph obtained from $G(r)$ by adding a vertex $u$ constituting $(r-1)$-st partition of $G(r)$, which is the fixed point of $\sigma^* = (u)\sigma$, also has a hamiltonian path. The last two results generalize the result of Clapham [2].

The proof technique employed in proving the results in this paper is essentially similar to the proof technique in Clapham [2].

2. Paths in bi-p.s.c. graphs

**Theorem 2.1.** Every connected bi-p.s.c. graph $G(2)$ of order $p$ with $\mathcal{C}_m \neq \emptyset$ has a $(p-3)$-path: this statement is best possible.

**Proof.** Let $\sigma \in \mathcal{C}_m$ and $\sigma = \sigma_1 \cdot \cdot \cdot \sigma_t$ be its disjoint cycle form. We then consider two cases according as (i) $\lambda = 1$, and (ii) $\lambda > 1$.

**Case 1.** $\lambda = 1$. Let $\sigma = (1 \ 2 \ \ldots \ n)$ where $n = 4t$, $t \geq 1$ ($n \equiv 0 \pmod{4}$) by Theorem 1.2). Without loss to generality, we can assume that $(1, 2) \in E$ (for otherwise, $(2, 3) \in E$ and we can consider $\sigma = (2 \ 3 \ \ldots \ n 1)$). Since $\sigma^2 \in \text{Aut} G(2)$, we get that $(i, i+1) \in E$ for all $i$ odd.

If $t = 1$, then $G$ consists of two copies of $K_4$.

Suppose that, $t > 1$. Then two cases arise according as $(1, 4) \in E$ or not.

If $(1, 4) \in E$, then $(i, i+3) \in E$ for all odd $i$. In this case,

$$1, 4, 3, 6, \ldots, 4t, 4t-1, 2, 1$$

is a hamiltonian cycle.

If $(1, 4) \not\in E$, then $(2, 5) \in E$ and hence $(i, i+3) \in E$ for all $i$ even. In this case, $G(2)$ has two disjoint $2t$-cycles as follows:

$$C^1_{2t} : 1, 2, 5, 6, \ldots, 4t-7, 4t-6, 4t-3, 4t-2, 1$$

and

$$C^2_{2t} : 3, 4, 7, 8, \ldots, 4t-5, 4t-4, 4t-1, 4t, 3.$$
Remark. The cycle \( C_{2r}^1 \) (or \( C_{2r}^2 \)) has the vertex labels \( \equiv 1 \) or \( 2 \) (mod 4) (\( \equiv 0 \) or \( 3 \) (mod 4)) and they appear alternatingly.

Since \( G(2) \) is connected there must exist an edge from some vertex of \( C_{2r}^1 \) to some vertex of \( C_{2r}^2 \). Then \( G \) has a hamiltonian path.

Case 2. \( r > 1 \). Let

\[
\sigma = (u_1 u_2 \cdots u_{4r}). \quad (i = 1, \ldots , \lambda)
\]

where \(|\sigma| = 4t_i \) (\( i = 1, \ldots , \lambda \)). Then each \( \sigma_i \) (\( i = 1, \ldots , \lambda \)) is one of the following three types:

1. \( \langle \sigma_i \rangle \) is hamiltonian.
2. \( \langle \sigma_i \rangle \) has two disjoint \( 2t \)-cycles.
3. \( \langle \sigma_i \rangle \equiv 2K_2 \).

Let \( \sigma \) have \( \lambda_i \) cycles of type \( i \) (\( i = 1, 2, 3 \)) and accordingly arrange the cycles of \( \sigma \) such that the first \( \lambda_1 \) cycles are of type 1, the next \( \lambda_2 \) cycles are of type 2 and the last \( \lambda_3 \) cycles are of type 3, as follows:

\[
\sigma = \alpha_1 \cdots \alpha_{\lambda_1} \beta_1 \cdots \beta_{\lambda_2} \gamma_1 \cdots \gamma_{\lambda_3},
\]

where \( \lambda = \lambda_1 + \lambda_2 + \lambda_3 \).

We now define an ordering between two cycles of certain type. First, for any \( \alpha_i \), \( \alpha_j \) cycles of type 1, we write \( \alpha_i < \alpha_j \) if there is an edge from some even vertex of \( \alpha_i \) to some odd vertex of \( \alpha_j \) where \( i \neq j \), \( 1 \leq i, j \leq \lambda_1 \), and \( u_i \in \alpha_i \), \( u_j \in \alpha_j \) (\( i = 1, \ldots , \lambda \)) is said to be odd or even according as \( j \) is odd or even. Notice that, if \( \alpha_i < \alpha_j \) then every even vertex of \( \alpha_i \) is adjacent to some odd vertex of \( \alpha_j \), and every odd vertex of \( \alpha_i \) is adjacent to some even vertex of \( \alpha_j \).

Now, if \( \alpha_i < \alpha_n \), then \((u_{i2}, u_{i3}) \notin E \), which implies that \((u_{i1}, u_{i2}) \in E \) and hence \( \alpha_i < \alpha_n \). Thus for any two cycles \( \alpha_i, \alpha_j \) of type 1 either \( \alpha_i < \alpha_j \) or \( \alpha_j < \alpha_i \) holds.

Observation 4. (i) \( \alpha_i < \alpha_j \) and \( \alpha_j < \alpha_i \) may both hold.

(ii) \( \alpha_i < \alpha_j \) and \( \alpha_j < \alpha_i \) do not imply \( \alpha_i < \alpha_j \), where \( \alpha_i, \alpha_j, \alpha_k \) are cycles of type 1.

By Rédei's theorem the cycles of type 1 may be arranged, after suitable relabelling, as follows:

\[
\alpha_1 < \alpha_2 < \cdots < \alpha_{\lambda_1}.
\]

Consider now \( \beta_i, \beta_j \) cycles of type 2, where \( i \neq j \), \( 1 \leq i, j \leq \lambda_2 \). Recall that each such cycle \( \beta_i \) of length \(|\beta| = 4s_i \) (\( i = 1, \ldots , \lambda_2 \)) induces a subgraph \( \langle \beta_i \rangle \) which contains two disjoint \( 2s_i \)-cycles as follows:

\[
C_{2s_1}^1 : v_{i1}, v_{i2}, v_{i5}, v_{i6}, \ldots, v_{i,4s_1-3}, v_{i,4s_1-2}, v_{i1}
\]

and

\[
C_{2s_2}^2 : v_{i3}, v_{i4}, v_{i7}, v_{i8}, \ldots, v_{i,4s_2-4}, v_{i,4s_2-1}, v_{i3}.
\]
We shall write $\beta_i > \beta_j$ if there is an edge from some odd vertex of $C_{2x_i}^1$ to some even vertex $\delta$ of $C_{2x_j}^1$, and in this case we write $C_{2x_i}^2 > C_{2x_j}^2$. Also, it can be easily seen that, $C_{2x_i}^2 > C_{2x_j}^2$.

Now, if $\beta_i \not> \beta_j$, that is, if $C_{2x_i}^2 \not> C_{2x_j}^1$ then $(v_{i1}, v_{i2}) \notin E$ and hence $(v_{i2}, v_{i3}) \in E$ which implies that $C_{2x_i}^2 > C_{2x_j}^1$. Thus interchanging the roles of $C_{2x_i}^2$ and $C_{2x_j}^2$ we get $\beta_i > \beta_j$. Thus for any two cycles $\beta_i$ and $\beta_j$ of type 2 either $\beta_i > \beta_j$ or $\beta_j > \beta_i$ holds. (Observation 4 is true for $\beta_i$'s $(i = 1, \ldots, A_2)$.) Hence, by Rédei's Theorem the cycles of type 2 may be arranged after suitable relabelling as follows: $\beta_1 > \beta_2 > \cdots > \beta_{A_2}$.

Lastly, let $\gamma_i$ and $\gamma_j$ be cycles of type 3, where $i \neq j$, $1 \leq i, j \leq A_3$, each $\gamma_i$ consists of two copies of $K_2$, say, $K_{2x_i}^1 = (w_{i1}, w_{i2})$ and $K_{2x_i}^2 = (w_{i3}, w_{i4})$, $(i = 1, \ldots, A_3)$. We write $\gamma_i > \gamma_j$ if $K_{2x_i}^1 > K_{2x_j}^1$, that is, when $(w_{i1}, w_{i2}) \in E$. This implies that $K_{2x_j}^1 > K_{2x_i}^2$ since $(w_{i3}, w_{i4}) \in E$. If $\gamma_i \not> \gamma_j$ then $(w_{i2}, w_{i3}) \notin E$, that is $(w_{i2}, w_{i4}) \in E$ and hence $K_{2x_j}^1 > K_{2x_i}^2$. In this case, by interchanging the roles of $K_{2x_i}^1$ and $K_{2x_j}^2$ we obtain that $\gamma_j > \gamma_i$. Thus for any two cycles $\gamma_i$ and $\gamma_j$ of type 3 either $\gamma_i > \gamma_j$ or $\gamma_j > \gamma_i$ holds. (Notice that Observation 4 is true for $\gamma_i$'s $(i = 1, \ldots, A_3)$.)

Hence by Rédei's Theorem cycles of type 3 may be arranged after suitable relabelling as follows:

$$\gamma_1 > \gamma_2 > \cdots > \gamma_{A_3}.$$

Now, let $\beta_i$ and $\gamma_i$ be cycles of type 2 and 3 respectively. We write $\beta_i > \gamma_i$ if $C_{2x_i}^1 > K_{2x_i}^1$, that is, there is an edge from some odd vertex of $C_{2x_i}^1$ to $w_{i2}$. Also it follows that $C_{2x_i}^2 > K_{2x_i}^1$. Analogously, $\gamma_i > \beta_i$ means that there is an edge from $w_{i1}$ to some even vertex of $C_{2x_i}^1$. If $\beta_i \not< \gamma_i$, then $(v_{i1}, w_{i2}) \notin E$ that is $(v_{i2}, w_{i3}) \in E$ and hence $K_{2x_j}^1 > C_{2x_i}^1$. Now, by interchanging the roles of $K_{2x_i}^1$ and $K_{2x_j}^2$ we get that $\gamma_i > \beta_i$. Thus, for any cycles $\beta_i$ of type 2 and $\gamma_i$ of type 3 either $\beta_i > \gamma_i$ or $\gamma_i > \beta_i$ holds. Hence, by Rédei's Theorem the cycles of type 2 and 3 may be arranged after suitable relabelling as follows:

$$\delta_{i+1} > \delta_{i+2} > \cdots > \delta_{i+A_3},$$

where each $\delta_i$ is spanned by two cycles or copies of $K_2$, say, $\delta_i^1$ and $\delta_i^2$. Two cases arise according as (a) $\lambda_1 > 0$, $\lambda_2 + \lambda_3 \geq 0$, and (b) $\lambda_1 = 0$, $\lambda_2 + \lambda_3 > 0$.

Case (a) $\lambda_1 > 0$, $\lambda_2 + \lambda_3 \geq 0$. Choose $B = \alpha_1$, $C = \alpha_i$, and if $\lambda_2 + \lambda_3 > 0$, $A = \delta_i^1$, and $D = \delta_{i+1}^2$. We write $A > B$ ($C > D$) if every odd vertex of $A(C)$ is adjacent to some even vertex of $B(D)$. Note that if $A \not> B$, then every even vertex of $A$ is adjacent to some odd vertex of $B$ and if $C \not> D$, then given any even vertex $v$ of $C$ either $v$ or $\sigma(v)$ is adjacent to some odd vertex of $D$. We also observe that if $u$ is an odd vertex of $C$ and $u$ is adjacent to $v, w$ in the hamiltonian cycle in $C$, then either $v = \sigma^2(w)$ or $w = \sigma^2(v)$. Thus if $C \not> D$, then there is a hamiltonian path in $C$ which starts at any given odd vertex of $C$ and ends at an even vertex of $C$ which is adjacent to some odd vertex of $D$. We now consider the following four cases. In each of these cases we shall specify a $(p-3)$-path in $G(2)$. 

Case (a.i). $A > B, C > D$. $A > B$ implies that each odd vertex of $A$ is adjacent to some even vertex of $B$. Since, $C > D$ the same thing holds between them. The $(p-3)$-path is as follows (see also Fig. 1):

```
            \delta_{\lambda_{i+1}}                    \delta_{\lambda_{i+2}}
               ............................... s
             \gamma_0 0 0 t,~ t~ B ~" ............
               \gamma_1 \gamma_2

Fig. 1.
```

Start with any even vertex in $\delta_{\lambda_{i+1}}$, cover all the vertices of it traversing along the cycle (or $K_2$) ending up in an odd vertex. This odd vertex leads to an even vertex of $\delta_{\lambda_{i+2}}$ and cover all the vertices of it from this even vertex. Proceed until an even vertex of $\delta_{\lambda_{i}}$ is reached and cover all its vertices, the end vertex being odd. Since $A > B$, this odd vertex leads to some even vertex of $B$. Cover all the vertices of $B$ from this even vertex, except the last odd vertex, the end vertex being even. This even vertex leads to an odd vertex of $\alpha_{2}$, from which cover all its vertices, the end vertex being even. Thus proceed until an odd vertex of $\alpha_{1}$ is reached, from which cover all its vertices except the last even vertex. The last odd vertex leads to an even vertex of $D$ from which cover all its vertices, the last vertex being odd. This in turn leads to an even vertex of $\delta_{\lambda_{i+2}}$ from which cover all its vertices ending in an odd vertex. In this way all the vertices of $\delta_{\lambda_{i+1}}, \ldots, \delta_{\lambda_{2}}$ can be covered.

The path described above is a $(p-3)$-path which misses exactly two vertices, one from each of $\alpha_{1}$ and $\alpha_{i}$.

In all the other three cases we shall describe the $(p-3)$-path through figures.

Case (a.ii). $A > B, C \not> D$. The $(p-3)$-path in this case is as shown in Fig. 2, which misses exactly two vertices, one from each of $\delta_{\lambda_{i+1}}$ and $\alpha_{1}$.

Note that, if the vertex missed by the path is in a part of cycle of type 3 then simply we cover one vertex of the corresponding $K_2$ and go to the succeeding cycle. We assume the same in what follows, whenever such a situation arises.

Case (a.iii). $A \not> B, C > D$. The $(p-3)$-path in this case misses exactly two vertices of $G(2)$, one from each of $\delta_{\lambda_{i}}$ and $\alpha_{1}$, and is as shown in Fig. 3.
Case (a.iv). $A \not> B$, $C \not> D$, or $\lambda_2 + \lambda_3 = 0$. If $\lambda_2 + \lambda_3 > 0$, the $(p-3)$-path misses exactly one vertex from each of $\delta_{\lambda_1}$ and $\delta_{\lambda_1+1}$, and is as shown in Fig. 4.

If $\lambda_2 + \lambda_3 = 0$, that is if all cycles of $\sigma$ are of type 1, then, clearly, $G$ has a hamiltonian path, see Fig. 4 (only the type 1 part).

Case (b). $\lambda_1 = 0$, $\lambda_2 + \lambda_3 > 0$. In this case, the cycles $\delta_i$ ($i = 1, \ldots, \lambda$) of $\sigma$ may be arranged as follows:

$$\delta_1 > \delta_2 > \cdots > \delta_\lambda$$
where $\lambda = \lambda_2 + \lambda_3$ and each $(\delta_k)$ is spanned by two cycles or copies of $K_2$, viz. $\delta_1^1$, $\delta_2^2$ ($k = 1, \ldots, \lambda$). We now consider the following three cases:

**Case (b. i).** An odd vertex of $\delta_1^1$ is adjacent to some even vertex of $\delta_2^2$. Then every odd vertex of $\delta_k^1$ is adjacent to some even vertex of $\delta_2^2$. In this case we have a Hamiltonian path as exhibited in Fig. 5.

**Case (b. ii).** An odd vertex of $\delta_1^1$ is adjacent to some even vertex of $\delta_2^2$. Then every even vertex of $\delta_1^1$ is adjacent to some odd vertex of $\delta_2^2$ and every odd vertex of $\delta_k^2$ is adjacent to some even vertex of $\delta_2^2$. Let $V_m = \bigcup_{k=1}^{\lambda} \delta_m^k$ ($m = 1, 2$). Since $G$ is connected, there is a vertex $u_m \in V_m$ ($m = 1, 2$) such that $(u_1, u_2) \in E$. Let $u_1 \in \delta_1^1$ and $u_2 \in \delta_2^2$ for some $(i, j = 1, \ldots, \lambda)$. Without loss of generality, we can take $u_1$ to be odd and $u_2$ to be even. Otherwise, we can interchange the roles of $\delta_1^1$ and $\delta_2^2$ for each ($k = 1, \ldots, \lambda$). We now construct a Hamiltonian path, as exhibited in Fig. 6.
Case (b.iii). Every even vertex of $\delta^1$ is adjacent to some odd vertex of $\delta^2$. In this case we exhibit a $(p-3)$-path, which misses exactly one vertex each in $\delta^1$ and $\delta^2$, as exhibited in Fig. 7.

This completes the proof of the first part of the theorem.

In order to show that the result is best possible, we exhibit an infinite class of bi-p.s.c. graphs having a $(p-3)$-path and no $(p-2)$-path. For this, by Observation 2, in connection with the hypothesis $\mathcal{C}_m \neq \emptyset$, it is enough to construct such examples for the order $p = 4t$ where $n_1 = 2t = n_2$.

Let $H = H_1$ be the graph shown on Fig. 8(a). Define, $H_i$ (See $H_2$ of Fig. 8(b)) such that $V(H_i) = V(H_{i-1}) \cup \{u_i, v_i, w_i, x_i\}$ and that $H_i$ contains $H_{i-1}$ as an induced graph on $V(H_{i-1})$ with the additional edges as follows:

$(u_i, b), (v_i, b)$ for all $b \in B_{i-1}$.

$(u_i, w_i)$ and $(v_i, x_i)$,

where $(A_{i-1}, B_{i-1})$ is the bipartition of $V(H_{i-1})$. Then, $H_i$ is of order $p = 4i + 4$ and is a bi-p.s.c. graph. For each $i \geq 1$ and the following permutation $\sigma_i$ is a...
bi-p.c.p. of $H_i$:

$$\sigma_i: (u, v, x, u_i, v_i) \cdots (u_1, v_1, x_1, u_1, v_1)(u \times v \times w).$$

A $(p-3)$-path of $H_i$ is as given below:

$$v, x, v_1, x_1, v_2, x_2, v_3, \ldots, v_{i-1}, x_{i-1}, v_i, w, u_1, w_1, u_2, w_2, u_3, w_3, \ldots, u_{i-1}, w_{i-1}, u_i, w_i.$$  

Notice that, the vertices $x_i$ and $u$ are missing in the above path.

Finally, since $H_i$ contains 4 vertices of valency 1 (namely, $u$, $v$, $x_i$, $w_i$) there cannot be a $(p-2)$-path. This completes the proof of the theorem.

**Theorem 2.2.** Let $G(2)$ be a bi-p.s.c. graph with bi-p.c.p. $\sigma \in \mathbb{E}_m \neq \emptyset$ such that the graph induced on each cycle of $\sigma$ is connected. Then $G(2)$ has a hamiltonian path.

**Proof.** Let $\sigma = \sigma_1 \cdots \sigma_\lambda \in \mathbb{E}_m$. (i) $\sigma_i$ is connected implies that $G$ is connected and that $|\sigma_i| \geq 8$, and $|\sigma_i| = 0 \pmod{4}$ for each $(i = 1, \ldots, \lambda)$. Let $|\sigma_i| = 4t_i$, $(i = 1, \ldots, \lambda)$. Further, let

$$\sigma_i = (u_{i_1}, u_{i_2}, \ldots, u_{i_{4t_i}}).$$

Without loss to generality, we can assume that $(u_{i_1}, u_{i_2}) \in E$. Since $\sigma^2 \in \text{Aut } G(2)$, $(u_{i_j}, u_{i_{j+1}}) \in E$ for all odd $j$. Suppose now that, $(u_{i_1}, u_{i_4}) \in E$. Then $(u_{i_j}, u_{i_{j+1}}) \in E$ for all odd $j$. In this case, we have the following hamiltonian cycle:

$$u_{i_1}, u_{i_4}, u_{i_3}, u_{i_6}, u_{i_5}, u_{i_8}, u_{i_7}, \ldots, u_{i_{4t_i-3}}, u_{i_{4t_i-4}}, u_{i_{4t_i-1}}, u_{i_2}, u_{i_1}.$$  

Such a cycle $\sigma_i$ is called of type 1.

In the other case, that is if $(u_{i_1}, u_{i_4}) \notin E$, then $(u_{i_2}, u_{i_5}) \in E$ and hence
(u_{ij} u_{i,j+3}) \in E \text{ for all even } j. \text{ In this case, we have two disjoint } 2t_i\text{-cycles as follows:}

\begin{align*}
C_{2t_1}^1: & \, u_{t_1}, u_{t_2}, u_{t_3}, u_{t_4}, u_{t_5}, u_{t_6}, u_{t_7}, u_{t_8}, \ldots, u_{t_i}, u_{t_i+1}, u_{t_i+2}, u_{t_i+3}, u_{t_i+4}, u_{t_i+5}, u_{t_i+6}, u_{t_i+7}, u_{t_i+8}, \ldots, u_{t_{i+1}}, u_{t_{i+2}}, u_{t_{i+3}}, u_{t_{i+4}}. \\
C_{2t_2}^2: & \, u_{2t_2}, u_{2t_2+1}, u_{2t_2+2}, u_{2t_2+3}, u_{2t_2+4}, u_{2t_2+5}, u_{2t_2+6}, u_{2t_2+7}, u_{2t_2+8}, \ldots, u_{t_{i+1}}, u_{t_{i+2}}, u_{t_{i+3}}, u_{t_{i+4}}.
\end{align*}

Such a cycle \( t_i \) is called of type 2. In this case, since \( \langle \sigma_i \rangle \) is connected, there is \( u_{ij} \in \langle C_{2t_i}^1 \rangle \) and \( u_{ik} \in \langle C_{2t_i}^2 \rangle \) such that \( (u_{ij}, u_{ik}) \in E \), where either \( j \) is odd and \( k \) is even, or, \( j \) is even and \( k \) is odd. In either case, since \( \sigma^2 \in \text{Aut G}(2) \), we have that for any \( u_{ij} \in \langle C_{2t_i}^1 \rangle \) with \( j \) odd, there is an \( u_{ik} \in \langle C_{2t_i}^2 \rangle \) with \( k \) even and for any \( u_{ij} \in \langle C_{2t_i}^2 \rangle \) with \( j \) odd, there is an \( u_{ik} \in \langle C_{2t_i}^1 \rangle \) with \( k \) even such that \( (u_{ij}, u_{ik}) \in E \). Now, given any even vertex of \( \langle C_{2t_i}^1 \rangle \) (resp. \( \langle C_{2t_i}^2 \rangle \)) there is a path along \( C_{2t_i}^1 \) (resp. \( C_{2t_i}^2 \)) which covers all the vertices of \( \langle C_{2t_i}^1 \rangle \) (resp. \( \langle C_{2t_i}^2 \rangle \)) and ends up in an odd vertex say \( u_{ij} \) of \( \langle C_{2t_i}^1 \rangle \) (resp. \( \langle C_{2t_i}^2 \rangle \)); this odd vertex is adjacent to some even vertex say \( u_{ik} \) of \( \langle C_{2t_i}^1 \rangle \) (resp. \( \langle C_{2t_i}^2 \rangle \)) and one can continue along \( C_{2t_i}^1 \) (resp. \( C_{2t_i}^2 \)) in a path which ends up in an odd vertex of \( \langle C_{2t_i}^1 \rangle \) (resp. \( \langle C_{2t_i}^2 \rangle \)). Thus, if \( \sigma_i \) is a cycle of type 2, given any even vertex \( u_{ik} \) of \( \sigma_i \), there is a hamiltonian path in \( \sigma_i \) which starts with \( u_{ik} \) and ends up in an odd vertex of \( \sigma_i \). Note that the last observation also holds if \( \sigma_i \) is a cycle of Type 1. Thus, given any cycle \( \sigma_i \), and an even vertex \( u_{ik} \) in \( \langle \sigma_i \rangle \), there is a hamiltonian path in \( \langle \sigma_i \rangle \) which starts from \( u_{ik} \) and ends up in an odd vertex of \( \langle \sigma_i \rangle \).

Now, we order \( \sigma_1, \ldots, \sigma_x \) in the following manner. We shall write \( \sigma_i > \sigma_j \) if an odd vertex of \( \sigma_i \) is adjacent to an even vertex of \( \sigma_j \). Evidently, if \( \sigma_i > \sigma_j \), then \( \sigma_j > \sigma_i \) follows. Hence, by Rédei's Theorem, the cycles of \( \sigma \) may be ordered by suitable relabelling as follows: \( \sigma_1 > \sigma_2 > \ldots > \sigma_x \).

We now start with any even vertex \( u_{ik} \) of \( \sigma_1 \). We know that there is a hamiltonian path in \( \langle \sigma_1 \rangle \) which starts with \( u_{ik} \), and ends up in an odd vertex of \( \sigma_1 \). Since \( \sigma_1 > \sigma_2 \), this odd vertex is adjacent to some even vertex \( u_{ik} \) of \( \sigma_2 \). There is a hamiltonian path in \( \langle \sigma_2 \rangle \) which starts with \( u_{ik} \) and ends up in an odd vertex of \( \sigma_2 \). From this odd vertex we proceed to an even vertex \( u_{ik} \) of \( \sigma_3 \) and so on. This gives us a hamiltonian path.

Thus, \( G \) has a hamiltonian path with any even vertex of \( \sigma_i \) as an end vertex. Similarly, reversing the procedure, we can get a hamiltonian path with any odd vertex of \( \sigma_x \) as an end vertex.

### 3. Hamiltonian paths in \( r \)-p.s.c. graphs with \( C^* \neq \emptyset \)

**Theorem 3.1.** Let \( G(r) \) be an \( r \)-p.s.c. graph, \( r \geq 4 \) with an \( r \)-p.c.p. \( \sigma \in C^* \neq \emptyset \) where each cycle of \( \sigma \) has non-empty intersection with at least four partitions of \( G(r) \). Then \( G(r) \) has a hamiltonian path.

**Proof.** The proof goes on similar lines as that of Clapham [2].
Let \( \sigma = \sigma_1 \cdots \sigma_\lambda \in \mathbb{C}^* \). Then by Theorems 1.2 and 1.3 we conclude that each \( \sigma_i \) of \( \sigma \) is a \((k_i, \alpha_i)\)-cycle, where \( k_i \geq 4 \), by hypothesis.

To begin with, we consider two cases according as (1) \( \lambda = 1 \), and (2) \( \lambda > 1 \).

**Case 1. \( \lambda = 1 \).** For convenience, let \( \sigma \) be a \((k, \alpha)\)-cycle. \( k \geq 4 \) and \( \sigma = (1 \, 2 \, 3 \cdots k\alpha) \), where \( |\sigma| = k\alpha \) is even. We first assume that \((1, 2) \notin E \).

If \( k = 4 \) and \( \alpha = 1 \), then \( 2, 1, 3, 4 \) or \( 1, 2, 4, 3 \) is the required hamiltonian path. Otherwise, since \( k\alpha \) is even, it follows that \( k \geq 4 \) and \( \alpha \geq 2 \). Then \((1, 4) \in E \) if and only if \((4, 7) \notin E \). Hence, we may suppose that either (a) \((i, i + 3) \in E \) for all odd \( i \), or (b) \((j, j + 3) \in E \) for all even \( j \).

In case (a), we consider the hamiltonian path: \( P_1 \) or \( P_2 \) according as \((1, 3) \in E \) or \((2, 4) \in E \) where

\[
P_1: 2, 1, 4, 3, 6, 5, \ldots, k\alpha - 2, k\alpha - 3, k\alpha.
\]

\[
P_2: 1, 4, 3, 6, 5, 8, \ldots, k\alpha - 3, k\alpha, 2, k\alpha - 1.
\]

In case (b), we construct a hamiltonian path \( P \) as follows. Let \( P_1 \) be the path

\[
1, 2, 5, 6, 9, 10, \ldots, \text{the last term being } k\alpha - 2 \text{ or } k\alpha \text{ according as } k\alpha \equiv 0 \text{ or } 2 \pmod{4},
\]

and \( P_2 \) be the path

\[
3, 4, 7, 8, 11, 12, \ldots, \text{the last term being } k\alpha \text{ or } k\alpha - 2 \text{ according as } k\alpha \equiv 0 \text{ or } 2 \pmod{4}.
\]

Then \( P \) is obtained by combining \( P_1 \) and \( P_2 \), using the edge \((1, 3) \) or \((k\alpha - 2, k\alpha) \) whichever exists. (Note that since \( \sigma^2 \in \text{Aut } G(r) \), either \((1, 3) \in E \) or \((k\alpha - 2, k\alpha) \in E \).)

If \((1, 2) \notin E \), then \((2, 3) \in E \) and the proof is similar. In any case, since \( \sigma^2 \in \text{Aut } G(r) \), we have the following

**Remark.** Either (i) for any two consecutive odd vertices of \( \sigma \), there is a hamiltonian path in which they appear consecutively and (ii) for any two consecutive even vertices of \( \sigma \), there is a hamiltonian path in which they are end vertices.

or, (i)' for any two consecutive even vertices of \( \sigma \), there is a hamiltonian path in which they appear consecutively and (ii)' for any two consecutive odd vertices of \( \sigma \), there is a hamiltonian path in which they are end vertices.

**Case 2. \( \lambda > 1 \).** Then by the Remark made in Case 1, it follows that any cycle \( \sigma_i \) of \( \sigma \) satisfies either (i) and (ii) or (i)' and (ii)' . A cycle \( \sigma_i \) of \( \sigma \) is said to be of type 1 if it satisfies (i) and (ii), and is of type 2 if it satisfies (i)' and (ii)'.

We now define an ordering between any two cycles of \( \sigma \) as follows:

Let \( \sigma_i \) and \( \sigma_j \) be cycles of \( \sigma \) of type 1. Then we write \( \sigma_i < \sigma_j \) if some even vertex of \( \sigma_i \) is adjacent to some odd vertex of \( \sigma_j \). Then, it can be easily seen that if \( \sigma_j < \sigma_i \) then \( \sigma_i < \sigma_j \).

Hence, for any two cycles \( \sigma_i, \sigma_j \) of \( \sigma \) of type 1 either \( \sigma_i < \sigma_j \), or \( \sigma_j < \sigma_i \) holds.

Let \( \sigma_i \) and \( \sigma_j \) be cycles of type 2. We write \( \sigma_i < \sigma_j \) if an odd vertex of \( \sigma_i \) is adjacent to some even vertex of \( \sigma_j \). Again, it follows, with this ordering, that either \( \sigma_i < \sigma_j \), or \( \sigma_j < \sigma_i \) holds.

Lastly, if \( \sigma_i \) and \( \sigma_j \) are of types 1 and 2 respectively, we write \( \sigma_i < \sigma_j \) if an even
vertex of \( \sigma_i \) is adjacent to some even vertex of \( \sigma_i \), and \( \sigma_i < \sigma_i \) if an odd vertex of \( \sigma_i \) is adjacent to some odd vertex of \( \sigma_i \). Then if \( \sigma_i \not< \sigma_i \) it follows that \( \sigma_i < \sigma_i \).

Hence, either \( \sigma_i < \sigma_i \) or \( \sigma_i < \sigma_i \) holds in this case also.

Thus for any two cycles \( \sigma_i \) and \( \sigma_i \) of \( \sigma \) either \( \sigma_i < \sigma_i \), or \( \sigma_i < \sigma_i \) holds. Hence by Rédei's Theorem the cycles of \( \sigma \) can be ordered by an appropriate relabelling of \( \sigma_i \)'s as follows:

\[
\sigma_1 < \sigma_2 < \cdots < \sigma_\lambda.
\]

Now, each \( \langle \sigma_i \rangle \) is a \( k \)-p.s.c. graph for some \( k \geq 4 \) and by Case 1 there is a hamiltonian path in each \( \langle \sigma_i \rangle \) (\( i = 1, \ldots, \lambda \)). For \( \sigma_1 \) and \( \sigma_2 \) we consider the following four cases according to their types:

**Case 2(a).** \( \sigma_1 \) and \( \sigma_2 \) are both of type 1. Then there is a hamiltonian path in \( \langle \sigma_1 \rangle \) with its end vertices at consecutive even vertices, say, \( u_{11} \) and \( u_{11+2} \) of \( \sigma_1 \). Since \( \sigma_1 < \sigma_2 \), \( (u_{11}, u_{1j}) \in E \) for some odd \( j \) and hence \( (u_{11+j}, u_{1j+2}) \in E \) where the second suffix of a vertex is reduced modulo the length of the cycle containing it. Now, since \( \sigma_2 \) is of type 1, there is a hamiltonian path in \( \langle \sigma_2 \rangle \) with \( u_{2j} \) and \( u_{2j+2} \), with odd \( j \), appearing consecutively. We can now obtain a hamiltonian path in \( \langle \sigma_1 \cup \sigma_2 \rangle \) by inserting the hamiltonian path of \( \langle \sigma_1 \rangle \) between \( u_{2j} \) and \( u_{2j+2} \) in the hamiltonian path of \( \langle \sigma_2 \rangle \).

The remaining cases are dealt with in an analogous way:

**Case 2(b).** \( \sigma_1 \) is of type 1 and \( \sigma_2 \) is of type 2. In this case, we can obtain a hamiltonian path in \( \langle \sigma_1 \cup \sigma_2 \rangle \) by inserting the hamiltonian path of \( \langle \sigma_1 \rangle \) (its end vertices being consecutive even vertices \( u_{1i} \) and \( u_{1i+2} \) of \( \sigma_1 \)) between \( u_{2j} \) and \( u_{2i+2} \) of the hamiltonian path of \( \langle \sigma_2 \rangle \) (this is possible since, \( \sigma_1 < \sigma_2 \) implies that, for some even \( j \), \( (u_{1i}, u_{2j}), (u_{1i+2}, u_{2j+2}) \in E \) and \( u_{2j}, u_{2j+2} \) appear consecutively).

**Case 2(c).** \( \sigma_1 \) and \( \sigma_2 \) are of types 2 and 1 respectively. In this case, a hamiltonian path of \( \langle \sigma_1 \cup \sigma_2 \rangle \) can be obtained by inserting the hamiltonian path of \( \langle \sigma_1 \rangle \) (its end vertices at consecutive odd vertices \( u_{1i} \) and \( u_{1i+2} \) of \( \sigma_1 \)) between \( u_{2j} \) and \( u_{2j+2} \) the consecutive odd vertices appearing consecutively in the hamiltonian path of \( \langle \sigma_2 \rangle \) with odd \( j \).

**Case 2(d).** \( \sigma_1 \) and \( \sigma_2 \) are both of type 2. In this case, the hamiltonian path of \( \langle \sigma_1 \rangle \), with its end vertices being consecutive odd vertices \( u_{1i} \) and \( u_{1i+2} \), is inserted between \( u_{2j} \) and \( u_{2j+2} \)—the consecutive even vertices in the hamiltonian path of \( \langle \sigma_2 \rangle \) which gives a hamiltonian path in \( \langle \sigma_1 \cup \sigma_2 \rangle \).

Thus it is possible to construct a hamiltonian path in \( \langle \sigma_1 \cup \sigma_2 \rangle \), where \( \sigma_1 < \sigma_2 \). Next, in a similar way this hamiltonian path can be inserted into a hamiltonian path of \( \langle \sigma_3 \rangle \), and so on. Theorem 3.1 is now proved.

Note that the hamiltonian path constructed in the above proof has the following properties:

(1) It has two consecutive odd (even) vertices of \( \sigma_1 \) appearing consecutively if \( \sigma_1 \) is of type 1 (type 2).
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(2) Its end vertices are consecutive even (odd) vertices of $\sigma_i$, if $\sigma_i$ is of type 1 (type 2).

(3) For $\sigma_i < \sigma_{i+1}$ ($i = 1, \ldots, \lambda$), it has some $u_i$ and $u_{i+1,k}$ appearing consecutively where $j$ and $k$ are as follows:

- $j$ even and $k$ odd, if $\sigma_i$ and $\sigma_{i+1}$ are both of type 1,
- $j$ even and $k$ even, if $\sigma_i$ is of type 1 and $\sigma_{i+1}$ is of type 2,
- $j$ odd and $k$ even, if $\sigma_i$ and $\sigma_{i+1}$ are both of type 2, and
- $j$ odd and $k$ odd, if $\sigma_i$ is of type 2 and $\sigma_{i+1}$ is of type 1.

Let $G(r+1)$ be an $(r+1)$-p.s.c. graph with an $(r+1)$-p.c.p. $\sigma^* = (u)_{\sigma \in \mathcal{E}^{*}}$ where $u$ is a fixed vertex. $A_{r+1} = \{u\}$ and all other cycles of $\sigma$ has non-empty intersections with at least four partitions of $G(r+1)$. Then we have the following:

**Theorem 3.2.** $G(r+1)$ has a hamiltonian path.

**Proof.** Let $G(r)$ be the subgraph induced by $\sigma$ in $G(r+1)$. Then $G(r)$ satisfies the conditions of Theorem 3.1.

Now consider the hamiltonian path $h$ in $G(r)$ as described in Theorem 3.1. It is composed of several paths each being hamiltonian within a cycle of $\sigma$. By the properties (1), (2) and (3), consecutive vertices (say) $u_{i,j}$, $u_{i,j+2}$ of the same parity appear consecutively within hamiltonian path of $\sigma$ and consecutive vertices of opposite parity to $u_i$ and $u_{j+2}$ appear as end vertices within that hamiltonian path $h$. Further, if $u$ is adjacent to a vertex $u_i$ of $\sigma$, then $u$ is adjacent to all the vertices of $\sigma$ with the same parity as that of $u_i$ and $u$ is not adjacent to all other vertices of $\sigma$.

Now, let $u_{i,j}$, $u_{i,j+2}$ be two vertices of the same parity appearing consecutively in $h_i$ where $h_i$ denotes a hamiltonian path in $\langle \sigma_i \rangle$ ($i = 1, \ldots, \lambda$). Suppose that, $u$ is adjacent to $u_{i,j}$. Then $(u,u_{i,j+2}) \in E$ and hence $u$ can be incorporated in between $u_{i,j}$ and $u_{i,j+2}$ in $h$ and we get a hamiltonian path of $G(r+1)$.

Now, let $u_i$ be an end vertex of $h_i$ (and hence by (2) also that of $h$). If $(u,u_i) \in E$, we extend $h$ so as to include $u$.

If neither of the above two cases is possible, then $u$ is adjacent to vertices in $\sigma$ having opposite parity with $u_{i,j}$. But, $h_i$ has one such vertex as its end vertex. Therefore, $u$ is adjacent to this end vertex of $h_i$. By (3) this vertex is adjacent to a vertex of $\sigma$ in $h$. If $u$ is adjacent to this vertex of $\sigma$ also, then we are through. Otherwise, $u$ is adjacent to vertices of opposite parity in $\sigma$. As $h_2$ has one such end vertex, $u$ is adjacent to it. Thus we proceed; we may find an $i$, $1 \leq i \leq \lambda - 1$ such that $(u,u_{i,k}) \in E$ and $u$ is also adjacent to the consecutive vertices $u_{i+1,t}$ and $u_{i+1,t+2}$ appearing consecutively in $h_{i+1}$.

But $(u_{i+1},u_{i+1,t+2}) \in E$ or $(u_{i+1},u_{i+1,t+2}) \in E$ in $h$ and we accommodate $u$ between either $u_{i+1}$ and $u_{i+1,t}$, or $u_{i+1}$ and $u_{i+1,t+2}$ as the case may be. If there is no such $i$, then finally we get that $u$ is adjacent to the end vertices of $h_{i+1}$ and $u$ is not adjacent to the vertices appearing consecutively within $h_i$. Then $u$ must be adjacent to the end vertices of $h_i$ contrary to the assumption. This proves the theorem.
Corollary 3.3. (Clapham [2]). Every s.c. graph has a hamiltonian path.

Lastly, we remark that, the results of this paper are not direct consequences of the sufficient condition given by Chvátal in terms of degree sequences and hence the special proof technique of Clapham is needed.

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