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PATHS IN r-PARTITE SELF-COMPLEMENTARY GRAPHS

T. GANGOPADHYAY and S.P. RAO HEBBARE

Stat.-Math. Division, Indian Statistical Institute, Calcutta, India

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This paper aims at finding best possible paths in r-partite self-complementary (r-p.s.c.) graphs $G(r)$. It is shown that, every connected bi-p.s.c. graphs $G(2)$ of order p, with a bi-partite complementing permutation (bi-p.c.p) σ having mixed cycles, has a (p-3)-path and this result is best possible. Further, if the graph induced on each cycle of bi-p.c.p. of $G(2)$ is connected then $G(2)$ has a hamiltonian path. Lastly the fact that every r-p.s.c. graph with an r-partite complementing permutation (r-p.c.p.) σ which permutes the partitions and for which each cycle of σ has non-empty intersection with at least four partitions of $G(r)$, has a hamiltonian path, is established. The graph obtained from $G(r)$ by adding a vertex u constituting $(r+1)$-st partition of $G(r)$, which is the fixed point of $\sigma^* = (u)\sigma$ also has a hamiltonian path. The last two results generalize the result that every self-complementary graph has a hamiltonian path.

Introduction

The word "graph" will mean a finite, undirected graph without loops and multiple edges. For the notation and terminology not defined here we refer to Harary [4].

An "r-partite graph" $G(r)$ is a graph whose vertex set $V = V(G(r))$ can be partitioned into $r \geq 1$ non-empty subsets, also called partitions, so that no edge has both ends in any one subset. Let $A_1, \ldots, A_r$ constitute an r-partition of V with $|A_i| = n_i$, $n_i \geq 1$ $(i = 1, \ldots, r)$.

An r-partite graph $G(r)$ is said to be "complete r-partite" if each vertex is joined to every other vertex that is not in the same subset. Such a graph is denoted by $K_{n_1,\ldots,n_r}$. Clearly, $K_{n_1,\ldots,n_r}$ has $\sum_{i=1}^{r} n_i$ vertices and $\sum_{i>j=1}^{r} n_i n_j$ edges.

Bipartition of a connected graph, if exists, is unique. But, in general, r-partition of a graph need not be unique. Henceforth, if $G(r)$ is given to be an r-partite graph, we assume that an r-partition of $G(r)$ is prescribed.

The "r-partite complement" $\bar{G}(r)$ of an r-partite graph $G(r)$ is again an r-partite graph with vertex set $V(G(r))$, satisfying the following conditions:

(i) for $u, v \in A_i$, $1 \leq i \leq r : (u, v) \notin E(\bar{G}(r))$,
(ii) for $u \in A_i$, $v \in A_j$, $1 \leq i \neq j \leq r : (u, v) \in E(\bar{G}(r))$ iff $(u, v) \notin E(G(r))$.

An r-partite graph $G(r)$ is said to be "r-partite self-complementary" (r-p.s.c.) if there exists an r-partition of $V(G(r))$ with respect to which $G(r)$ and $\bar{G}(r)$ are isomorphic.

The concepts r-partite complement and r-p.s.c. graph are first defined and studied in Hebbare [5].

Remark. The class of classical self-complementary (s.c.) graphs, first studied by Ringel [7], and Sachs [8], is included in the class of r-p.s.c. graphs, with $r \geq 1$ and $n_1 = \cdots = n_r = 1$. We refer to a survey article by Bhaskara Rao [1] and the references given in there, for most of the existing literature on s.c. graphs.

Let $G(r)$ be an r-p.s.c. graph with the vertex set $V(G(r)) = \{1, 2, \ldots, p\}$. Then the isomorphism between $G(r)$ and $\bar{G}(r)$ can be represented as a permutation σ in the set $V(G(r))$. We then write, $\sigma(G(r)) = \bar{G}(r)$, and call σ an "r-partite complementing permutation" (r-p.c.p) for $G(r)$. We assume that, all permutations are expressed as the product of disjoint cycles. Further, we do not distinguish the symbols of the permutation and vertices of the graph. Now, let $\sigma = \sigma_1 \cdots \sigma_\lambda$ be the disjoint cycle representation of σ. A cycle, σ_i $(i = 1, \ldots, \lambda)$ of σ is said to be "pure" if $\sigma_i \subseteq A_j$, for some $j \in \{1, 2, \ldots, r\}$ and "mixed" otherwise. In other words, a mixed cycle of σ contains vertices from at least two partitions of $G(r)$. Let $\mathscr{C}(G(r))$, $\mathscr{C}_{\rm p}(G(r))$, and $\mathscr{C}_{\rm m}(G(r))$ denote the set of all r-p.c.p., r-p.c.p. each of whose cycles is pure, and r-p.c.p. each of whose cycles is mixed, of $G(r)$. We simply write $\mathscr{C}$, $\mathscr{C}_{\rm p}$ and $\mathscr{C}_{\rm m}$ for the above sets when there is no confusion. We list here some observations and theorems from Gangopadhyay and Hebbare [3] which will be useful in what follows.

Observation 1. Let $G(r)$ be an r-p.s.c. graph and $\sigma \in \mathscr{C}$. Then for any two vertices u and v belonging to different partitions of $G(r)$, $(u, v) \in E$ if and only if $(\sigma(u), \sigma(v)) \notin E$ where (u, v) denotes an edge of $G(r)$.

Observation 2. For an r-p.s.c. graph $G(r)$, $\sum_{i>j=1}^{r} n_i n_j$ must be even. In particular, when $r = 2$, n_1 or n_2 must be even and when $r = 3$, at least two of n_1, n_2 and n_3 must be even.

Observation 3. Let $\{\sigma_{i_1}, \ldots, \sigma_{i_\mu}\}$ be a subset of the set of cycles of σ where $1 \leq \mu \leq \lambda$ such that the union of $\sigma_{i_1}, \ldots, \sigma_{i_\mu}$ has non-empty intersection with k partitions $(1 \leq k \leq r)$ of $G(r)$ and with no other. Then the graph induced on the vertices of $\sigma_{i_1}, \ldots, \sigma_{i_\mu}$ is a k-p.s.c. graph with a k-p.c.p. being $\sigma^* = \sigma_{i_1} \cdots \sigma_{i_\mu}$.

An r-p.c.p. σ of an r-p.s.c. graph $G(r)$ is said to be periodic if σ maps each A_i into some A_j. The class of all periodic r-p.c.p.'s of $G(r)$ is denoted by $\mathscr{C}^*(G(r))$.

Theorem 1.1. *Let $G(r)$ be an r-p.s.c. graph and let $\sigma \in \mathscr{C}^*$. Then $\sigma^2 \in \operatorname{Aut} G(r)$, where $\operatorname{Aut} G(r)$ denotes the group of all automorphisms of $G(r)$.*

In particular, if σ is a bi-p.c.p. of a connected bi-p.s.c. graph $G(2)$, then $\sigma^2 \in \operatorname{Aut} G(2)$, and if σ is a p-p.c.p. of a p-p.s.c. (i.e., s.c.) graph $G(p)$, then $\sigma^2 \in \operatorname{Aut} G(p)$.

Let $\sigma \in \mathscr{C}$ and $\sigma = \sigma_1 \cdots \sigma_\lambda$. A mixed cycle σ_i of σ, with $|\sigma_i| = k\alpha$, is said to be a "(k, α)-cycle" if σ_i has exactly $\alpha \geqslant 1$ vertices from each of the $k \geqslant 2$ partitions, say, $A_1, \ldots, A_k$ of $G(r)$ in the following order:

$$\sigma_i = (u_{11} \cdots u_{k1} u_{12} \cdots u_{k2} \cdots u_{1\alpha} \cdots u_{k\alpha})$$

where

$$u_{lm} \in A_l \ (l = 1, \ldots, k;\ m = 1, \ldots, \alpha).$$

Theorem 1.2. *Let $G(r)$ be an r-p.s.c. graph and let $\sigma \in \mathscr{C}^*$. Let σ_i be a mixed cycle of σ having non-empty intersection with k of the partitions of $G(r)$ and with no other. Then $|\sigma_i| = k\alpha$, for some $k \geqslant 2$, $\alpha \geqslant 1$ and σ_i is a (k, α)-cycle. Further,* $k \equiv 0 \pmod 4$ when α is odd.

Theorem 1.3. *Let $G(r)$ be an r-p.s.c. graph and let $\sigma \in \mathscr{C}^*$. Let σ_1 be a (k, α_1)-cycle of σ having non-empty intersection with $A_1, \ldots, A_k$ in the same order. Then the following hold:*

(a) *Any other cycle σ_2 of σ having non-empty intersection with any of the partitions $A_1, \ldots, A_k$ is again a (k, α_2)-cycle, for some $\alpha_2 \geqslant 1$ and $\sigma_2 \subset \bigcup_{i=1}^{k} A_i$.*

(b) *The order of the partitions of σ_2 is same as that of σ_1 upto a cyclic permutation.*

As a consequence of Theorems 1.2 and 1.3 it follows that cycles of any connected bi-p.c.p. of a bi-p.s.c. graph are either all pure or all mixed.

Theorem (Rédei [6]). *Let C be a set of n elements with a relation $<$ such that, for all a and b $(a \neq b)$ in C, either $a < b$ or $b < a$. Then the elements of C may be arranged in a sequence $a_1 < a_2 < \cdots < a_n$.*

Note that, Rédei's theorem is equivalent to saying that every finite tournament has a hamiltonian path.

S.c. graphs by their very nature enjoy nice properties such as that every s.c. graph G has a hamiltonian path, a fact proved by Clapham [2]; if $p \geqslant 8$, for every integer l, $3 \leqslant l \leqslant p-2$, G has an l-cycle and furthermore, if G is hamiltonian then G is pancyclic. Hence, the class of s.c. graphs can be classified into three classes according as the circumference being p-2, p-1 and p. Further, each of the above three classes of s.c. graphs is characterized in terms of degree sequences, in particular, the case p characterizes the class of hamiltonian s.c. graphs. All these facts are proved by Bhaskara Rao and we refer to [1] for the relevant references.

The class of r-p.s.c. graphs is a natural generalization of s.c. graphs in the class of simple (without loops and multiple edges) graphs. In particular, we have the feeling that most of the results in s.c. graphs may be generalized or extended to r-p.s.c. graphs, especially to r-p.s.c. graphs with an r-p.c.p. $\in \mathscr{C}^*$ consisting of only (k, α)-cycles.

Structural properties of r-p.c.p. of r-p.s.c. graphs are considered in Gangopadhyay and Hebbare [3] wherein, besides the results stated above, a generalization of Ringel and Sachs' Theorem (See [1]) for s.c. graphs to r-p.s.c. graphs is given.

This paper aims at determining the maximum length of a path that exists in any r-p.s.c. graph. It is shown that, every connected bi-p.s.c. graph G with $\mathscr{C}_m \neq \emptyset$ has a (p-3)-path, (i.e. a path consisting of exactly p-3 edges and p-2 vertices) and that this result is best possible and that G has a hamiltonian path if the graph induced on each cycle of a $\sigma \in \mathscr{C}_m$ is connected. Lastly, the fact that for $r \geqslant 4$, every r-p.s.c. graph $G(r)$ with a $\sigma \in \mathscr{C}^*$ such that each cycle σ_i of σ has non-empty intersection with at least 4 partitions of $G(r)$, has a hamiltonian path, is also established.

The graph obtained from $G(r)$ by adding a vertex u constituting $(r+1)$-st partition of $G(r)$, which is the fixed point of $\sigma^* = (u)\sigma$, also has a hamiltonian path. The last two results generalize the result of Clapham [2].

The proof technique employed in proving the results in this paper is essentially similar to the proof technique in Clapham [2].

2. Paths in bi-p.s.c. graphs

Theorem 2.1. *Every connected bi-p.s.c. graph $G(2)$ of order p with $\mathscr{C}_m \neq \emptyset$ has a (p-3)-path; this statement is best possible.*

Proof. Let $\sigma \in \mathscr{C}_m$ and $\sigma = \sigma_1 \cdots \sigma_\lambda$ be its disjoint cycle form. We then consider two cases according as (i) $\lambda = 1$, and (ii) $\lambda > 1$.

Case 1. $\lambda = 1$. Let $\sigma = (1\ 2 \ldots n)$ where $n = 4t$, $t \geqslant 1$ ($n \equiv 0 \pmod 4$) by Theorem 1.2). Without loss of generality, we can assume that $(1, 2) \in E$ (for otherwise, $(2, 3) \in E$ and we can consider $\sigma = (2\ 3 \cdots n\ 1)$). Since $\sigma^2 \in \operatorname{Aut} G(2)$, we get that $(i, i+1) \in E$ for all i odd.

If $t = 1$, then G consists of two copies of K_2.

Suppose that $t > 1$. Then two cases arise according as $(1, 4) \in E$ or not.

If $(1, 4) \in E$, then $(i, i+3) \in E$ for all odd i. In this case,

$$1, 4, 3, 6, \ldots, 4t, 4t\text{-}1, 2, 1$$

is a hamiltonian cycle.

If $(1, 4) \notin E$, then $(2, 5) \in E$ and hence $(i, i+3) \in E$ for all i even. In this case, $G(2)$ has two disjoint $2t$-cycles as follows:

$$C^1_{2t}: 1, 2, 5, 6, \ldots, 4t\text{-}7, 4t\text{-}6, 4t\text{-}3, 4t\text{-}2, 1$$

and

$$C^2_{2t}: 3, 4, 7, 8, \ldots, 4t\text{-}5, 4t\text{-}4, 4t\text{-}1, 4t, 3.$$

Remark. The cycle $C^1_{2t}(C^2_{2t})$ has the vertex labels $\equiv 1$ or $2 \pmod 4$ ($\equiv 0$ or $3 \pmod 4$) and they appear alternatingly.

Since $G(2)$ is connected there must exist an edge from some vertex of C^1_{2t} to some vertex of C^2_{2t}. Then G has a hamiltonian path.

Case 2. $\lambda > 1$. Let

$$\sigma_i = (u_{i1}u_{i2} \cdots u_{i,4t_i}), \quad (i = 1, \ldots, \lambda)$$

where $|\sigma_i| = 4t_i$, $(i = 1, \ldots, \lambda)$. Then each σ_i $(i = 1, \ldots, \lambda)$ is one of the following three types:

(1) $\langle\sigma_i\rangle$ is hamiltonian.
(2) $\langle\sigma_i\rangle$ has two disjoint $2t_i$-cycles.
(3) $\langle\sigma_i\rangle \cong 2K_2$.

Let σ have λ_i cycles of type i $(i = 1, 2, 3)$ and accordingly arrange the cycles of σ such that the first λ_1 cycles are of type 1, the next λ_2 cycles are of type 2 and the last λ_3 cycles are of type 3, as follows:

$$\sigma = \underbrace{\alpha_1 \cdots \alpha_{\lambda_1}}_{\textit{Type } 1}\underbrace{\beta_1 \cdots \beta_{\lambda_2}}_{\textit{Type } 2}\underbrace{\gamma_1 \cdots \gamma_{\lambda_3}}_{\textit{Type } 3}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

We now define an ordering between two cycles of certain type. First, for any α_i, α_j cycles of type 1, we write $\alpha_i < \alpha_j$ if there is an edge from some even vertex of α_i to some odd vertex of α_j where $i \neq j$, $1 \leq i, j \leq \lambda_1$, and $u_{ij} \in \sigma_i$, $(i = 1, \ldots, \lambda)$ is said to be odd or even according as j is odd or even. Notice that, if $\alpha_i < \alpha_j$ then every even vertex of α_i is adjacent to some odd vertex of α_j, and every odd vertex of α_j is adjacent to some even vertex of α_i.

Now, if $\alpha_i \not< \alpha_j$, then $(u_{i2}, u_{j3}) \notin E$, which implies that $(u_{i1}, u_{j2}) \in E$ and hence $\alpha_j < \alpha_i$. Thus for any two cycles α_i, α_j of type 1 either $\alpha_i < \alpha_j$ or $\alpha_j < \alpha_i$ holds.

Observation 4. (i) $\alpha_i < \alpha_j$ and $\alpha_j < \alpha_i$ may both hold.

(ii) $\alpha_i < \alpha_j$ and $\alpha_j < \alpha_k$ do not imply $\alpha_i < \alpha_k$, where α_i, α_j, α_k are cycles of type 1.

By Rédei's theorem the cycles of type 1 may be arranged, after suitable relabelling, as follows:

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{\lambda_1}.$$

Consider now β_i, β_j, cycles of type 2, where $i \neq j$, $1 \leq i, j \leq \lambda_2$. Recall that each such cycle β_i of length $|\beta_i| = 4s_i$ $(i = 1, \ldots, \lambda_2)$ induces a subgraph $\langle\beta_i\rangle$ which contains two disjoint $2s_i$-cycles as follows:

$$C^1_{2s_i}: v_{i1}, v_{i2}, v_{i5}, v_{i6}, \ldots, v_{i,4s_i-6}, v_{i,4s_i-3}, v_{i,4s_i-2}, v_{i1}$$

and

$$C^2_{2s_i}: v_{i3}, v_{i4}, v_{i7}, v_{i8}, \ldots, v_{i,4s_i-4}, v_{i,4s_i-1}, v_{i,4s_i}, v_{i3}.$$

We shall write $\beta_i > \beta_j$ if there is an edge from some odd vertex of $C^1_{2s_i}$ to some even vertex of $C^1_{2s_j}$, and in this case we write that $C^1_{2s_i} > C^1_{2s_j}$. Also, it can be easily seen that, $C^2_{2s_i} > C^2_{2s_j}$.

Now, if $\beta_i \not> \beta_j$, that is, if $C^1_{2s_i} \not> C^1_{2s_j}$ then $(v_{i1}, v_{j2}) \notin E$ and hence $(v_{i2}, v_{j3}) \in E$ which implies that $C^2_{2s_i} > C^1_{2s_j}$. Thus interchanging the roles of $C^1_{2s_j}$ and $C^2_{2s_j}$ we get $\beta_j > \beta_i$. Thus for any two cycles β_i and β_j of type 2 either $\beta_i > \beta_j$ or $\beta_j > \beta_i$ holds. (Observation 4 is true for β_i's $(i = 1, \ldots, \lambda_2)$.) Hence, by Rédei's Theorem the cycles of type 2 may be arranged after suitable relabelling as follows: $\beta_1 > \beta_2 > \cdots > \beta_{\lambda_2}$.

Lastly, let γ_i and γ_j be cycles of type 3, where $i \neq j$, $1 \leq i, j \leq \lambda_3$, each $\langle \gamma_i \rangle$ consists of two copies of K_2, say, $K^1_{2,i} = (w_{i1}, w_{i2})$ and $K^2_{2,i} = (w_{i3}, w_{i4})$, $(i = 1, \ldots, \lambda_3)$. We write $\gamma_i > \gamma_j$ if $K^1_{2,i} > K^1_{2,j}$, that is, when $(w_{i1}, w_{j2}) \in E$. This implies that $K^2_{2,i} > K^2_{2,j}$ since $(w_{i3}, w_{j4}) \in E$. If $\gamma_i \not> \gamma_j$ then $(w_{i1}, w_{j2}) \notin E$, that is $(w_{i2}, w_{j3}) \in E$ and hence $K^2_{2,j} > K^1_{2,i}$. In this case, by interchanging the roles of $K^1_{2,j}$ and $K^2_{2,j}$ we obtain that $\gamma_j > \gamma_i$. Thus for any two cycles γ_i and γ_j of type 3 either $\gamma_i > \gamma_j$ or $\gamma_j > \gamma_i$ holds. (Notice that Observation 4 is true for γ_i's $(i = 1, \ldots, \lambda_3)$.)

Hence by Rédei's Theorem cycles of type 3 may be arranged after suitable relabelling as follows:

$$\gamma_1 > \gamma_2 > \cdots > \gamma_{\lambda_3}.$$

Now, let β_i and γ_j be cycles of type 2 and 3 respectively. We write $\beta_i > \gamma_j$ if $C^1_{2s_i} > K^1_{2,j}$, that is, there is an edge from some odd vertex of $C^1_{2s_i}$ to w_{j2}. Also it follows that $C^2_{2s_i} > K^2_{2,j}$. Analogously, $\gamma_j > \beta_i$ means that there is an edge from w_{j1} to some even vertex of $C^1_{2s_i}$. If $\beta_i \not> \gamma_j$, then $(v_{i1}, w_{j2}) \notin E$ that is $(v_{i2}, w_{j3}) \in E$ and hence $K^2_{2,j} > C^1_{2s_i}$. Now, by interchanging the roles of $K^1_{2,j}$ and $K^2_{2,j}$ we get that $\gamma_j > \beta_i$. Thus, for any cycles β_i of type 2 and γ_j of type 3 either $\beta_i > \gamma_j$ or $\gamma_j > \beta_i$ holds. Hence, by Rédei's Theorem the cycles of type 2 and 3 may be arranged after suitable relabelling as follows:

$$\delta_{\lambda_1+1} > \delta_{\lambda_1+2} > \cdots > \delta_{\lambda},$$

where each $\langle \delta_i \rangle$ is spanned by two cycles or copies of K_2, say, δ^1_i and δ^2_i. Two cases arise according as (a) $\lambda_1 > 0$, $\lambda_2 + \lambda_3 \geq 0$, and (b) $\lambda_1 = 0$, $\lambda_2 + \lambda_3 > 0$.

Case (a) $\lambda_1 > 0$, $\lambda_2 + \lambda_3 \geq 0$. Choose $B = \alpha_1$, $C = \alpha_{\lambda_1}$ and, if $\lambda_2 + \lambda_3 > 0$, $A = \delta^1_{\lambda}$, and $D = \delta^2_{\lambda_1+1}$.

We write $A > B$ $(C > D)$ if every odd vertex of $A(C)$ is adjacent to some even vertex of $B(D)$. Note that if $A \not> B$, then every even vertex of A is adjacent to some odd vertex of B and if $C \not> D$, then given any even vertex v of C either v or $\sigma^2(v)$ is adjacent to some odd vertex of D. We also observe that if u is an odd vertex of C and u is adjacent to v, w in the hamiltonian cycle in C, then either $v = \sigma^2(w)$ or $w = \sigma^2(v)$. Thus if $C \not> D$, then there is a hamiltonian path in C which starts at any given odd vertex of C and ends at an even vertex of C which is adjacent to some odd vertex of D. We now consider the following four cases. In each of these cases we shall specify a $(p$-$3)$-path in $G(2)$.

Case (a.i). $A > B, C > D$. $A > B$ implies that each odd vertex of A is adjacent to some even vertex of B. Since, $C > D$ the same thing holds between them. The (p-3)-path is as follows (see also Fig. 1):

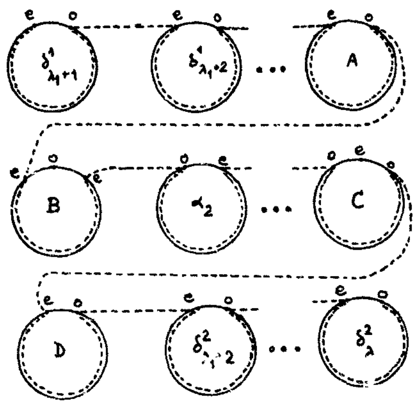

Fig. 1.

Start with any even vertex in $\delta^1_{\lambda_1+1}$, cover all the vertices of it traversing along the cycle (or K_2) ending up in an odd vertex. This odd vertex leads to an even vertex of $\delta^1_{\lambda_1+2}$ and cover all the vertices of it from this even vertex. Proceed until an even vertex of δ^1_{λ} is reached and cover all its vertices, the end vertex being odd. Since $A > B$, this odd vertex leads to some even vertex of B. Cover all the vertices of B from this even vertex, except the last odd vertex, the end vertex being even. This even vertex leads to an odd vertex of α_2, from which cover all its vertices, the end vertex being even. Thus proceed until an odd vertex of α_{λ_1} is reached, from which cover all its vertices except the last even vertex. The last odd vertex leads to an even vertex of D from which cover all its vertices, the last vertex being odd. This in turn leads to an even vertex of $\delta^2_{\lambda_1+2}$ from which cover all its vertices ending in an odd vertex. In this way all the vertices of $\delta^2_{\lambda_1+3,\ldots,}$ δ^2_{λ} can be covered.

The path described above is a (p-3)-path which misses exactly two vertices, one from each of α_1 and α_{λ_1}.

In all the other three cases we shall describe the (p-3)-path through figures.

Case (a.ii). $A > B$, $C \not> D$. The (p-3)-path in this case is as shown in Fig. 2, which misses exactly two vertices, one from each of $\delta^2_{\lambda_1+1}$ and α_1.

Note that, if the vertex missed by the path is in a part of cycle of type 3 then simply we cover one vertex of the corresponding K_2 and go to the succeeding cycle. We assume the same in what follows, whenever such a situation arises.

Case (a.iii). $A \not> B$, $C > D$. The (p-3)-path in this case misses exactly two vertices of $G(2)$, one from each of δ^1_{λ} and α_{λ_1} and is as shown in Fig. 3.

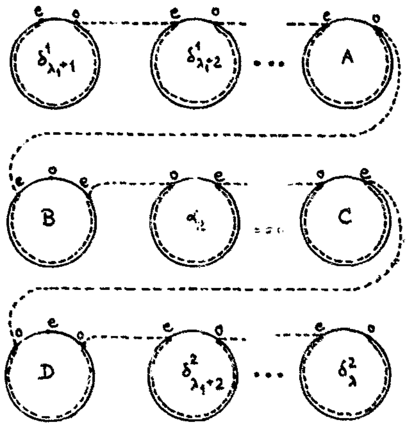

Fig. 2.

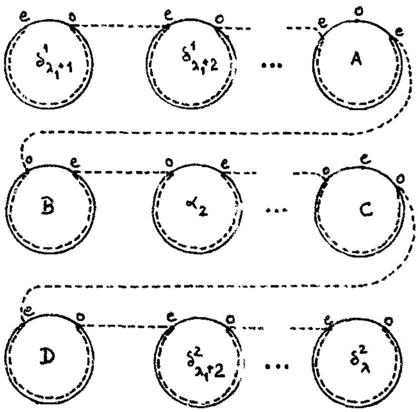

Fig. 3.

Case (a.iv). $A \not> B$, $C \not> D$, or $\lambda_2+\lambda_3=0$. If $\lambda_2+\lambda_3>0$, the (p-3)-path misses exactly one vertex from each of δ^1_λ and $\delta^2_{\lambda_1+1}$, and is as shown in Fig. 4.

If $\lambda_2+\lambda_3=0$, that is if all cycles of σ are of type 1, then, clearly, G has a hamiltonian path, see Fig. 4 (only the type 1 part).

Case (b). $\lambda_1=0$, $\lambda_2+\lambda_3>0$. In this case, the cycles δ_i $(i=1,\ldots,\lambda)$ of σ may be arranged as follows:

$$\delta_1>\delta_2>\cdots>\delta_\lambda$$

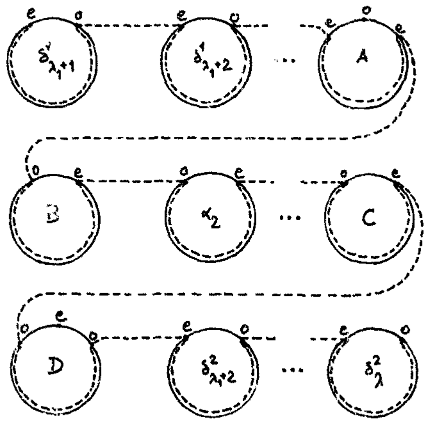

Fig. 4.

where $\lambda = \lambda_2 + \lambda_3$ and each $\langle \delta_k \rangle$ is spanned by two cycles or copies of K_2, viz. δ_k^1, δ_k^2 ($k = 1, \ldots, \lambda$). We now consider the following three cases:

Case (b. i). An odd vertex of δ_λ^1 is adjacent to some even vertex of δ_1^2. Then every odd vertex of δ_λ^1 is adjacent to some even vertex of δ_1^2. In this case we have a hamiltonian path as exhibited in Fig. 5.

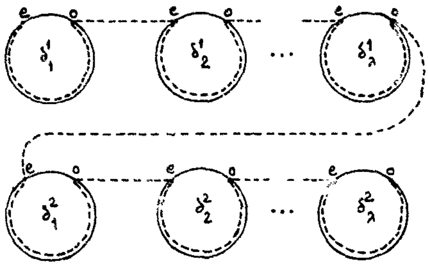

Fig. 5.

Case (b.ii). An odd vertex of δ_λ^1 is adjacent to some even vertex of δ_1^1. Then every even vertex of δ_1^1 is adjacent to some odd vertex of δ_λ^1 and every odd vertex of δ_λ^2 is adjacent to some even vertex of δ_1^2. Let $V_m = \bigcup_{k=1}^{\lambda} \delta_k^m$ ($m = 1, 2$). Since G is connected, there is a vertex $u_m \in V_m$ ($m = 1, 2$) such that $(u_1, u_2) \in E$. Let $u_1 \in \delta_i^1$ and $u_2 \in \delta_j^2$ for some ($i, j = 1, \ldots, \lambda$). Without loss of generality, we can take u_1 to be odd and u_2 to be even. Otherwise, we can interchange the roles of δ_k^1 and δ_k^2 for each ($k = 1, \ldots, \lambda$). We now construct a hamiltonian path, as exhibited in Fig. 6.

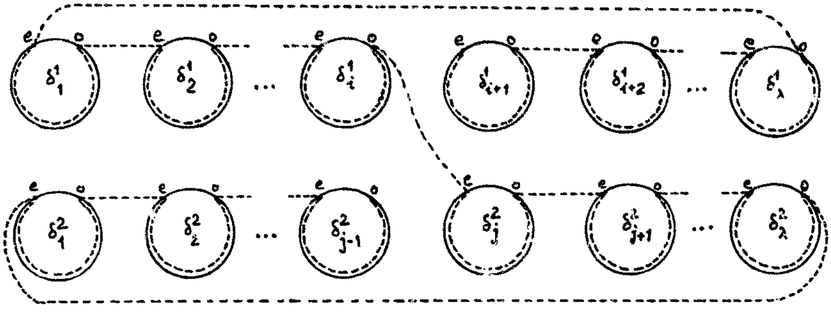

Fig. 6.

Case (b.iii). Every even vertex of δ^1_λ is adjacent to some odd vertex of δ^2_1. In this case we exhibit a (p-3)-path, which misses exactly one vertex each in δ^1_λ and δ^2_1, as exhibited in Fig. 7.

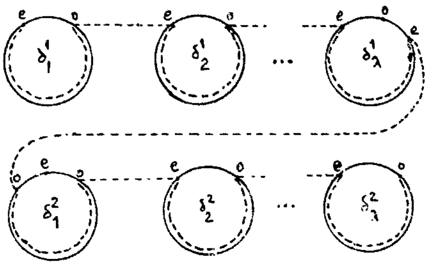

Fig. 7.

This completes the proof of the first part of the theorem.

In order to show that the result is best possible, we exhibit an infinite class of bi-p.s.c. graphs having a (p-3)-path and no (p-2)-path. For this, by Observation 2, in connection with the hypothesis $\mathscr{C}_m \neq \emptyset$, it is enough to construct such examples for the order $p = 4t$ where $n_1 = 2t = n_2$.

Let $H = H_1$ be the graph shown on Fig. 8(a). Define, H_i (See H_2 of Fig. 8(b)) such that $V(H_i) = V(H_{i-1}) \cup \{u_i, v_i, w_i, x_i\}$ and that H_i contains H_{i-1} as an induced graph on $V(H_{i-1})$ with the additional edges as follows:

$$(u_i, b), (v_i, b) \quad \text{for all } b \in B_{i-1}.$$

$$(u_i, w_i) \quad \text{and} \quad (v_i, x_i),$$

where (A_{i-1}, B_{i-1}) is the bipartition of $V(H_{i-1})$. Then, H_i is of order $p = 4i + 4$ and is a bi-p.s.c. graph, for each $i \geq 1$ and the following permutation σ_i is a

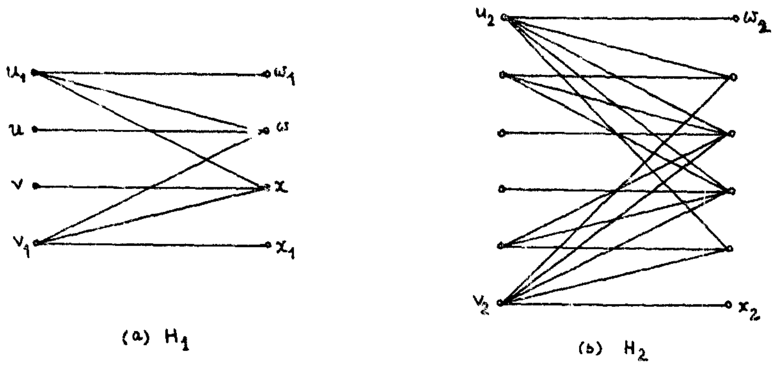

Fig. 8.

bi-p.c.p. of H_i:

$$\sigma_i : (u_i\, x_i\, v_i\, w_i) \cdots (u_1\, x_1\, v_1\, w_1)(u\, x\, v\, w).$$

A $(p\text{-}3)$-path of H_i is as given below:

$$v,\ x,\ v_1,\ x_1,\ v_2,\ x_2,\ v_3,\ \ldots,\ v_{i-1},\ x_{i-1},\ v_i,\ w,\ u_1,$$
$$w_1,\ u_2,\ w_2,\ u_3,\ w_3,\ \ldots,\ u_{i-1},\ w_{i-1},\ u_i,\ w_i.$$

Notice that, the vertices x_i and u are missing in the above path.

Finally, since H_i contains 4 vertices of valency 1 (namely, u, v, x_i, w_i) there cannot be a $(p\text{-}2)$-path. This completes the proof of the theorem.

Theorem 2.2. *Let $G(2)$ be a bi-p.s.c. graph with bi-p.c.p. $\sigma \in \mathscr{C}_m \neq \emptyset$ such that the graph induced on each cycle of σ is connected. Then $G(2)$ has a hamiltonian path.*

Proof. Let $\sigma = \sigma_1 \cdots \sigma_\lambda \in \mathscr{C}_m$. $\langle \sigma_i \rangle$ is connected implies that G is connected and that $|\sigma_i| \geq 8$, and $|\sigma_i| \equiv 0 \pmod 4$ for each $(i = 1, \ldots, \lambda)$. Let $|\sigma_i| = 4t_i$, $(i = 1, \ldots, \lambda)$. Further, let

$$\sigma_i = (u_{i1}, u_{i2}, \ldots, u_{i,4t_i}).$$

Without loss to generality, we can assume that $(u_{i1}, u_{i2}) \in E$. Since $\sigma^2 \in \operatorname{Aut} G(2)$, $(u_{ij}, u_{i,j+1}) \in E$ for all odd j. Suppose now that, $(u_{i1}, u_{i4}) \in E$. Then $(u_{ij}, u_{i,j+3}) \in E$ for all odd j. In this case, we have the following hamiltonian cycle:

$$u_{i1},\ u_{i4},\ u_{i3},\ u_{i6},\ u_{i5},\ u_{i8},\ u_{i7},\ \ldots,\ u_{i,4t_i-3} u_{i,4t_i},\ u_{i,4t_i-1},\ u_{i2},\ u_{i1}.$$

Such a cycle σ_i is called of type 1.

In the other case, that is if $(u_{i1}, u_{i4}) \notin E$, then $(u_{i2}, u_{i5}) \in E$ and hence

$(u_{ij}, u_{i,j+3}) \in E$ for all even j. In this case, we have two disjoint $2t_i$-cycles as follows:

$$C^1_{2t_i}: u_{i1}, u_{i2}, u_{i5}, u_{i6}, u_{i9}, u_{i,10}, \ldots, u_{i,4t_i-3}, u_{i,4t_i-2}, u_{i1}$$

$$C^2_{2t_i}: u_{i3}, u_{i4}, u_{i7}, u_{i8}, u_{i,11}, u_{i,12} \ldots, u_{i,4t_i-1}, u_{i,4t_i}, u_{i3}.$$

Such a cycle t_i is called of type 2. In this case, since $\langle\sigma_i\rangle$ is connected, there is $u_{ij} \in \langle C^1_{2t_i}\rangle$ and $u_{ik} \in \langle C^2_{2t_i}\rangle$ such that $(u_{ij}, u_{ik}) \in E$, where either j is odd and k is even, or, j is even and k is odd. In either case, since $\sigma^2 \in \operatorname{Aut} G(2)$, we have that for any $u_{ij} \in \langle C^1_{2t_i}\rangle$ with j odd, there is an $u_{ik} \in \langle C^2_{2t_i}\rangle$ with k even and for any $u_{ij} \in \langle C^2_{2t_i}\rangle$ with j odd, there is an $u_{ik} \in \langle C^1_{2t_i}\rangle$ with k even such that $(u_{ij}, u_{ik}) \in E$. Now, given any even vertex of $\langle C^1_{2t_i}\rangle$ (resp. $\langle C^2_{2t_i}\rangle$) there is a path along $C^1_{2t_i}$ (resp. $C^2_{2t_i}$) which covers all the vertices of $\langle C^1_{2t_i}\rangle$ (resp. $\langle C^2_{2t_i}\rangle$) and ends up in an odd vertex say u_{ij} of $\langle C^1_{2t_i}\rangle$ (resp. $\langle C^2_{2t_i}\rangle$); this odd vertex is adjacent to some even vertex say u_{iK} of $\langle C^2_{2t_i}\rangle$ (resp. $\langle C^1_{2t_i}\rangle$) and one can continue along $C^2_{2t_i}$ (resp. $C^1_{2t_i}$) in a path which ends up in an odd vertex of $\langle C^2_{2t_i}\rangle$ (resp. $\langle C^1_{2t_i}\rangle$). Thus, if σ_i is a cycle of type 2, given any even vertex u_{iK} of σ_i there is a hamiltonian path in σ_i which starts with u_{iK} and ends up in an odd vertex of σ_i. Note that the last observation also holds if σ_i is a cycle of Type 1. Thus, given any cycle σ_i, and an even vertex u_{iK} in $\langle\sigma_i\rangle$, there is a hamiltonian path in $\langle\sigma_i\rangle$ which starts from u_{iK} and ends up in an odd vertex of $\langle\sigma_i\rangle$.

Now, we order $\sigma_1, \ldots, \sigma_\lambda$ in the following manner. We shall write $\sigma_i > \sigma_j$ if an odd vertex of σ_i is adjacent to an even vertex of σ_j. Evidently, if $\sigma_i \not> \sigma_j$, then $\sigma_j > \sigma_i$ follows. Hence, by Rédei's Theorem, the cycles of σ may be ordered by suitable relabelling as follows: $\sigma_1 > \sigma_2 > \cdots > \sigma_\lambda$.

We now start with any even vertex u_{iK_1} of σ_1. We know that there is a hamiltonian path in $\langle\sigma_1\rangle$ which starts with u_{iK_1} and ends up in an odd vertex of σ_1. Since $\sigma_1 > \sigma_2$, this odd vertex is adjacent to some even vertex u_{iK_2} of σ_2. There is a hamiltonian path in $\langle\sigma_2\rangle$ which starts with u_{iK_2} and ends up in an odd vertex of σ_2. From this odd vertex we proceed to an even vertex u_{iK_3} of σ_3 and so on. This gives us a hamiltonian path.

Thus, G has a hamiltonian path with any even vertex of σ_1 as an end vertex. Similarly, reversing the procedure, we can get a hamiltonian path with any odd vertex of σ_λ as an end vertex.

3. Hamiltonian paths in r-p.s.c. graphs with $\mathscr{C}^* \neq \emptyset$

Theorem 3.1. *Let $G(r)$ be an r-p.s.c. graph, $r \geq 4$ with an r-p.c.p. $\sigma \in \mathscr{C}^* \neq \emptyset$ where each cycle of σ has non-empty intersection with at least four partitions of $G(r)$. Then $G(r)$ has a hamiltonian path.*

Proof. The proof goes on similar lines as that of Clapham [2].

Let $\sigma = \sigma_1 \cdots \sigma_\lambda \in \mathscr{C}^*$. Then by Theorems 1.2 and 1.3 we conclude that each σ_i $(i = 1, \ldots, \lambda)$ of σ is a (k_i, α_i)-cycle, where $k_i \geq 4$, by hypothesis.

To begin with, we consider two cases according as (1) $\lambda = 1$, and (2) $\lambda > 1$.

Case 1. $\lambda = 1$. For convenience, let σ be a (k, α)-cycle. $k \geq 4$ and $\sigma = (1\,2\,3 \cdots k\alpha)$, where $|\sigma| = k\alpha$, is even. We first assume that $(1, 2) \in E$.

If $k = 4$ and $\alpha = 1$, then 2, 1, 3, 4 or 1, 2, 4, 3 is the required hamiltonian path. Otherwise, since $k\alpha$ is even, it follows that $k \geq 4$ and $\alpha \geq 2$. Then $(1, 4) \in E$ if and only if $(4, 7) \notin E$. Hence, we may suppose that either (a) $(i, i+3) \in E$ for all odd i, or (b) $(j, j+3) \in E$ for all even j.

In case (a), we consider the hamiltonian paths P_1 or P_2 according as $(1, 3) \in E$ or $(2, 4) \in E$ where

$$P_1 : 2, 1, 4, 3, 6, 5, \ldots, k\alpha - 2, k\alpha - 3, k\alpha.$$

$$P_2 : 1, 4, 3, 6, 5, 8, \ldots, k\alpha - 3, k\alpha, 2, k\alpha - 1.$$

In case (b), we construct a hamiltonian path P as follows. Let P_1 be the path $1, 2, 5, 6, 9, 10, \ldots,$ the last term being $k\alpha - 2$ or $k\alpha$ according as $k\alpha \equiv 0$ or $2 \pmod 4$, and P_2 be the path $3, 4, 7, 8, 11, 12, \ldots,$ the last term being $k\alpha$ or $k\alpha - 2$ according as $k\alpha \equiv 0$ or $2 \pmod 4$. Then P is obtained by combining P_1 and P_2, using the edge $(1, 3)$ or $(k\alpha - 2, k\alpha)$ whichever exists. (Note that since $\sigma^2 \in \operatorname{Aut} G(r)$, either $(1, 3) \in E$ or $(k\alpha - 2, k\alpha) \in E$.

If $(1, 2) \notin E$, then $(2, 3) \in E$ and the proof is similar. In any case, since $\sigma^2 \in \operatorname{Aut} G(r)$, we have the following

Remark. Either (i) for any two consecutive odd vertices of σ, there is a hamiltonian path in which they appear consecutively and (ii) for any two consecutive even vertices of σ, there is a hamiltonian path for which they are end vertices,

or, (i)$'$ for any two consecutive even vertices of σ, there is a hamiltonian path in which they appear consecutively and (ii)$'$ for any two consecutive odd vertices of σ, there is a hamiltonian path for which they are end vertices.

Case 2. $\lambda > 1$. Then by the Remark made in Case 1, it follows that any cycle σ_i of σ satisfies either (i) and (ii) or (i)$'$ and (ii)$'$. A cycle σ_i of σ is said to be of type 1 if it satisfies (i) and (ii), and is of type 2 if it satisfies (i)$'$ and (ii)$'$.

We now define an ordering between any two cycles of σ as follows:

Let σ_i and σ_j be cycles of σ of type 1. Then we write $\sigma_i < \sigma_j$ if some even vertex of σ_i is adjacent to some odd vertex of σ_j. Then, it can be easily seen that if $\sigma_i \not< \sigma_j$ then $\sigma_j < \sigma_i$.

Hence, for any two cycles σ_i, σ_j of σ of type 1 either $\sigma_i < \sigma_j$, or $\sigma_j < \sigma_i$ holds.

Let σ_i and σ_j be of type 2. We write $\sigma_i < \sigma_j$ if an odd vertex of σ_i is adjacent to some even vertex of σ_j. Again, it follows, with this ordering, that either $\sigma_i < \sigma_j$, or $\sigma_j < \sigma_i$ holds.

Lastly, if σ_i and σ_j are of types 1 and 2 respectively, we write $\sigma_i < \sigma_j$ if an even vertex of σ_i is adjacent to some even vertex of σ_j, and $\sigma_j < \sigma_i$ if an odd vertex of σ_i is adjacent to some odd vertex of σ_j. Then if $\sigma_i \nless \sigma_j$ it follows that $\sigma_j < \sigma_i$. Hence, either $\sigma_i < \sigma_j$ or $\sigma_j < \sigma_i$ holds in this case also.

Thus for any two cycles σ_i and σ_j of σ either $\sigma_i < \sigma_j$, or $\sigma_j < \sigma_i$ holds. Hence by Rédei's Theorem the cycles of σ can be ordered by an appropriate relabelling of σ_i's as follows:

$$\sigma_1 < \sigma_2 < \cdots < \sigma_\lambda.$$

Now, each $\langle\sigma_i\rangle$ is a k-p.s.c. graph for some $k \geq 4$ and by Case 1 there is a hamiltonian path in each $\langle\sigma_i\rangle$ $(i = 1, \ldots, \lambda)$. For σ_1 and σ_2 we consider the following four cases according to their types:

Case 2(a). σ_1 and σ_2 are both of type 1. Then there is a hamiltonian path in $\langle\sigma_1\rangle$ with its end vertices at consecutive even vertices, say, u_{1i} and u_{1i+2} of σ_1. Since $\sigma_1 < \sigma_2$, $(u_{1i}, u_{2j}) \in E$ for some odd j and hence $(u_{1,i+2}, u_{2,j+2}) \in E$ where the second suffix of a vertex is reduced modulo the length of the cycle containing it. Now, since σ_2 is of type 1, there is a hamiltonian path in $\langle\sigma_2\rangle$ with u_{2j} and $u_{2,j+2}$, with odd j, appearing consecutively. We can now obtain a hamiltonian path in $\langle\sigma_1 \cup \sigma_2\rangle$ by inserting the hamiltonian path of $\langle\sigma_1\rangle$ between u_{2j} and $u_{2,j+2}$ in the hamiltonian path of $\langle\sigma_2\rangle$.

The remaining cases are dealt with in an analogous way:

Case 2(b). σ_1 is of type 1 and σ_2 is of type 2. In this case, we can obtain a hamiltonian path in $\langle\sigma_1 \cup \sigma_2\rangle$ by inserting the hamiltonian path of $\langle\sigma_1\rangle$ (its end vertices being consecutive even vertices u_{1i} and $u_{1,i+2}$ of σ_1) between u_{2j} and $u_{2,j+2}$ of the hamiltonian path of $\langle\sigma_2\rangle$ (this is possible since, $\sigma_1 < \sigma_2$ implies that, for some even j, (u_{1i}, u_{2j}), $(u_{1,i+2}, u_{2,j+2})$ E and u_{2j}, $u_{2,j+2}$ appear consecutively).

Case 2(c). σ_1 and σ_2 are of types 2 and 1 respectively. In this case, a hamiltonian path of $\langle\sigma_1 \cup \sigma_2\rangle$ can be obtained by inserting the hamiltonian path of $\langle\sigma_1\rangle$ (its end vertices at consecutive odd vertices u_{1i} and $u_{1,i+2}$ of σ_1) between u_{2j} and $u_{2,j+2}$, the consecutive odd vertices appearing consecutively in the hamiltonian path of $\langle\sigma_2\rangle$ with odd j.

Case 2(d). σ_1 and σ_2 are both of type 2. In this case, the hamiltonian path of $\langle\sigma_1\rangle$, with its end vertices being consecutive odd vertices u_{1i} and $u_{1,i+2}$, is inserted between u_{2j} and $u_{2,j+2}$—the consecutive even vertices in the hamiltonian path of $\langle\sigma_2\rangle$ which gives a hamiltonian path in $\langle\sigma_1 \cup \sigma_2\rangle$.

Thus it is possible to construct a hamiltonian path in $\langle\sigma_1 \cup \sigma_2\rangle$, where $\sigma_1 < \sigma_2$. Next, in a similar way this hamiltonian path can be inserted into a hamiltonian path of $\langle\sigma_3\rangle$, and so on. Theorem 3.1 is now proved.

Note that the hamiltonian path constructed in the above proof has the following properties:

(1) It has two consecutive odd (even) vertices of σ_1 appearing consecutively if σ_1 is of type 1 (type 2).

(2) Its end vertices are consecutive even (odd) vertices of σ_λ if σ_λ is of type 1 (type 2).

(3) For $\sigma_i < \sigma_{i+1}$ $(i = 1, \ldots, \lambda)$, it has some u_{ij} and $u_{i+1,k}$ appearing consecutively where j and k are as follows:

j even and k odd, if σ_i, σ_{i+1} are both of type 1,

j even and k even, if σ_i is of type 1 and σ_{i+1} is of type 2,

j odd and k even, if σ_i and σ_{i+1} are both of type 2, and

j odd and k odd, if σ_i is of type 2 and σ_{i+1} is of type 1.

Let $G(r+1)$ be an $(r+1)$-p.s.c. graph with an $(r+1)$-p.c.p. $\sigma^* = (u)\sigma \in \mathscr{C}^*$ where u is a fixed vertex, $A_{r+1} = \{u\}$ and all other cycles of σ has non-empty intersections with at least four partitions of $G(r+1)$. Then we have the following:

Theorem 3.2. *$G(r+1)$ has a hamiltonian path.*

Proof. Let $G(r)$ be the subgraph induced by σ in $G(r+1)$. Then $G(r)$ satisfies the conditions of Theorem 3.1.

Now consider the hamiltonian path h in $G(r)$ as described in Theorem 3.1. It is composed of several paths each being hamiltonian within a cycle of σ. By the properties (1), (2) and (3), consecutive vertices (say) $u_{i,j}$, $u_{i,j+2}$ of same parity appear consecutively within hamiltonian path of σ_i and consecutive vertices of opposite parity to u_{ij} and $u_{i,j+2}$ appear as end vertices within that hamiltonian path h. Further, if u is adjacent to a vertex u_{ij} of σ_i, then u is adjacent to all the vertices of σ_i with the same parity as that of u_{ij} and u is not adjacent to all other vertices of σ_i.

Now, let u_{1j}, $u_{1,j+2}$ be two vertices of same parity appearing consecutively in h_1 where h_i denotes a hamiltonian path in $\langle\sigma_i\rangle$ $(i = 1, \ldots, \lambda)$. Suppose that, u is adjacent to u_{1j}. Then $(u, u_{1,j+2}) \in E$ and hence u can be incorporated in between u_{1j} and $u_{1,j+2}$ in h and we get a hamiltonian path of $G(r+1)$.

Now, let $u_{\lambda l}$ be an end vertex of h_λ (and hence by (2) also that of h). If $(u, u_{\lambda l}) \in E$, we extend h so as to include u.

If neither of the above two cases is possible, then u is adjacent to vertices in σ_1 having opposite parity with u_{1j}. But, h_1 has one such vertex as its end vertex. Therefore, u is adjacent to this end vertex of h_1. By (3) this vertex is adjacent to a vertex of σ_2 in h. If u is adjacent to this vertex of σ_2 also, then we are through. Otherwise, u is adjacent to vertices of opposite parity in σ_2. As h_2 has one such end vertex, u is adjacent to it. Thus we proceed; we may find an i, $1 \leq i \leq \lambda - 1$ such that $(u, u_{i,s}) \in E$ and u is also adjacent to the consecutive vertices $u_{i+1,t}$ and $u_{i+1,t+2}$ appearing consecutively in h_{i+1}. But $(u_{is}, u_{i+1,t}) \in E$ or $(u_{is}, u_{i+1,t+2}) \in E$ in h and we accommodate u between either u_{is} and $u_{i-1,t}$, or u_{is} and $u_{i+1,t+2}$ as the case may be. If there is no such i, then finally we get that u is adjacent to the end vertices of $h_{\lambda-1}$ and u is not adjacent to the vertices appearing consecutively within h_λ. Then u must be adjacent to the end vertices of h_λ, contrary to the assumption. This proves the theorem.

Corollary 3.3. (Clapham [2]). *Every s.c. graph has a hamiltonian path.*

Lastly, we remark that, the results of this paper are not direct consequences of the sufficient condition given by Chvátal in terms of degree sequences and hence the special proof technique of Clapham is needed.

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