Analysis in Space-Time Bundles,
II. The Spinor and Form Bundles

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The structures of the spin and form bundles over the universal cosmos \( \mathcal{M} \), and their relations with corresponding bundles over the Minkowski space \( \mathcal{M}_0 \) canonically imbedded in \( \overline{\mathcal{M}} \), are treated. Wave equations covariant with respect to the causal group \( G \) of \( \overline{\mathcal{M}} \) are studied, their solution manifolds and other stable (essentially positive-energy) invariant subspaces of the section spaces of the bundles are determined, and the indecomposability of relevant invariant subspace chains is shown. Explicit parallelizations of the bundles are applied to the Dirac and Maxwell equations on \( \overline{\mathcal{M}} \). A basis for spinor fields that diagonalizes a complete set of \( \mathcal{K} \)-covariant quantum numbers (\( \mathcal{K} = \text{maximal essentially compact subgroup of } G \)) is developed. Local multilinear invariants of bundles over \( \overline{\mathcal{M}} \) are treated and specialized to convergent mathematical versions of the Fermi and Yukawa interaction Lagrangians that are \( G \)-invariant for the appropriate conformal weights.

INTRODUCTION

The physically fundamental cases of spinors and forms on the model for space-time that is particularly natural from the standpoints of causality, symmetry, and universality are developed here from the same general standpoint as in [1]. Invariant wave equations—those of Dirac and Maxwell—arise automatically from the bundle structure as specifications of

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the temporal subgroup action on invariant subspaces of the spaces of sections of the corresponding bundles. Explicit parallelizations are given that facilitate analysis of concrete issues regarding these spaces, which in physical terms represent spinor, electromagnetic, and other fields. Among such issues are stability (positivity of various forms of the "energy"); the analytic relations between the relevant wave equations in the universal cosmos \( \tilde{M} \), the basic space-time in this treatment, and in Minkowski space \( M_0 \), which is invariantly contained in \( \tilde{M} \); the structure of the composition series of the stable spinor bundle; the spectral resolution of a basic complete set of quantum numbers (or \( \mathbf{K} \)-covariant state vector labels).

Since the background and motivation of the study of which this is Part II were indicated in [1], it suffices here to indicate more specifically the organization of this part. With a view to later applications to the mathematical study of interactions, a general study of multilinear invariants of section spaces of bundles, representative of mathematical counterparts to some of the formal interaction Lagrangians used in physical theory, is made in the first section. The role of the dual bundle physically representative of the anti-particle to that represented by the bundle is emphasized here.

Proceeding then in order of increasing spin, the spin-1/2 bundle, i.e., spinor fields, are treated in a fashion parallel to that for scalar fields in [1]. However, a variety of complications develop as a consequence of the non-triviality of the fiber. Remarkably, the qualitative picture that emerges, which was in part summarized in [2], is quite parallel to that in the scalar case. Thus, the stable subspace has a two-step composition series, and the extension is an essential one. Unlike the most distinctive conformally invariant subspace in the scalar case, which lived on a 2-fold cover of the conformal compactification of Minkowski space, the corresponding subspace here lives on the 4-fold cover. These notable subspaces arise only for specific conformal weights (sometimes called conformal or canonical dimensions). However, other values of the weights determine fields that satisfy physically indicated stability and causality criteria, and are studied here.

Following the treatment of the spinor bundle, the form bundle is studied in the next section by a direct \( \mathbf{K} \)-invariant parallelization. Parallelized formulations of Maxwell's equations are developed both for 2-forms and 1-forms in a \( \mathbf{K} \)-invariant Lorentz gauge (i.e., in theoretical physical terms, both for electromagnetic fields and potentials). The final section illustrates and applies the earlier ones by a treatment of the conformal invariance of mathematical versions of the Fermi and Yukawa interaction Lagrangians. The integrated Lagrangians exist here, unlike the case of Minkowski space, as convergent integrals for sections that, e.g., satisfy the Dirac and/or Maxwell equations.

Being part of a systematic study, the present work is not designed to seek idiosyncratic features, but some further points of comparison between field
theory for these bundles in $\tilde{M}$ and $M_0$ are of physical and mathematical interest. Unlike the situation in $M_0$, there is a clear-cut distinction in $\tilde{M}$ between the “neutrino” representation,—which gives the conformal transformation rules for a perfectly finite (i.e., non-infinitesimal) invariant subspace of the spinor bundle,—and a natural candidate for the “electron” representation (inclusive of all masses). The former is a “small” representation in, e.g., the physically relevant sense that sections are determined by their Cauchy data at fixed times, while the latter is of normal size; the “mass” must be fixed in order for the Cauchy data to determine the section.

Moreover, both the neutrino and photon fields have non-vanishing lowest (“universal”) energies, as a consequence of which so-called infra-red singularities that are troublesome in Minkowski space are avoided. Photon fields live on the conformal compactification $\tilde{M}$, and the usual trilinear Lagrangian for the charged spinor field–photon interaction, in its conformally invariant form (for “massless” spinors), has a finite and conformally invariant space-time integral over a 4-fold cover of $M$, as a special case of the (slightly generalized) Yukawa interaction earlier cited.

In general the present results indicate considerable rigidity in the structure of elementary particle models, which in apparent fundamental forms change less than might have been expected on passage from $M_0$ to $\tilde{M}$; but that quite material, potentially empirically significant differences do exist. In addition, while there are algebraic and geometrical complications in the treatment in $\tilde{M}$ relative to that in $M_0$, analytically the fields on $\tilde{M}$ are better behaved. Thus, in addition to the earlier-indicated convergence of integrated Lagrangians for fields that vanish almost nowhere on $M_0$, scattering theory is much simplified, propagation from time $-\infty$ to time $+\infty$ being transformed effectively into propagation from universal time $-\pi$ to universal time $+\pi$; operators whose trace is in question (e.g., the exponentiated hamiltonians involved in the Planck and Fermi laws) do indeed have absolutely convergent trace (which they lack in $M_0$); all smooth everywhere-defined solutions to the Dirac and Maxwell equations are normalizable (which they are not in $M_0$), and distribution solutions are simply derivatives of normalizable ones.

This Part II of a series continues the numbering of Part I, referred to below as (I), as regards sections, theorems, etc., and the notation is consistent with that introduced in (I), q.v. for notations not developed below.

VI. Duals, Invariants, Weights

6.1. Introduction

Induction from a given representation of an isotropy subgroup to a bundle, followed by formation of the induced representation on the space of sections,
or briefly, formation of the *induced field*, does not quite commute with canonical operations on representations. Thus the dual or anti-dual of the induced field is not exactly the field induced from the dual or anti-dual of the inducing representation. Relatedly, the local invariants of the induced fields are not precisely in correspondence with the invariants of the inducing representations, but involve also the transformation group context.

Duality of fields is involved in the modelling of the physical concept of anti-particle. Multilinear invariants are commonly used in the theoretical description of interactions. These notions will be dealt with in a preliminary way in this section. Section 9 will apply the results to spinor fields and forms and treat mathematical counterparts of the Fermi and Yukawa Lagrangians.

6.2. Duals and Anti-duals

Given a topological linear space \( L \), the dual \( \hat{L} \) (resp. anti-dual \( \tilde{L} \)) is defined as the linear vector space consisting of all continuous linear (resp. anti-linear) functionals \( f \) on \( L \). ("Anti-linear" means that \( f(\lambda x) = \overline{\lambda} f(x) \) for all \( \lambda \in \mathbb{C} \) and \( x \in L \).) The dual \( \hat{R} \) (resp. anti-dual \( \tilde{R} \)) of a representation \( R \) of a group \( G \) on \( L \) is the canonical representation of \( G \) defined by the equation:

\[
(R^\dagger f)(x) = f(R(g)x) \quad \text{for } g \in G, \ x \in L, \ \text{and } f \in \hat{L} \]  

In the relevant case in which \( L \) is a Hilbert space \( H \), the dual and anti-dual representations are canonically (anti-linearly or linearly, respectively) equivalent to representations in \( H \). For applications it is important to include the case of a *quasi-Hilbert space*, by which is meant a topological linear vector space \( H \) endowed with a non-degenerate hermitian sesquilinear form \( \langle \cdot, \cdot \rangle \) (which is not necessarily positive definite), and topologized in such a manner that the only continuous linear functionals \( f \) on \( H \) are those of the form \( x \rightarrow \langle x, y \rangle \) for some \( y \) in \( H \). (Such a topology is always available. e.g., the weak topology defined by these functionals.)

For arbitrary \( x \in H \), let \( x^* \) denote the linear functional on \( H \) given by the equation \( x^*(y) = \langle y, x \rangle \). The canonical anti-linear mapping \( C_{\text{abs}}: x \rightarrow x^* \) from \( H \) onto its dual carries the representation \( \hat{R} \) into the representation \( g \rightarrow R(g^{-1})^\dagger \), where the superscript * denotes the Hilbert space adjoint (cf. below). Similarly, there exists a canonical linear equivalence between the representation \( \tilde{R} \) and the representation \( g \rightarrow R(g^{-1})^\dagger \) of \( G \) on \( H \). It will normally involve no essential loss of clarity to identify the abstract dual and anti-dual representations with their given concrete forms in \( H \).

Somewhat more generally, two continuous representations \( R \) and \( R' \) of a topological group \( G \) on respective topological linear spaces \( L \) and \( L' \) are said to be in duality (resp. anti-duality) in case there is given a continuous non-degenerate bilinear (resp. sesquilinear) form \( F \) on \( L \times L' \) such that

\[
F(R(g)x, R'(g)x') = F(x, x') \quad \text{for arbitrary } x \in L \text{ and } x' \in L'.
\]

In the case of a representation in a quasi-Hilbert space and its anti-dual representation, the
form $F(x, x') = \langle x, x' \rangle$ is such an invariant. The notation $\langle \cdot, \cdot \rangle$ may be used for general spaces in anti-duality or duality when the definition of $\langle \cdot, \cdot \rangle$ is clear from the context. If $A$ and $A'$ are continuous linear operators on $L$ and $L'$ such that $\langle Ax, y \rangle = \langle x, A'y \rangle$ for all $x$ in $L$ in and $y$ in $L'$, then $A$ and $A'$ are mutually adjoint, and $A'$ may be denoted as $A^T$ or $A^*$, consistently with normal usage, depending on whether the spaces are in duality or anti-duality.

A representation $R$ in a quasi-Hilbert space is said to be quasi-unitary in case each $R(g)$, $g \in G$, is quasi-unitary, i.e., preserves the quasi-unitary structure. Such a representation is anti-unitarily equivalent to its dual (or contragredient) representation via the mapping $C_{abs}$ and unitarily equivalent to its anti-dual via the map $x \to x^\#$.

6.3. Bundle Duality

The general context here is that of a bundle on a homogeneous space $G/H$ of a Lie group $G$ modulo a closed subgroup $H$, induced from a given representation $R$ of $H$. Because it may be appropriate in applications to induce from points of $G/H$ other than the coset $H$ itself, as exemplified in (I), it is convenient to develop the treatment in terms of the (abstractly equivalent) notions of a given transitive transformation group $G$ of a manifold $M$ together with induction from a given representation of the subgroup $G_{x_0}$ fixing an arbitrary given point $x_0$ of $M$. For any such given $R$, and arbitrary complex $A$, the equation

$$h(g) = \eta g \circ R(g)$$

defines another representation, assuming that $G_{x_0}$ is simply-connected and that the indicated power is defined so as to give a local representation near $g = e$. This representation $R_A$ induces a bundle that is said to be reweighted relative to that induced from $R$, of relative (bundle) weight $\lambda$. For reasons made clear by Theorem 6.1, the dual (resp. anti-dual) bundle to that induced from $R$ is defined as that induced from $(R)_1$ (resp. $(R)_1$).

Throughout this article the terms "parallelized" and "parallelization" refer to the specific type of parallelization treated in Section 4 of (I). "Parallelizing subgroup" will mean either the subgroup $N$ of Theorem 4.1 or the subgroup $N_0$ of Theorem 4.3. The element of a regular measure invariant under the parallelizing subgroup (and so equivalent to the left-invariant measure on this subgroup) will be denoted as $dx$. "Smooth" will mean infinitely differentiable unless otherwise indicated.

**Theorem 6.1.** Let $R$ and $R$ denote the (spaces of) smooth parallelized fields of compact support induced from the finite-dimensional representation $R$ of $G_{x_0}$ and its bundle dual, respectively.
Then the bilinear form

$$\langle \Phi, \Psi \rangle = \int_M \langle \Phi(x), \Psi(x) \rangle \, dx \quad (\Phi \in \mathbb{R}, \Psi \in \bar{\mathbb{R}})$$

is $G$-invariant.

Proof: The following somewhat more general result is the basis of the proof and later corollaries.

**Lemma 6.1.1.** Let $S(g, x)$ denote the multiplier for the $R$-bundle over $M$ in parallelized form. Then the multiplier $S_\lambda(g, x)$ for the $R_\lambda$-bundle, i.e., that induced by the reweighted representation of relative weight $\lambda$, is given by the equation

$$S_\lambda(g, x) = S(g, x) J(g^{-1}, x)^\lambda,$$

where $J(g, x)$ denotes the Jacobian $d(gx)/dx$ of $g$ at $x$. The multiplier $\hat{S}(g, x)$ for the $\hat{R}$-bundle is given by the equation

$$\hat{S}(g, x) = S(g^{-1}, g^{-1}x)^\tau.$$

**Proof of Lemma.** Consider first the case of a global parallelization given by Theorem 4.1. Then if $R_\lambda$ is as above

$$S_\lambda(g, x) = R_\lambda(g^*) = R(g^*)(\det d_{x_0}(g^*)^{-1})^\lambda$$

$$= S(g, x)(\det d_{x_0}(g^*)^{-1})^\lambda.$$ 

It suffices therefore to show that

$$\det d_{x_0}(x_0(\phi(g^{-1})x)^{-1} g^{-1}xx_0^{-1}) = d(g^{-1})/dx.$$ 

The left side of this putative equation is a product of five Jacobians, of which only the third, $d(g^{-1}x)/dx$, is not clearly identically one, the others being Jacobians of translations by elements of $N$, which preserve $dx$.

Now consider the case of a possibly local parallelization by a subgroup $N_0$ in accordance with Theorem 4.3. The analysis is similar except that in place of the left side of the last putative equation one has the expression

$$\det d_x((J^{-1}(\phi(g^{-1})x))^{-1} g^{-1}(J^{-1}x)), $$

which is a product of three Jacobians. The first and third of these are identically one since $J^{-1}(N) \subset N_0$ and $N_0$ leaves invariant the element of measure $dx$.

To show that (6.2b) is valid, note that $S(g, x)^{-1} = S(g^{-1}, g^{-1}x)$, whence $\hat{S}(g, x) = \hat{R}(g^*) = (R(g^*)^{-1})^\tau = (S(g, x)^{-1})^\tau = S(g^{-1}, g^{-1}x)^\tau$. 
Completion, proof of theorem. Let $\Phi$ and $\Psi$ be arbitrary smooth compactly-supported sections of the parallelized $R$- and $(\tilde{R})_1$-bundles over $M$, and let the actions of $G$ on these bundles be denoted as $U$ and $\tilde{U}$. Then for arbitrary $g$ in $G$,

$$\langle U(g)\Phi, \tilde{U}(g)\Psi \rangle = \int_M \langle U(g)\Phi(x), \tilde{U}(g)\Psi(x) \rangle \, dx$$

$$= \int_M \langle \mathcal{S}(g, x)\Phi(g^{-1}x), \tilde{\mathcal{S}}(g, x)\Psi(g^{-1}x) \rangle \, dx.$$  

By Lemma 6.1.1, $\tilde{\mathcal{S}}(g, x) = J(g^{-1}, x) \mathcal{S}(g^{-1}, g^{-1}x)^T$, implying that the last expression may be written as

$$\int_M \langle \mathcal{S}(g, x)\Phi(g^{-1}x), \mathcal{S}(g^{-1}, g^{-1}x)^T \Psi(g^{-1}x) \rangle J(g^{-1}, x) \, dx$$

$$= \int_M \langle \Phi(x), \Psi(x) \rangle \, dx.$$  

**Corollary 6.1.1.** With the same hypothesis and notation as in Theorem 6.1, suppose the finite-dimensional representations $R_1, ..., R_n$ of $G_{x_0}$ have the invariant

$$F(x_1, ..., x_n) = F(R_1(x_1), ..., R_n(x_n)), \quad g \in G_{x_0},$$

$F$ being multilinear (or sesquilinear) on $R_1 \times \cdots \times R_n$. Let $R'_j$ denote $R_j$ after reweighting with real relative weight $\lambda_j$ and suppose that $\lambda_1 + \cdots + \lambda_n = 1$. Let $B'_j$ denote the space of parallelized smooth sections of compact support of the bundle induced from $R'_j$. Then the multi- (or sesqui-) linear form on $B'_1 \times \cdots \times B'_n$

$$(\Phi_1, ..., \Phi_n) \to \int_M F(\Phi_1(x), ..., \Phi_n(x)) \, dx, \quad \Phi_j \in B'_j,$$

is $G$-invariant.

**Proof:** The proof is a slight variant of that for the theorem.

**Corollary 6.1.2.** With the same hypothesis and notation as Corollary 6.1.1, suppose that there are two parallelizing subgroups, $N$ and $N_0$, as treated in Theorem 4.3, and that the $N$- and $N_0$-invariant volume elements $dx$ and $d\tilde{x}$ are scaled so as to be equal at the point $e$ in Theorem 4.3. Then

$$\int_M F(\Phi_1(x), ..., \Phi_n(x)) \, dx = \int_M F(\Phi_1(x), ..., \Phi_n(x)) \, d\tilde{x}.$$

whenever $\Phi_j$ and $\tilde{\Phi}_j$ are the fields parallelized by $N$ and $N_0$, respectively, for a given abstract section $\Phi_j$ having support in the open region where the $N_0$-parallelization is defined.

**Proof.** By Theorem 4.3,

$$\tilde{\Phi}_j(x) = R'_j(x_0(J^{-1}x)^{-1}x x_0^{-1}) \Phi_j(x),$$

and

$$R'_j(g) = (\det d_x g)^{-\lambda_j} R_j(g).$$

Since $F$ is multilinear and an invariant for the representations $R_j$, and since $\lambda_1 + \cdots + \lambda_n = 1$, it suffices to show that the following relation holds at the point $x$:

$$\{\det d_x(x_0(J^{-1}x)^{-1}x x_0^{-1})\}^{-1} d\tilde{x} = dx.$$

This relation is equivalent to the relation

$$d\tilde{x} = \det d_x((J^{-1}x)^{-1}x) dx.$$

Since $dx$ and $d\tilde{x}$ agree at $e$, and are invariant under the actions of $x$ in $N$ and $J^{-1}x$ in $N_0$, this last equation is valid.

### 6.4. Conformal Weight

A related but distinct notion of weight plays an important part in the study of conformal bundles. A representation $\hat{R}$ of $\hat{\Pi}$ is said to be of (conformal) weight $W$, $w$ being a given complex number, in case $\hat{R}(S_\lambda) = \lambda^w I$ for $\lambda$ real and positive, $S_\lambda$ denoting the transformation $x \rightarrow \lambda x$ in Minkowski space, and $\hat{\Pi}$ here denoting the physical Poincaré group. A useful example is given by

**Corollary 6.1.3.** Let $\hat{R}$ denote the one-dimensional representation of $\hat{\Pi}$ of conformal weight $w$ that is trivial on homogeneous Lorentz transformations and vector translations in $\mathbb{M}_0$. Its bundle dual $\hat{\hat{R}}_1$ then has conformal weight $4 - w$.

**Proof.** This consists essentially of the computation of $dg$ at the inducing point, which here is $-I$ in $\mathbb{U}(2)$. By multiplicativity it suffices to treat three cases: (i) $g = S_\lambda$; (ii) $g$ is a Minkowski space translation; (iii) $g$ is a homogeneous Lorentz transformation. Using the symbol $\sim$ to mean equality within terms of second order in $W = Z + I$, $Z$ being arbitrary in $\mathbb{U}(2)$, observe that if $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, then

$$AZ + B \sim (B - A)(I + (B - A)^{-1}AW),$$

$$CZ + D \sim (D - C)(I + (D - C)^{-1}CW)$$
whence
\[(AZ + B)(CZ + D)^{-1}\]
\[\sim (B - A)(I + (B - A)^{-1} AW)(I - (D - C)^{-1} CW)(D - C)^{-1}.\]

Thus \(dg\) evaluated at \(-I\) is the linear transformation
\[W \rightarrow (B - A)((B - A)^{-1} A - (D - C)^{-1} C) W(D - C)^{-1}.\]

In the case of the scale transformation \(S_\lambda\), if \(\lambda = e^t\) then 
\[g = \begin{pmatrix} 1 & -e^{-t/2} \cosh(t/2) \\ -e^{t/2} \sinh(t/2) & 1 \end{pmatrix}.\]
It follows that \(dg\) acts as follows:
\[W \rightarrow -e^{t/2}(-e^{-t/2} \cosh(t/2) + e^{-t/2} \sinh(t/2)) W e^{-t/2} = e^{-t} W.\]

Thus \(dg\) evaluated at \(-I\) is multiplication by \(e^{-t}\), so that \(\det dg^{-1} = e^{4t} = \lambda^4\).

For translations in \(M_0\), \(g\) takes the form 
\[g = \begin{pmatrix} I + f & f \\ -f & I + f \end{pmatrix} \]
with \(f\) in \(h(2)\). It follows that \(dg_{Z=-I}\) carries \(W\) into \((-(-I + f) + f)W = W\), so \(dg_{Z=-I}\) and has determinant 1. In the case of homogeneous Lorentz transformations, it is clear from the fact that open simple Lie groups have no non-trivial onedimensional representations that \(dg_{Z=-I}\) must be of unit determinant. (Alternatively, the above expression for \(dg\) shows that it carries \(W\) into \(L^* WL^{-1}\), where \(L\) is the element of \(SL(2, C)\) representing the homogeneous Lorentz transformation in question.)

Thus the conformal weight of \(\hat{R}_1\) is 4 more than that of \(\hat{R}\). Since the conformal weight of \(\hat{R}\) must be the negative of that of \(R\), \(\hat{R}_1\) is of conformal weight \(4 - w\).

Remark 6.1. Note that the conformal weight of a reweighted representation of \(\hat{P}\) of relative weight \(\lambda\) is the sum of the original conformal weight and \(4\lambda\).

Remark 6.2. It follows that the scalar bundle of conformal weight 2 is self-dual; the action of \(G\) is in fact the standard unitarization obtained from the scalar action of \(G\) on functions by using as multiplier the square root of the Radon–Nikodym derivative (of the transformed to the original measure).

6.5. Conformal Invariants

In theoretical physics the "Lagrangian" is commonly assumed to be a scalar invariant. Representative conformally invariant Lagrangians of scalar fields may be subsumed under the formalism just given as follows.

a. "Power" Lagrangians. Let the weights \(w_1, ..., w_n\) be arbitrary; let \(\Phi_j\) denote a generic section of the \(R_{w_j}\)-bundle over \(\tilde{M}\) in the curved (left) parallelization. If \(w_1 + \cdots + w_n = 4\) then the \(n\)-fold multilinear form
\[(\Phi_1, ..., \Phi_n) \rightarrow \int_{\tilde{M}} \Phi_1 \cdots \Phi_n \, d_3 u.\]
where $\tilde{M}_f$ is a finite covering of $\tilde{M}$, defined on the direct product of the spaces of $C^\infty$ sections in question, is $\tilde{G}$-invariant.

By Remark 6.1 the representation $R_w$ is obtained from the weightless representation $R_0$ by reweighting with weight $w/4$. It follows that for any positive integer $n$ the map $\Phi \to \int_{\tilde{M}_f} \Phi(u)^n d_4 u$ is $\tilde{G}$-invariant on the $C^\infty$ fields induced from $R_{4/n}$, but not in general otherwise. In particular, if $n = 4$ the conformal invariance of a mathematical version of the interaction Lagrangian for the unique scalar wave equation that is conformally invariant in a common sense follows. This is the equation $\Box_c \Phi + \Phi + \Phi^3 = 0$. Conformal invariance is, however, limited essentially to causal group of $\tilde{M}_f$, which is only locally isomorphic to $\tilde{G}$, and there is no a priori assurance that there exist any non-trivial solutions to this equation defined on $\tilde{M}_f$. The Cauchy problem is globally soluble on $\tilde{M}$ itself for arbitrary $C^\infty$ Cauchy data, but the integrated Lagrangian is not defined for this space.

Corollary 6.1.2 is illustrated by the relation to the case of the flat parallelization. If indeed $\Phi$ is a section in the curved parallelization of the scalar bundle of weight $w = 4/n$, then by Theorem 5.2 of (I) the flat-parallelized form $\Phi_0$ of the abstract section determined by $\Phi$ has the form $\Phi_0 = p^n \Phi$. $p$ is non-vanishing on $M_0$, and it follows that if the section has compact support contained in $M_0$, then

$$\int_{M_0} (p(x)^{-w} \Phi_0(x))^n p(x)^4 d_4 x = \int_{\tilde{M}} \Phi(u)^n d_4 u.$$  

b. The "scalar massless free Lagrangian." Let $\Phi$ be a smooth section in the curved parallelization of the scalar bundle of weight 1 having compact support in $M_0$ (imbedded as usual in $\tilde{M}$); the corresponding flat-parallelized section is $\Phi_0 = p \Phi$. The expression

$$\int_{M_0} \left\{ \sum_{i=1}^3 \left( -|\partial \Phi_0/\partial x_i|^2 + |\partial \Phi_0/\partial x_0|^2 \right) \right\} d_4 x$$

equals $\int_{M_0} (\Box_c \Phi_0) \Phi_0 d_4 x$; by Theorems 5.4 and 5.6 of (I) this may be expressed as

$$\langle \Phi, \Phi \rangle = \int_{\tilde{M}} \left\{ (\Box_c + 1) \Phi \right\} \Phi d_4 u$$

$$= \int_{\tilde{M}} \left\{ |\text{grad} \Phi|^2 + |\Phi|^2 - |\Phi|^2 \right\} d_4 u$$

(6.4)

and is conformally invariant. Now recalling that

$$dU_w(S)\Phi = -S\Phi - wu_{-1}u_4 \Phi,$$

$$dU_w(S)\Phi_0 = -S\Phi_0 - w\Phi_0,$$
it follows as in the proof of Theorem 5.4 that expressions (6.3) and (6.4) are conformally invariant, and that for other weights this will not in general be the case.

The expression just treated is a mathematical version of the "scalar massless free Lagrangian" of conventional relativistic theory. Together with the treatment in (a), extended in the obvious way to the sesquilinear case, it follows that the total Lagrangian for a self-interacting scalar field of weight 1 with power interaction is conformally invariant if and only if the power is 4.

VII. THE SPINOR BUNDLE

7.1. Introduction

The spinor bundle is of special interest from both mathematical and physical standpoints. The treatment here via induced representations facilitates parallelization and relevant calculations. The results are naturally consistent with those obtainable from a differential-geometric approach.

This section develops the basic properties of spinor fields in connection with the program indicated in (I) and [2] emphasizing stability and causality considerations. In addition to proofs of the essentiality of the composition series of the stable subspace and other features indicated in [2], covariance relations of flat and curved Dirac operators are established, and the spectral resolution of a complete set of quantum numbers (inclusive of helicity, and substantially the same as the generic set indicated in [2]) is detailed.

7.2. The Inducing Representations

To begin with, the generalized "Poincaré" group \( \tilde{P}_n \cong (R^1 \times SL(n, C)) \tilde{\times} H(n) \) will be considered, although it is only when \( n = 2 \) that the inducing representations coincide with conventional spinor representations. These pseudo-spinor representations will be denoted as \( \mathcal{R}_w^+ \) and \( \mathcal{R}_w^- \); \( w \) designates the conformal weight; the \( \pm \) superscripts are entirely conventional, like an orientation, and are not correlated with real positivity or negativity. Later in this section they will be extended to projective representations of an extension of \( \tilde{P}_n \) by discrete symmetries in the case \( n = 2 \), but initially they are strict representations.

Specifically, \( \mathcal{R}_w^\pm \) are the \( n \)-dimensional representations defined as follows. Let \( \chi \) denote the isomorphism of \( \tilde{P} \) into \( \tilde{G}_p \), where \( p = 0 \times -I \), defined in (I). Recall that the covering of \( G \) by \( \tilde{G} \) restricts to an isomorphism of \( \tilde{G}_p \) onto the identity component of \( G_{-1} \); the composition of the isomorphism of \( \tilde{P} \) onto \( \tilde{G}_p \), with the latter isomorphism was denoted \( \beta \). Moreover if \( g = \beta((t \times L) \times F) = (t^p \, \chi \, p) \), where \( t \in R^1 \), \( L \in SL(n, C) \), and \( F \in H(n) \), then

\[
\Omega^{-1} g \Omega = \begin{pmatrix} A - B & B - C \\ 0 & A + C \end{pmatrix} - \begin{pmatrix} e^{i/2}L & (i/2) e^{-i/2}FL^*^{-1} \\ 0 & e^{-i/2}L^*^{-1} \end{pmatrix}.
\]
Now define

\[ R^+_w(g) = (A - B)(\det(A - B))^{2w_1 - n_1 - 1}, \]

\[ R^-_w(g) = (A + C)(\det(A - B))^{2w_0 + n_0 - 1}. \]

It is evident that \( R^+_w \) are representations of \( \bar{G}_n \). For \( g = \beta((t \times t) \times 0) \), \( R^+_w(g) = R^-_w(g) = e^{w_1} \), which implies that \( R^+_w \) are of weight \( w \), which may be complex. The relation to the conventional spin representations is clarified by the observation:

for \( g = \beta((0 \times L) \times 0) \), \( R^+_w(g) = L \) and \( R^-_w(g) = L^{-1} \). (7.1)

The representations of \( \bar{G} \) induced from \( R^+_w \) will be denoted as \( U^+_w \) in this section. Given an arbitrary pair of weights \( w = (w_1, w_2) \), \( R_w \) will denote the direct sum \( R^+_w \oplus R^-_w \), and \( U_w \) will denote \( U^+_w \oplus U^-_w \). When \( n = 2 \), \( U_w \) is the spinor representation of weights \( w = (w_1, w_2) \). For physical applications it is important to extend these representations to discrete symmetries. These were considered in (I) only when \( n = 2 \), to which case the remainder of this subsection is limited; moreover the weights are not involved in this extension and the subscript “\( w \)” is consequently omitted: the weights \( (w_1, w_2) \) will be assumed real and equal.

The elements \( T \) and \( P \) of \( \bar{G}_n \), which play the roles of time and space reversal in the physical interpretation, were defined in (I) so that the following relations hold in \( \bar{G}_n \):

\[ TP = PT; \quad T^2 = P^2 = e. \]

In order to treat a mathematical version of anti-particle conjugation in the physical interpretation, it is convenient to enlarge the isotropy group \( \bar{G}_p^+ \) to \( \bar{G}_p^+ \times Z_2 \), where the generator of the \( Z_2 \) group is denoted \( C \) and defined to act trivially on \( \bar{M} \). The actions of \( T \), \( P \), and \( C \) on bundles over \( \bar{M} \) will be obtained in later sections by induction from \( \bar{G}_p^+ \times Z_2 \) to \( \bar{G}_p^+ \times Z_2 \).

Conjugation by \( T \) and \( P \) in \( \bar{P} \) define outer automorphisms \( \tau \) and \( \pi \) of \( \bar{P} \). A projective extension of \( R \) to the discrete symmetries \( T \), \( P \), and \( C \) is a projective representation by linear or antilinear transformations such that

\[ R(T) R(g) R(T)^{-1} = R(g^\tau), \]

\[ R(P) R(g) R(P)^{-1} = R(g^\pi), \]

\[ R(C) R(g) R(C)^{-1} = R(g) \] (7.2)

for all \( g \) in \( \bar{P} \). The physical interpretation (deriving from positive energy considerations) requires that \( R(T) \) and \( R(C) \) be antilinear, and that \( R(P) \) be linear, and only such extensions are considered here. By the standard
extension of \( R \) from \( \tilde{P} \) to \( \tilde{P}^+ \times \mathbb{Z}_2 \) will be meant that defined by the following assignments, where \( \kappa \) denotes complex conjugation:

\[
R(T) = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \kappa, \quad R(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R(C) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \kappa. \tag{7.3}
\]

**Scholium 7.1.** The most general projective extension of \( R \) to \( \tilde{P}^+ \times \mathbb{Z}_2 \) that is linear on \( P \) and anti-linear on \( T \) and \( C \) differs from the standard one, apart from constant factors, by left multiplications by the diagonal matrices

\[
\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & ea|a|^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & e'|a| \end{pmatrix} \tag{7.4}
\]

for \( P, T, \) and \( C \), respectively, \( a \) being an arbitrary nonvanishing complex number and \( e \) and \( e' \) being \( \pm 1 \) (independently).

**Proof.** First conditions (7.2) are imposed to determine possible forms of \( C, P, \) and \( T \). Next, the desideratum of commutativity for the \( R(C), R(P), \) and \( R(T) \) within scalar factors completes the proof.

Suppose then that

\[
R(C) = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \kappa,
\]

where \( X, Y, Z, \) and \( W \) are \( 2 \times 2 \) matrices. Then by a defining relation for \( C \)

\[
\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \kappa \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \kappa.
\]

It follows that for all \( L \) in \( SL(2, \mathbb{C}) \)

\[
X\overline{L} = LX, \quad YL^{-1} = LY, \quad Z\overline{L} = L^{-1}Z, \quad WL^{-1} = L^{-1}W. \tag{7.5}
\]

The constraints on \( X \) and \( W \) imply that they commute with all elements of \( SL(2, \mathbb{R}) \) and hence are scalar matrices, implying that they vanish. The constraints on \( Y \) and \( Z \) mean that they transform \( SL(2, \mathbb{C}) \) in exactly the same fashion as \( \sigma_2 \), and so each can differ from \( \sigma_2 \) only by a scalar multiplicative factor. It follows that the linear factor in the above expression for \( R(C) \) is of the form

\[
\begin{pmatrix} 0 & \sigma_2 \\ a\sigma_2 & 0 \end{pmatrix}
\]

for some non-vanishing scalar \( a \), apart from an overall scalar factor.
To treat $R(P)$ and $R(T)$, note that if $g = \beta((0 \times L) \tilde{\times} 0)$ with $L$ in $SL(2, C)$ then $g^\dagger = g^* = \beta((0 \times L^*^{-1}) \tilde{\times} 0)$. The primary constraint on $R(P)$ when applied to a homogeneous Lorentz transformation therefore implies that

$$R(P) \begin{pmatrix} L & 0 \\ 0 & L^*^{-1} \end{pmatrix} R(P)^{-1} = \begin{pmatrix} L^*^{-1} & 0 \\ 0 & L \end{pmatrix}.$$ 

Taking $R(P)$ to have the form

$$R(P) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

gives the equations

$$AL = L^*^{-1}A, \quad BL^*^{-1} = L^*^{-1}B, \quad CL = LC, \quad DL^*^{-1} = LD$$

for all $L$ in $SL(2, C)$. Assuming $L$ unitary in the condition on $A$ shows that $A$ must be a scalar, and hence incapable of transforming $L$ into $L^*^{-1}$ for all $L$. Thus $A = 0$ and the same result shows also that $D = 0$. The conditions on $B$ and $C$ imply that they are scalars, say, $b$ and $c$. Thus apart from an overall scalar factor

$$R(P) = \begin{pmatrix} 0 & I_2 \\ bI_2 & 0 \end{pmatrix},$$

where $b$ is a non-vanishing complex number.

Now assuming that

$$R(T) = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \kappa,$$

the primary constraint when restricted to the homogeneous Lorentz group states that for all $L$ in $SL(2, C)$

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \kappa \begin{pmatrix} L & 0 \\ 0 & L^*^{-1} \end{pmatrix} = \begin{pmatrix} L^*^{-1} & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \kappa.$$ 

Multiplying out and equating matrix entries gives the equations

$$XL = L^*^{-1}X, \quad YL^*^{-1} = L^*^{-1}Y, \quad ZL = LZ, \quad WL^*^{-1} = LW.$$ 

The same arguments as earlier show that $T = Z = 0$ and that $X$ and $W$ are scalar multiples of $\sigma_2$. It follows that $R(T)$ has the form

$$\text{const.} \begin{pmatrix} \sigma_2 & 0 \\ 0 & c\sigma_2 \end{pmatrix} \kappa,$$
where $c$ is a complex number. Squaring $R(T)$ yields a scalar multiple of the matrix

$$\begin{pmatrix} I & 0 \\ 0 & |c|^2 \end{pmatrix},$$

whence $|c| = 1$.

The required projective commutativity of $R(C)$, $R(P)$, and $R(T)$, i.e., their commutativity within non-vanishing scalar factors, now give additional relations. Thus the condition $R(C) R(P) = \text{const.} \ R(P) R(C)$ gives the matrix equation

$$\begin{pmatrix} \sigma_2 \bar{b} & 0 \\ 0 & a \sigma_2 \end{pmatrix} = \text{const.} \ \begin{pmatrix} a \sigma_2 & 0 \\ 0 & b \sigma_2 \end{pmatrix},$$

whence $\bar{b}/a = a/b$. The condition $R(P) R(T) = \text{const.} \ R(T) R(P)$ similarly implies that

$$\begin{pmatrix} 0 & c \sigma_2 \\ b \sigma_2 & 0 \end{pmatrix} = \text{const.} \ \begin{pmatrix} 0 & \sigma_2 \\ c b \sigma_2 & 0 \end{pmatrix},$$

whence $c/b = 1/(c \bar{b})$. Finally the condition that $R(T) R(C) = \text{const.} \ R(C) R(T)$ implies that

$$\begin{pmatrix} 0 & I \\ c \bar{a} & 0 \end{pmatrix} = \text{const.} \ \begin{pmatrix} 0 & \bar{c} \\ a & 0 \end{pmatrix},$$

whence $1/(c \bar{a}) = \bar{c}/a$.

Collecting the relations just obtained in the forms $a^2 = b \bar{b}$, $c^2 = b/\bar{b}$, $c \bar{c} = a/\bar{a}$, it follows from the first relation that $a = e_1 |b|$, where $e_1 = \pm 1$, and from the second relation that $c = e_2 b/|b|$, where $e_2 = \pm 1$. This shows that $R(C)$, $R(P)$, and $R(T)$ must have the forms given in order to define the projective representation indicated. It is straightforward to check that conversely the forms given in Scholium 7.1 satisfy the conditions for defining this projective representation.

Remark 7.1. Thus the standard $C$, $P$, and $T$ satisfy the equations

$$R(C)^2 = 1, \quad R(P)^2 = I, \quad R(T)^2 = -I,$$

$$R(C) R(T) = R(T) R(C), \quad R(T) R(P) = R(P) R(T),$$

$$R(P) R(C) = -R(C) R(P).$$

Remark 7.2. It seems noteworthy that there are only two assignments of $C$, $P$, and $T$ for spinors that are inequivalent under conjugation by transfor-
mations commuting with the spin representation on the connected group, and satisfy the constraint \( C^2 = I \) in addition to Eqs. (7.2). The spin representation \( R = R^+ \oplus R^- \), as defined originally on the connected group \( \mathbb{P} \), commutes with all matrices of the form

\[
\begin{pmatrix}
\lambda I_2 & 0 \\
0 & \lambda' I_2
\end{pmatrix},
\]

where \( \lambda \) and \( \lambda' \) are scalars. Further, conjugation by a suitable such matrix reduces any of the indicated possibilities for \( T, C, \) and \( P \) in Scholium 7.1 to another such for which the complex constant \( a \) is 1, and leaves only the signs \( e \) and \( e' \) ambiguous. In fact, let \( b \) denote a square root of \( a^{-1} \); then

\[
\begin{pmatrix}
1 & 0 \\
0 & b
\end{pmatrix}
\begin{pmatrix}
\sigma_2 & 0 \\
0 & a \sigma_2 |a|^{-1}
\end{pmatrix}
\kappa
\begin{pmatrix}
1 & 0 \\
0 & b^{-1}
\end{pmatrix}
= \begin{pmatrix}
\sigma_2 & 0 \\
0 & a \sigma_2
\end{pmatrix}
\kappa;
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & b
\end{pmatrix}
\begin{pmatrix}
0 & \sigma_2 \\
-e' |a| \sigma_2 & 0
\end{pmatrix}
\kappa
\begin{pmatrix}
1 & 0 \\
0 & b^{-1}
\end{pmatrix}
= b^{-1}
\begin{pmatrix}
0 & \sigma_2 \\
-e' \sigma_2 & 0
\end{pmatrix}
\kappa;
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & b
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
a & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & b^{-1}
\end{pmatrix}
= b^{-1}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

The present choice \( e = 1 \) is purely conventional. The choice \( e' = 1 \) is, however, the result of requiring that \( R(C) \) be an (anti-linear) conjugation, i.e., have square \( I \). This shows directly that the spinor representation \( R^+ \oplus R^- \) is a real representation.

### 7.3. Parallelization of Spinor Fields

This section develops the transformation properties of spinor fields in their parallelized forms. Both curved and flat parallelizations are treated in both global and infinitesimal forms. Later, specialization is made to the physical case \( n = 2 \), and the dual actions on spinor fields of the causal group are determined.

Recall the notation from (1) according to which \( \Psi_0 \) and \( \Psi \) denote the flat and curved parallelized forms of a generic section \( \Psi \).

**Theorem 7.2.** In the curved parallelization on \( \tilde{M} \), \( \tilde{G} \) transforms spinor field sections as follows, where \( Z \) denotes the element of \( U(n) \) canonically covered by the element \( z \) of \( \tilde{M} \) (thus \( Z = e^{it}V \) if \( z = t \times V \)), and for any element \( g \) of \( \tilde{G} \)

\[
\begin{pmatrix}
A' & B' \\
C' & D'
\end{pmatrix}
\]
denotes the element of $SU(n, n)$ canonically covered by $g^{-1}$ as in (I):

$$U^+(g) : \Psi(z)$$

$$= |\det(C'Z + D')|^{-2w-1}(\det(C'Z + D'))^{-1} \Psi(g^{-1}z);$$

$$U^-(g) : \Psi(z)$$

$$= |\det(C'Z + D')|^{-2w-1}(\det(C'Z + D'))^{-1} \Psi(g^{-1}z),$$

where the indicated powers of determinants are defined as the exponentials of the (continuous) branches of their logarithms defined uniquely on $\hat{G} \times \hat{H}$ by stipulating their triviality when $g = e$.

Within the canonical image of $M_0$ (also denoted as $M_0$) in $\tilde{M}$, the curved and flat parallelizations are related as follows. For arbitrary sections $\Psi$ of the bundle induced from $R^+$,

$$\Psi_0(Z) = (\det Z)^{-w-1}(\det(\frac{1}{2} + \frac{1}{2}Z))^{2w-1-n-1} (I + Z) \Psi(Z); \quad (7.6)$$

similarly for the bundle induced from $R^-$,

$$\Psi_0(Z) = (\det Z)^{-w-1-n-1}(\det(\frac{1}{2} + \frac{1}{2}Z))^{2w-1+n-1} (I + Z^{-1})^{-1} \Psi(Z). \quad (7.7)$$

In the flat parallelization the spinor field actions take the following forms (for $g$ sufficiently close to $e$ and identified with the corresponding element of $G$ in the canonical local isomorphism):

$$U^+(g) : \Psi_0(Z) \rightarrow (\frac{1}{2}C_1 h + D_1)^{-1} (\det(\frac{1}{2}C_1 h + D_1))^{-2w-1+n-1} \Psi_0(g^{-1}Z),$$

$$U^-(g) : \Psi_0(Z) \rightarrow (\frac{1}{2}C_1 h + D_1)^{-1} (\det(\frac{1}{2}C_1 h + D_1))^{-2w-1+n-1} \Psi_0(g^{-1}Z),$$

where $c(h) = Z$ and

$$\Omega^{-1}g^{-1}\Omega = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$ 

Proof: The subgroup $\tilde{N}$ of $\tilde{G}$ on which the curved parallelization is based is that defined in Corollary 4.1.4: $\tilde{N} = \{ t \times U \times I : t \in R^1, U \in SU(n) \}$. In the application of Theorem 4.1, $\tilde{N}$ corresponds to $\tilde{M}$ via the mapping $t \times U \times I \rightarrow t \times U$, and the inducing point is taken as $x_0 = 0 \times -I \times I$ in $\tilde{N}$, corresponding to the point $0 \times -I$ in $\tilde{M}$. According to Theorem 4.1, the multiplier at $z = t \times U$ in $\tilde{M}$ is $R^\pm (g^*)$, for the action $U^\pm (g)$ on parallelized sections of the arbitrary element $g$ of $\tilde{G}$, where

$$g^* = (0 \times -I \times I)(-t \times U^{-1} \times I) \times g(\phi(g^{-1})(t \times U \times I))(0 \times -I \times I). \quad (\S)$$
Now $g^*$ is an element of $\tilde{G}_{x_0}$; as noted in the previous section, the canonical covering of $SU(n, n)$, when restricted to $\tilde{G}_{x_0}$, is an isomorphism of $\tilde{G}_{x_0}$ onto the connected component of $G_{-1}$. Thus the image of $g^*$ via this covering transformation can be evaluated by application of this transformation to each factor in Eq. (§), followed by their multiplication. The resulting product was evaluated, within an element of the center of $G$, in Corollary 4.1.2. Using the notation there, the respective multipliers for $U^\pm_w(g)$ are

$$R^+_w(g^*) = (\det Z W^{-1})^{(1/2)n} (Z^{-1} A W + Z^{-1} B)$$
$$\times \det((\det Z W^{-1})^{(1/2)n-1}(Z^{-1} A W + Z^{-1} B))^{(2n-1)/n},$$

$$R^-_w(g^*) = (\det Z W^{-1})^{(1/2)n-1}(Z^{-1} A W - C W)(\det(Z W^{-1}))^{(2n-1)/n}$$
$$\times \det(Z^{-1} A W + Z^{-1} B)^{(2n+1)/n}. $$

To simplify the non-scalar factors in these expressions, note that as regards $R^+_w$

$$Z^{-1} A W + Z^{-1} B = (C'Z + D')^{-1} \quad (\ast)$$

by Lemma 5.1.1. For $R^-_w$, note that

$$Z^{-1} A W - C W = D - Z^{-1} B,$$

since the element $g$ in Corollary 4.1.2 is in $G_{-1}$, and

$$D - Z^{-1} B = (C'Z + D')^*$$

by the equations relating the components of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \quad \text{and} \quad \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

at the end of Section 2.1.

The determinant factors in the multipliers involve $\det(Z W^{-1})$ and $\det(Z^{-1} A W + Z^{-1} B)$. These may be conveniently expressed in terms of $\det((C'Z + D')^*$ and its complex conjugate through the use of Eq. (\ast) together with the equation

$$\det(Z W^{-1}) = \det(C'Z + D') \det((C'Z + D')^{*-1}),$$

which is Eq. (5.3) with $g^{-1}$ substituted for $g$. 

Next, Corollary 4.3.2 of (I) is applied to the comparison of the flat and curved parallelizations. This involves the evaluation of $R^\pm$ on

\[
(d\det Z)^{-2n^{-1}} \begin{pmatrix}
(I - \frac{h}{4}Z & \frac{h}{4} \\
-\frac{h}{4}Z & I + \frac{h}{4}
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix},
\]
say. Evidently

\[
a - b = (d\det Z)^{-2n^{-1}} \frac{1}{2}(I + Z), \quad a + c = (d\det Z)^{-2n^{-1}}((\frac{1}{2}(I + Z^{-1}))^{-1}.
\]

Using these equations it is straightforward to evaluate $R^\pm$ on the requisite elements, obtaining the results given in Theorem 7.2.

Now applying Corollary 4.3.2 and the definition of $R^\pm$, the multipliers for $U^\pm_w$ in the flat parallelization are

\[
(\frac{1}{2}C''f + D'') \det(\frac{1}{2}C''f + D'')^{(2w-1)/n}, \quad (A'' - \frac{1}{2}hC'') \det(\frac{1}{2}C''f + D'')^{(2w+1)/n}.
\]

Using the following equation given in Lemma 5.2.1,

\[
\frac{1}{2}C''f + D'' = (\frac{1}{2}C_1h + D_1)^{-1},
\]

together with the equation

\[
(A'' - \frac{1}{2}hC'')^* = \frac{1}{2}C_1h + D_1
\]

which follows by further use of the equations at the end of Section 2.1, the equality of the expressions just given with the multipliers given in Theorem 7.2 follows.

**Corollary 7.2.1.** The infinitesimal multipliers for the representations $U^\pm_w$ of the theorem are as follows, with the notation

\[
X = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

cf. Section 3.1 of (I)) for an arbitrary element of $su(2, 2)$, and

\[
\begin{pmatrix}
a'' & b'' \\
c'' & d''
\end{pmatrix} \quad \text{for} \quad \Omega^{-1}X\Omega.
\]
In the curved parallelization:

\[
(d/dt) R_w^+ (g^*(e^tX)) \big|_{t=0} = cZ + d - n^{-1} \text{tr}(cZ + d) + wn^{-1} \text{tr}(bZ^{-1} + cZ);
\]

\[
(d/dt) R_w^- (g^*(e^tX)) \big|_{t=0} = d - Z^{-1}b - n^{-1} \text{tr}(d - Z^{-1}b) + wn^{-1} \text{tr}(bZ^{-1} + cZ).
\]

In the flat parallelization:

\[
(d/dt) R_w^+ (g^{**}(e^tX)) \big|_{t=0} = -\frac{1}{2}c''h + d'' + (2wn^{-1} - n^{-1}) \text{tr}(\frac{1}{2}c''h + d''),
\]

\[
(d/dt) R_w^- (g^{**}(e^tX)) \big|_{t=0} = a'' - \frac{1}{2}hc'' + (2wn^{-1} + n^{-1}) \text{tr}(\frac{1}{2}c''h + d'').
\]

**Proof.** This follows by straightforward linearization of the multipliers in Theorem 7.2 together with the relations \( b^* = c \), \( (a'')^* = -d \), and the skew-hermitian character of \( a \), \( d \), \( b'' \), and \( c'' \).

From this point onwards in the present section, \( n \) is specialized to have the value 2.

**Corollary 7.2.2.** When \( n = 2 \), Eqs. (7.6) and (7.7) take the forms

\[
\Psi_0(Z) = p^{w-1/2} e^{-it/2} \frac{1}{2}(I + Z) \Psi(Z) \tag{7.8}
\]

resp.

\[
\Psi_0(Z) = p^{w-1/2} e^{it/2} \frac{1}{2}(I + Z \det Z^{-1}) \Psi(Z). \tag{7.9}
\]

**Proof.** For Eq. (7.8) it suffices to compute

\[
\det(\frac{1}{2} + \frac{1}{2}Z) = e^{it}p = e^{it} \frac{1}{2}(u_{-1} + u_4) \tag{7.10}
\]

using Section 3.3 of (I). Equation (7.9) requires in addition the relation

\[
\left(\frac{1}{2}(I + Z^{-1})\right)^{-1} = e^{it}p^{-1} \frac{1}{2}(I + Z(\det Z)^{-1}), \quad Z \in U(2). \tag{7.11}
\]

To prove this, write \( Z = e^{it}U \) with \( U \in SU(2) \); then \( U + U^{-1} = 2u_4 \) and

\[
\frac{1}{2}(I + e^{-it}U^{-1}) \frac{1}{2}(I + e^{-it}U) = \frac{1}{2}(I + 2u_4 e^{-it} + e^{-2it})
\]

\[
= e^{-it} \frac{1}{2}(u_{-1} + u_4).
\]

In connection with the next result the notation \( U_b = u_1 b_1 + u_2 b_2 + u_3 b_3 \) is recalled from (I).
Corollary 7.2.3. The (internal) infinitesimal actions of the scale generator

\[ S = -(1/2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

in the spinor representations \( U_w^+ \) are as follows.

In the curved parallelization they are

\[
\begin{align*}
(d/dt)R_w^+(g(e^iS))|_{t=0} &= (1/2) e^{it} U_b \quad wu = u_4, \\
(d/dt)R_w^-(g(e^iS))|_{t=0} &= -(1/2) e^{-it} U_b - wu = u_4.
\end{align*}
\]

In the flat parallelization they are

\[
(d/dt)R_w^+(g(e^iS))|_{t=0} = (d/dt)R_w^-(g(e^iS))|_{t=0} = -w.
\]

Proof. This is by application of Corollary 7.2.1 to \( S \), noting that \( a = d = b'' = c'' = 0 \), \( b = c = d'' = -a'' = -(1/2) \), and that \( e^{-it} U_b = Z - (1/2) \text{tr} Z \).

Corollary 7.2.4. In the curved parallelization

\[
\begin{align*}
dU_w^+(X_0) &= -X_0, \\
dU_w^+(Y_k) &= -Y_k, \\
dU_w^+(X_k) &= -X_k - b_k 
\end{align*}
\]

for all \( w \) and \( k = 1, 2, 3 \).

In the flat parallelization

\[
\begin{align*}
dU_w^+(T_k) &= -T_k, \\
dU_w^+(L_{12}) &= -L_{12} + (1/2) b_3 \quad (+ \text{cyclic permutations})
\end{align*}
\]

for all \( w \) and \( k = 0, 1, 2, 3 \).

Proof. This follows from Corollary 7.2.1 and the relations

\[
\begin{align*}
X_0 &= (i/2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
X_k &= \begin{pmatrix} 0 & 0 \\ 0 & -b_k \end{pmatrix}, \\
Y_k &= \begin{pmatrix} b_k & 0 \\ 0 & 0 \end{pmatrix}
\end{align*}
\]

for \( k = 1, 2, 3 \), and \( L_{12} = (1/2)(Y_1 - X_1) \), etc.

Scholium 7.3. The representation dual to \( U_w^+ \) is \( U_{4-w}^+ \).

Proof. By Theorem 6.2 it suffices to show that the bundle dual of \( R_w^+ \) is equivalent to \( R_{4-w}^+ \). Given an arbitrary element \( g = (A B) \) in \( G_{-I} \), set \( \delta = \text{det}(A - B) \). Then the representation \( g \to ((A - B)^{-1})^{T} \delta^{w+1/2} \) is the dual of \( R_w^+ \); hence the bundle dual is the representation \( g \to ((A - B)^{-1})^{T} \delta^{4-w+1/2} \). Now for any \( 2 \times 2 \) matrix \( M \),

\[
M^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{det} M.
\]
Therefore the bundle dual of $R^+_w$ is equivalent via $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to $R^+_{-w}$. The proof for $R^-_w$ is similar.

7.4. Covariance of the Curved and Flat Dirac Operators

This section establishes the conformal covariance of the operators

$$\mathcal{D}_c = \gamma_0 X_0 + \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3 - (3/2) \gamma_4 \gamma_5$$

and

$$\mathcal{D}_f = \gamma_0 T_0 + \gamma_1 T_1 + \gamma_2 T_2 + \gamma_3 T_3$$

which act on $C^4$-valued functions on $\tilde{M}$. These are, respectively, the canonical Dirac operators relative to the standard curved and flat metrics (cf. Section 3.7 of (I)), but this fact is not used here.

The matrix-valued function $F(z) = F(t \times V)$ on $\tilde{M}$ defined by the equation (where $\otimes = e^{itV}$)

$$F(z) = \begin{pmatrix} pe^{-(i/2)t} & \frac{1}{2}(I + Z) \\ 0 & pe^{(i/2)t} \frac{1}{2}(I + Z)(\det Z)^{-1} \end{pmatrix}$$

is involved in this covariance. Indeed, given a pair of weights $w = (w_1, w_2)$, and denoting as $p^w$ the $4 \times 4$ matrix

$$\begin{pmatrix} p^{w_1}I_2 & 0 \\ 0 & p^{w_2}I_2 \end{pmatrix},$$

then by Corollary 7.2.2,

$$\Psi_0 = p^{w - 3/2} F \Psi$$

(7.12)

for an arbitrary section $\Psi$ of the $R_w$-bundle; here $w - 3/2$ denotes the pair of weights $(w_1 - 3/2, w_2 - 3/2)$.

**Theorem 7.4.** The operators $\mathcal{D}_c$ and $\mathcal{D}_f$ are related by the equation

$$p F \mathcal{D}_c - \mathcal{D}_f F.$$

**Proof.** This is facilitated by the observation that if $c(h) = Z$, then

$$F = \begin{pmatrix} e^{(i/2)t}p^2(I - h/2) & 0 \\ 0 & e^{-(i/2)t}p^2(I + h/2) \end{pmatrix}$$

(7.13)

and

$$p^{-1}F^{-1} = \begin{pmatrix} p^{-2}e^{(i/2)t}(I - h/2) & 0 \\ 0 & p^{-2}e^{-(i/2)t}(I + h/2) \end{pmatrix},$$

(7.14)
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where \( \tilde{h} = \text{tr} h - h \). This observation follows from Eq. (7.11) and the identity \(((I + Z)/2)^{-1} = I - h/2 \).

In addition some definitions and identities regarding \( h(2) \) will be useful. Let the quadratic form \( \langle h, h \rangle \) be defined on \( h(2) \) as \( x_0^2 - x_1^2 - x_2^2 - x_3^2 = x^2 \), and let \((\sigma \cdot h) \) be defined as \( \sigma_2 x_3 - \sigma_3 x_2 \), with \((\sigma \cdot h) \) defined for \( j = 2 \) and \( 3 \) by cyclic permutation. Then the following identities hold, by easy arguments:

\[
\begin{align*}
[\sigma_j, h] &= 2(\sigma \cdot h) \quad (j = 1, 2, 3), \\
h\tilde{h}h &= -2h\langle h, k \rangle + \langle h, h \rangle k; \\
\end{align*}
\]

useful special cases of (7.16) are

\[
\begin{align*}
h\tilde{h}h &= -h\langle h, h \rangle, \\
h^2 &= 2ix_0h + \langle h, h \rangle, \\
h\sigma_jh &= 2ix_jh - \langle h, h \rangle \sigma_j \quad (j = 1, 2, 3). \\
\end{align*}
\]

Turning now to the proof proper, it suffices to show that the stated conclusion is valid on \( M_0 \), since \( \mathcal{D}_c \) and \( \mathcal{D}_j \) are real-analytic differential operators and \( F \) and \( p \) are real-analytic functions on \( \tilde{M} \). Now \( p^{-1}F^{-1}\mathcal{D}_jF \) is evidently a differential operator of first order. From Eqs. (7.13) and (7.14) it follows that the "vector field part" of \( p^{-1}F^{-1}\mathcal{D}_jF \), i.e., the part that is homogeneous of first order, takes the form

\[
\begin{bmatrix}
0 \\
-iBC \\
0
\end{bmatrix}
\begin{bmatrix}
\partial/\partial x_0 \\
\partial/\partial x_j
\end{bmatrix}
+ \sum_{j=1}^{3}
\begin{bmatrix}
0 \\
iB\sigma_jC \\
0
\end{bmatrix}
\begin{bmatrix}
-iA\sigma_jD \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\partial/\partial x_j
\end{bmatrix},
\]

where \( A = I - h/2, \quad B = I + \tilde{h}/2, \quad C = I - \tilde{h}/2, \quad D = I + h/2 \). The matrix products in expression (7.20) may be evaluated from Eqs. (7.15), (7.18), and (7.19). Some computation using the expressions for the \( X_j \) (\( j = 0, 1, 2, 3 \)) in Table I of [2] as linear combinations of the \( \partial/\partial x_j \) then reduces expression (7.20) to the operator \( \sum_{j=0}^{3} \gamma_j X_j \).

It remains to show that the "multiplicative part" of \( p^{-1}F^{-1}\mathcal{D}_jF \) (i.e., the 0-order term in this first-order differential operator) is \( (-2 - \tilde{h})^3/2 \). By Eqs. (7.13) and (7.14) one contribution is

\[
\begin{bmatrix}
A \\
0
\end{bmatrix}
\begin{bmatrix}
\mathcal{D}_j \\
0
\end{bmatrix}
\begin{bmatrix}
C \\
D
\end{bmatrix}
=
\begin{bmatrix}
(I - h/2) \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1/2
\end{bmatrix}
+ \sum_{j=1}^{3}
\begin{bmatrix}
(I - h/2) \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1/2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
-2 - \tilde{h}
\end{bmatrix};
\]

where \( \tilde{h} = \text{tr} h - h \). This observation follows from Eq. (7.11) and the identity \(((I + Z)/2)^{-1} = I - h/2 \).

In addition some definitions and identities regarding \( h(2) \) will be useful. Let the quadratic form \( \langle h, h \rangle \) be defined on \( h(2) \) as \( x_0^2 - x_1^2 - x_2^2 - x_3^2 = x^2 \), and let \((\sigma \cdot h) \) be defined as \( \sigma_2 x_3 - \sigma_3 x_2 \), with \((\sigma \cdot h) \) defined for \( j = 2 \) and \( 3 \) by cyclic permutation. Then the following identities hold, by easy arguments:

\[
\begin{align*}
[\sigma_j, h] &= 2(\sigma \cdot h) \quad (j = 1, 2, 3), \\
h\tilde{h}h &= -2h\langle h, k \rangle + \langle h, h \rangle k; \\
\end{align*}
\]

useful special cases of (7.16) are

\[
\begin{align*}
h\tilde{h}h &= -h\langle h, h \rangle, \\
h^2 &= 2ix_0h + \langle h, h \rangle, \\
h\sigma_jh &= 2ix_jh - \langle h, h \rangle \sigma_j \quad (j = 1, 2, 3). \\
\end{align*}
\]

Turning now to the proof proper, it suffices to show that the stated conclusion is valid on \( M_0 \), since \( \mathcal{D}_c \) and \( \mathcal{D}_j \) are real-analytic differential operators and \( F \) and \( p \) are real-analytic functions on \( \tilde{M} \). Now \( p^{-1}F^{-1}\mathcal{D}_jF \) is evidently a differential operator of first order. From Eqs. (7.13) and (7.14) it follows that the "vector field part" of \( p^{-1}F^{-1}\mathcal{D}_jF \), i.e., the part that is homogeneous of first order, takes the form

\[
\begin{bmatrix}
0 \\
-iBC \\
0
\end{bmatrix}
\begin{bmatrix}
\partial/\partial x_0 \\
\partial/\partial x_j
\end{bmatrix}
+ \sum_{j=1}^{3}
\begin{bmatrix}
0 \\
iB\sigma_jC \\
0
\end{bmatrix}
\begin{bmatrix}
-iA\sigma_jD \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\partial/\partial x_j
\end{bmatrix},
\]

where \( A = I - h/2, \quad B = I + \tilde{h}/2, \quad C = I - \tilde{h}/2, \quad D = I + h/2 \). The matrix products in expression (7.20) may be evaluated from Eqs. (7.15), (7.18), and (7.19). Some computation using the expressions for the \( X_j \) (\( j = 0, 1, 2, 3 \)) in Table I of [2] as linear combinations of the \( \partial/\partial x_j \) then reduces expression (7.20) to the operator \( \sum_{j=0}^{3} \gamma_j X_j \).

It remains to show that the "multiplicative part" of \( p^{-1}F^{-1}\mathcal{D}_jF \) (i.e., the 0-order term in this first-order differential operator) is \( (-2 - \tilde{h})^3/2 \). By Eqs. (7.13) and (7.14) one contribution is

\[
\begin{bmatrix}
A \\
0
\end{bmatrix}
\begin{bmatrix}
\mathcal{D}_j \\
0
\end{bmatrix}
\begin{bmatrix}
C \\
D
\end{bmatrix}
=
\begin{bmatrix}
(I - h/2) \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1/2
\end{bmatrix}
+ \sum_{j=1}^{3}
\begin{bmatrix}
(I - h/2) \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1/2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
-2 - \tilde{h}
\end{bmatrix};
\]
the other is
\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\begin{pmatrix}
\log(e^{it/2}p^2) & 0 \\
0 & \log(e^{-it/2}p^2)
\end{pmatrix}
\begin{pmatrix}
C & 0 \\
0 & D
\end{pmatrix}.
\]

The resulting putative equalities reduce to
\[
-1/2 + h = (I - h/2) \left(-i(\partial/\partial x_0) \log(e^{-it/2}p^2) \right.
- \sum_{j=1}^{3} b_j(\partial/\partial x_j) \log(e^{-it/2}p^2) \left)(I + h/2\right)
\]
and
\[
1/2 + \tilde{h} = (I + \tilde{h}/2) \left(-i(\partial/\partial x_0) \log(e^{it/2}p^2) \right.
+ \sum_{j=1}^{3} b_j(\partial/\partial x_j) \log(e^{it/2}p^2) \left)(I - \tilde{h}/2\right).
\]

Since the latter equation is the transform of Eq. (7.21) under $T$, it suffices to establish (7.21).

To this end let $q$ denote the function on $M_0$: $q = (1 - x^2/4) + ix_0$. Then
\[
(\partial/\partial x_0)q = -(x_0/2) + i \quad \text{and} \quad (\partial/\partial x_j)q = x_j/2 \quad \text{for} \quad j > 0.
\]
Since $q$ may also be expressed as $e^{it/2}p^{-1}$ (cf. Section 3.3 of (I)), $\exp(-it/2)p^2 = q^{-5/4}q^{-3/4}$. It follows that the left side of Eq. (7.21) is
\[
(I - \frac{1}{2}h)(\frac{3}{4}q^{-1} - \frac{5}{4}q^{-2} - \frac{1}{4}\tilde{h}(\frac{3}{4}q^{-1} + \frac{5}{4}q^{-2}))(I + \frac{1}{2}h)
= (\frac{3}{4}q^{-1} \quad \frac{5}{4}q^{-2})(1 \quad \frac{1}{4}x_0^2 \quad \frac{1}{4}ix_0h) \quad \frac{1}{4}(\frac{3}{4}q^{-1} + \frac{5}{4}q^{-2})(\tilde{h} + \frac{1}{4}x^2h),
\]
using Eqs. (7.17) and (7.18). Thus Eq. (7.21) is equivalent to the equation
\[
(-\frac{1}{2} + h)((1 - \frac{1}{4}x^2)^2 + x_0^2) = (1 - \frac{1}{4}x^2 - \frac{1}{4}ix_0h)(-\frac{1}{4}(1 - \frac{1}{4}x^2) + 2ix_0)
- \frac{1}{4}(\tilde{h} + \frac{1}{4}x^2h)(2(1 - \frac{1}{4}x^2) - \frac{1}{4}ix_0)
\]
(multiplying by $qq$), which is straightforward to check.

Next, as in the scalar case ((I), Corollaries 5.3.2 and 5.3.4), covariance relations between the curved and flat operators in unparallelized form are derived. Given any pair of weights $w$ as before, define $\mathcal{D}_c$ and $\mathcal{D}_f$ on (abstract sections $\Psi$ of the $R_w$-bundle by the equations
\[
\mathcal{D}_c \Psi = \Phi \quad \text{provided that} \quad \mathcal{D}_c \Psi = \Phi,
\]
\[
\mathcal{D}_f \Psi = \Phi \quad \text{provided that} \quad \mathcal{D}_f \Psi_o = \Phi_o.
\]
COROLLARY 7.4.1. For any weights \( w = (w_1, w_2) \),
\[
p^{5/2-w} \mathcal{D}_c - \mathcal{D}_f p^{3/2-w}.
\]  
(7.22)

**Proof.** If \( \mathcal{D}_c \Psi = \Phi \), then by Theorem 7.4 and Eq. (7.12),
\[
p^{5/2-w} \Phi_0 = p^{5/2-w} F p^{w-3/2} \mathcal{D}_c \Psi = p F \mathcal{D}_c \Psi
\]  
\[= \mathcal{D}_f F \Psi = \mathcal{D}_f (p^{3/2-w} \Psi_0). \]

The stated conclusion is now the flat parallelized form of Eq. (7.22).

A special case is

COROLLARY 7.4.2. Within the bundle induced from \( R_{3/2}^+ \oplus R_{3/2}^- \),
\[
p \mathcal{D}_c = \mathcal{D}_f.
\]

Moreover the only weights \( w \) such that \( \mathcal{D}_c \) and \( \mathcal{D}_f \) have the same kernel in the bundle induced from \( R_w \) are \((3/2, 3/2)\).

In fact, \((3/2, 3/2)\) is also the only pair of weights \( w \) for which \( \mathcal{D}_c \) is conformally quasi-invariant. In order to prove a slightly more general result (Theorem 7.6), the operator \( \mathcal{D}_c^r \), where \( r \) is any given constant, is defined by the equation
\[
\mathcal{D}_c^r \Psi = \Phi \quad \text{provided that} \quad (\mathcal{D}_c + r \gamma_4 \gamma_5) \Psi = \Phi.
\]

SCHOLIUM 7.5. All the \( \mathcal{D}_c^r \) commute with \( dU_w(K) \).

**Proof:** \( dU_w(K) \) is determined in Corollary 7.2.4. Thus \( \mathcal{D}_c^r \) evidently commutes with \( X_0 \) and the \( Y_j \). Commutativity with \( dU_w(X_j) \) follows from the relation \([\gamma_2 X_2 + \gamma_3 X_3, X_1 + b_1] = 0\), which is a consequence of the relations \([X_1, X_2] = -2X_3, [b_1, b_2] = -2b_3\), and cyclic permutations.

THEOREM 7.6. There exists a scalar function \( m(S) \) such that
\[
[\mathcal{D}_c^r, dU_{w_1}(S) \oplus dU_{w_2}(S)] = m(S) \mathcal{D}_c^r,
\]  
(7.23)

either locally or globally, if and only if \( r = 0 \) and \( w_1 = w_2 = 3/2 \). In this event, \( m(S) = -u_{-1} u_4 \).

**Proof:** The terms in Eq. (7.23) will first be computed in the flat parallelization, using Corollary 7.2.3 and Eq. (7.12). Given any \( \Psi \),
\[
(\mathcal{D}_c^r dU_w(S) \Psi)_0 = p^{w-3/2} F (\mathcal{D}_c + r \gamma_4 \gamma_5) F^{-1} p^{w-3/2} (-S - w) \Psi_0,
\]  
\[-(dU_w(S) \mathcal{D}_c^r \Psi)_0 = (S + w) F p^{w-3/2} (\mathcal{D}_c + r \gamma_4 \gamma_5) F^{-1} p^{3/2-w} \Psi_0,
\]
and

\[(\mathcal{R}_c \Psi)_0 = \mathbf{F} p^{w-3/2}(\mathcal{R}_c + r\gamma_4 \gamma_5) \mathbf{F}^{-1} p^{3/2-w} \Psi_0.\]

From this and Theorem 7.4 it follows that Eq. (7.23) is equivalent to the equation

\[
[S, p^{w-3/2} p^{-1} \mathcal{R}_f p^{3/2-w}] \Psi_0 + m_1 \Psi_0 = mp^{w-3/2} p^{-1} \mathcal{R}_f p^{3/2-w} \Psi_0 + m_2 \Psi_0, \tag{7.24}
\]

where

\[
m_1 = rS(p^{w-\hat{w}} \gamma_4 \gamma_5 \mathbf{F}^{-1}), \quad m_2 = mrp^{w-\hat{w}} \gamma_4 \gamma_5 \mathbf{F}^{-1},
\]

\[
m = m(S) - w + \hat{w}, \quad \hat{w} = (w_2, w_1),
\]

and \(w - \hat{w}\) denotes

\[
\begin{pmatrix}
  w_1 - w_2 & 0 \\
  0 & w_2 - w_1
\end{pmatrix}.
\]

At this point Eq. (7.24) will be shown to hold in the case \(r = 0, w_1 = w_2 = 3/2\). By Scholium 3.1 of (I),

\[
[S, p^{-1} \mathcal{R}_f] = (Sp^{-1}) \mathcal{R}_f + p^{-1}[S, \mathcal{R}_f] = (p^{-1}(1 - u_{-1} u_4) - p^{-1}) \mathcal{R}_f = -u_{-1} u_4(p^{-1} \mathcal{R}_f).
\]

Returning to the general case, the covariance just established is used to evaluate the commutator in Eq. (7.24). The equation becomes

\[
p^{w-3/2} p^{-1} \mathcal{R}_f ((Sp^{3/2-w}) \Psi_0) + m_1 \Psi_0 = (m_3 - p^{3/2-w} Sp^{w-3/2}) p^{w-3/2} p^{-1} \mathcal{R}_f p^{3/2-w} \Psi_0 + m_2 \Psi_0, \tag{7.25}
\]

where \(m_3 = m + u_{-1} u_4\).

Evaluation of the vector field terms of Eq. (7.25) gives the equation

\[
p^{w-3/2} p^{-1} (Sp^{3/2-\hat{w}}) = (m_3 - p^{3/2-w} Sp^{w-3/2}) p^{w-3/2} p^{-1} \mathcal{R}_f p^{3/2-w} \Psi_0 + m_2 \Psi_0,
\]

or

\[
m_3 = p^{\hat{w}-w} Sp^{w-\hat{w}}.
\]

Resubstituting into Eq. (7.25), it results that

\[
p^{w-5/2} \mathcal{R}_f ((Sp^{3/2-w}) \Psi_0) + m_1 \Psi_0 = -p^{w-\hat{w}} S(p^{\hat{w}-3/2}) p^{-1} \mathcal{R}_f (p^{3/2-w} \Psi_0) + m_2 \Psi_0. \tag{7.26}
\]
In order to conclude that \( r = 0 \) and \( w = (3/2, 3/2) \) it will suffice to examine the multiplicative part of Eq. (7.26) on the time-like line where \( u_4 = 1 \) near the origin. There

\[
\mathbf{F} = I + O(t^2), \quad p = 1 + O(t^2), \quad Sp = -p(1 - u_{-1}u_4) = O(t^2),
\]

and

\[
Sp^{3/2 - w} = (w - 3/2)p^{3/2 - w}(1 - u_{-1}u_4), \quad 1 - u_{-1} = O(t^2). \quad (7.27)
\]

It follows that the left side and the first term on the right side of Eq. (7.26) are \( O(t) \), and \( m = -1 + O(t^2) \), so that \( m_2 = -r \gamma_4 \gamma_5 + O(t^2) \). Thus \( r = 0 \), so that \( m_1 = m_2 = 0 \). This implies that the right side of Eq. (7.26) is \( O(t^2) \); and by Eq. (7.27) the multiplicative part of the left side of Eq. (7.26) is

\[
t \begin{pmatrix}
0 & i(3/2 - w_1) \\
i(3/2 - w_1) & 0
\end{pmatrix}
+ O(t^2) + \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix},
\]

where \( tr A = tr B = 0 \). Hence \( w_1 = w_2 = 3/2 \).

**Corollary 7.6.1.** For each \( X \) in \( \mathcal{S} \), there exist smooth functions \( m(X) \) and \( m_0(X) \) on \( \overline{\mathcal{M}} \) and \( \mathcal{M}_0 \) such that

\[
\mathcal{D}_c, dU_{3/2}(X) \otimes dU_{3/4}(X) - m(X) \mathcal{D}_c
\]

and

\[
\mathcal{D}_f, dU_{3/2}(X) \otimes dU_{3/2}(X) = m_0(X) \mathcal{D}_f.
\]

In particular, \( m(S) = -u_{-1}u_4 \), \( m_0(S) = -1 \), and \( m_0(X) = m(X) + X(\log p) \).

**Proof.** The proof follows that of Corollary 5.3.5 of (I) and is otherwise based on Scholium 7.5 and Theorem 7.6.

### 7.5. Invariance of Hermitian Forms

By Corollary 7.6.1 the kernel of \( \mathcal{D}_c \) in the bundle induced from \( R_{3/2}^+ \oplus R_{3/2}^- \) is invariant under the corresponding action of \( \mathbf{G} \) (spinor representation in this bundle). It will be shown below that this invariant subspace admits an invariant Hilbert-space structure, i.e., one in which the action of \( \mathbf{G} \) becomes unitary. In addition it will be shown that the quotient modulo this subspace admits an invariant hermitian form. It will be convenient from this point to identify sections of this bundle with their left parallelizations unless otherwise indicated.

Given smooth sections \( \Psi \) and \( \Phi \) of the bundle over \( \overline{\mathcal{M}}^{(4)} \), define

\[
\left\langle \left\langle \Psi, \Phi \right\rangle \right\rangle = \int_{\overline{\mathcal{M}}^{(4)}} \left\langle \left\langle \mathcal{D}_c \Psi, \Phi \right\rangle \right\rangle' \, du.
\]
where \( \langle \cdot, \cdot \rangle' \) is the hermitian form on \( C^4 \) defined by the equation

\[
\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle' = x_1 \bar{y}_3 + x_2 \bar{y}_4 + x_3 \bar{y}_1 + x_4 \bar{y}_2.
\]

This form may also be described as the transform under \( \Omega \) of the form \( \langle \cdot, \cdot \rangle \) defined on \( C^2 \oplus C^2 \) in (1), Section 2.1, and can also be written as \( Y^*X \) if \( Y^* \) is defined as \( \bar{Y} \gamma_4 \) (\( Y^* \) is commonly denoted in physics literature as \( \bar{Y} \)). It is straightforward to check that

\[
\langle \mathcal{D}_c \psi, \phi \rangle = \langle \psi, \mathcal{D}_c \phi \rangle,
\]

so that \( \langle \cdot, \cdot \rangle \) is an hermitian form. There has been anticipation here of a result of subsequent sections in defining the form on sections over \( \mathcal{M}^{(4)} \), but in the present section any finite covering of \( \mathcal{M} \) could equally be used, with the same proofs.

If \( \psi \) and \( \phi \) are in the kernel of \( \mathcal{D}_c \), i.e., are solutions of the curved Dirac equation, the form \( \langle \psi, \phi \rangle \) is defined by the equation

\[
\langle \psi, \phi \rangle = \int_{S^3} \phi^* \gamma_4 \psi d_3 u = \int_{S^3} \phi^* \psi d_3 u.
\]

This form is obviously hermitian and positive definite; it appears to depend on the time \( t \), integration being only over space, but differentiation with respect to \( t \) together with integration by parts shows that it is in fact independent of \( t \).

**Theorem 7.7.** \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle' \) are invariant under the spinor representation of \( G \) of weights \((3/2, 3/2)\).

**Proof:** It must be shown that

\[
\langle \mathcal{D}_c dU(S) \psi, \phi \rangle + \langle \mathcal{D}_c \psi, dU(S) \phi \rangle
\]

vanishes for all sections \( \psi \) and \( \phi \) over \( \mathcal{M}^{(4)} \). By Theorem 7.6 this is equivalent to the equation

\[
0 = \int_{\mathcal{M}^{(4)}} \langle dU(S) \mathcal{D}_c \psi, \phi \rangle' d_4 u - \int_{\mathcal{M}^{(4)}} u_{-1} u_4 \langle \mathcal{D}_c \psi, \phi \rangle' d_4 u
\]

\[
+ \int_{\mathcal{M}^{(4)}} \langle \mathcal{D}_c \psi, dU(S) \phi \rangle' d_4 u.
\]

To show this apply Corollary 7.2.3 and replace \( \mathcal{D}_c \psi \) by \( \psi \); it suffices to show that
0 = \int \langle -S\Psi, \Phi \rangle' - \int u_{-1} u_{-1} \Psi, \Phi \rangle'
+ \int \langle \Psi, -S\Phi \rangle' - 2 \int \left( \frac{3}{2} u_{-1} u_{-1} \Psi, \Phi \rangle' \right.
- \frac{1}{2} \int \left. \langle \left( e^{it} u_{b} \begin{pmatrix} 0 \\ e^{-it} u_{b} \end{pmatrix} \right) \Psi, \Phi \rangle' \right.
- \frac{1}{2} \int \left. \langle \Psi, \left( e^{it} u_{b} \begin{pmatrix} 0 \\ e^{-it} u_{b} \end{pmatrix} \right) \Phi \rangle' \right.

The last two terms in fact cancel pointwise; and the remaining terms cancel by the same arguments used to establish equation (5.8) in (I).

\langle \cdot, \cdot \rangle' is evidently SU(2) x SU(2) invariant. Next it is shown to be temporally invariant. For the equation \( \partial_{c} \Psi = 0 \) is equivalent to the equation

\[
\partial \Psi/\partial t = \left[ \sum_{j=1}^{3} \left( \sigma_j \begin{pmatrix} 0 \\ -\sigma_j \end{pmatrix} \right) X_j + (3/2) \begin{pmatrix} i \\ 0 \end{pmatrix} \right] \Psi \equiv H\Psi. \quad (7.28)
\]

Since the operator \( H \) defined by Eq. (7.28) is skew-hermitian, temporal invariance follows.

It remains to prove scale invariance. Now when \( t = 0, u_{-1} = 1 \) and \( dU(S)\Psi = -S\Psi - (3/2) u_{-1} \Psi - (1/2) U_{b} \Psi \), by Corollary 7.2.3. Skew-symmetry of \( dU(S) \) with respect to \( \langle \cdot, \cdot \rangle'\) is equivalent to

\[
0 = \int \Phi^* (-S\Phi) + 2 \int (-3/2) \Phi^* \Psi + \int (-S\Phi)^* \Psi
- \frac{1}{2} \int \Phi^* (U_{b} \Psi) - \frac{1}{2} \int (U_{b} \Phi)^* \Psi.
\]

But again the last two terms cancel pointwise, and the remaining three equal 0 by the argument which established Eq. (5.9) in (I).

Finally, as in Theorem 5.6 of (I), we show that the above hermitian forms become locally identical with the usual flat forms on Minkowski space, when expressed in the flat parallelization, and evaluate the curved and flat energies (which are indefinite for solutions of the Dirac equation) in the curved and flat formats.

**Theorem 7.8.** Let \( \Psi \) and \( \Phi \) be smooth sections of the \( R^{1/2} \oplus R^{1/2} \) bundle over \( \tilde{M}^{(4)} \), and define \( \Psi_0 = F\Psi \) and \( \Phi_0 = F\Phi \) on \( M_0 \). Then \( \Psi_0 \) and \( \Phi_0 \) are the flat parallelized forms of \( \Psi \) and \( \Phi \), by (7.12).
(i) If \( \Psi \) and \( \Phi \) have support in \( \mathbf{M}_0 \), then

\[
\langle \Psi, \Phi \rangle = \int_{\mathbf{M}_0} \langle \mathcal{D}_f \Psi_0, \Phi_0 \rangle' \, d_4x.
\]

Now let \( \Psi \) and \( \Phi \) satisfy the curved Dirac equation.

(ii) Then \( \Psi_0 \) and \( \Phi_0 \) satisfy the flat Dirac equation, and

\[
\langle \Psi, \Phi \rangle = \int_{\mathbf{R}^3} (\Phi_0)^* \Psi_0 \, d_3x.
\]

In (iii) and (iv) below let \( dU \) denote \( dU_{3/2}^+ \oplus dU_{3/2}^- \).

(iii)

\[
i \langle \langle dU(T_0) \Psi, \Psi \rangle \rangle = \int \Psi^* \left( -\frac{1}{2} (1 + u_4) \sum_{j=1}^{3} b_j \gamma_5 x_j \Psi + \left( \frac{3}{4} (1 + u_4) + \frac{1}{4} U_b \right) \gamma_5 \Psi \right)
\]

and

\[
i \langle \langle dU(-\hat{T}_0) \Psi, \Psi \rangle \rangle = \int \Psi^* \left( -\frac{1}{2} (1 - u_4) \sum_{j=1}^{3} b_j \gamma_5 x_j \Psi + \left( \frac{3}{4} (1 - u_4) - \frac{1}{4} U_b \right) \gamma_5 \Psi \right).
\]

where \( \int \) denotes integration over \( S^3 \) at \( t = 0 \) with respect to \( d_3u \).

(iv)

\[
i \langle \langle dU(T_0) \Psi, \Psi \rangle \rangle = \int_{\mathbf{R}^3} (\Psi_0)^* \left( -\sum_{j=1}^{3} b_j \gamma_5 (\partial \Psi_0 / \partial x_j) \right) \, d_3x
\]

and

\[
i \langle \langle dU(-\hat{T}_0) \Psi, \Psi \rangle \rangle = \int_{\mathbf{R}^3} (\Psi_0)^* \left( -\frac{1}{4} \sum_{j=1}^{3} b_j \gamma_5 \right)
\]

\[
\times \left[ r^2 (\partial \Psi_0 / \partial x_j) + x_j \Psi_0 \right] \, d_3x
\]

(integrations at \( x_0 = 0 \)).

Proof. (i) By Theorem 7.4 and Section 3.7 of (1),

\[
\langle \Psi, \Phi \rangle = \int_{\mathbf{R}^4} \langle p^{-1} F^{-1} \mathcal{D}_f FF^{-1} \Psi_0, F^{-1} \Phi_0 \rangle' p^4 \, d_4x
\]

\[
= \int_{\mathbf{R}^4} \langle F^{-1} \mathcal{D}_f \Psi_0, F^{-1} \Phi_0 \rangle' p^3 \, d_4x.
\]
But by Eqs. (7.13) and (7.14), the adjoint of $F^{-1}$ relative to $\langle \cdot, \cdot \rangle$ is $Fp^{-1}$, hence the stated form of $\langle \langle \Psi, \Phi \rangle \rangle$.

(ii) $\mathcal{D}_f \Psi_0 = \mathcal{D}_f \Phi_0 = 0$ follow from Theorem 7.4. We have

$$\langle \langle \Psi, \Phi \rangle \rangle = \int_{S^3} (F^{-1} \Phi_0)^* (F^{-1} \Psi_0) p^1 d_3 x.$$ 

Now

$$F^{-1} = p^{-1} \begin{pmatrix} \frac{1}{2} (I + Z)^{-1} & 0 \\ 0 & \frac{1}{2} (I + Z)^{-1} \end{pmatrix}$$

and

$$\left(\frac{1}{2} (I + Z^{-1})\right)^{-1} \left(\frac{1}{2} (I + Z)\right)^{-1} = \left(\frac{1}{2} (1 + u_4)\right)^{-1} = p^{-1}$$

when $t = 0$, whence the stated form of $\langle \langle \Psi, \Phi \rangle \rangle$.

(iii) Since the matrix form of $T_0$ is $(i/4)(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix})$, the internal part of $dU(T_0)$ is $(-i/4) \gamma_3 U_0$ by Corollary 7.2.1. The first equation then follows from $T_0 = \frac{1}{2}(1 + u_4)X_0$ when $t = 0$, and Eq. (7.28). The second follows from the first, $X_0 = T_0 - \hat{T}_0$, and

$$i \langle \langle dU(X_0) \Psi, \Psi \rangle \rangle = \int \Psi^* \left( - \sum_{j=1}^{3} (b_j \gamma_3 X_j \Psi) + \frac{3}{2} \gamma_3 \Psi, \Psi \right),$$

which follows easily from (7.28).

(iv) Since $dU(T_0) = -T_0$ in the flat parallelization (cf. Corollary 7.2.4), and $(\partial \Psi_0/\partial x_0) = \sum_{j=1}^{3} \gamma_3 (\partial \Psi_0/\partial x_j)$, the first equation is clear. For the second, note that $\Omega^{-1}(-\hat{T}_0)\Omega = (\begin{smallmatrix} 0 & 0 \\ \sigma_3 & 0 \end{smallmatrix})$, so by Corollary 7.2.1 the internal part of $dU(-\hat{T}_0)$ in the flat parallelization is

$$\frac{1}{4} \gamma_3 \sum_{j=1}^{3} x_j \sigma_j.$$ 

Since also $-\hat{T}_0 = \frac{1}{4} r^2 (\partial / \partial x_0)$ when $x_0 = 0$, the last equation follows.

7.6. Basis for Spinor Fields over $S^3$

This section introduces a basis for 2-component spinor fields over $S^3$ that will be used throughout this work. The basis is constructed using the scalar basis elements $\beta_{klm}$ defined in Section 5.4 of (I). The present basis will also be used to define (i) a basis for 4-component spinor fields over $\tilde{\mathbf{M}}$ in Section 7.7, and (ii) bases for the higher spin representations in later work.

In this section $U$ will denote the representation of the simply connected Lie group (isomorphic to $SU(2) \times SU(2)$) generated by the $X_j$ and $Y_j$
(j = 1, 2, 3), on smooth $C^2$-valued functions over $S^1$, given in infinitesimal form by

$$dU(Y_j) = -Y_j, \quad dU(X_j) = -X_j - b_j$$

as in Corollary 7.2.4. A basis element (or rather 1-dimensional subspace) will be specified as a joint eigenspace of $dU(\lambda)$, $dU(\rho)$, $dU(\mu)$, and $dU(L_{12})$, as in Scholium 7.9. The enveloping algebra elements $\lambda$, $\rho$, $\mu$, and $\xi$ were defined in Section 5.4 of (I).

The irreducible constituents of $U$ turn out to be all of multiplicity 1. To see this, recall from (I) the decomposition

$$L_2(S^3) = 0 \otimes 0 + \frac{1}{2} \otimes \frac{1}{2} + 1 \otimes 1 + \cdots = \sum j \otimes j$$

of scalar functions: each $(2j + 1)^2$-dimensional space $j \otimes j$ is spanned by basis elements $\beta_{klm}$, where $2j = k + l$, $-l \leq m \leq l$, $k, l \geq 0$, and $k, l, m$ are integral. As discussed there, each space $j \otimes j$ can be regarded as a tensor product of two spin $j$ representation spaces, where the $Y_j$ act on the left factors and the $X_j$ act on the right factors.

Now corresponding to each $j \otimes j$ is a $2(2j + 1)^2$-dimensional space $j \otimes j \otimes j = j \otimes j \otimes (7.29)$ of $C^2$-valued functions, a triple tensor product. It clearly splits under the $dU(Y_j)$ and $dU(X_j)$ into irreducible subspaces

$$j \otimes (j \otimes j) = \sum_{l = -1}^l j \otimes (j + h) + j \otimes (j - h), \quad (7.30)$$

which establishes the multiplicity 1 claim above.

Further decomposition of $j \otimes (j + h)$ results by restricting $dU$ to the infinitesimal rotations $Y_j - X_j$. Thus $dU(\mu)$ takes the values $-l(l + 1)$, where $\frac{1}{2} \leq l \leq j + (j - \frac{1}{2})$ if $h = -1$ and $\frac{1}{2} \leq l \leq j + (j + \frac{1}{2})$ if $h = 1$; for a given $l$, an eigenvalue of $dU(-iL_{12})$ is denoted $m$, so that $-l \leq m \leq l$.

$l$ and $m$ are always half-integral. A more convenient set of integral parameters, alternative to $(j, h, l, m)$, is $(k, h, l, m)$, which is defined by $l = \frac{1}{2}$, $k = 2j - 1$, and $m = m - \frac{1}{2}$. The ranges of $(k, h, l, m)$ are $h = \pm 1$, $k \geq 0$ if $h = 1$ and $k \geq 1$ if $h = -1$, $l \geq 0$, and $-l - 1 \leq m \leq l$.

In the following scholium these one-dimensional subspaces are determined in terms of the

$$\beta_{k1m} = \begin{pmatrix} \beta_{k1m} \\ 0 \end{pmatrix} \quad \text{and} \quad \beta^{-1}_{k1m} = \begin{pmatrix} 0 \\ \beta_{k1m} \end{pmatrix} \quad (7.31)$$
It is understood that $\beta_{k,l,m}$ denotes 0 if $(k, l, m)$ are not in the above-stated ranges $k \geq 0$ and $-l \leq m \leq l$.

**Scholium 7.9.** Define

$$\Psi_{k,l,m}^h = i\hbar(k + l + 1 + h(l + 1))\beta_{k,l,m}^{-1} + i\hbar(k + l + 1 + h(l + 1))(l + m + 1)\beta_{k,l,m}^{h} + 2(l + 1)\beta^{-1}_{k-1,l+1,m-1} - 2(l + 1)(l - m + 1)\beta_{k-1,l+1,m}.$$  \hfill (7.32)

Then the $\Psi_{k,l,m}^h$ diagonalize the quantum numbers $k, h, l, m$: with $\Psi$ denoting $\Psi_{k,l,m}^h$ below,

$$dU(\lambda)\Psi = -(k + l)(k + l + 2)\Psi,$$

$$dU(\rho)\Psi = -(k + l + h)(h + l + h + 2)\Psi,$$

$$dU(\mu)\Psi = -(l + \frac{1}{2})(l + 3/2)\Psi,$$

$$dU(L_{12})\Psi = i(m + \frac{1}{2})\Psi,$$

so that

$$dU(\frac{1}{2}(\lambda - \rho))\Psi = dU(L_{14}L_{12} + L_{24}L_{31} + L_{14}L_{23})\Psi = (\frac{1}{2}h(k + l + 1) + \frac{1}{2})\Psi$$

and

$$dU(\xi)\Psi = (-k^2 - k(21 + 2 + h) - h(l + 1) + \frac{1}{2})\Psi.$$  

For the proof, see below. The terms in the r.h.s. of (7.32) refer to the splitting (7.29), in which $j \otimes j \otimes \frac{1}{2}$ is associated as $(j \otimes j) \otimes \frac{1}{2}$; the left side of (7.32) refers to the splitting (7.30), in which $j \otimes (j \otimes \frac{1}{2})$. The relation between the two bases expresses a sort of "nonassociativity" and is determined quantitatively by the "$6j$ symbols." To facilitate present and later applications, we briefly recall their definition.

Consider a tensor product $j_1 \otimes j_2 \otimes j_3$ of three representations of $SU(2)$; let the copies of $su(2)$ acting on each factor be spanned over the reals by $J^j_1, J^j_2, J^j_3$ ("left," "right," and "internal"); $j = 1, 2, 3$ satisfying the commutation relations stated in Section 5.4 of (I), e.g., $[J^j_1, J^j_2] = iJ^j_3$. Take normalized basis elements $|j_1, m_1\rangle, |j_2, m_2\rangle, |j_3, m_3\rangle$ spanning this tensor product space and satisfying (5.10-12) of (I). Now there are two natural bases of $j_1 \otimes j_2 \otimes j_3$, which diagonalize the third component $J^3_1 + J^3_2 + J^3_3$ of the total angular momentum, i.e.,

$$|j_1, (j_2, j_3)_{j_{23}}; l m\rangle = \sum_{m_1, m_2, m_3} \langle j_1, m_1; j_{23}, m_{23} | l m\rangle |j_1, m_1\rangle |j_{23}, m_{23}\rangle.$$
where
\[ |j_{23}m'\rangle = \sum_{m_2, m_2} \langle j_2 m_2 ; j_3 m_3 | j_{23} m' \rangle | j_2 m_2 \rangle | j_3 m_3 \rangle \]
whose elements are indexed by the triples \((j_2, l, m)\); and an analogous basis, denoted \(|(j_1, j_2) j_{12}, j_3 ; \ell m\rangle\), equal to
\[ \sum_{m', m_3} \langle j_{12} m'; j_3 m_3 | j_{12} m' \rangle | j_3 m_3 \rangle, \]
where
\[ |j_{12}m'\rangle = \sum_{m_1, m_2} \langle j_1 m_1 ; j_2 m_2 | j_{12} m' \rangle | j_1 m_1 \rangle | j_2 m_2 \rangle, \]
and indexed by the triples \((j_{12}, l, m)\). The equations relating the two bases are [6]
\[ \langle j_1, (j_2, j_3) j_{23} ; \ell m \rangle = (-1)^{j_1 + j_2 + j_3 + l} \sum_{j_{12}} ((2j_{12} + 1)(2j_{23} + 1))^{1/2} \]
\[ \cdot \begin{vmatrix} j_1 & j_2 & j_{12} \\ j_3 & l & j_{23} \end{vmatrix} \langle (j_1, j_2) j_{12}, j_3 ; \ell m \rangle, \] (7.33)
where
\[ \begin{vmatrix} j_1 & j_2 & j_{12} \\ j_3 & l & j_{23} \end{vmatrix} \]
are the 6j symbols, which are real.

Proof of Scholium 7.9. In the present case, \(j_1 = j_2 = j, j_3 = \frac{1}{2}, j_{23} = j + \frac{1}{2}h\), and \(l, m\) are the \(l = l + \frac{1}{2}, m = m + \frac{1}{2}\) defined before the statement of the scholium. As in (I), we take \(j_j = -\frac{\pi}{j} Y_{ij}\) and \(j_{ij} = \frac{\pi}{j_i} X_{ij}\); also \(j_{j_{ij}} = \frac{1}{2}j_{ij}\).

The left side of (7.33) diagonalizes \(\lambda, \rho, \mu,\) and \(L_{12}\) by definition. In addition,
\[ |(j_1, j_2) j_{12}, j_3 ; \ell m \rangle = \sum_{m_{12}, m_3} \langle j_{12} m_{12} ; j_3 m_3 | \ell m \rangle | j_{12} m_{12} \rangle | j_3 m_3 \rangle \]
and \(|j_{12} m_{12}\rangle\) equals the scalar function \(|k' j_{12} m_{12}\rangle\) defined in (5.20) in (I), where \(k' + j_{12} = 2j_1\). Also \(|j_3 m_3\rangle = (\lambda) (\lambda)\) if \(m_3 = \frac{1}{2} (-\frac{1}{2})\), so that \(|j_{12} m_{12}\rangle | j_3 m_3 \rangle\) is proportional to
\[ \beta_{k'j_{12}m_{12}}^{2m_3} \]
In this way the combination (7.33) and (7.34) will give the right side of (7.32), up to a computed constant factor, after insertion of the required Clebsch-Gordan and 6j coefficients and simplification.
Since \( j_3 = \frac{1}{2} \) and \( l = l + \frac{1}{2} \), the sum over \( j_{12} \) in (7.33) is over \( j_{12} = l \) and \( j_{12} = l + 1 \). We list the required coefficients, from [6, pp. 98, 89]:

\[
\begin{aligned}
\left\{ \begin{array}{c}
j \quad j \quad l \\
\frac{1}{2} \quad l + \frac{1}{2} \quad j + \frac{1}{2}h
\end{array} \right\} &= (-1)^{k+1} \frac{k + l + 1 + h(l+1)}{2(2l+1)(l+k+1)(l+k+1+h)}^{1/2}, \\
\left\{ \begin{array}{c}
j \quad j \quad l + 1 \\
\frac{1}{2} \quad l + \frac{1}{2} \quad j + \frac{1}{2}h
\end{array} \right\} &= (-1)^{k+1} \frac{k + l + 1 - h(l+1)}{2(2l+3)(l+k+1)(l+k+1+h)}^{1/2},
\end{aligned}
\]

\[
\begin{aligned}
\langle l m + 1; \frac{1}{2} \frac{1}{2} l + \frac{1}{2} m + \frac{1}{2} | l + \frac{1}{2} m + \frac{1}{2} \rangle &= \left( \frac{l-m}{2l+1} \right)^{1/2}, \\
\langle l m; \frac{1}{2} \frac{1}{2} l + \frac{1}{2} m + \frac{1}{2} | l + \frac{1}{2} m + \frac{1}{2} \rangle &= \left( \frac{l-m+1}{2l+1} \right)^{1/2}, \\
\langle l + 1 m + 1; \frac{1}{2} \frac{1}{2} l + \frac{1}{2} m + \frac{1}{2} | l + \frac{1}{2} m + \frac{1}{2} \rangle &= \left( \frac{l+m+2}{2l+3} \right)^{1/2},
\end{aligned}
\]

and

\[
\langle l + 1 m; \frac{1}{2} \frac{1}{2} l + \frac{1}{2} m + \frac{1}{2} | l + \frac{1}{2} m + \frac{1}{2} \rangle = - \left( \frac{l-m+1}{2l+3} \right)^{1/2}.
\]

Substitution into (7.33) and (7.34) gives

\[
\begin{aligned}
| j, (j, \frac{1}{2}) j + \frac{1}{2} h; l + \frac{1}{2} m &+ \frac{1}{2} \rangle \\
&= (-1)^{k+1} (k + l + 1 + h)^{1/2} \\
&\times \bigg\{ (-1)^{k+1} h \left( \frac{k + l + 1 + h(l+1)}{2(l+k+1)(l+k+1+h)} \right)^{1/2} \\
&\times \bigg[ \left( \frac{l-m}{2l+1} \right)^{1/2} | k l m + 1 \rangle | \frac{1}{2} \frac{1}{2} - \frac{1}{2} \rangle + \left( \frac{l+m+1}{2l+1} \right)^{1/2} | k l m \rangle | \frac{1}{2} \frac{1}{2} \rangle \bigg] \\
&+ (-1)^{k+1} \left( \frac{k + l + 1 - h(l+1)}{2(l+k+1)(k+l+1+h)} \right)^{1/2} \\
&\times \bigg[ \left( \frac{l+m+2}{2l+3} \right)^{1/2} | k+1 l + 1 m + 1 \rangle | \frac{1}{2} \frac{1}{2} \rangle \\
&- \left( \frac{l-m+1}{2l+3} \right)^{1/2} | k+1 l + 1 m \rangle | \frac{1}{2} \frac{1}{2} \rangle \bigg] \bigg\}.
\end{aligned}
\]

(7.35)
After substituting definition (5.20) (from (I)) of the $|k \ell m\rangle$,

$$|j, (j, \frac{1}{2}) j + \frac{1}{2} h; l + \frac{1}{2} m + \frac{1}{2}\rangle$$

$$= i^{-(l+1)} \frac{2^{l!}(k!)^{1/2} \Gamma((l-m)!)^{1/2} \cdot \psi_{k\ell m}^h}{(2k+2l+1)! (l+m+1)! (k+l+1+h(l+1)))^{1/2}},$$  \hspace{1cm} (7.36)

where the $\psi_{k\ell m}^h$ were defined in (7.32).

As an abbreviation, define

$$|k \ell m\rangle = |j, (j, \frac{1}{2}) j + \frac{1}{2} h; l + \frac{1}{2} m + \frac{1}{2}\rangle,$$  \hspace{1cm} (7.37)

where $k = 2j - l$ as always. Since the $|k \ell m\rangle$ are normalized, so are the $|k \ell m\rangle$, whence the

**Corollary 7.9.1.**

$$\int_{S^1} |\psi_{k\ell m}^h|^2 \omega_3 = \frac{(k+2l+1)! (l+m+1)! (k+l+1+h(l+1))}{2^{l+1} (l!)^2 k!(l-m)!}.$$  \hspace{1cm} (7.36)

7.7. Actions of $dU(L_{j4})$ on Spinor Basis

By (5.10-12) of (I), we have

$$dU(L_{12}) |k \ell m\rangle = i(m + \frac{1}{2}) |k \ell m\rangle,$$

$$dU(L_{31} - iL_{23}) |k \ell m\rangle = (l - m)^{1/2} (l + m + 2)^{1/2} |k \ell m + 1\rangle,$$  \hspace{1cm} (7.38)

$$dU(-L_{31} - iL_{23}) |k \ell m\rangle = (l + m + 1)^{1/2} (l - m + 1)^{1/2} |k \ell m - 1\rangle.$$  \hspace{1cm} (7.38)

The purpose of this section is to derive analogous formulas for the $dU(L_{j4})$, which are somewhat similar to those in Table VIII in (I).

Some additional notation from (I) for linear combinations of the $X_j$ and $Y_j$ will be useful. Besides the previous $J_j^a$ and $J_j^p$, (I) defined

$$J_j = J_j^a + J_j^p, \quad J_+ = J_1 + iJ_2, \quad J_- = J_1 - iJ_2.$$  \hspace{1cm} (7.39)

and also similarly defined $J_j^a$ and $J_j^p$ by replacing $J_j$ by $J_j^a$ and $J_j^p$ ($j = 1, 2$) in (7.39). Thus

$$J_+ = -\frac{1}{2}iY_1 - \frac{1}{2}Y_2 \quad \text{and} \quad J_+^a = -\frac{1}{2}iY_1 + \frac{1}{2}Y_2.$$  \hspace{1cm} (7.39)

Recall also $Y_1 = L_{14} + L_{23}$, $X_1 = L_{14} - L_{23}$, etc., so that

$$J_+ = L_{31} - iL_{23}, \quad J_- = -L_{31} - iL_{23}, \quad J_3 = -iL_{12},$$

$$L_{24} + iL_{14} = -2J_+ + J_-, \quad L_{24} - iL_{14} = 2J_+ - J_+.$$
SCHOLIUM 7.10. In any left-parallelized representation $dU$ of the $X_j$ and $Y_j$ in an induced bundle over $S^3$:

1. $\text{id}U(L_{34}) = -(L_{34} + \text{id}L_{12}) - \text{id}U(L_{12}),$

2. $dU(L_{24} - iL_{14}) = -(L_{31} + iL_{23}) - (L_{24} - iL_{14}) - dU(I_+),$

3. $dU(L_{24} + iL_{14}) = -(L_{31} + iL_{23}) - (L_{24} + iL_{14}) + dU(I_-)$. 

Proof. The only fact used is that $dU(Y_j) = -Y_j$ in any left-parallelized representation.

For convenience the commutation relations of the various basis elements are listed.

SCHOLIUM 7.11.

\[
\begin{align*}
[L_{ij}, L_{jk}] &= L_{ik} \quad (1 \leq i, j, k \leq 4), \\
[J_3, J_+] &= J_+,
[J_3, J_-] &= -J_-, \quad [J_+, J_-] = 2J_3; \\
[J_3, iL_{34}] &= 0,
[J_3, L_{24} - iL_{14}] &= L_{24} - iL_{14}, \quad [J_3, L_{24} + iL_{14}] = -(L_{24} + iL_{14}); \\
[J_+, iL_{34}] &= L_{24} - iL_{14},
[J_+, L_{24} - iL_{14}] &= 0, \quad [J_+, L_{24} + iL_{14}] = 2iL_{34}; \\
[J_-, iL_{24}] &= L_{24} + iL_{14},
[J_-, L_{24} - iL_{14}] &= 2iL_{34}, \quad [J_-, L_{24} + iL_{14}] = 2iL_{34}, \\
[L_{24} + iL_{14}, L_{24} - iL_{14}] &= 2J_3,
[iL_{34}, L_{24} - iL_{14}] &= -J_+,
[iL_{34}, L_{24} + iL_{14}] &= -J_-.
\end{align*}
\]

By Scholium 7.10 it follows that we must compute the external actions of all of the $L_{ij}$ $(1 \leq i, j \leq 4)$ on the $|k \, h \, l \, m\rangle$; this appears practical only by reduction to the scalar case and use of Table VIII of (I). By (7.35) we have

\[
|k \, 1 \, l \, m\rangle = \frac{(k + 2l + 2)^{1/2}}{(2(k + l + 1))^{1/2}} W_1(k, l, m) + \frac{k^{1/2}}{(2(k + l + 1))^{1/2}} W_{-1}(k, l, m),
\]

\[
|k \, -1 \, l \, m\rangle = -\frac{k^{1/2}}{(2(k + l + 1))^{1/2}} W_1(k, l, m) + \frac{(k + 2l + 2)^{1/2}}{(2(k + l + 1))^{1/2}} W_{-1}(k, l, m),
\]
where

\[ W_{1}(k, l, m) = \frac{(l-m)^{1/2}}{(2l+1)^{1/2}} |k \ l \ m + 1\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{(l+m+1)^{1/2}}{(2l+1)^{1/2}} |k \ l \ m\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

\[ W_{-1}(k, l, m) = \frac{(l+m+2)^{1/2}}{(2l+3)^{1/2}} |k - 1 \ l + 1 \ m + 1\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{(l-m+1)^{1/2}}{(2l+3)^{1/2}} |k - 1 \ l + 1 \ m\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \]

these relations are readily reversible. In this way the vector field actions of the $L_{ij}$ on the $W_{x_{1}}(k, l, m)$, and then on the $|k \pm 1 \ l \ m\rangle$, were computed. The results of the latter are given in Table I. Table II results from this and Scholium 7.10.

7.8. Action of Dirac Operator on Basis

A convenient basis for spinor fields may be defined as follows. First define $C^{2}$-valued functions on $\tilde{\mathbb{M}}$ as follows:

\[ \psi^{h}_{klmn} = e^{int} \psi^{h}_{klm}, \quad |n \ k \ h \ l \ m\rangle = e^{int} |k \ h \ l \ m\rangle. \]

where $n$ is real and otherwise arbitrary. In terms of these, define $C^{4}$-valued functions on $\tilde{\mathbb{M}}$ by the following equations:

\[ \psi^{h} = \begin{pmatrix} \psi^{h}_{klmn} \\ 0 \end{pmatrix}, \quad \psi^{h-1} = \begin{pmatrix} 0 \\ \psi^{h}_{klmn} \end{pmatrix}, \]

\[ |n \ k \ h \ l \ m\rangle = \begin{pmatrix} |n \ k \ h \ l \ m\rangle \\ 0 \end{pmatrix}, \quad |n \ k \ h \ l \ m - 1\rangle = \begin{pmatrix} 0 \\ |n \ k \ h \ l \ m\rangle \end{pmatrix}. \]

Relative to the curved parallelization for spinor fields, the latter functions form a basis for such fields. To treat the action of the Dirac operator on them it is convenient to use an expression for it in terms largely of enveloping algebra elements.

**Scholium 7.12.** Let $U$ denote one of the induced representations $U^{+}_{w_{1}} \oplus U^{-}_{w_{2}}$ in the curved parallelization. Then

\[ \mathcal{D}_{c} = i\gamma_{4} dU(X_{0}) + (1/2) \gamma_{4} \gamma_{5} dU(\rho - \lambda). \]

**Proof.** This follows from Corollary 7.2.4.
TABLE I
External Actions of the $L_{ij}$ on the $|kilm\rangle$

$$
(iL_{12})|k 1 l m\rangle = \frac{(2m+1)[k(l+1)+(l+2)+l(l+2)]}{2(k+1)(2l+1)(2l+3)} |k 1 l m\rangle
$$

$$
+ \frac{k^{1/2}(k+2l+2)^{1/2}(l+1)(2m+1)}{(k+1)(2l+1)(2l+3)} |k-1 l m\rangle
$$

$$
+ \frac{(k+2l+2)^{1/2}(k+1)^{1/2}[(l-m)^{1/2}(l+m+1)^{1/2}]}{2(k+1)(2l+1)} |k+1 l-1 l-1 m\rangle
$$

$$
+ \frac{(k+2l+3)^{1/2}(l-m+1)^{1/2}(l+m+2)^{1/2}}{2(k+1)(2l+3)} |k-1 l+1 l+1 m\rangle
$$

$$
- \frac{k^{1/2}(k-1)^{1/2}(l-m+1)^{1/2}(l+m+2)^{1/2}}{2(k+1)(2l+3)} |k-1 l-1 l+1 m\rangle
$$

$$
(iL_{13})|k-1 l m\rangle = \frac{(2m+1)[kl(l+1)+k(2l+2)-(l+2)(2l+1)]}{2(k+1)(2l+1)(2l+3)} |k-1 l m\rangle
$$

$$
+ \frac{k^{1/2}(k+2l+2)^{1/2}(l+1)(2m+1)}{(k+1)(2l+1)(2l+3)} |k 1 l m\rangle
$$

$$
- \frac{k^{1/2}(k+1)^{1/2}(l-m)^{1/2}(l+m+1)^{1/2}}{2(k+1)(2l+1)} |k+1 l-1 l-1 m\rangle
$$

$$
- \frac{k^{1/2}(k+2l+1)^{1/2}(l-m)^{1/2}(l+m+1)^{1/2}}{2(k+1)(2l+1)} |k+1 l-1 l-1 m\rangle
$$

$$
- \frac{(k-1)^{1/2}(k+2l+2)^{1/2}(l-m+1)^{1/2}(l+m+2)^{1/2}}{2(k+1)(2l+3)} |k-1 l-1 l+1 m\rangle
$$

$$
+ \frac{(k+2l+2)^{1/2}(k+2l+3)^{1/2}(l-m+1)^{1/2}(l+m+2)^{1/2}}{2(k+1)(2l+3)} |k-1 l+1 l+1 m\rangle
$$

Table continued
\[ (-L_{31} + iL_{23}) |k \ 1 \ m \rangle = \frac{(l - m)^{1/2} (l + m + 2)^{1/2} [(k + 2l + 2)(l + 3) + k(l + 2)(2l + 1)]}{(k + l + 1)(2l + 1)(2l + 3)} |k - 1 \ 1 \ m + 1 \rangle \]

\[ + \frac{2k^{1/2}(k + 2l + 2)^{1/2} (l + 1)(l - m)^{1/2} (l + m + 2)^{1/2}}{(k + l + 1)(2l + 1)(2l + 3)} |k - 1 \ l \ 1 \ m + 1 \rangle \]

\[ - \frac{k^{1/2}(k + 2l + 3)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2}}{2(k + l + 1)(2l + 3)} |k - 1 \ 1 \ l + 1 \ m + 1 \rangle \]

\[ + \frac{k^{1/2}(k - 1)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2}}{2(k + l + 1)(2l + 3)} |k - 1 \ - 1 \ l + 1 \ m + 1 \rangle \]

Table continued
(L_{31} + iL_{23}) | k \, l \, m = \frac{2k^{1/2}(k + 2l + 2)^{1/2} (l + 1)(l - m + 1)^{1/2} (l + m + 1)^{1/2}}{(k + l + 1)(2l + 1)(2l + 3)} | k - 1 \, l \, m - 1 \rangle

+ \frac{(l - m + 1)^{1/2} (l + m + 1)^{1/2} [(k + 2l + 2) l(2l + 3) + k(l + 2)(2l + 1)]}{(k + l + 1)(2l + 1)(2l + 3)} | k - 1 \, l \, m - 1 \rangle.

\times | k \, l \, m - 1 \rangle

+ \frac{k^{1/2}(k + 2l + 3)^{1/2} (l - m + 2)^{1/2} (l - m + 1)^{1/2}}{2(k + l + 1)(2l + 3)} | k - 1 \, l \, l + 1 \, m - 1 \rangle

- \frac{k^{1/2}(k - 1)^{1/2} (l - m + 1)^{1/2} (l - m - 2)^{1/2}}{2(k + l + 1)(2l + 3)} | k - 1 \, l - 1 \, l + 1 \, m - 1 \rangle

- \frac{(k + 1)^{1/2} (k + 2l + 2)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2}}{2(k + l + 1)(2l + 1)} | k + 1 \, l - 1 \, l - 1 \, m - 1 \rangle

- \frac{(k + 2l + 2)^{1/2} (k + 2l + 1)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2}}{2(k + l + 1)(2l + 1)} | k + 1 \, l - 1 \, l - 1 \, m - 1 \rangle

(L_{31} + iL_{23}) | k - 1 \, l \, m = \frac{2k^{1/2}(k + 2l + 2)^{1/2} (l + 1)(l + m + 1)^{1/2} (l - m + 1)^{1/2}}{(k + l + 1)(2l + 1)(2l + 3)} | k - 1 \, l \, m - 1 \rangle

+ \frac{(l + m + 1)^{1/2} (l - m + 1)^{1/2} [kl(2l + 3) + (k + 2l + 2)(l + 1)(2l + 1)]}{(k + l + 1)(2l + 1)(2l + 3)} | k - 1 \, l \, m - 1 \rangle

+ \frac{k^{1/2}(k + 1)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2}}{2(k + l + 1)(2l + 1)} | k + 1 \, l - 1 \, l - 1 \, m - 1 \rangle

+ \frac{k^{1/2}(k + 2l + 1)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2}}{2(k + l + 1)(2l + 1)} | k + 1 \, l - 1 \, l - 1 \, m - 1 \rangle

+ \frac{(k + 2l + 2)^{1/2} (k + 2l + 3)^{1/2} (l - m + 1)^{1/2} (l - m + 2)^{1/2}}{2(k + l + 1)(2l + 3)} | k - 1 \, l + 1 \, l + 1 \, m - 1 \rangle

- \frac{(k - 1)^{1/2} (k + 2l + 2)^{1/2} (l - m + 1)^{1/2} (l - m + 2)^{1/2}}{2(k + l + 1)(2l + 3)} | k - 1 \, l + 1 \, l + 1 \, m - 1 \rangle
\[(iL_{34}) | k l m \rangle = \frac{(2m + 1) k(k + 2l + 2)}{(k + l + 1)(2l + 1)(2l + 3)} | k l m \rangle - \frac{(2m + 1) k^{1/2}(2l + k + 1)^{1/2} (l + 1)}{(k + l + 1)(2l + 1)(2l + 3)} | k - 1 l m \rangle + \frac{k^{1/2} (k - 1)^{1/2} (l - m + 1)^{1/2} (l + m + 2)^{1/2}}{2(k + l + 1)(2l + 3)} | k - 1 - 1 l + 1 m \rangle + \frac{k^{1/2} (k + 2l + 3)^{1/2} (l - m + 1)^{1/2} (l + m + 2)^{1/2}}{2(k + l + 1)(2l + 1)} | k - 1 1 l + 1 m \rangle + \frac{(k + 2l + 2)^{1/2} (k + 1)^{1/2} (l - m + 1)^{1/2} (l + m + 1)^{1/2} (2k + 2l + 1)}{2(k + l + 1)(2l + 1)} | k + 1 1 l - 1 m \rangle - \frac{(k + 2l + 2)^{1/2} (k + 2l + 1)^{1/2} (l - m)^{1/2} (l + m + 1)^{1/2} (2k + 2l + 1)}{2(k + l + 1)(2l + 1)} | k + 1 1 l - 1 m \rangle + \frac{k^{1/2} (k + 1)^{1/2} (l - m)^{1/2} (l + m + 1)^{1/2} (2k + 2l + 3)}{2(k + l + 1)(2l + 1)} | k + 1 1 - 1 l + 1 m \rangle + \frac{(k + 2l + 2)^{1/2} (k + 2l + 3)^{1/2} (l - m - 1)^{1/2} (l + m + 2)^{1/2}}{2(k + l + 1)(2l + 3)} | k - 1 1 l + 1 m \rangle + \frac{(k - 1)^{1/2} (k + 2l + 2)^{1/2} (l - m + 1)^{1/2} (l + m + 2)^{1/2} (2k + 2l + 3)}{2(k + l + 1)(2l + 3)} | k - 1 - 1 l + 1 m \rangle \times | k - 1 - 1 l + 1 m \rangle \]

Table continued
ANALYSIS IN THE SPINOR AND FORM BUNDLES

Table I—(continued)

\[(L_{24} - iL_{14}) |k \ 1 \ l \ m \rangle\]
\[= \frac{2k(k + 2l + 2)(l - m)^{1/2} (l + m + 2)^{1/2} }{(k + l + 1)(2l + 1)(2l + 3)} |k \ 1 \ l \ m + 1 \rangle \]
\[+ \frac{2k^{1/2}(k + 2l + 2)^{1/2} (l + 1)(l - m)^{1/2} (l + m + 2)^{1/2} }{(k + l + 1)(2l + 1)(2l + 3)} |k - 1 \ l \ m + 1 \rangle \]
\[+ \frac{k^{1/2}(k - 1)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2} }{2(k + l + 1)(2l + 3)} |k - 1 - 1 \ l \ m + 1 \rangle \]
\[+ \frac{k^{1/2}(k + 2l + 3)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2} (2k + 2l + 1) }{2(k + l + 1)(2l + 3)} |k - 1 1 \ l \ m + 1 \rangle \]
\[- \frac{(k + 1)^{1/2}(k + 2l + 2)^{1/2} (l - m)^{1/2} (l - m - 1)^{1/2} (2k + 2l + 1) }{2(k + l + 1)(2l + 1)} |k + 1 - 1 \ l \ m + 1 \rangle \]
\[\times |k + 1 1 \ l - 1 \ m + 1 \rangle \]
\[+ \frac{(k + 2l + 2)^{1/2} (k + 2l + 1)^{1/2} (l - m)^{1/2} (l - m - 1)^{1/2} }{2(k + l + 1)(2l + 1)} |k + 1 - 1 \ l - 1 \ m + 1 \rangle \]

\[(L_{24} - iL_{14}) |k - 1 \ l \ m \rangle\]
\[= \frac{2k^{1/2}(k + 2l + 2)^{1/2} (l + 1)(l - m)^{1/2} (l + m + 2)^{1/2} }{(k + l + 1)(2l + 1)(2l + 3)} |k \ 1 \ l \ m + 1 \rangle \]
\[\frac{- 2k(k + 2l + 2)(l - m)^{1/2} (l + m + 2)^{1/2} }{(k + l + 1)(2l + 1)(2l + 3)} |k - 1 \ l \ m + 1 \rangle \]
\[\frac{- k^{1/2}(k + 2l + 1)^{1/2} (l - m)^{1/2} (l - m - 1)^{1/2} (2k + 2l + 3) }{2(k + l + 1)(2l + 1)} |k + 1 - 1 \ l - 1 \ m + 1 \rangle \]
\[- \frac{k^{1/2}(k + 1)^{1/2} (l - m)^{1/2} (l - m - 1)^{1/2} }{2(k + l + 1)(2l + 1)} |k + 1 1 \ l - 1 \ m + 1 \rangle \]
\[- \frac{(k + 2l + 2)^{1/2} (k + 2l + 3)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2} }{2(k + l + 1)(2l + 3)} |k - 1 1 \ l \ m + 1 \rangle \]
\[+ \frac{(k + 1)^{1/2}(k + 2l + 2)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2} (2k + 2l + 3) }{2(k + l + 1)(2l + 3)} |k - 1 - 1 \ l + 1 \ m + 1 \rangle \]
\[\times |k - 1 - 1 \ l + 1 \ m + 1 \rangle \]

Table continued
\( (L_{24} + iL_{14}) \mid k \, 1 \, l \, m \rangle 
\begin{align*}
\left( L_{24} + iL_{14} \right) | k \, 1 \, l \, m \rangle &= \frac{-2k^{1/2}(k + 2l + 2)^{\frac{1}{2}} (l + m + 1)^{\frac{1}{2}} (l - m + 1)^{\frac{1}{2}} (l + 1)}{(k + l + 1)(2l + 1)(2l + 3)} | k - 1 \, l \, m - 1 \rangle \\
&+ \frac{-2k(k + 2l + 2)(l + m + 1)^{\frac{1}{2}} (l - m + 1)^{\frac{1}{2}}}{(k + l + 1)(2l + 1)(2l + 3)} | k \, 1 \, l \, m - 1 \rangle \\
&+ \frac{k^{1/2}(k - 1)^{1/2} (l - m + 1)^{1/2} (l - m + 2)^{1/2}}{2(k + l + 1)(2l + 3)} | k - 1 - 1 \, l + 1 \, m - 1 \rangle \\
&+ \frac{k^{1/2}(k + 2l + 3)^{1/2} (l - m + 1)^{1/2} (l - m + 2)^{1/2} (2k + 2l + 1)}{2(k + l + 1)(2l + 3)} | k - 1 \, 1 \, l + 1 \, m - 1 \rangle \\
&- \frac{(k + 1)^{1/2} (k + 2l + 2)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2} (2k + 2l + 1)}{2(k + l + 1)(2l + 3)} \\
&\times | k + 1 \, 1 \, l - 1 \, m - 1 \rangle \\
&+ \frac{(k + 2l + 2)^{1/2} (k + 2l + 1)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2}}{2(k + l + 1)(2l + 1)} | k + 1 - 1 \, l - 1 \, m - 1 \rangle \\
&+ \frac{k^{1/2}(k + 2l + 1)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2} (2k + 2l + 3)}{2(k + l + 1)(2l + 3)} | k + 1 - 1 \, l - 1 \, m - 1 \rangle \\
&+ \frac{k^{1/2}(k + 2l + 1)^{1/2} (l + m)^{1/2} (l + m + 1)^{1/2} (2k + 2l + 3)}{2(k + l + 1)(2l + 3)} | k + 1 - 1 \, l - 1 \, m - 1 \rangle \\
&\times | k + 1 - 1 \, l - 1 \, m - 1 \rangle
\end{align*}
ANALYSIS IN THE SPINOR AND FORM BUNDLES

TABLE II

Action of the $dU(L_{j4})$ on the $|k h l m\rangle$

\[
i dU(L_{j4}) |k \bar{1} l m\rangle = -\frac{(k + 2l + 2)^{1/2} (l - m)^{1/2} (l + m + 1)^{1/2}}{2l + 1} |k + 1 \bar{1} l - 1 m\rangle
\]
\[
+ \frac{(2m + 1)(2k + 2l + 3)}{2(2l + 1)(2l + 3)} |k \bar{1} l m\rangle
\]
\[
- \frac{k^{1/2}(k + 2l + 3)^{1/2} (l - m + 1)^{1/2} (l + m + 2)^{1/2}}{2l + 3} |k - 1 \bar{1} l + 1 m\rangle
\]

\[
i dU(L_{j4}) |k - \bar{1} l m\rangle = -\frac{k^{1/2}(k + 2l + 1)^{1/2} (l - m)^{1/2} (l + m + 1)^{1/2}}{2l + 1} |k + 1 - 1 l - 1 m\rangle
\]
\[
- \frac{(2m + 1)(2k + 2l + 1)}{2(2l + 1)(2l + 3)} |k - \bar{1} l m\rangle
\]
\[
- \frac{(k - 1)^{1/2} (k + 2l + 2)^{1/2} (l - m + 1)^{1/2} (l + m + 2)^{1/2}}{2l + 3} |k - 1 - 1 l + 1 m\rangle
\]

\[
dU(L_{24} - iL_{14}) |k \bar{1} l m\rangle = \frac{(k + 1)^{1/2} (k + 2l + 2)^{1/2} (l - m)^{1/2} (l + m + 1)^{1/2}}{2l + 1} |k + 1 \bar{1} l - 1 m + 1\rangle
\]
\[
- \frac{(2k + 2l + 3)}{(2l + 1)(2l + 3)} |k \bar{1} l m + 1\rangle
\]
\[
- \frac{k^{1/2}(k + 2l + 3)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2}}{2l + 3} |k - 1 \bar{1} l + 1 m + 1\rangle
\]

\[
dU(L_{14} - iL_{24}) |k \bar{1} l m\rangle = \frac{k^{1/2}(k + 2l + 1)^{1/2} (l - m)^{1/2} (l - m - 1)^{1/2}}{2l + 1} |k + 1 - 1 l - 1 m + 1\rangle
\]
\[
+ \frac{(2k + 2l + 1)}{(2l + 1)(2l + 3)} |k - 1 l m + 1\rangle
\]
\[
- \frac{(k - 1)^{1/2} (k + 2l + 2)^{1/2} (l + m + 2)^{1/2} (l + m + 3)^{1/2}}{2l + 3} |k - 1 - 1 l + 1 m + 1\rangle
\]

Table continued
\[ \begin{align*}
\mathcal{D}_c \Psi_{klnm}^{he} &= (n - he(k + l + 1 + 1/2)) \Psi_{klnm}^{he - e}.
\end{align*} \]

**Proof.** This follows from Scholia 7.12 and 7.9.

The next result treats the relation between the square of the Dirac operator and the (scalar) wave operator.

**Corollary 7.12.2.** With the notation of Scholium 7.12, for any constant c:

\[ \mathcal{D}_c[X_0^2 - c\lambda - (1 - c)p + 1 - c(1 - c)] = -(\mathcal{D}_c + (c - \frac{1}{2})\gamma_4\gamma_5)^2. \]

In particular,

\[ -(\mathcal{D}_c + \frac{1}{2}\gamma_4\gamma_5)^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2 + 1 \]

\((c = 1)\), and

\[ \mathcal{D}_c[X_0^2] = -(\mathcal{D}_c)^2 \]
(c = 1/3). The value of the left side of Eq. (7.40) on $\Psi_{klmn}^{he}$ is

$$n^2 (k + l + 1 + (1 - c)h)^2.$$

**Proof.** It suffices to check that the results of applying the two sides of Eq. (7.40) to a basis agree. By Corollary 7.12.1,

$$(D_c + (c - \frac{1}{2}) \gamma_4 \gamma_5) \Psi_{klmn}^{he} = (n - he(k + l + 1 + (1 - c)h)) \Psi_{klmn}^{h-e}.$$ 

Now using Scholium 7.9, the proof is concluded.

The following theorem shows that the kernel of $D_c$ consists of four distinct subspaces, on each of which the energy is one-sided. It will appear in Section 7.10 that these spaces are precisely the minimal (conformally) invariant subspaces of $U^+_{3/2} \oplus U^-_{3/2}$; and further, that this spinor representation is the unique one admitting such minimal invariant subspaces of the type specified in Theorem 7.22.

**Theorem 7.13.** All distribution solutions of the equation $D_c \Psi = 0$ on $R^1 \times S^3$ are invariant under $\xi^4$ (or equivalently, are periodic in time with period $4\pi$), and the temporal evolution group defined by this equation acts continuously on arbitrary $L^2$-Sobolev spaces over $S^3$.

Explicitly, any such solution has the form

$$\left( e^{it/2} 0 \
0 e^{-it/2} \right) \left( 1 - A \right)^k \left( e^{-it/2} 0 \
0 e^{it/2} \right) \Psi$$

for some solution $\Psi$ of class $C^1$ and positive integer $k$. Moreover for any solution $\Psi$ of the equation $D_c \Psi = 0$, the components of the matrix

$$\left( e^{-it/2} 0 \
0 e^{it/2} \right) \Psi$$

satisfy the curved wave equation

$$(\square_c + 1) \left( e^{-it/2} 0 \
0 e^{it/2} \right) \Psi = 0.$$ 

Such solutions of the equation $D_c \Psi = 0$ are spanned by those $\Psi_{klmn}^{he}$, where

- (1) $e = 1, h = 1, n = k + 1 + 3/2$;
- (2) $e = 1, h = -1, n = -k - 1 - 1/2$;
- (3) $e = -1, h = 1, n = -k - 1 - 3/2$;
- (4) $e = -1, h = -1, n = k + 1 + 1/2$.

The totality of solutions is not periodic with period less than $4\pi$, or equivalently, cannot be lifted up from any lower covering of $\tilde{M}$ than $\tilde{M}^{(4)}$. 

Proof. In view of the equation
\[
\left( \begin{array}{cc}
e^{it/2} & 0 \\ 0 & e^{-it/2} \end{array} \right) \left( \mathcal{D}_{c} + \frac{1}{2} \gamma_{4} \gamma_{5} \right) \left( \begin{array}{cc} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{array} \right) = \mathcal{D}_{c}
\]
and Corollary 7.12.2, the proof is essentially the same as that of Theorem 5.5 of (I). The assertions regarding the basis follow from Corollary 7.12.1.

**Corollary 7.13.1.** Let \( n \) be integral or half-integral. Then
\[
\mathbf{Y} = \mathbf{y} \left( k \right) \mathbf{y} \mathbf{h} \mathbf{m} \mathbf{e} \text{ is lifted up from } \mathbf{M}^{(4)}, \text{ and}
\]
\[
\langle \langle \mathbf{Y}, \mathbf{Y} \rangle \rangle = (n - \hbar k + l + 1 + \frac{1}{2} \hbar) \cdot 8\pi^{3}.
\]
If furthermore \( \mathbf{Y} \) appears in the list (1)–(4) in Theorem 7.13, then
\[
\langle \langle \mathbf{Y}, \mathbf{Y} \rangle \rangle = 2\pi^{2}.
\]

Proof. \( \langle \langle \cdot, \cdot \rangle \rangle \) and \( \langle \langle \cdot, \cdot \rangle \rangle \) were defined with the measures \( d_{4}u \) and \( d_{3}u \), while the \( |k,l,m,e> \) and so forth were normalized with \( \omega_{3} \) (cf. Lemma 5.4.2 of (I)); \( d_{3}u = 2\pi^{2}\omega_{3} \) by Section 3.7 of (I).

The eigenspaces of the flat Dirac operator are important in conventional relativistic theory. The spectral decomposition of \( \mathcal{D}_{c} \) is easily exhibited in terms of the basis defined here. The eigenvalues of \( \mathcal{D}_{c} \) are either real or purely imaginary; as will be seen in Section 7.10, the physically relevant (positive energy) parts of the spinor bundles are spanned by the real eigenspaces of \( \mathcal{D}_{c} \), which are spanned by those \( \mathbf{Y} \), such that \( |n| \geq k + l + 1 + \frac{1}{2} \hbar \). We introduce the notation \( p = h(k + l + 1 + \frac{1}{2} \hbar) \), and the following notation for the real eigenspaces of \( \mathcal{D}_{c} \) (determined by Corollary 7.12.1):

The \( +(n^{2} - p^{2})^{1/2} \) eigenspace of \( \mathcal{D}_{c} \) is spanned by
\[
E_{klmn}^{h+1} = (n + p)^{1/2} \mathbf{y}_{klmn}^{h+1} + (n - p)^{1/2} \mathbf{y}_{klmn}^{h-1} \quad (n > 0)
\]
and
\[
P_{klmn}^{h+1} = (-n - p)^{1/2} \mathbf{y}_{klmn}^{h+1} - (-n + p)^{1/2} \mathbf{y}_{klmn}^{h-1} \quad (n < 0);
\]
the \( -(n^{2} - p^{2})^{1/2} \) eigenspace of \( \mathcal{D}_{c} \) is spanned by
\[
E_{klmn}^{h-1} = (n + p)^{1/2} \mathbf{y}_{klmn}^{h+1} - (n - p)^{1/2} \mathbf{y}_{klmn}^{h-1} \quad (n > 0)
\]
and
\[
P_{klmn}^{h-1} = (-n - p)^{1/2} \mathbf{y}_{klmn}^{h+1} + (-n + p)^{1/2} \mathbf{y}_{klmn}^{h-1} \quad (n < 0).
\]
(E is for "electron" and P is for "positron.") We also write $E_{klmn}^{hs}$ and $P_{klmn}^{hs}$, where $s = \pm 1$; then $s = 1$ (resp. $s = -1$) if and only if $E_{klmn}^{hs}$, or $P_{klmn}^{hs}$, is in a positive (resp. negative) eigenspace of $\mathcal{D}_c$. Note that

$$\gamma_5 E_{klmn}^{hs} = E_{klmn}^{h-s} \quad \text{and} \quad \gamma_5 P_{klmn}^{hs} = P_{klmn}^{h-s}; \quad (7.41)$$

however, the spectral decomposition of $\gamma_4 \mathcal{D}_c \gamma_4$ is entirely different from that of $\mathcal{D}_c$.

7.7. Actions of Discrete Symmetries

This section determines the actions on the spinor field basis of the discrete symmetries $T$, $C$, and $P$. These actions are obtained by inducing to $\tilde{G}^+ \times Z_2$ the standard form of the projective extension of the spin representation $R_w = R_w^+ \oplus R_w^-$ defined by Eq. (7.2). As earlier the weights $(w, w)$ are assumed real and fields are taken in their curved parallelized forms unless otherwise indicated. The induced representation $U_w^+ \oplus U_w^-$ is denoted in this section as $U$. Recalling the standardizing equations (7.3), the actions $U(T)$, $U(C)$, and $U(P)$ on fields are then given by Corollaries 4.1.5 and 4.1.7:

$$U(T): \Psi(t \times W) \rightarrow \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \Psi(-t \times W), \quad (7.42)$$

$$U(C): \Psi(t \times W) \rightarrow \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \Psi(t \times W), \quad (7.43)$$

$$U(P): \Psi(t \times W) \rightarrow \begin{pmatrix} 0 & -W^{-1} \\ -W^{-1} & 0 \end{pmatrix} \Psi(t \times W^{-1})$$

It follows that Remark 7.2 applies equally if $R$ is replaced by $U$.

Scholium 7.14. The actions of $T$, $C$, and $P$ on the spinor basis are

$$U(T) \Psi_{klmn}^{he} = i(-1)^{m+1} \frac{(l+m+1)!}{(l-m)!} \Psi_{k+l-m-1,n}^{he},$$

$$U(C) \Psi_{klmn}^{he} = i(-1)^m e \frac{(l+m+1)!}{(l-m)!} \Psi_{k-l-m-1,n}^{he},$$

$$U(P) \Psi_{klmn}^{he} = i(-1)^l \Psi_{k+l+1,n}^{he},$$

$$U(T) |nklme\rangle = i(-1)^{l+m} |nkhl-m-1,e\rangle,$$

$$U(C) |nklme\rangle = ie(-1)^{l+m+1} |nkhl-m-1,-e\rangle,$$

$$U(P) |nkhlme\rangle = (-1)^l |n k + h - h l m - e\rangle.$$
Proof. We compute $U(P) \Psi_{klmn}^{he}$ first; this case will require the following formulas from Table III:

$$
(u_4 - U_b) \Psi_{klm}^4 = \frac{2l + k + 2}{k + l + 1} \Psi_{k-1lm}^4 - \frac{l + 1}{k + l + 1} \Psi_{k+1lm}^4,
$$

$$
(u_4 - U_b) \Psi_{klm}^{-1} = \frac{l + 1}{k + l + 1} \Psi_{k-1lm}^{-1} + \frac{k}{k + l + 1} \Psi_{k+1lm}^{-1},
$$

which are proved in the next section.

Setting

$$
V_1(k, l, m) = i\beta_{k+1lm}^{-1} + i(l + m + 1) \beta_{klm}^1
$$

and

$$
V_{-1}(k, l, m) = 2(l + 1) \beta_{k-1lm+1}^{-1} - 2(l + 1)(l - m + 1) \beta_{k-1lm+1}^1,
$$

it is easily seen from (7.32) that

$$
V_{-1}(k, l, m) = \frac{k \Psi_{klm}^1 + (k + 2l + 2) \Psi_{klm}^{-1}}{2(k + l + 1)},
$$

$$
V_1(k, l, m) = \frac{\Psi_{klm}^1 - \Psi_{klm}^{-1}}{2(k + l + 1)},
$$

so that

$$
\Psi_{klm}^1 = (k + 2l + 2) V_1(k, l, m) + V_{-1}(k, l, m)
$$

and

$$
\Psi_{klm}^{-1} = -k V_1(k, l, m) + V_{-1}(k, l, m).
$$

Now by Corollary 5.4.5 of (I), $\beta_{klm}(W^{-1}) = (-1)^{l} \beta_{klm}(W)$ for all $W \in SU(2)$, so that by the above formulas involving the $V_{\pm 1}(k, l, m)$,

$$
-W^{-1} \Psi_{klm}^1(W^{-1})
$$

$$
= (-1)^{l+1} (u_4 - U_b) \left[ \frac{l + 1}{k + l + 1} \Psi_{klm}^1 - \frac{k + 2l + 2}{k + l + 1} \Psi_{klm}^{-1} \right]
$$

$$
= (-1)^l \Psi_{k+1lm}^{-1}(W) \quad \text{by (7.44)}.\]
### TABLE III

Multiplicative and Scale Actions on the $\Psi^h_{kln}$

\[
\begin{align*}
\mu_{-1} \Psi^h_{kln} &= \frac{1}{2} \Psi^h_{kln} + \frac{1}{2} \Psi^h_{kln-1} \\
\mu_4 \Psi^l_{kln} &= \frac{1}{2} \frac{2l + k + 2}{l + k + 1} \Psi^l_{k-l+1} + \frac{1}{2} \frac{k + 1}{l + k + 2} \Psi^l_{k+l+1} \\
&\quad - \frac{1}{2} \frac{l + 1}{(l + k + 1)(l + k + 2)} \Psi^{-1}_{k+l+1} \\
\mu_4 \Psi^{-1}_{kln} &= \frac{1}{2} \frac{l + 1}{(l + k)(l + k + 1)} \Psi^{-1}_{k-l+1} + \frac{1}{2} \frac{2l + k + 1}{l + k} \Psi^{-1}_{k-l+1} \\
&\quad + \frac{1}{2} \frac{k}{l + k + 1} \Psi^{-1}_{k+l+1} \\
\mu_{-1} \mu_4 \Psi^l_{kln} &= \frac{1}{4} \frac{2l + k + 2}{l + k + 1} \Psi^l_{k-l+mn+1} + \frac{1}{4} \frac{k + 1}{l + k + 2} \Psi^l_{k+l+mn+1} \\
&\quad - \frac{1}{4} \frac{l + 1}{(l + k + 1)(l + k + 2)} \Psi^{-1}_{k-l+mn+1} + \frac{1}{4} \frac{2l + k + 2}{l + k + 1} \Psi^{-1}_{k-l+mn+1} \\
&\quad + \frac{1}{4} \frac{k + 1}{l + k + 2} \Psi^{-1}_{k+l+mn+1} \\
&\quad - \frac{1}{4} \frac{l + 1}{(l + k + 1)(l + k + 2)} \Psi^{-1}_{k+l+mn+1} \\
\mu_{-1} \mu_4 \Psi^{-1}_{kln} &= \frac{1}{4} \frac{l + 1}{(l + k)(l + k + 1)} \Psi^{-1}_{k-l+mn+1} + \frac{1}{4} \frac{2l + k + 1}{l + k} \Psi^{-1}_{k-l+mn+1} \\
&\quad + \frac{1}{4} \frac{k}{l + k + 1} \Psi^{-1}_{k+l+mn+1} \\
&\quad - \frac{1}{4} \frac{l + 1}{(l + k + 1)(l + k + 2)} \Psi^{-1}_{k+l+mn+1} \\
U_4 \Psi^l_{kln} &= \frac{1}{2} \frac{2l + k + 2}{l + k + 1} \Psi^l_{k-l+1} + \frac{1}{2} \frac{k + 1}{l + k + 2} \Psi^l_{k+l+1} \\
&\quad + \frac{1}{2} \frac{(l + 1)(2l + 2k + 3)}{(l + k + 1)(l + k + 2)} \Psi^{-1}_{k+l+1} \\
U_4 \Psi^{-1}_{kln} &= \frac{1}{2} \frac{l + 1}{(l + k)(l + k + 1)} \Psi^{-1}_{k-l+1} \\
&\quad + \frac{1}{2} \frac{2l + k + 1}{l + k} \Psi^{-1}_{k-l+1} - \frac{1}{2} \frac{k}{l + k + 1} \Psi^{-1}_{k+l+1}
\end{align*}
\]

*Table continued*
Similarly,

\[ -W^{-1}\Psi^{-1}_{kln}(W^{-1}) = (-1)^l \Psi^l_{k-1ln}(W) \quad (W \in SU(2)); \]

this, together with (7.36) and (7.37), gives the stated actions of \( U(P) \).

Since by (7.43) the forms of \( U(T) \) and \( U(C) \) are so similar, it suffices to prove that

\[ \sigma_2 \overline{\Psi^h_{kln}} = i(-1)^{m+1} \frac{(l+m+1)!}{(l-m)!} \psi_{k l m}^h, \quad (7.45) \]
and again apply (7.36) and (7.37), noting the anti-linearity of $U(T)$ and $U(C)$. We prove (7.45) in the case $h = 1$; the other case is similar. By Corollary 5.4.5, $\beta_{klm} = (-1)^m ((l + m)!/(l - m)!) \beta_{kl - m - 1}$, so that $\Psi_{klm}$ equals

$$i(-1)^m (k + 2l + 2) \frac{(l + m + 1)!}{(l - m - 1)!} \beta_{kl - m - 1}^{-1}$$

$$- i(-1)^m (k + 2l + 2) \frac{(l + m + 1)!}{(l - m)!} \beta_{kl - m}^1$$

$$+ 2(l + 1)(-1)^m \frac{(l + m + 2)!}{(l - m)!} \beta_{k-l+1+m-1}^{-1}$$

$$- 2(l + 1)(-1)^m \frac{(l + m + 1)!}{(l - m)!} \beta_{k-l+m-1}^1.$$  

$\sigma_2$ takes $\beta_{klm}^1$ into $i\beta_{klm}^{-1}$ and $\beta_{klm}^{-1}$ into $-i\beta_{klm}^1$, and thus takes the previous expression into the right side of (7.45).

Consider again the list (1)–(4) of subspaces of ker $\mathcal{D}_c$ in Theorem 7.13. It follows that $U(T)$ leaves each invariant; $U(C)$ exchanges (1) and (3) and exchanges (2) and (4); and $U(P)$ exchanges (1) and (4) and exchanges (2) and (3). Thus $\mathcal{G}^+ \times Z_2$ will act irreducibly on ker $\mathcal{D}_c$ (cf. Theorem 7.22). Finally, $U(\zeta) = i$ on (1) and (2), and $U(\zeta) = -i$ on (3) and (4).

**Corollary 7.14.1.** With assignments (7.43),

(a) $U(T) U(C) U(P) \Psi_{klimn}^{he} = (-1)^{l+1} e^{\Psi_{k+hlmn}^{he}}$;

(b) $\mathcal{D}_c$ commutes with $U(T)$, $U(C)$, and $U(P)$.

**Proof:** This follows from Scholium 7.14 and Corollary 7.12.1.

It follows from (b) above that $U(T)$, $U(C)$, and $U(P)$ leave the positive eigenspaces of $\mathcal{D}_c$ invariant. This is also exhibited in the following corollary, in connection with which it should be recalled that $F_{klimn}^{hs}$ and $P_{klimn}^{hs}$ are in a positive (resp. negative) subspace of $\mathcal{D}_c$ if and only if $s = 1$ (resp. $s = -1$).

**Corollary 7.14.2.** Define $f(l, m) = (l + m + 1)!/(l - m)!$. Then

$$U(T) E_{klimn}^{hs} = i(-1)^{m+1} f(l, m) E_{kl-m-1,n}^{hs},$$

$$U(T) P_{klimn}^{hs} = i(-1)^{m+1} f(l, m) P_{kl-m-1,n}^{hs},$$

$$U(C) E_{klimn}^{hs} = si(-1)^{m+1} f(l, m) E_{kl-m-1,n}^{hs},$$

$$U(C) P_{klimn}^{hs} = si(-1)^{m+1} f(l, m) P_{kl-m-1,n}^{hs},$$

$$U(P) E_{klimn}^{hs} = s(-1)^l E_{k+hlnm}^{hs},$$

$$U(P) P_{klimn}^{hs} = s(-1)^{l+1} P_{k+hlnm}^{hs}.$$
Proof. This follows from Scholium 7.14 and the definitions of $E_{klmn}^{hs}$ and $P_{klmn}^{hs}$.

7.10. Scale and Multiplicative Actions

The transformations of spinor fields that have thus far been considered (e.g., those given by $dU(K)$, $D_c$, and the discrete symmetries) have basically left invariant the $K$-types of the spinor representations. That is, a given basis element $\Psi_{klmn}^h$ has been transformed into linear combinations of others such that the parameters $n$ and ordered pair $(k + i, k + l + h)$ have been left invariant. The exceptions have been $U(P)$ and $U(C)$, which reverse the ordering in the ordered pair and change the sign of $n$.

This section examines other transformations, namely, multiplication by $u_j$, $u_{-1}$, $U_b$, and (infinitesimal) scale transformations, which besides possibly changing $h$, all increment or decrement $k$ and $n$ by 1 while leaving $l$ and $m$ fixed.

Scholium 7.15. $u_{-1} \Psi_{klmn}^h$, $u_4 \Psi_{klmn}^h$, and $-S \Psi_{klmn}^h$ have the forms listed in Table III.

Proof. This follows from (7.32) and Table IX of (I).

According to Corollary 7.2.3, it is necessary to evaluate $U_b \Psi_{klm}^h$. This involves multiplication of the $\beta_{klm}$ by $u_3$ and $u_2 \pm iu_1$, which has not been considered thus far. In fact, some recursion relations for Gegenbauer polynomials and associated Legendre functions additional to those considered in (I) are involved. Recall first

$$\cos \theta P_l^m(\cos \theta) = \frac{l + m}{2l + 1} P_{l-1}^m(\cos \theta) + \frac{l - m + 1}{2l + 1} P_{l+1}^m(\cos \theta) \quad (7.46)$$

from (5.23) of (I), and

$$\sin \theta P_l^m(\cos \theta) = \frac{1}{2l + 1} P_{l-1}^{m+1}(\cos \theta) - \frac{1}{2l + 1} P_{l+1}^{m+1}(\cos \theta) \quad (7.47)$$

from [4, p. 161, Eq. (13)].

Scholium 7.16. $U_b \Psi_{klm}^h$ has the form listed in Table III.

Lemma 7.16.1.

$$\sin^2 \rho C_{k+1}^l(\cos \rho) = \frac{(2l + k + 1)(2l + k)}{4l(l + k + 1)} C_k^l(\cos \rho)$$

$$-\frac{(k + 2)(2k + 1)}{4l(l + k + 1)} C_{k+2}^l(\cos \rho) \quad (7.48)$$
and

\[ C^{l+1}_k(\cos \rho) = -\frac{l+1}{l+k+1} C^{l+2}_k(\cos \rho) + \frac{l+1}{l+k+1} C^{l+2}_{k-2}(\cos \rho). \]  

(7.49)

**Proof.** Equation (7.48) follows from (5.24) and (5.27) of (I). Equation (7.49) follows from the equation

\[ C^{l+1}_k(x) = C^{l+2}_k(x) + C^{l+2}_{k-2}(x) - 2x C^{l+2}_{k-1}(x) \]

[5, p. 182, Eq. (10.41)], and then use of (5.26) of (I).

By Sections 3.2 and 3.3 of (I),

\[ U_b = \begin{pmatrix} i \sin \rho \cos \theta & i e^{-i \phi} \sin \rho \sin \theta \\ i e^{i \phi} \sin \rho \sin \theta & -i \sin \rho \cos \theta \end{pmatrix}. \]  

(7.50)

**Lemma 7.16.2.**

\[ \sin \rho \cos \theta \beta_{klm} = \frac{(2l+k+1)(2l+k)(l+m)}{4l(l+k+1)(2l+1)} \beta_{kl-1m} \\
- \frac{(k+1)(k+2)(l+m)}{4l(l+k+1)(2l+1)} \beta_{k+2l-1m} \\
- \frac{(l+1)(l-m+1)}{(2l+1)(l+k+1)} \beta_{k-2l+1m} \\
+ \frac{(l+1)(l-m+1)}{(2l+1)(l+k+1)} \beta_{kl+1m}, \]  

(7.51)

\[ e^{i \phi} \sin \rho \sin \theta \beta_{klm} = \frac{(2l+k+1)(2l+k)}{4l(2l+1)(l+k+1)} \beta_{kl-1m+1} \\
- \frac{(k+l)(k+2)}{4l(2l+1)(l+k+1)} \beta_{k+2l-1m+1} \\
\frac{l+1}{(l+k+1)(2l+1)} \beta_{kl+1m+1} \\
+ \frac{l+1}{(l+k+1)(2l+1)} \beta_{k-2l+1m+1}. \]  

(7.52)
and

\[ e^{-i\phi} \sin \rho \sin \theta \beta_{klm} = -\frac{(2l + k + 1)(2l + k)(l + m - 1)(l + m)}{4l(2l + 1)(l + k + 1)} \beta_{kl-1,m-1} \]

\[ + \frac{(k + 1)(k + 2)(l + m - 1)(l + m)}{4l(2l + 1)(l + k + 1)} \beta_{k+2l-1,m-1} \]

\[ + \frac{(l + 1)(l - m + 1)(l - m + 2)}{(l + k + 1)(2l + 1)} \beta_{kl+1,m-1} \]

\[ - \frac{(l + 1)(l - m + 1)(l - m + 2)}{(l + k + 1)(2l + 1)} \beta_{k-2l+1,m-1}. \quad (7.53) \]

**Proof.** Equation (7.51) follows from (7.46), and then (7.48) and (7.49). Equations (7.52) and (7.53) follow from use of (7.47), and then again (7.48) and (7.49).

**Proof of Scholium 7.16.** Application of matrix (7.50) to (7.32) and use of Lemma 7.16.2 produces 32 terms, each a multiple of some $\beta^s_{par}$. Sixteen of these cancel in pairs, and the rest combine as stated.

**Scholium 7.17.** $dU_w^e(S) \psi^e_{klmn} (e = \pm 1)$ is given in Table IV.

**Proof.** This follows from Corollary 7.2.3. and Scholia 7.15 and 7.16. Naturally Tables V and VI result from applying (7.36) and (7.37) to Tables III and IV.

7.11 Action of $D_f$ and Composition Series

By combining Table III and Corollaries 7.4.2 and 7.12.1, one obtains

**Scholium 7.18.** Within the bundle induced from $R^+_{3/2} \oplus R^-_{3/2}$,

\[ D_f \psi^e_{klmn} = (n \quad e(k + l + 3/2)) \left[ \frac{1}{4} \psi^1_{klmn+1} - e \right] + \frac{1}{4} \psi^1_{klmn-1} \]

\[ + \frac{1}{4} \frac{k + 2l + 2}{k + l + 1} \psi^1_{k-l+1,mn} + \frac{1}{4} \frac{k + 1}{k + l + 2} \psi^1_{k+1,mn} \]

\[ - \frac{1}{4} \frac{l + 1}{(k + l + 1)(k + l + 2)} \psi^1_{k+1,mn} \]
### Table IV

Actions of $dU(S)$ on the $\Psi_{k l m n}^{\nu}$

\[
dU_{\nu}^{\alpha}(S) \Psi_{k l m n}^{11} = \frac{1}{4} \frac{2l+k+2}{l+k+1} (-n+l+k+3-w) \Psi_{k-1l/mn+1}^{11}
\]

\[
- \frac{1}{4} \frac{k+1}{l+k+2} (n+l+k+1+w) \Psi_{k+1l/mn+1}^{11}
\]

\[
+ \frac{1}{4} \frac{(l+1)(n-l-k-3+w)}{(l+k+1)(l+k+2)} \Psi_{k+1l/mn+1}^{11}
\]

\[
+ \frac{1}{4} \frac{2l+k+2}{l+k+1} (n+l+k+2-w) \Psi_{k-1l/mn-1}^{11}
\]

\[
+ \frac{1}{4} \frac{k+1}{l+k+2} (n-l-k-w) \Psi_{k+1l/mn-1}^{11}
\]

\[
+ \frac{1}{4} \frac{l+1}{(l+k+1)(l+k+2)} (-n+l+k+w) \Psi_{k+1l/mn-1}^{11}
\]

\[
dU_{\nu}^{\alpha}(S) \Psi_{k l m n}^{11} = \frac{1}{4} \frac{l+1}{(l+k)(l+k+1)} (n+l+k-1+w) \Psi_{k-1l/mn+1}^{11}
\]

\[
+ \frac{1}{4} \frac{2l+k+1}{l+k} (-n+l+k+1-w) \Psi_{k+1l/mn+1}^{11}
\]

\[
- \frac{1}{4} \frac{k}{l+k+1} (n+l+k+w-1) \Psi_{k+1l/mn+1}^{11}
\]

\[
- \frac{1}{4} \frac{l+1}{(l+k)(l+k+1)} (n+l+k+2-w) \Psi_{k-1l/mn-1}^{11}
\]

\[
+ \frac{1}{4} \frac{2l+k+1}{l+k} (n+l+k+2-w) \Psi_{k-1l/mn-1}^{11}
\]

\[
+ \frac{1}{4} \frac{k}{l+k+1} (n-l-k-w) \Psi_{k+1l/mn-1}^{11}
\]
### TABLE IV—(continued)

\[
dU_{\omega}^- (S) \Psi_{k l m n}^{l-1} = \frac{1}{4} \frac{2l + k + 2}{l + k + 1} (-n + l + k + 2 - w) \Psi_{k+1 l m n+1}^{l-1} \\
- \frac{1}{4} \frac{k + 1}{l + k + 2} (n + l + k + w) \Psi_{k+1 l m n+1}^{l-1} \\
+ \frac{1}{4} \frac{l + 1}{(l + k + 1)(l + k + 2)} (n + l + k + w) \Psi_{k+1 l m n+1}^{l-1} \\
+ \frac{1}{4} \frac{2l + k + 2}{l + k + 1} (n + l + k + 2 - w) \Psi_{k+1 l m n-1}^{l-1} \\
+ \frac{1}{4} \frac{k + 1}{l + k + 2} (n - l - k - 1 - w) \Psi_{k+1 l m n-1}^{l-1} \\
+ \frac{1}{4} \frac{l + 1}{(l + k + 1)(l + k + 2)} (-n - l - k + w - 3) \Psi_{k+1 l m n-1}^{l-1} \\
\]

\[
dU_{\omega}^- (S) \Psi_{k l m n}^{l-1} = \frac{1}{4} \frac{l + 1}{(l + k)(l + k + 1)} (n - l - k - 2 - w) \Psi_{k-1 l m n+1}^{l-1} \\
+ \frac{1}{4} \frac{2l + k + 1}{l + k} (-n + l + k + 2 - w) \Psi_{k-1 l m n+1}^{l-1} \\
- \frac{1}{4} \frac{k}{l + k + 1} (n + l + k + w) \Psi_{k+1 l m n+1}^{l-1} \\
- \frac{1}{4} \frac{l + 1}{(l + k)(l + k + 1)} (n - l - k + 1 - w) \Psi_{k-1 l m n-1}^{l-1} \\
+ \frac{1}{4} \frac{2l + k + 1}{l + k} (n + l + k + 1 - w) \Psi_{k-1 l m n-1}^{l-1} \\
+ \frac{1}{4} \frac{k}{l + k + 1} (n - l - k + 1 - w) \Psi_{k+1 l m n-1}^{l-1} \\
\]
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TABLE V

Multiplicative and Scale Actions on Normalized Spinor Basis

\[ u_{-1} |nkhlm\rangle = \frac{1}{2} |n + 1khlm\rangle + \frac{1}{2} |n - 1khlm\rangle \]

\[ u_0 |nkhlm\rangle = -\frac{1}{2} i |n + 1khlm\rangle + \frac{1}{2} i |n - 1khlm\rangle \]

\[ u_a |k1lm\rangle = \frac{1}{2} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} |k - 11lm\rangle \]

\[ + \frac{1}{2} \frac{(k + 1)^{1/2}(k + 2l + 3)^{1/2}}{k + l + 2} |k + 11lm\rangle \]

\[ - \frac{1}{2} \frac{l + 1}{(k + l + 1)(k + l + 2)} |k + 1 - 1lm\rangle \]

\[ u_a |k-1lm\rangle = -\frac{1}{2} \frac{l + 1}{(k + l)(k + l + 1)} |k - 11lm\rangle \]

\[ + \frac{1}{2} \frac{(k - 1)^{1/2}(k + 2l + 1)^{1/2}}{k + 1} |k - 1 - 1lm\rangle \]

\[ + \frac{1}{2} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} |k + 1 - 1lm\rangle \]

\[ U_b |k1lm\rangle = -\frac{1}{2} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{l + k + 1} |k - 11lm\rangle \]

\[ + \frac{1}{2} \frac{(k + 1)^{1/2}(k + 2l + 3)^{1/2}}{k + l + 2} |k + 11lm\rangle \]

\[ + \frac{1}{2} \frac{(l + 1)(2l + 2k + 3)}{(l + k + 1)(l + k + 2)} |k + 1 - 1lm\rangle \]

\[ U_b |k-1lm\rangle = -\frac{1}{2} \frac{(l + 1)(2l + 2k + 1)}{(l + k)(l + k + 1)} |k - 11lm\rangle \]

\[ + \frac{1}{2} \frac{(k - 1)^{1/2}(2l + k + 1)^{1/2}}{l + k} |k - 1 - 1lm\rangle \]

\[ + \frac{1}{2} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{l + k + 1} |k + 1 - 1lm\rangle \]

Table continued
TABLE V—(continued)

\[ -S \mid n k l m \rangle = \frac{1}{4} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} (-n + k + l + 2) \mid n + 1 k - 1 l m \rangle \]

\[ -S \mid n k l m \rangle = \frac{1}{4} \frac{(k + 1)^{1/2}(k + 2l + 3)^{1/2}}{k + l + 2} (n + k + l) \mid n + 1 k + 1 l m \rangle \]

\[ + \frac{1}{4} \frac{l + 1}{(k + l + 1)(k + l + 2)} (n + k + l) \mid n + 1 k + 1 l m \rangle \]

\[ + \frac{1}{4} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} (n + k + l + 2) \mid n - 1 k - 1 l m \rangle \]

\[ + \frac{1}{4} \frac{(k + 1)^{1/2}(k + 2l + 3)^{1/2}}{k + l + 2} (n - l - k) \mid n - 1 k + 1 l m \rangle \]

\[ + \frac{1}{4} \frac{l + 1}{(k + l + 1)(k + l + 2)} (-n + k + l) \mid n - 1 k + 1 l m \rangle \]

\[ -S \mid n k - 1 l m \rangle = \frac{1}{4} \frac{l + 1}{(k + l)(k + l + 1)} (n - k - l - 2) \mid n + 1 k - 1 l m \rangle \]

\[ + \frac{1}{4} \frac{(k - 1)^{1/2}(k + 2l + 1)^{1/2}}{k + l} (-n + k + l + 1) \mid n + 1 k - 1 l m \rangle \]

\[ - \frac{1}{4} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} (n + k + l + 1) \mid n + 1 k - 1 l m \rangle \]

\[ - \frac{1}{4} \frac{l + 1}{(l + k)(k + l + 1)} (n + k + l + 2) \mid n - 1 k - 1 l m \rangle \]

\[ + \frac{1}{4} \frac{(k - 1)^{1/2}(k + 2l + 1)^{1/2}}{k + 1} (n + k + l + 2) \mid n - 1 k - 1 l m \rangle \]

\[ + \frac{1}{4} \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} (n - k - l) \mid n - 1 k + 1 l m \rangle \]
### TABLE VI

Actions of $dU(S)$ on Normalized Spinor Basis

<table>
<thead>
<tr>
<th>$dU^\nu_\nu(S)</th>
<th>n k l m \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= \frac{1}{4} k^{1/2}(k + 2l + 2)^{1/2} \frac{l}{l + k + 1} (-n + l + k + 3 - w)</td>
<td>n + 1 \ k + 1 \ l \ m \rangle$</td>
</tr>
<tr>
<td>$- \frac{1}{4} (k + 1)^{1/2} (k + 2l + 3)^{1/2} \frac{l}{l + k + 2} (n + l + k + 1 + w)</td>
<td>n + 1 \ k + 1 \ l \ m \rangle$</td>
</tr>
<tr>
<td>$+ \frac{1}{4} \frac{l + 1}{(l + k + 1)(l + k + 2)} (n - l - k - 3 + w)</td>
<td>n + 1 \ k + 1 \ l \ m \rangle$</td>
</tr>
<tr>
<td>$+ \frac{1}{4} k^{1/2}(k + 2l + 2)^{1/2} \frac{l}{l + k + 1} (n + l + k + 2 - w)</td>
<td>n + 1 \ k + 1 \ l \ m \rangle$</td>
</tr>
<tr>
<td>$+ \frac{1}{4} (k + 1)^{1/2} (k + 2l + 3)^{1/2} \frac{l}{l + k + 2} (n - k - l - w)</td>
<td>n + 1 \ k + 1 \ l \ m \rangle$</td>
</tr>
<tr>
<td>$+ \frac{1}{4} \frac{l + 1}{(l + k + 2)(l + k + 1)} (-n + l + k + w)</td>
<td>n + 1 \ k + 1 \ l \ m \rangle$</td>
</tr>
</tbody>
</table>

$\frac{1}{4} (l + k + 1)(l + k) (n + l + k - 1 + w) | n + 1 \ k + 1 \ l \ m \rangle$

$+ \frac{1}{4} (k - 1)^{1/2} (k + 2l + 1)^{1/2} \frac{l}{l + k} (-n + k + l + 1 - w) | n + 1 \ k - 1 \ l \ m \rangle$

$- \frac{1}{4} k^{1/2}(k + 2l + 2)^{1/2} \frac{l}{l + k + 1} (n + l + k + w - 1) | n + 1 \ k + 1 \ l \ m \rangle$

$- \frac{1}{4} \frac{l + 1}{(l + k + 1)(l + k)} (n + l + k + 2 - w) | n - 1 \ k - 1 \ l \ m \rangle$

$+ \frac{1}{4} (k - 1)^{1/2} (k + 2l + 1)^{1/2} \frac{l}{l + k} (n + l + k + 2 - w) | n - 1 \ k - 1 \ l \ m \rangle$

$+ \frac{1}{4} k^{1/2}(k + 2l + 2)^{1/2} \frac{l}{l + k + 1} (-n - l - k - w) | n - 1 \ k + 1 \ l \ m \rangle$

Table continued
TABLE VI—(continued)

\[ dU_w^+(S) | n k 1 l m \rangle = \]

\[
\frac{1}{4} \left\{ \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} \right\} (-n + l + k + 2 - w) | n + 1 k - 1 l m \rangle \\
- \frac{1}{4} \left\{ \frac{(k + 1)^{1/2}(k + 2l + 3)^{1/2}}{k + l + 2} \right\} (n + l + k + w) | n + 1 k + 1 l m \rangle \\
+ \frac{1}{4} \left\{ \frac{l + 1}{(k + l + 2)(k + l + 1)} \right\} (n + l + k + w) | n + 1 k + 1 - 1 l m \rangle \\
+ \frac{1}{4} \left\{ \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} \right\} (n + l + k + 3 - w) | n - 1 k - 1 l m \rangle \\
+ \frac{1}{4} \left\{ \frac{(k + 1)^{1/2}(k + 2l + 3)^{1/2}}{k + l + 2} \right\} (n - l - k - 1 - w) | n - 1 k + 1 l m \rangle \\
+ \frac{1}{4} \left\{ \frac{l + 1}{(k + l + 2)(l + k + 1)} \right\} (-n - l - k + w - 3) | n - 1 k + 1 - 1 l m \rangle \\
\]

\[ dU_w^-(S) | n k - 1 l m \rangle = \]

\[
\frac{1}{4} \left\{ \frac{l + 1}{(k + l + 1)(k + l)} \right\} (n - k - l - 2 + w) | n + 1 k - 1 l m \rangle \\
+ \frac{1}{4} \left\{ \frac{(k - 1)^{1/2}(k + 2l + 1)^{1/2}}{k + l} \right\} (n | k + l + 2 w) | n + 1 k - 1 - 1 l m \rangle \\
- \frac{1}{4} \left\{ \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} \right\} (n + k + l + w) | n + 1 k + 1 - 1 l m \rangle \\
- \frac{1}{4} \left\{ \frac{l + 1}{(k + l + 1)(k + l)} \right\} (n - k - l + 1 - w) | n - 1 k - 1 l m \rangle \\
+ \frac{1}{4} \left\{ \frac{(k - 1)^{1/2}(k + 2l + 1)^{1/2}}{k + l} \right\} (n + k + l + 1 - w) | n - 1 k - 1 - 1 l m \rangle \\
+ \frac{1}{4} \left\{ \frac{k^{1/2}(k + 2l + 2)^{1/2}}{k + l + 1} \right\} (-n - l - k + 1 - w) | n - 1 k + 1 - 1 l m \rangle \\
\]
ANALYSIS IN THE SPINOR AND FORM BUNDLES

and

\[ \mathcal{D}_f \psi_{klnn}^{-1} = (n + e(k + l + 1/2)) \left[ \frac{1}{4} \psi_{klnn+1}^{-1} - \frac{1}{4} \psi_{klnn-1}^{-1} \right. \\
\left. - \frac{1}{4} \frac{l + 1}{(k + l)(k + l + 1)} \psi_{k-lmn}^{-1} \right. \\
\left. + \frac{1}{4} \frac{k + 2l + 1}{k + l + 1} \psi_{k-lmn}^{-1} \right. \\
\left. + \frac{1}{4} \frac{k}{k + l + 1} \psi_{k+l+1}^{-1} \right]. \]

Let \( E^\pm \) be the space of all finite linear combinations of the \( \psi_{klmn}^{he} \), for \( e = \pm 1 \). Each of the representations \( dU^\pm_w \) of the Lie algebra of \( G \) acts on \( E^\pm \); this section determines all of the invariant subspaces of these representations. It is well known that these representations on \( K \)-finite vectors determine the global reducibility properties of corresponding representations of at least the finite covering groups of the adjoint groups. These representation spaces derive from the induced bundles over finite coverings of \( \tilde{M} \), and are spanned by those \( \psi_{klmn}^{he} \) for which \( n \) is rational.

It is found in particular that the solution manifold of the Dirac equation (of mass 0) is naturally described purely in bundle and transformation group terms, the wave equation being derivative rather than fundamental here. The spaces (1)–(4) of Theorem 7.13 are the unique infinite-dimensional causally invariant minimal subspaces with the property that each joint eigenspace of \( dU(p) \) and \( dU(\lambda) \) is an eigenspace of \( dU(X_0) \) (or alternatively and physically quite significantly, that each smooth element of the subspace is determined by its restriction to any space-like surface \( \{t\} \times S^3 \); cf. Theorem 7.22).

As in Lemma 5.5.4 of (I); the eigenspaces of the central element \( \zeta \) provide an initial decomposition of \( E^\pm \).

**Scholium 7.19.** For \( e = \pm 1 \) and all \( w \),

\[ U^\pm_w(\zeta) \psi_{klmn}^{he} = e^{-i\pi(n-k-l)} \psi_{klmn}^{he}. \]

**Proof.** Recall from Corollary 2.1.1 of (I) that \( \zeta = \pi \times -I \times I \in \tilde{K} \); by Corollary 4.1.4,

\[ U(\zeta): \Psi(t \times W) \rightarrow \Psi(\zeta^{-1}(t \times W)) \]

\[ = \Psi((t-\pi) \times -W) \quad (t \in R, W \in SU(2)). \]

Then by Lemma 5.4.3 of (I) and (7.32),

\[ U(\zeta): e^{\tau n} \psi_{klm}^h \rightarrow e^{(t-\pi)n}(\pm 1)^{k+1} \psi_{klm}^h. \]
Given any real number \( \lambda \), define \( E^\pm_\lambda \) to be the span of all \( \Psi_{k_{mn}}^{\pm} (e = \pm 1) \) such that \( n - k - l - \lambda \) is an even integer. Clearly \( E^\pm \) is the direct sum of all \( E^\pm_\lambda \), and by Scholium 7.19 each \( E^\pm_\lambda \) is invariant under any \( dU_w^\pm \). As in (1), elements of \( E^\pm_\theta \) are lifted up from \( \mathbf{M} \), those of \( E^\pm_0 + E^\pm_1 \) from \( \overline{\mathbf{M}} \) and so on. Concerning solutions (Theorem 7.13) of the Dirac equation, it is relevant to observe that \( U_w^\pm(\zeta) \) equals \( i \) on \( E^\pm_{3/2} \) and \( -i \) on \( E^\pm_{1/2} \). Finally, define \( H^\pm_w = i dU_w^\pm(\mathbf{X}_0) \).

Scholium 7.3 now implies

**Scholium 7.20.** For any weight \( w \) and rational \( \lambda \), the representation \( dU_w^\pm \) on \( E^\pm_\lambda \) is contragredient to \( dU_{4-w}^\pm \) on \( E^\pm_{-\lambda} \).

Suppose \( w \) is real. Then \( dU_w^\pm \) acts irreducibly on all the \( E^\pm_\lambda \) except \( E^+_w \) and \( E^-_{w+1} \), \( dU_w^- \) acts irreducibly on all the \( E^-_\lambda \) except \( E^-_{w} \) and \( E^-_{w+1} \).

\( w + \frac{1}{2} \) is integral if and only if \( E^-_{1-w} = E^+_w \), which in turn is valid if and only if \( E^+_w = E^-_{1-w} \).

**Proof:** Under the assumptions stated on \( w \) and \( \lambda \), the coefficients in the terms for \( dU_w^\pm(\mathbf{S}) \) \( \Psi_{k_{mn}}^{\pm} \) in Table IV are always nonzero, for \( \Psi_{k_{mn}}^{\pm} \) in \( E^\pm_\lambda \). Clearly \( E^\pm_a = E^\pm_b \) if and only if \( \frac{1}{2} (a - b) \) is integral.

**Theorem 7.22.** Let \( w \) be real.

Suppose first that \( w + \frac{1}{2} \) is nonintegral. Then \( E^+_w \) and \( E^-_{1-w} \) (resp. \( E^-_w \) and \( E^-_{w+1} \)) have 3-step composition series under \( dU_w^+ \) (resp. \( dU_w^- \)):

\[
E^+_w \supseteq M^+ \supseteq V_+ \supseteq \{0\},
E^+_w \supseteq N^+ \supseteq W^+ \supseteq \{0\},
E^-_{w+1} \supseteq M^- \supseteq V^- \supseteq \{0\},
E^-_{w} \supseteq N^- \supseteq W^- \supseteq \{0\}.
\]

\( H^+_w \) is bounded from below (above) on \( V^+ \) (resp. \( W^+ \)), and this numerical bound can be taken to be 0 provided \( w > 0 \), in all cases.

Now let \( w \) be an integer plus \( \frac{1}{2} \). Suppose first that \( w = 9/2, 11/2, 13/2, \ldots, \) and let \( dU_w^+ \) act on \( E^+_w \). Then \( E^+_w \) contains three minimal invariant subspaces \( V_+, \ V, \ V_- \); \( H^+_w \) is strictly positive (negative) on \( V_+ \) (resp. \( V_- \)). The quotient of \( E^+_w \) by the sum of \( V_+ \), \( V \), and \( V_- \) contains two minimal invariant subspaces \( W_+ \) and \( W_- \) on which the induced action of \( H^+_w \) is strictly positive and negative. Finally, the further quotient by \( W_+ \oplus W_- \) is nonzero and finite-dimensional.

If \( w = 7/2 \) or \( w = 5/2 \), the same is true (for the actions of \( dU_w^+ \) on \( E^+_w \) and \( E^+_w \), respectively), except that the last finite-dimensional quotient is zero.
Now let \( w = 3/2 \), and let \( dU_w^+ \) act on \( E_w^+ \). Then \( E_{3/2}^+ \) has two minimal invariant subspaces \( W_+ \), equal to the span of all \( \Psi_{klmn}^{11} \) such that \( n = k + 1 + 3/2 \), and \( W_- \), equal to the span of all \( \Psi_{klmn}^{-11} \) such that \( n = -k - l - 3/2 \). \( E_{3/2}^+ \) is the sum (not direct) of invariant subspaces \( V_+ \), \( V \), and \( V_- \), and

\[
V \cap V_+ = W_+ \quad V \cap V_- = W_- \quad V_+ \cap V_- = \{0\}.
\]

\( W_+ + W_- \) is the kernel of \( \mathcal{D}_\mathcal{C} \) (or \( \mathcal{D}_\mathcal{P} \)) restricted to \( E_+ = +1 \) eigenspace of \( \gamma_5 \). 
\( dU_{3/2}^+ \) is infinitesimally unitarizable on \( W_+ \) and \( W_- \), with the unitary structure \( (\langle \cdot, \cdot \rangle) \) of Section 7.4. The induced actions of \( dU_{3/2}^+ \) on the quotients \( V_+ / W_+ \) and \( V_- / W_- \) (but not \( V/(W_+ + W_-) \)) are infinitesimally unitarizable, with the unitary structures \( (\langle \cdot, \cdot \rangle) \) and \( -(\langle \cdot, \cdot \rangle) \), respectively.

\( H_{3/2}^+ \) is positive (negative) on \( W_+ \) (resp. \( W_- \)), with the proper values \( 3/2 \), \( 5/2 \), \( 7/2 \), \( \ldots \). 

Now let \( w = -1/2 \), \( -3/2 \), \( -5/2 \), \ldots, and let \( dU_w^+ \) act on \( E_w^+ \). Then \( E_w^+ \) contains a nonzero finite-dimensional minimal invariant subspace \( F \); \( E_w^+ / F \) contains two minimal invariant subspaces \( W_+ \) and \( W_- \), on which \( H_w^+ \) is strictly positive and negative. The further quotient by \( W_+ \oplus W_- \) is the direct sum of three irreducible subspaces \( V_+ \), \( V \), and \( V_- \), and \( H_w^+ \) is positive (negative) on \( V_+(V_-) \).

If \( w = 1/2 \), the same is true except that the initial finite-dimensional space is zero.

The composition series for \( dU_{1/2}^- \) acting on \( E_{-1/2}^- \), in the case \( w + 1/2 \) is integral, has the same description as that above of \( dU_{3/2}^+ \) acting on \( E_{3/2}^+ \), except that if \( w = 3/2 \), \( W_+ \) is equal to the span of all \( \Psi_{klmn}^{-11} \) such that \( n = k + l + 1/2 \); \( W_- \) is equal to the span of all \( \Psi_{klmn}^{11} \) such that \( n = -k - l - 3/2 \); and \( W_+ + W_- \) is equal to the kernel of the Dirac operator acting on \( E_- \).

The four irreducible invariant subspaces of \( E_{3/2}^+ \) and \( E_{1/2}^- \) that span the kernel of \( \mathcal{D}_\mathcal{C} \) as noted above are the only subspaces \( X \) of any of the \( E_{1/2}^\pm \) invariant under any of the \( dU_{1/2}^\pm (\mathcal{S}) \), with the property that any \( f \) in \( X \) is determined by its restriction to a space-like surface \( t \times S^3 \).

Proof. The proof follows that of Corollary 5.5.6 of (I), and is based on the graphical representations in Figs. 1 and 2 of the scale actions \( dU_{1/2}^\pm (\mathcal{S}) \) in Table IV. This graphical summary is necessarily somewhat more elaborate than that in Section 5 of (I) for the scalar case, and is described below.

\( dU_{1/2}^\pm (\mathcal{S}) \Psi_{klmn}^{11} \) is generically a linear combination of four terms involving \( \Psi_{pqrs}^{11} \), where \( p = k \pm 1 \), \( s = n \pm 1 \), plus two terms involving \( \Psi_{pqrs}^{-11} \) (\( h = +1 \) changes to \( h = -1 \)), where \( p = k + 1 \) and \( s = n \pm 1 \). The genericity of these latter two terms is indicated by the \( < \) near "\( h = +1 \)" in Fig. 1. The situation is similar for \( dU_{1/2}^+ (\mathcal{S}) \Psi_{klmn}^{-11} \), which also involve \( \Psi_{pqrs}^{11} \), where \( p = k - 1 \) and \( s = n \pm 1 \), explaining the \( > \) near "\( h = -1 \)" in Fig. 1. The six arrows on the
Fig. 1. Scale actions in $U^+_w$.  

Fig. 2. Scale actions in $U^-_w$.  

four rays in each half-plane \(((k + l, n)\)-plane) graphs, indicate when the coefficients of these terms vanish, and the notation is an embellishment of that for the scalar case. The only new convention are the parallel double arrows, e.g., \(\triangleleft\triangleleft\) on the line \(L: n = k + l + w, h = +1\), indicate that if \((k + l, n)\) is on \(L\), \(dU^+_w(S) \Psi_{k l m n}^1\) does not involve \(\Psi_{k+1, \ldots, \ldots, 1+m, \ldots, n-1}^+\) or \(\Psi_{k+1, \ldots, \ldots, 1+m, \ldots, n-1}^-\). The two arrows on the line \(L': n = k + l + 3 - w, h = +1\), indicate that if \((k + l, n)\) is on \(L'\), \(dU^+_w(S) \Psi_{k l m n}^1\) does not involve \(\Psi_{k-1, \ldots, \ldots, 1+m, \ldots, n-1}^+\) or \(\Psi_{k+1, \ldots, \ldots, 1+m, \ldots, n+1}^-\).

The nonunitarity of the induced action of \(dU^+_w(S)\) on \(V/(W_+ + W_-)\) (unlike the scalar case) follows from the uniqueness, within a scalar factor, of an invariant hermitian form on the space of an irreducible representation, and the evident indefiniteness of \(\langle \cdot, \cdot \rangle\) on this space. Finally, the last assertion follows from inspection of the \(K\)-types of the composition series.

### VIII. The Form Bundle

#### 8.1. Preliminaries

Diffeomorphisms of smooth manifolds act canonically on smooth forms over the manifold, and this action defines a representation of the diffeomorphism group. The restriction of this representation to the essential causal group \(\hat{G}\) of the manifold \(\hat{M}\) defines the representation \(U\) that this section primarily concerns. According to the next result, this is an induced representation of the type earlier considered; however, this fact will not be used until the next section, a direct treatment of parallelization being quite convenient in the present section.

**Scholium 8.1.** *The action \(U\) of \(\hat{G}\) on the space of differential forms of degree \(d\) is induced from the representation of \(\hat{P}\) in the space of antisymmetric tensors of rank \(d\) and conformal weight \(d\).*

**Proof:** It will suffice to treat the case of 1-forms, since other inducing representations of \(U\) on forms of other degrees will be corresponding antisymmetric tensor products of this representation with itself, and since the scalar case \((d = 0)\) is evident. To compute the inducing representation it is convenient to use the isotropy group \(\hat{P}\) of the fixed point \(0 \times -1\) of \(\hat{M}\), relative to any formulation as \(R^1 \times S^3\). The local action of \(\hat{G}\) at this point, which is all that is involved here, is identical to that of \(G\) at the point \(-1\) of \(U(2)\), by virtue of the covering relations earlier indicated. Using the expression given in (I) for the imbedding of \(\hat{P}\) in \(SU(2, 2)\) (p. 85) it is straightforward to check the validity of the stated conclusion.

Parallelization of forms, relations to Maxwell’s equations, etc., can be conveniently treated through the use of the standard 1-forms \(\beta_j\) defined by the equations \(\beta_j(X_k) = \delta_{jk}\) \((j, k = 0, 1, 2, 3)\), and introduced in (I), to which
reference is made for some preliminaries. Some further development of the calculus of forms on $\mathbf{M}$ in terms of this basis will facilitate the later analysis in this section.

Noting that for arbitrary generators $X$ and $Y$ of $G$, $(dU(X)\beta)(Y) = \beta([X, Y]) + X(\beta(Y))$, $\beta$ being an arbitrary 1-form, the following relations then follow directly from the definition of the $\beta_j$: for arbitrary $j, k = 0, 1, 2, 3$,

\[ dU(Y_k)\beta_j = 0, \quad dU(X_j)\beta_j = 0, \quad dU(X_i)\beta_j = 2\beta_k, \quad dU(X_0)\beta_j = 0, \]
\[ dU(X_i)\beta_k = -2\beta_j \quad \text{if} \quad i, j, k \text{ are in cyclic order}. \]

Since vector fields are canonically dual to 1-forms, the action of $U$ on 1-forms induces a corresponding action on vector fields, taking the form (and denoting the extended action also as $U$):

\[ dU(X)Y = -[X, Y], \]

where $X$ is arbitrary in $G$ and $Y$ is an arbitrary smooth vector field. Using this fact and earlier relations for the $\beta_j$, actions of $dU(S)$ on the $\beta_j$ and alternative presentations of the $X_j$ and $Y_j$ are readily computed, and are collected in Table VII for later use.

TABLE VII
Relevant Basis Elements

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_4$</td>
<td>$u_3$</td>
<td>$u_2$</td>
<td>$u_4$</td>
<td>$u_3$</td>
<td>$u_2$</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial u_4}$</td>
<td>$\frac{\partial}{\partial u_3}$</td>
<td>$\frac{\partial}{\partial u_2}$</td>
<td>$\frac{\partial}{\partial u_4}$</td>
<td>$\frac{\partial}{\partial u_3}$</td>
<td>$\frac{\partial}{\partial u_2}$</td>
</tr>
</tbody>
</table>

\[ dU(S)\beta_0 = -u_1u_4\beta_0 + u_0u_4\beta_1 + u_0u_2\beta_2 + u_0u_3\beta_3, \]
\[ dU(S)\beta_1 = u_0u_4\beta_0 - u_1u_4\beta_1 + u_1u_3\beta_2 - u_1u_2\beta_3, \]
\[ dU(S)\beta_2 = u_0u_3\beta_0 - u_1u_3\beta_1 - u_1u_2\beta_2 + u_1u_1\beta_3, \]
\[ dU(S)\beta_3 = u_0u_2\beta_0 + u_1u_2\beta_1 - u_1u_1\beta_2 - u_1u_0\beta_3. \]
The **standard components** of differential forms on $\tilde{M}$ are defined as those relative to the basis formed by products of the $\beta_j$. Thus for a 1-form $A$ they are the functions $A_j$ such that $A = A_0 \beta_0 + A_1 \beta_1 + A_2 \beta_0 + A_3 \beta_3$. Associated with any 1-form $A$ is the 0-form $N(A)$ defined by the equation $N(A) = -\ast d \ast A$. It is useful to note the straightforward

**Scholium 8.2.** For any 1-form $A$, $N(A)$ may be expressed in terms of standard components as

$$N(A) = X_0 A_0 - X_1 A_1 - X_2 A_2 - X_3 A_3,$$

and $d \ast A = -N(A) \beta_0 \beta_1 \beta_2 \beta_3$. If $A = d\phi$ for a smooth function $\phi$, then

$$N(d\phi) = -\Box \phi. \quad (8.1)$$

It follows from the bi-orthogonality of the $\beta_j$ and the $X_k$ that for any smooth function $f$ on $M$,

$$df = \sum_{j=0}^{3} (X_j f) \beta_j.$$

Using this relation and those developed above it follows that for any 1-form $A$ (assumed smooth, as are all other in this section unless otherwise indicated)

$$dU(X_1)A = -(X_1 A_0) \beta_0 - (X_1 A_1) \beta_1 - (X_1 A_2 + 2A_1) \beta_2$$

$$+ (-X_1 A_3 + 2A_2) \beta_3,$$

which remains valid with cyclic permutations of 1, 2, 3. Using these relations repeatedly, the following relation results:

$$dU(X_1^2 + X_2^2 + X_3^2)A = (\Delta A_0) \beta_0 + (\Delta A_1 - 4F_{23}) \beta_1$$

$$+ (\Delta A_2 - 4F_{31}) \beta_2 + (\Delta A_3 - 4F_{12}) \beta_3, \quad (8.2)$$

where the $A_j$ are the standard components of $A$ and the $F_{ij}$ are the standard components of $dA$. The standard components of any given 2-form $\omega$ are the $F_{ij}$ defined by the equation

$$\omega = \sum_{j=1}^{3} F_{0j} \beta_0 \beta_j + \sum' F_{ij} \beta_i \beta_j,$$

where $\sum'$ signifies summation over all values of $i$ and $j$ in cyclic order (i.e., 12, 23, and 31) and $F_{ji} = -F_{ij}$ for all $i$ and $j$.
A similar computation yields the expression for an arbitrary 2-form $\omega$

$$dU(X_1)\omega = -(X_1 F_{01}) \beta_0 \beta_1 - (X_1 F_{02} + 2F_{03}) \beta_0 \beta_2$$

$$+ (-X_1 F_{03} + 2F_{02}) \beta_0 \beta_3 + (-X_1 F_{12} + 2F_{31}) \beta_1 \beta_2$$

$$- (X_1 F_{23}) \beta_2 \beta_3 - (X_1 F_{31} + 2F_{12}) \beta_3 \beta_1.$$

Repeated application of this yields the equation

$$dU(X_1^2 + X_2^2 + X_3^2)\omega = (\Delta F_{01} + 4(X_1 F_{02} - X_2 F_{03} - 2F_{01})) \beta_0 \beta_1$$

$$+ \text{corresp. terms in } \beta_0 \beta_2 \text{ and } \beta_0 \beta_3$$

$$+ (\Delta F_{12} + 4(X_1 F_{23} - X_1 F_{31} - 2F_{12})) \beta_1 \beta_2$$

$$+ \text{corresp. terms in } \beta_2 \beta_3 \text{ and } \beta_3 \beta_1. \quad (8.3)$$

In the case of a 3-form $j$ the standard components are the $j_k$ defined by the equation

$$j = j_0 \beta_1 \beta_2 \beta_3 + j_1 \beta_0 \beta_2 \beta_3 + j_2 \beta_0 \beta_3 \beta_1 + j_3 \beta_0 \beta_1 \beta_2.$$

It is straightforward to verify that

$$dj = (X_0 j_0 - X_1 j_1 - X_2 j_2 - X_3 j_3) \beta_0 \beta_1 \beta_2 \beta_3$$

for any 3-form $j$.

### 8.2. Maxwell's Equations

These equations, taking the form

$$d\omega = 0, \quad d* \omega = 0$$

for a 2-form $\omega$, were shown by Bateman and Cunningham [3] to be conformally invariant in $M_0$. Here they are treated on $\tilde{M}$ in terms of their parallelization via standard components.

**Theorem 7.1.** A two-form $\omega$ satisfies Maxwell's equations if and only if its standard components $F_{ij}$ satisfy

$$X_0 F_{12} + X_1 F_{20} + X_2 F_{01} = 2F_{03}, \quad X_0 F_{01} + X_2 F_{12} + X_3 F_{13} = -2F_{23},$$

$$X_0 F_{23} + X_2 F_{30} + X_3 F_{02} = 2F_{01}, \quad X_0 F_{02} + X_3 F_{23} + X_1 F_{21} = -2F_{31},$$

$$X_0 F_{31} + X_3 F_{10} + X_1 F_{03} = 2F_{02}, \quad X_0 F_{03} + X_1 F_{31} + X_2 F_{32} = -2F_{12}$$

$$X_1 F_{23} + X_2 F_{31} + X_3 F_{12} = 0, \quad X_1 F_{01} + X_2 F_{02} + X_3 F_{03} = 0. \quad (8.4)$$

In terms of the complex quantities

$$M_p = e^{-i\theta}(F_{0p} + iF_{qr}), \quad \tilde{M}_p = e^{i\theta}(F_{0p} - iF_{qr}),$$
where \((p, q, r)\) is a cyclic permutation of \((1, 2, 3)\), Eqs. (8.4) are equivalent to
\[
\begin{align*}
i\dot{M}_1 &= M_1 + X_2M_3 + X_3M_2, \\
i\dot{M}_2 &= M_2 + X_3M_1 - X_1M_3, \\
i\dot{M}_3 &= M_3 + X_1M_2 - X_2M_1, \\
X_1M_1 + X_2M_2 + X_3M_3 &= 0, \tag{8.5}
\end{align*}
\]
and the four other equations obtained by replacing \(M_j\) by \(\tilde{M}_j\) and \(-i\) by \(i\) in (8.5). If the two-form \(\omega\) is real, then systems (8.4) and (8.5) are equivalent.

**Proof.** This follows by use of the formulas
\[
\begin{align*}
d\omega &= (X_2F_{01} - X_1F_{02} + X_0F_{12} - 2F_{03})\beta_0\beta_1\beta_2 \\
&\quad + (X_3F_{02} - X_2F_{03} + X_0F_{23} - 2F_{01})\beta_0\beta_2\beta_3 \\
&\quad + (X_1F_{03} - X_3F_{01} + X_0F_{31} - 2F_{02})\beta_0\beta_3\beta_1 \\
&\quad + (X_3F_{12} + X_1F_{23} + X_2F_{31})\beta_1\beta_2\beta_3, \\
&\quad \ast \omega = F_{01}\beta_2\beta_3 + F_{02}\beta_3\beta_1 + F_{03}\beta_1\beta_2 - F_{23}\beta_0\beta_1 - F_{31}\beta_0\beta_2 - F_{12}\beta_0\beta_3, \\
d\ast \omega &= (X_0F_{03} - X_2F_{23} + X_1F_{31} + 2F_{12})\beta_0\beta_1\beta_2 \\
&\quad + (X_0F_{01} - X_3F_{31} + X_2F_{12} + 2F_{23})\beta_0\beta_2\beta_3 \\
&\quad + (X_0F_{02} - X_1F_{12} + X_3F_{23} + 2F_{31})\beta_0\beta_3\beta_1 \\
&\quad + (X_1F_{01} + X_2F_{02} + X_3F_{03})\beta_1\beta_2\beta_3.
\end{align*}
\]

**Theorem 7.2.** The standard components \(F_{ij}\) of a solution \(\omega\) of Maxwell's equations satisfy the relations
\[
\Box_c F_{01} = -2X_0F_{23}, \quad \Box_c F_{12} = 2X_0F_{01},
\]
and their cyclic permutations. The quantities \(M_j\) and \(\tilde{M}_j\), defined in Theorem 7.1, satisfy the curved wave equation
\[
(\Box_c + 1)M_j = (\Box_c + 1)\tilde{M}_j = 0.
\]

**Proof.** By combining Eqs. (8.4), it results that
\[
\begin{align*}
X_0^2F_{01} &= -X_0X_2F_{12} - X_0X_3F_{13} - 2X_0F_{23}, \\
X_1^2F_{01} &= -X_1X_2F_{02} - X_1X_3F_{03}, \\
X_2^2F_{01} &= -X_0X_2F_{12} + X_2X_1F_{02} + 2X_2F_{03}, \\
X_3^2F_{01} &= X_0X_3F_{31} + X_3X_1F_{03} - 2X_3F_{02}.
\end{align*}
\]
Use now of \( |X_0, X_1| = 0, |X_1, X_2| = -2X_3, \) etc., implies \( \Box c F_{01} = -2X_0 F_{23} \). The other equations follow similarly. For example,

\[
(\Box c + 1) M_i = -e^{-it}(F_{01} + iF_{23}) - 2ie^{-it}X_0(F_{01} + iF_{23})
+ e^{-it}c_i(F_{01} + iF_{23}) + e^{-it}(F_{01} + iF_{23})
= 0.
\]

**Corollary 7.2.1.** Any distribution solution of Maxwell’s equations on \( \bar{M} \) exists as a continuous map from \( R \) to a fixed (i.e., time-independent) \( L_2 \)-Sobolev space on \( S^3 \), and is invariant under \( \zeta \) (thus it is lifted up from \( \bar{M} \)).

**Proof:** The first statement follows from the previous theorem and the same conclusion for the curved wave equation, Theorem 5.5. It was further shown in (I) that any solution of the curved wave equation on \( \bar{M} \) is antisymmetric under the central element \( \zeta \) (defined to act on \( R^1 \times S^3 \) by translation in \( R^1 \) by \( \pi \) and the antipodal map in \( S^3 \)). Since \( e^{\pm it} \) are also antisymmetric under \( \zeta \), it follows that the components \( F_{ij} \) are \( \zeta \)-invariant; since the basis forms \( \beta_j \) are also \( \zeta \)-invariant, the same is true of the two-form solution.

For certain purposes it is more convenient to treat Maxwell’s equations in terms of a 1-form \( A \), the “potential,” whose derivative \( dA \) is the 2-form solution. Thus the equations take the form \( d \ast dA = 0 \).

**Theorem 7.3.** A 1-form \( A \) on \( \bar{M} \) satisfies \( d \ast dA = 0 \) if and only if

\[
\Box c A_1 = X_1 N - 2F_{23}, \quad \Box c A_3 = X_1 N - 2F_{12},
\]

\[
\Box c A_2 = X_2 N - 2F_{31}, \quad \Box c A_0 = X_0 N,
\]

where \( N = N(A) = X_0 A_0 - X_1 A_1 - X_2 A_2 - X_2 A_3 \), and \( A_j \) and \( F_{ij} \) are the standard components of \( A \) and \( dA \).

Conversely, given any distribution 2-form \( \omega \) on \( \bar{M} \) satisfying \( d\omega = d \ast \omega = 0 \), there exists a 1-form \( A \) on \( \bar{M} \) such that \( dA = \omega \) and such that \( N(A) = 0 \); \( A \) can be taken \( C^\infty \) if \( \omega \) is such.

**Proof:** Given \( A \) and \( d\omega \) with the usual components \( A_j \) and \( F_{ij} \), one computes

\[
F_{0j} = X_0 A_j - X_j A_0 \quad \text{for } j = 1, 2, 3,
\]

and, e.g.,

\[
F_{12} = X_1 A_2 - X_2 A_1 + 2A_3,
\]

so that

\[
d \ast dA = (\Box c A_0 - X_0 N) \beta_1 \beta_2 \beta_3 + (\Box c A_1 - X_1 N + 2F_{23}) \beta_0 \beta_2 \beta_3
+ (\Box c A_2 - X_2 N + 2F_{31}) \beta_0 \beta_3 \beta_1 + (\Box c A_3 - X_3 N + 2F_{12}) \beta_0 \beta_1 \beta_2.
\]
Conversely, given any such solution \(\omega\), it may be assumed that \(\omega\) is \(C^n\) for arbitrarily large \(n\), by applying a sufficiently high inverse power of \(1 - \chi_1 - \chi_2 - \chi_3\). Since the relevant cohomology of \(U(2)\) vanishes, there exists a \(C^n\) 1-form \(A\) on \(\mathfrak{M}\) such that \(\omega = dA\). Now modify \(A\) by adding \(d\phi\) for some function \(\phi\); \(\phi\) will be chosen so that \(N(d\phi) = -N(A)\). Expand \(N(A)\) and \(\phi\) in terms of the complete orthogonal basis \(\beta_{k1mn}\). By (8.1) it follows that the above equation holds if and only if the coefficients \(\phi_{k1mn}\) can be chosen so that

\[
(-n^2 + (k + l)(k + l + 2)) \phi_{k1mn} = N_{k1mn},
\]

the \(N_{k1mn}\) being the expansion coefficients for \(N(A)\). Now the coefficient of \(\phi_{k1mn}\) above is nonvanishing with an inverse bounded by 1, except when \(n = k + l = 0\). But \(N_{0000}\) is proportional to the integral of \(N(A)\) over \(\mathfrak{M}\), which vanishes on \(XJF\) for any distribution \(F\). Thus the \(\phi_{k1mn}\) can be solved for, and \(\phi\) will lie in the same \(L_2\)-Sobolev space as \(N(A)\); in particular, it will be smooth if \(A\) is smooth.

**Definition.** A 1-form solution \(A\) of Maxwell’s equations is said to be in the Lorentz gauge in case \(N(A) = 0\).

**Corollary 7.3.1.** Any 1-form solution of Maxwell’s equations on \(\mathfrak{M}\) that is in the Lorentz gauge and is periodic in \(t\) with period some rational fraction of \(\pi\) is necessarily invariant under \(\zeta\).

**Proof.** For any such \(A\) and \(j = 0, 1, 2, 3\)

\[
\Box_c A_j = f_j
\]

by Theorem 7.3, where \(f_j\) is invariant under \(\zeta\). By assumption, each \(A_j\) can be expanded in terms of the \(\phi_{k1mn}\), and, as in the previous proof, \(\Box_c\) (and \(\zeta\)) acts on such Fourier coefficients by multiplication by a strictly nonvanishing function. Thus, except for constant functions on \(\mathfrak{M}\), which are \(\zeta\)-invariant, it follows that the Fourier components of \(A_j\) vanish whenever the corresponding ones of \(f_j\) do. Thus \(A_j\) is also \(\zeta\)-invariant.

The next theorem evaluates the enveloping algebra element \(L_c\) in the representation \(U\), restricted to 1- and 2-forms, particularly in the Maxwell subrepresentations.

**Theorem 7.4.** Given any 1- and 2-forms \(A\) and \(\omega\) with the standard components \(A_j\) and \(F_{ij}\),

\[
dU(L_c)\omega = (\Box_c F_{01} - 2(X_3F_{02} - X_2F_{03} - 2F_{01}))\beta_0\beta_1
\]

\[
+ \text{corresp. terms in } \beta_0\beta_2 \text{ and } \beta_0\beta_3
\]

\[
+ (\Box_c F - 2(X_2F_{23} - X_1F_{31} - 2F_{12}))\beta_1\beta_2
\]

\[
+ \text{corresp. terms in } \beta_2\beta_3 \text{ and } \beta_3\beta_1,
\]
and

\[ dU(L_c)A = (\square_c A_0) \beta_0 + (\square_c A_1 + 2F_{23}) \beta_1 + (\square_c A_2 + 2F_{13}) \beta_2 + (\square_c A_3 + 2F_{12}) \beta_3 \]

(the \( F_{ij} \) in the latter equation being components of \( dA \)). In particular, if \( \omega \) satisfies the Maxwell equations, then

\[ dU(L_c)\omega = 0, \]

and if \( d \ast dA = 0 \), then

\[ dU(L_c)A = d(N(A)). \]

**Proof.** This follows directly from Eq. (8.2), (8.3), (8.4), and Theorems 7.2 and 7.3.

**IX. THE FERMI AND YUKAWA LAGRANGIANS**

9.1. Introduction

Local multilinear or sesquilinear invariants of fields, known in the physics literature as Lagrangians, are important in the theoretical treatment of interacting systems. Conventionally the integrated Lagrangian, the integral over \( M_0 \) of a "Lagrangian density" and a more physical and unique quantity than the latter, is a formal scalar invariant. The integral, however, lacks well-defined mathematical existence; it evidently diverges for generic smooth sections and indeed its convergence over all of \( M_0 \) is quasionable even when sections satisfying partial differential equations associated with the Lagrangian are involved. Here rigorous and conceptually simple counterparts to the well-known Lagrangians involved, quite formally, in the work of Fermi and of Yukawa, are treated using the compactness of \( \bar{M} \) and its finite covers.

In the formal theory the invariance of the integrated Lagrangian is inherently not a rigorous result. In the present context it is mathematically clear-cut and follows from the general considerations of Section VI together with corresponding features of the spinor and form bundles. With appropriate limitations on conformal weights, invariance under \( \bar{G} \) and anticipated transformation properties under discrete symmetries are readily established. For general conformal weights there is naturally only \( \bar{\mathbb{K}} \)-invariance.

Since \( M_0 \) is canonically imbedded in \( \bar{M} \) as an open dense submanifold, integration over \( M_0 \) and \( \bar{M} \) are closely connected. The stable subspaces of the
bundles thus far treated are lifted up from finite covers of $\tilde{\mathbb{M}}$ (e.g., from $\tilde{\mathbb{M}}^{(4)}$ in the case of the stable spinor bundle with "massless" subspace treated in Section 7). Such finite covers, denoted here as $\tilde{\mathbb{M}}_f$, contain canonically a finite number of disjoint copies of $\mathbb{M}_0$, whose union is dense. This number, equal to the degree of the covering, may be arbitrarily large. The infinite cover $\tilde{\mathbb{M}}$ is thus in a sense approximable by the $\tilde{\mathbb{M}}_f$, but it is not yet known whether it is the locus of non-trivial stable bundles; it will not be treated in this section.

9.2. The General Fermi and Yukawa Lagrangians

Both the Fermi and Yukawa Lagrangians involve spinor currents, which in the present context are sections of the tensor product of the spinor bundle with its dual. For mathematical intelligibility and generality they are presented primarily in slightly abstract forms requiring some preliminary definitions.

**Definitions.** Consider a bundle over $\tilde{\mathbb{M}}_f$, induced from a representation of $\tilde{\mathbb{P}}$ possibly extended by discrete symmetries. Such a bundle is quasi-unitary in case there is given in the representation space $\mathbb{R}$ of the inducing representation $R$ a non-degenerate (but not necessarily definite) sesquilinear and hermitian inner product that is invariant under the restriction of $R$ to the unimodular Poincaré group $\tilde{\mathbb{P}}_0$. It is required in addition that the inner product $\langle \cdot, \cdot \rangle$ invariant under $\tilde{\mathbb{P}}_0$ be invariant also under the discrete symmetries in the inducing group, within sign and/or complex conjugation.

Two quasi-unitary bundles are the same (resp. equivalent) except for weight if their inducing representations, when restricted to $\tilde{\mathbb{P}}_0$, and invariant inner products, are the same (resp. quasi-unitarily equivalent). A pseudo-form bundle is one equivalent in this sense to a form bundle.

The canonical interaction Lagrangian between two quasi-unitary bundles that are the same (resp. equivalent) is the sesquilinear form, defined relative to a given space-time factorization for $\tilde{\mathbb{M}}$, by the equation

$$L(\Psi, \Psi') = \int_{\tilde{\mathbb{M}}_f} \langle \Psi(u), \Psi'(u) \rangle \, du, \tag{9.1}$$

where $\Psi$ and $\Psi'$ are sections in the curved parallelization (resp. the same expression with $\Psi'$ replaced by its quasi-unitary transform in the first bundle).

Given a quasi-unitary bundle over $\tilde{\mathbb{M}}$, a current is a section of the tensor product of the bundle with its dual. More generally, a current relative to a given ordered pair of quasi-unitary bundles is a section of the tensor product of the first with the dual of the second, which tensor product is called the current bundle relative to the given ones.
A Fermi Lagrangian is the canonical one between currents of copies of spinor bundles. A Yukawa Lagrangian is the canonical one between the current bundle of copies of spinor bundles and a pseudo-form bundle.

The following scholium gives conditions under which Fermi and Yukawa Lagrangians are $\bar{G}$-invariant. To facilitate its statement, it is convenient to note the following variant of a well known result of Brauer and Weyl [7].

Lemma 9.1. The tensor product of the spin representation of the orthochronous unimodular Poincaré group $\tilde{\mathbf{P}}_0^{\text{ortho}}$ (i.e., $\tilde{\mathbf{P}}_0$ extended by $P$) with its dual is quasi-unitarily equivalent to a unique direct sum of reweighted pseudo-form bundles, each occurring with multiplicity 1, of degree 0, 1, ..., 4.

The projection of the tensor product space $\mathbf{R}_{\text{spin}} \otimes \hat{\mathbf{R}}_{\text{spin}}$ onto the subspace transforming under $\tilde{\mathbf{P}}_0^{\text{ortho}}$ equivalently to $i$-forms ($i = 0, 1, 2, 3, 4$) will be denoted as $P_i$.

Scholium 9.2. Let $\Psi_i$ denote arbitrary smooth sections of the spinor bundles of (real) weights $w_j$ over $\tilde{\mathbf{M}}_j$ in the curved parallelization relative to a given space-time factorization of $\tilde{\mathbf{M}}$ ($j = 1, 2, 3, 4$). Let the $g_i$ ($i = 0, 1, 2, 3, 4$) be arbitrary constants. Then the Fermi Lagrangian

$$L(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = \int_{\tilde{\mathbf{M}}} \sum_{l=0}^{4} g_l \langle P_i(\Psi_1(u) \otimes \Psi_2(u)), P_i(\Psi_3(u) \otimes \Psi_4(u)) \rangle \, d_4 u$$

is $\bar{K}$-invariant, and is $\bar{G}$-invariant provided $w_1 + w_2 + w_3 + w_4 = 4$.

If moreover $\Phi_i$ denotes an arbitrary smooth section of the pseudo-form bundle of weight $v_i$ and degree $i$ on $\tilde{\mathbf{M}}_i$, then the Yukawa Lagrangian

$$L(\Psi_1, \Psi_2, \Phi) = \int_{\tilde{\mathbf{M}}} \sum_{l=0}^{4} g_l \langle P_i(\Psi_1(u) \otimes \Psi_2(u)), \Phi_i(u) \rangle \, d_4 u,$$

where the prime denotes the element of the range of $P_i$ corresponding to $\Phi_i(x)$ via the equivalence of Lemma 9.1, is $\bar{K}$-invariant, and is $\bar{G}$-invariant provided $w_1 + w_2 + v_3 + v_4 = 4$ (for all $i$).

Proof. The $\bar{K}$-invariance is immediate from the $\bar{K}$-invariance of the measure together with Theorem 4.1 and Lemma 9.1. The full $\bar{G}$-invariance under the indicated condition on the weights follows now from Corollary 6.1.1.

9.3. Some Specific Fermi and Yukawa Lagrangians and Their Discrete Symmetry Transformation Properties

The foregoing general considerations may be exemplified by prototypical cases that are "specific" in the sense that all the $g_i$ except one are taken to
vanish. The Lagrangians take a more familiar form if the anti-linear map \( x \rightarrow x^\# \) from the representation space of \( R \) to that of \( \bar{R} \) is used, rendering the currents interpretable as between particles and anti-particles.

The original Fermi vector Lagrangian is the case \( i = 1 \). If \( \Psi_1 \) and \( \Psi_2 \) are sections of an \( R_w \)-bundle, then the projection \( P_1 \) carries \( \Psi_1 \otimes \Psi_2^w \) into the Lorentz vector with components \( j_\lambda = \langle \gamma_\lambda \Psi_1, \Psi_2 \rangle' \). The resulting Lagrangian takes the form \[ \int_{\mathbb{M}} \langle j(u), j'(u) \rangle \, du, \] where \( j_\lambda \) differs from \( j_\lambda \) only in the replacement of \( \Psi_1 \) and \( \Psi_2 \) by \( \Psi_3 \) and \( \Psi_4 \); the inner product between the resulting vectors is the Lorent-invariant one. The Lagrangian is \( \bar{G} \)-invariant only when the sum of the weights \( w_j \) is 4.

To determine the transformation properties of Lagrangians under discrete symmetries it is useful to note the

**Lemma 9.3.** For arbitrary \( x, y \) in \( C^4 \),

\[ \langle R(P)x, R(P)y \rangle' = \langle x, y \rangle', \quad \langle R(T)x, R(T)y \rangle' = \overline{\langle x, y \rangle'}, \]

\[ \langle R(C)x, R(C)y \rangle' = -\overline{\langle x, y \rangle}, \]

\[ \langle \gamma_5 R(P)x, R(P)y \rangle' = -\overline{\langle \gamma_5 x, y \rangle'}, \quad \langle \gamma_5 R(T)x, R(T)y \rangle' = \overline{\langle \gamma_5 x, y \rangle'}, \]

\[ \langle \gamma_5 R(C)x, R(C)y \rangle' = \overline{\langle \gamma_5 x, y \rangle}. \]

**Proof.** This is straightforward and is omitted.

**Corollary 9.2.1.** The original Fermi Lagrangian is invariant under \( P \), and transformed into its complex conjugate by \( T \) and \( C \).

**Proof.** The parallelized forms of \( T, C, \) and \( P \) were given in Section 7.8. The stated transformation properties of \( T \) and \( C \) then follow from Lemma 9.3, the commutativity of \( R(C) \) with the \( \gamma_j \), and the relations \( R(T) \gamma_0 = -\gamma_0 R(T), R(T) \gamma_j = \gamma_j R(T) \) for \( j = 1, 2, 3 \).

Invariance under \( P \) requires additional argument because \( U(P) \) has the space-dependent multiplier

\[ \begin{pmatrix} 0 & -W^{-1} \\ -W^{-1} & 0 \end{pmatrix} - R(P) \begin{pmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{pmatrix}, \quad W \in SU(2). \]

Noting that this multiplier commutes with \( \gamma \), and using Lemma 9.3, it suffices to show that the expression

\[ \sum_{j=1}^{3} \langle \gamma_j \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \Psi_1, \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \Psi_2 \rangle' \langle \gamma_j \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \Psi_3, \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \Psi_4 \rangle' \]

has a value that is independent of \( W \in SU(2) \) for arbitrary vectors \( \Psi_j \) \( (j = 1, 2, 3, 4) \) in \( C^4 \). This follows from the fact that conjugation by such
transforms the $\gamma_j$, and hence also their matrix elements, according to
an orthogonal transformation in $R^3$, which preserves the Euclidean inner
product.

The Yukawa Lagrangian in the case in which all $g_i$ except $i = 1$ vanish,
which was used originally in quantum electrodynamics (apart from the
kinetic energy of the electron), couples a pseudo-one-form, say $A$, with
standard components $A_\lambda$ ($\lambda = 0, 1, 2, 3$), with the earlier noted current
$j_1 = \langle \gamma_1 \Psi_1, \Psi_2 \rangle '$ determined by $\Psi_1 \otimes \Psi_2^\ast$. It then takes the form

$$\sum_{\lambda=0}^{3} \int_{M_r} \langle \gamma_\lambda \Psi_1(u), \Psi_2(u) \rangle ' A_\lambda(u) d_4 u. \quad (9.2)$$

Invariance under discrete symmetries is here discussed under the assumption
that their action on pseudo-forms is by induction from $P' \times Z_2$ to $G' \times Z_2$,
as treated generally in Corollary 4.1.3. The actions of $P$, $C$, and $T$
on the basic one-forms at the inducing point $0 \times -I$ are defined as follows:

$$R(P)p_j = -\delta_{ij} (j = 1, 2, 3); \quad R(C) = \text{complex conjugation}; \quad R(T) = -R(C) R(P).$$

**Corollary 9.2.3.** The Yukawa Lagrangian (9.2) is $P$-invariant;
transformed into its complex conjugate by $T$; and transformed into its
negative complex conjugate by $C$.

**Proof:** Corollaries 4.1.3 and 4.1.7 show that the action of $T$ and $C$ on
fields involve only constant multipliers, equal to those at the inducing point.
The indicated transformation properties for $T$ and $C$ then follow from
Lemma 9.3.

As earlier the action of $P$ on fields is space-dependent. Using Lemma 9.3
and the anti-commutativity of $R(P)$ with the $\gamma_j$ ($j = 1, 2, 3$), $P$-invariance
reduces to the independence from $W \in SU(2)$ of the expression

$$\sum_{j=1}^{3} \langle \gamma_j R(g) \Psi_1, R(g') \Psi_2 \rangle ' (R'(g)A),$$

where $g = \beta((0 \times W) \times 0)$ and $R$ and $R'$ denote the respective inducing
representations. As earlier this reduces in turn to the euclidean invariance of
scalar products of euclidean vectors.

**Remark.** If the spinor fields have the weights $3/2$ (sometimes called the
"canonical dimension") and $A$ is a strict one-form and so of weight 1,
Lagrangian (9.2) is $G$-invariant, by Scholium 9.2. This may be construed as
a mathematical version of what is known in the physical literature as the
conformal invariance of massless quantum electrodynamics. However, the
formal Lagrangian associated with the latter interacting fields, which differs from that given by Eq. (9.2) only in the replacement of $\hat{M}_f$ by $M_0$ and in the use of the flat in place of the curved parallelization, is not clearly convergent even when the $\Psi_1$ and $\Psi_2$ are taken as solutions of the massless Dirac equation and $A$ as a solution of the Maxwell equations. When formulated precisely as given by (9.2), however, the Lagrangian is convergent, conformally invariant, and dependent only on the "electromagnetic field" $dA$ associated with $A$, in accordance with "gauge invariance." It can moreover be shown that for any normalizable solutions of the massless Dirac and Maxwell equations on $M_0$, there are corresponding solutions of the corresponding equations on $\hat{M}^{(4)}$ or any higher covering.

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REFERENCES


ERRATA TO "ANALYSIS IN SPACE-TIME BUNDLES, I"

p. 82, line 15: Omit "with $U(n).""

p. 88, line 27: Change "principle" to "principal."

p. 88, lin 27: Change $P$ to $\hat{P}$.

p. 88, line 29: Change "contained in $O$" to "contained in $O$" (boldface $O$).
p. 98, line 7: The $E$ should be script $E$.
p. 98, line 12: "$L_r$ and $L_c$" should be "$L_r$ and $L_c$" (no boldface).
p. 99, lines 30 and 31: The $U$ in $U(g)$ should be boldface.
p. 100, line 1 of proof of Corollary 4.1.1: 3.1 should be 4.1.
p. 102, lines 3 and 4, and p. 105, lines 5–8: $\pi$ and $\tau$ should be $P$ and $T$, respectively.
p. 117, line 3 of Theorem 5.1: $U(g)$ should be $U_u(g)$.
p. 125, part (iii) and p. 126, part (iv) of Theorem 5.6: $P_0$ and $\tilde{P}_0$ should be $T_0$ and $\tilde{T}_0$.
p. 126, line 17, and p. 127, line 2: $P_0$ should be $T_0$.
p. 128: Following condition (5.14), insert "... the inner product being one relative to which the generators $J^I_j$ and $J^T_j$ are hermitian."

pp. 128–133: All occurrences of $J_i$, $J^I_i$, $J^T_i$ ($i = 1, 2, 3$), $J_+$, and $J_+$ should have boldface $J$'s.
p. 130, line 5: Change $\xi^{-1/2}$ to $(-\xi)^{-1/2}$.
p. 131, Eq. (5.20), and p. 134, line 16: "$i^{-1}$" should be "$i^{-l}$".
p. 131, line 2 of Lemma 5.4.4: Insert parentheses to read (5.11), (5.12).
p. 132, Table VII: The $L_{ij}$ should have boldface $L$'s.
p. 137, Corollary 5.5.2: The 1 in the factor \( n+1+k+2 \quad w \) should be changed to $l$.
p. 138, Table X: The factor 1/2 in the fourth term of the expression given for $-S|nklm\rangle$ should be changed to 1/4, like the previous three terms. A similar factor of 1/4 should be inserted in the fourth term of the expression for $u_{-1}u_4|nklm\rangle$. The $S$ in $U_u(S)$ should be boldface.

p. 139, line 9: Change "subspace" to "subspaces."
p. 140, line 6: $W$ should be $w$. The containing paragraph is part of Corollary 5.5.6 and should also be italicized.

**Errata to Ref. [2]**

p. 5262, Table 1, top row: The "$SU(2, 2)$ generator" for "$L_{-1,0}$" should be

\[
\frac{1}{2} \begin{pmatrix}
 b_0 & 0 \\
 0 & -b_0
\end{pmatrix}.
\]
p. 5262, Table 1, bottom line: $x_4$ should be $x_2$. 