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Groupoid quantales: A non-étale setting

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ABSTRACT

We establish a bijective correspondence involving a class of unital involutive quantales and a class of groupoids whose space of units is a sober space. This class includes equivalence relations that arise from group actions. The resulting axiomatization of the class of quantales, as well as the correspondence defined here, extend the theory of étale groupoids and their quantales (Resende (2007) [10]) to a point-set, non-étale setting.

1. Introduction

Important examples of groupoids that are non-étale abound. Typical examples are given by equivalence relations induced from group actions with fixed points. It is then natural to seek algebraic descriptions of these groupoids, analogously to what is done for instance in [9,11,13] in the context of groupoids that are étale. In this paper, we establish a bijective correspondence involving a class of unital involutive quantales and a class of groupoids whose set of units is a sober topological space. This correspondence extends, in a spatial setting, the correspondence between localic étale groupoids and inverse quantal frames defined in [13].

The correspondence in [13] has also been extended beyond the étale setting in [10], to a correspondence between open groupoids and *open quantal frames*. As their name suggests, these quantales satisfy the frame distributivity condition, but are not required to be unital (the inverse quantal frames in [13] are exactly the *unital* open quantal frames in [10]). The correspondence defined in this paper covers an alternative extension: the quantales considered here are unital, but do not need to be frames.

As already observed in [10], the essential difference between the groupoid-quantale correspondence in the étale and in the non-étale setting lays in the role played by the inverse semigroup of G-sets of a groupoid. Indeed, all the information needed to reconstruct any étale groupoid is encoded in the inverse semigroup formed by the germs of its local bisections. The quantales associated with both étale groupoids and inverse semigroups, i.e. the inverse quantal frames, being characterized as the free join completions of the inverse semigroups, contain no extra information than the inverse semigroups themselves. However, in the non-étale setting, the inverse semigroup of G-sets is not enough to reconstruct the groupoid: the missing information governs the various possible ways in which any two G-sets of the groupoid intersect one another (notice that this is exactly the information content that becomes trivial in étale groupoids, because G-sets are closed under finite intersection). This extra information is stored in the quantale, which is why quantales are essential to this setting. In this paper, the role of germs in the reconstruction process is played by the classes of an equivalence relation that we refer to as the *incidence* relation, which encodes information on the incidence of any two G-sets at a point, in the language of quantales.

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As mentioned earlier, the quantales considered here are not in general frames. Correspondingly, their associated groupoids do not have a topological (or localic) structure on their spaces of arrows G_1. In place of topologies, designated collections of G-sets are used, which we refer to as *selection bases* (cf. Definition 3.5). Indeed, rather than being purely between quantales and groupoids, our correspondence is between quantales and *pairs* $(\mathcal{G}, \mathcal{S})$ of groupoids and selection bases. In fact, these pairs can be regarded as categories on the topology of the space of units G_0 (cf. Remark 3.7). This observation paves the way to a pointfree generalization of this correspondence, which we develop in the companion paper [8].

The results we present in this paper find their main motivation in a much wider research program that seeks noncommutative extensions of the Gelfand–Naimark duality [1]. Interestingly, these results have also many points in common with and are potentially relevant to another area of research (which as far as our knowledge goes is disconnected from the first). This area belongs to order theory, algebra and logic, seeks representability results for classes of relation algebras, and its research program is well exemplified by [3], where a certain class of relation algebras is concretely represented via groupoids. We believe that presenting our results in the spatial setting and making use of purely order-theoretic and topological techniques is useful in making the connections with this area more transparent and in making these results more easily accessible to its community of researchers.

The paper is organized as follows: in Section 2 we give the basic definitions and properties of groupoids and quantales; in Section 3 we introduce our main groupoid setting of pairs $(\mathcal{G}, \mathcal{S})$ and their associated *groupoid quantales* $\mathcal{Q}(\mathcal{G}, \mathcal{S})$; in Section 4 we introduce the *SGF-quantales*: these quantales are to groupoid quantales what locales are to topologies. In the same section, a procedure is defined to associate a set groupoid $\mathcal{G}(\mathcal{Q})$ with every SGF-quantale $\mathcal{Q}$. This procedure is based on the *incidence relation* and its properties, which are detailed in Section 4.1. In Section 5 we introduce the *spatial* SGF-quantales, prove that this class includes the groupoid quantales $\mathcal{Q}(\mathcal{G}, \mathcal{S})$, and that if $\mathcal{Q}$ is spatial, then the set of units of $\mathcal{G}(\mathcal{Q})$ can be made into a sober space. In Section 6 we prove that the back-and-forth correspondence between spatial SGF-quantales and the pairs $(\mathcal{G}, \mathcal{S})$ is bijective. In Section 7 we explain in detail why, although it is not so by definition, this correspondence is compatible with the étale setting of [13]. In Section 8 we conclude with two concrete examples.

2. Preliminaries

2.1. Strongly Gelfand quantales

A *quantale* $\mathcal{Q}$ [5,14] is a complete join-semilattice endowed with an associative binary operation $\cdot$ that is completely distributive in each coordinate, i.e.

D1: $c \cdot \bigvee I = \bigvee\{c \cdot q : q \in I\}$
D2: $\bigvee I \cdot c = \bigvee\{q \cdot c : q \in I\}$

for every $c \in \mathcal{Q}, I \subseteq \mathcal{Q}$. Since it is a complete join-semilattice, $\mathcal{Q}$ is also a complete, hence bounded, lattice. Let 0, 1 be the lattice bottom and top of $\mathcal{Q}$, respectively. Conditions $D1$ and $D2$ readily imply that $\cdot$ is order-preserving in both coordinates and, as $\bigvee \varnothing = 0$, that $c \cdot 0 = 0 = 0 \cdot c$ for every $c \in \mathcal{Q}$. $\mathcal{Q}$ is *unital* if there exists an element $e \in \mathcal{Q}$ for which

U: $e \cdot c = c = c \cdot e$ for every $c \in \mathcal{Q}$,

and is *involutive* if it is endowed with a unary operation $*$ such that, for every $c, q \in \mathcal{Q}$ and every $I \subseteq \mathcal{Q}$,

I1: $c^{**} = c$.
I2: $(c \cdot q)^* = q^* \cdot c^*$.
I3: $(\bigvee I)^* = \bigvee\{q^* : q \in I\}$.

Relevant examples of unital involutive quantales are:

1. The quantale $\mathcal{P}(R)$ of subrelations of a given equivalence relation $R \subseteq X \times X$.
2. The quantale $\mathcal{P}(G)$, for every group G.
3. Any frame $\mathcal{Q}$, setting $\cdot := \wedge$, $* :=$ id and $e := 1_{\mathcal{Q}}$.

A *homomorphism* of (involutive) quantales is a map $\varphi : \mathcal{Q} \to \mathcal{Q}'$ that preserves $\bigvee, \cdot$ (and $*$). If $\mathcal{Q}, \mathcal{Q}'$ are unital quantales, then φ is *unital* if $e' \leq \varphi(e)$ and is *strictly unital* if $\varphi(e) = e'$. Notice that since every homomorphism is completely join-preserving, then $\varphi(0) = \varphi(\bigvee \varnothing) = \bigvee \varnothing = 0$. However, a homomorphism of quantales does not need to preserve the lattice top. For example, if $R \subset S$ are equivalence relations on X, then the inclusion $\mathcal{P}(R) \to \mathcal{P}(S)$ is a strictly unital homomorphism of quantales that does not. If $\varphi(1_{\mathcal{Q}}) = 1_{\mathcal{Q}'}$ then φ is *strong*.
Let $\mathcal{Q}$ be a unital involutive quantale. An element $f \in \mathcal{Q}$ is *functional* if $f^* \cdot f \leq e$ and is a *partial unit* if both f and f^* are functional[2]. The set of functional elements (resp. partial units) will be denoted by $\mathcal{F}(\mathcal{Q})$ (resp. $\mathcal{I}(\mathcal{Q})$). It is easy to verify that $e \in \mathcal{I}(\mathcal{Q})$ and $\mathcal{I}(\mathcal{Q})$ is closed under composition and involution of $\mathcal{Q}$. Moreover, if $f \leq g \in \mathcal{I}(\mathcal{Q})$ then $f \in \mathcal{I}(\mathcal{Q})$.

Let $\mathcal{Q}_e = \{c \in \mathcal{Q} : c \leq e\}$. $\mathcal{Q}_e \subseteq \mathcal{I}(\mathcal{Q})$, moreover, $\mathcal{Q}_e$ is a unital involutive subquantale of $\mathcal{Q}$.

2 If $\mathcal{Q} = \mathcal{P}(R)$ for some equivalence relation $R \subseteq X \times X$, then functional elements (partial units) are exactly the graphs of (invertible) partial maps f on X.

Definition 2.1. A unital involutive quantale $\mathcal{Q}$ is *strongly Gelfand* (or an *SG-quantale*) if

SG. $a \leq a \cdot a^* \cdot a$ for every $a \in \mathcal{Q}$.

Recall that $\mathcal{Q}$ is a *Gelfand* quantale (see also [14]) if $a = a \cdot a^* \cdot a$ for every right-sided element of $\mathcal{Q}$ ($a \in \mathcal{Q}$ being *right-sided* if $a = a \cdot 1$). It is immediate to see that every SG-quantale is Gelfand, and that $f = f \cdot f^* \cdot f$ for every SG-quantale $\mathcal{Q}$ and every $f \in \mathcal{F}(\mathcal{Q})$. We will simplify notation and write $a \cdot b$ as ab.
A quantale $\mathcal{Q}$ is *supported* if it is endowed with a *support*, which is a completely join-preserving map $\varsigma : \mathcal{Q} \to \mathcal{Q}_e$ s.t. $\varsigma(a) \leq aa^*$ and $a \leq \varsigma(a)a$ for every $a \in \mathcal{Q}$. For every supported quantale $\mathcal{Q}$, $\mathcal{Q}_e$ coincides with $\varsigma\mathcal{Q}$ and it is a locale with $ab = a \wedge b$ and trivial involution (cf. [12, Lemma II.3.3]). It is immediate to see that every supported quantale is an SG-quantale. Therefore the item 1 of the following proposition shows that the fundamental property of supported quantales mentioned above generalizes to SG-quantales. Even more importantly, the items 3 and 4 of the following proposition show that the crucial connection between supported quantales and inverse monoids [12, Theorem II.3.17.1] generalizes to SG-quantales[3]:

Proposition 2.2. *For every SG-quantale $\mathcal{Q}$,*

1. *the subquantale $\mathcal{Q}_e$ is a frame: in particular, involution $*$ coincides with the identity, and composition $\cdot$ with $\wedge$.*
2. *For every $f, g \in \mathcal{F}(\mathcal{Q})$ such that $f \leq g$, $f = g$ iff $ff^* = gg^*$.*
3. *$\mathcal{I}(\mathcal{Q})$ is an inverse monoid[4] whose set of idempotents coincides with $\mathcal{Q}_e$, and whose natural order coincides with the order inherited from $\mathcal{Q}$.*
4. *The assignment $\mathcal{Q} \mapsto \mathcal{I}(\mathcal{Q})$ extends to a functor $\mathcal{I}$ from the category of SG-quantales to the category of inverse monoids.*

Proof. 1. Let $d \leq e$. By SG, $d \leq dd^*d \leq ed^*e = d^*$, and likewise, $d^* \leq d$, hence involution is identity on $\mathcal{Q}_e$. If $c \leq e$, then $cc = c$: indeed, $cc \leq ce = c$, and by SG and the fact that involution is identity on $\mathcal{Q}_e$, $c = cc^*c = (cc)c \leq (cc)e = cc$. Let $d_1, d_2 \leq e$. Then $d_1d_2 \leq d_1e = d_1$ and $d_1d_2 \leq ed_2 = d_2$, so $d_1d_2 \leq d_1 \wedge d_2$. Conversely, if $c \leq d_1$ and $c \leq d_2$, then $c = cc \leq d_1d_2$, hence $d_1 \wedge d_2 \leq d_1d_2$.
2. By SG and since $f \leq g$ implies $f^* \leq g^*$, $g = gg^*g \leq ff^*g \leq fg^*g \leq fe = f$.
3. By SG, $ff^*f = f$ and $f^*ff^* = f^*$ for every $f \in \mathcal{I}(\mathcal{Q})$. Hence, it is enough to show that the restriction of the product to the idempotent elements of $\mathcal{I}(\mathcal{Q})$ is commutative. This follows from item 1 above and from the fact that for every $f \in \mathcal{I}(\mathcal{Q})$, $ff = f$ iff $f \leq e$: Indeed, if $f \leq e$, then by (1), $ff = f \wedge f = f$. Conversely, if $ff = f$, then $f^* = (ff)^* = f^*f^*$, hence $ff^* = ff^*f^* \leq ef^* = f^*$, and so $f = ff^*f \leq f^*f \leq e$. Since $\mathcal{Q}_e \subseteq \mathcal{I}(\mathcal{Q})$, this also shows that the set of idempotent elements of $\mathcal{I}(\mathcal{Q})$ coincides with $\mathcal{Q}_e$. Hence, the natural order of the inverse monoid $\mathcal{I}(\mathcal{Q})$ is defined as follows: $f \leq g$ iff $f = gh$ for some $h \in \mathcal{Q}_e$, and therefore it coincides with the order inherited from $\mathcal{Q}$.
4. Every strict homomorphism of unital involutive quantales maps partial units to partial units, hence it restricts to a homomorphism of inverse monoids. □

2.1.1. A natural action

For every SG-quantale $\mathcal{Q}$, a natural action[5] can be defined of the inverse semigroup $\mathcal{I}(\mathcal{Q})$ on $\mathcal{Q}_e$: indeed, for every $f \in \mathcal{I}(\mathcal{Q})$ and every $h \in \mathcal{Q}_e$ let $h^f = f^*hf$. This is indeed an action of $\mathcal{I}(\mathcal{Q})$ because of the identity $(h^f)^g = h^{fg}$.

Lemma 2.3. *For every $h \in \mathcal{Q}_e$ and $f \in \mathcal{I}(\mathcal{Q})$,*

1. *$hf = fh^f$ and $f^*h = h^f f^*$.*
2. *If $h \leq ff^*$ then $h = fh^f f^*$.*

Proof. 1. Since $f = ff^*f$ and because the product is commutative in $\mathcal{Q}_e$, we get $hf = h \cdot (ff^*)f = (ff^*) \cdot hf = fh^f$. The second equality goes analogously 2. Immediate. □

2.2. Groupoids

Definition 2.4. A *set groupoid* is a tuple $\mathcal{G} = (G_0, G_1, m, d, r, u, i)$, s.t.:

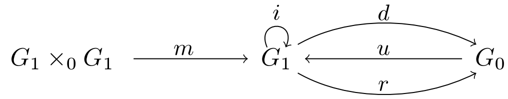
G1. G_0 and G_1 are sets;
G2. $d, r : G_1 \to G_0$ and $u : G_0 \to G_1$ s.t. $d(u(p)) = p = r(u(p))$ for every $p \in G_0$;
G3. $m : (x, y) \mapsto xy$ is an associative map defined on $G_1 \times_0 G_1 = \{(x, y) \mid r(x) = d(y)\}$ and s.t. $d(xy) = d(x)$ and $r(xy) = r(y)$;
G4. $xu(r(x))) = x = u(d(x))x$ for every $x \in G_1$.;

[3] We thank Pedro Resende for pointing to our attention this interpretation of items 1 and 3 of Proposition 2.2.

[4] An *inverse semigroup* (cf. [9]) is a semigroup such that for every element x there exists a unique inverse, i.e. an element y such that $x = xyx$ and $y = yxy$. Equivalently, an inverse semigroup is a semigroup such that every element has some inverse and any two idempotent elements commute. An *inverse monoid* is an inverse semigroup with a multiplicative unit.

[5] In [13] (discussion before Lemma 4.5) a similar action is defined on the whole of a stable quantale frame $\mathcal{Q}$ on $\mathcal{Q}_e = \varsigma\mathcal{Q}$ by the assignment $(a, h) \mapsto \varsigma(ah)$, which makes $\mathcal{Q}$ into a $\varsigma\mathcal{Q}$-module. So the action introduced here is a slight generalization of the action defined there.

G5. the map $i : G_1 \to G_1$ denoted as $i(x) = x^{-1}$ is s.t. $xx^{-1} = u(d(x))$, $x^{-1}x = u(r(x))$, $d(x^{-1}) = r(x)$ and $r(x^{-1}) = d(x)$ for every $x \in G_1$.



Example 2.5. 1. For any equivalence relation $R \subseteq X \times X$, the tuple $(X, R, \circ, \pi_1, \pi_2, \Delta, ()^{-1})$ defines a groupoid. Of particular interest are versions of this examples where X is a topological space: for instance, the space of Penrose tilings [1,6,7] is such an example and its associated groupoid is étale.

2. For any group $(G, \cdot, e, ()^{-1})$, the tuple $(\{e\}, G, \cdot, d, r, u, ()^{-1})$ is a groupoid, and the equalities G4 and G5 just restate the group axioms.
3. The following example is a special but important case of the first one: every topological space X can be seen as a groupoid by setting $G_1 = G_0 = X$ and identity structure maps. In this case, $G_1 \times_0 G_1 = \{(x, x) \mid x \in X\}$ and $xx = x$ for every $x \in X$.
4. A groupoid can be associated with any action[6] $G \times X \to X$ of a group G on a set X, by setting $G_1 = G \times X$, $G_0 = X$, and for all $g, h \in G$ and $x, y \in X$, $d(g, x) = x$, $r(g, x) = gx$, $u(x) = (e, x)$ ($e \in G$ being the identity element), and $(g, x) \cdot (h, y) = (hg, x)$ whenever $y = gx$.
5. To a group action as above, another groupoid can be associated, which is given by the equivalence relation $R \subseteq X \times X$ defined by xRy iff there exists some $g \in G$ such that $y = gx$.

Some useful facts about groupoids are reported in the following:

Lemma 2.6. *For all $p \in G_0$, $x, y \in G_1$,*

1. $u(p)^{-1} = u(p)$,
2. $x = xx^{-1}x$ *and* $x^{-1} = x^{-1}xx^{-1}$,
3. *if* $xy^{-1}, x^{-1}y \in u[G_0]$ *then* $x = y$,
4. *if* $x = xyx$ *and* $yxy = y$, *then* $y = x^{-1}$,
5. $(x^{-1})^{-1} = x$,
6. $(xy)^{-1} = y^{-1}x^{-1}$.

For every groupoid $\mathcal{G}$, $\mathcal{P}(G_1)$ can be given the structure of a unital involutive quantale (see also [12] and [13] 1.1 for a more detailed discussion): indeed, the product and involution on G_1 can be lifted to $\mathcal{P}(G_1)$ as follows:

$$S \cdot T = \{x \cdot y \mid x \in S,\ y \in T \text{ and } r(x) = d(y)\} \qquad S^* = \{x^{-1} \mid x \in S\}.$$

Denoting by E the image of the structure map $u : G_0 \to G_1$, we get:

Fact 2.7. $\langle \mathcal{P}(G_1), \bigcup, \cdot, ()^*, E\rangle$ *is a strongly Gelfand quantale.*

Proof. SG follows from Lemma 2.6.2. □

3. SP-groupoids and their quantales

In what follows, a *groupoid* is a set groupoid $\mathcal{G} = (G_0, G_1)$ s.t. G_0 additionally carries a topology which makes it a sober space.[7] For every $p \in G_0$, let $\overline{p}$ denote the topological closure of $\{p\}$. The topology on G_0 will be denoted by $\Omega(G_0)$. We do not fix any a priori topology on G_1.

Definition 3.1. A *local bisection* of a groupoid $\mathcal{G}$ is a map $s : U \to G_1$ such that $d \circ s = \mathrm{id}_U$ and $t = r \circ s$ is a homeomorphism $t : U \to V$ between open sets of G_0. A *bisection image*[8] of $\mathcal{G}$ is the image of some local bisection of $\mathcal{G}$. Let $\mathcal{S}(\mathcal{G})$ be the collection of the bisection images of $\mathcal{G}$.

6 For any group G, a *(left) action* of G on a set X is a function $\cdot : G \times X \to X$ s.t. for all $g, h \in G$ and $x \in X$, $(gh)x = g(hx)$ and $ex = x$ (e being the identity of G). For any topological group G and any topological space X, a *continuous action* of G on X is a continuous map $G \times X \to X$ which is an action of G as a discrete group G on the underlying set of X.

7 For every topological space X, a closed set C is *irreducible* iff $C \neq \varnothing$ and for all closed sets K_1, K_2, $C \subseteq K_1 \cup K_2$ implies that $C \subseteq K_1$ or $C \subseteq K_2$. A *sober* space is a topological space s.t. its irreducible closed sets are exactly the topological closures of singletons.

8 Images of local bisections are sometimes referred to as *G-sets* (cf. [11]). However, since "*G*-sets" usually refers to sets equipped with a group action, we propose an alternative name here.

Notice that since $d \circ s = \mathrm{id}_U$, local bisections are completely determined by their corresponding bisection images. We will denote bisection images by S, T, possibly indexed, and their corresponding local bisections will be s, t, possibly indexed. Since G_1 is not endowed with any topology, the local bisections according to the definition above are not required to be continuous, as is the case e.g. in [10,13]. This design choice can be motivated as follows. First, there exists at least a topology on G_1 w.r.t. which the local bisections of Definition 3.1 are always continuous, and it is defined as follows: Let $R \subseteq G_0 \times G_0$ be the equivalence relation induced by G_1 and let $\pi : G_1 \to R$ be the map defined as $\pi(x) = (d(x), r(x))$; the open subsets of G_1 are those of the form $\pi^{-1}[A]$, for any open subset $A \subseteq R$ in the product topology inherited from $G_0 \times G_0$. This topology is in general not even T_0. However, even if $\mathcal{S}$ is defined as the family of local bisections that are continuous w.r.t. some given topologies on G_0 and on G_1, if the resulting topological groupoid $\mathcal{G}$ is not étale, then the quantale $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ in Definition 3.8 below will contain the topology of G_1 as a subquantale but will not coincide with it, nor will this topology be uniquely identifiable inside $\mathcal{Q}(\mathcal{G}, \mathcal{S})$. So the topology on G_1 is a piece of information that cannot be retained along the back-and-forth correspondence defined in this paper. On the other hand, the absence of topology on G_1 allows for a greater generality: for instance, G_1 can be taken as a set endowed with a measure (typically a Haar measure) and, correspondingly, the local bisections can be taken as measurable maps defined on open sets of G_0. This could be interesting in view of possible applications of this setting to the theory of C^*-algebras. Also, not assuming any topology on G_1 allows in principle for a greater choice of selection bases (cf. Example 3.6).

Finally, the groupoids as we understand them in this paper can always be made into étale topological groupoids, by endowing G_1 with the topology generated by taking the intersections of bisection images as a subbase. However, their associated inverse quantal frames turn out to be in general much larger than the quantales we associate with these (non-topological) groupoids. The comparison with [13], which we will discuss more in detail in Section 7, is based on a special case of this observation.

The statements in the following proposition are well known for other settings and readily follow from the definition of bisection image:

Proposition 3.2. *For every groupoid $\mathcal{G}$,*

1. $\mathcal{S}(\mathcal{G}) \subseteq \mathcal{I}(\mathcal{P}(G_1))$.
2. $(\mathcal{S}(\mathcal{G}), \cdot, ()^*, E)$ *is an inverse monoid.*

The following examples of groupoids and canonical families of local bisections are relevant for the theory developed in this article.

Example 3.3. 1. Let X be a topological space and G be a group endowed with the discrete topology. Let $G \times X \to X$ be a continuous group action, and let us consider the groupoid $(X, G \times X)$ as in Example 2.5.4. A canonical family of local bisections of this groupoid is given by those of the form $s_g : U \to G \times X$ defined by $s_g(x) = (g, x)$, for some $g \in G$. If $G_1 = G \times X$ is endowed with the product topology, then $d : G_1 = G \times X \to X$ is obviously étale.

2. Consider the groupoid $\mathcal{G} = (X, R)$, where $R \subseteq X \times X$ is the equivalence relation induced by a group action of $G \times X \to X$ as in Example 2.5.5. A distinguished subfamily of $\mathcal{S}(\mathcal{G})$ is the family of local bisections of the form $t_g : U \to R$ given by $t_g(x) = (x, gx)$, for some $g \in G$. If R is endowed with the quotient topology induced by the map $\pi : G \times X \to R$, defined by $(g, x) \mapsto (x, gx)$, then the first projection map $d : R \to X$ is not necessarily étale.

For example, let $X = \mathbb{C}$ and $G = \{z \in \mathbb{C} \mid z^n = 1\}$ be the group of the nth roots of the unity, for $n \geq 2$. Consider the action of G on X given by the product $(z, x) \mapsto zx$. Its induced equivalence relation is $R = \{(x, y) \mid y = zx,\ z \in G\}$. Any open neighborhood W of $(0, 0) \in R$ is of the form $W = \bigcup_{z \in G}\{(x, zx) \mid x \in U,\ z \in G\}$, for some open neighborhood U of $0 \in \mathbb{C}$. Indeed, W is open iff $\pi^{-1}(W)$ is open in $G \times X$. Since $(z, 0) \in \pi^{-1}(W)$ for all $z \in G$, $\pi^{-1}(W)$ must contain some open set of the form $\{z\} \times U$ for any $z \in G$. If $n > 1$, then, for any such open set $W \subseteq R$, the restriction $d : W \to d(W)$ is not injective, hence $d : R \to X$ is not a local homeomorphism, i.e. it is not étale.

Definition 3.4. A groupoid $\mathcal{G} = (G_0, G_1)$ as above has the *selection property*, or is an *SP-groupoid*, if G_1 is covered by bisection images.

Given a groupoid $\mathcal{G}$, we can associate a unital involutive quantale with every inverse monoid $\mathcal{S} \subseteq \mathcal{P}(G_1)$: namely, the quantale defined as the sub-join-semilattice of $\langle \mathcal{P}(G_1), \bigcup \rangle$ generated by $\mathcal{S}$. However, in our non-étale setting, we may not be able to reconstruct back the inverse semigroup from the quantale. For this, we need the following new, stronger definition:

Definition 3.5. A *selection base* for an SP-groupoid $\mathcal{G}$ is a family $\mathcal{S} \subseteq \mathcal{S}(\mathcal{G})$ verifying the following conditions:

SB1. $\mathcal{S}$ is a sub-inverse monoid of $\mathcal{S}(\mathcal{G})$;
SB2. $u[U] \in \mathcal{S}$ for every open set $U \in \Omega(G_0)$;
SB3. if $\{S_i\}_{i \in I} \subseteq \mathcal{S}$ and $S_i \cdot S_j^* \subseteq E$ and $S_i^* \cdot S_j \subseteq E$ for every $i, j \in I$, then $\bigcup_{i \in I} S_i \in \mathcal{S}$.
SB4. For every $S, T \in \mathcal{S}$, $\{p \in G_0 \mid s(p) = t(p)\}$ is the union of locally closed[9] subsets of G_0.

[9] For every topological space X, a subset $Y \subseteq X$ is *locally closed* if $Y = U \cap C$ for some open set U and some closed set C.

SB5. $\mathcal{S}$ covers G_1.

Selection bases are not in general topological bases, cf. Section 8.2 for an example.

Example 3.6. 1. A continuous group action $G \times X \to X$ as in Example 2.5.4 gives rise to a canonical selection base consisting of the bisection images corresponding to local bisections $s_g : U \to G_1$ defined by the assignment $x \mapsto g \cdot x$ for any $g \in G$.
2. If G_0 is a T_1 space, then the family $\mathcal{S}(\mathcal{G})$ of the local bisections is the greatest selection base. Notice that in this case the condition SB5 is trivially verified, since any subset of G_0 is the union of its singleton subsets, which are all closed.
3. Let X be a T_1-space with a continuous group action as above, and let R be the equivalence relation induced by the group action, as in Example 3.3.2. Then the groupoid $\mathcal{G} = (X, R)$ has the following, in general distinct, selection bases: the family $\mathcal{S}(\mathcal{G})$, and the one consisting of the local bisections of the form $x \mapsto (x, gx) = (x, g(x)x)$ for some constant map $g(x) = g \in G$. The latter family is in general strictly contained in the former: indeed, local bisections can be given by assignments of the form $x \mapsto (x, g(x)x)$, for some non-constant $g(x)$ s.t. the assignment $x \mapsto g(x)x$ defines a homeomorphism.

Remark 3.7. The pairs $(\mathcal{G}, \mathcal{S})$ can be regarded as categories on the topology of G_0, in the following way. Let $\mathcal{S}_{\mathcal{G}}$ be the category having the elements of $\Omega(G_0)$ as objects, and such that for every $U, V \in \Omega(G_0)$, $\mathrm{Hom}_{\mathcal{S}_{\mathcal{G}}}(U, V)$ is the set of those $s \in \mathcal{S}$ (identified with their associated local bisections) such that $r[s[U]] \subseteq V$. This category includes the frame $\Omega(G_0)$ as a sub-category, and axiom SB3 says that the functor $\mathrm{Hom}_{\mathcal{S}_{\mathcal{G}}}(-, U)$ is a sheaf on $\Omega(G_0)$.
This observation paves the way to a generalization of the present results to a setting of quantales associated with sheaves on locales, which will be developed in [8]. Axiom SB4, which is needed in the present setting (see proof of Proposition 5.3), will be always true in the localic setting. Indeed the subspace where two elements of $\mathcal{S}$ "intersect" each other can still be defined, but any subspace of a locale is a join of locally closed subspaces, (cf. [4], chapter IX pp. 504, 505 for a discussion on the canonical subspace associated with a given local operator).

3.1. Groupoid quantales

Definition 3.8. For every SP-groupoid $\mathcal{G}$ and every selection base $\mathcal{S}$ for $\mathcal{G}$, the *groupoid quantale* (*GQ* for short) $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ associated with the pair $(\mathcal{G}, \mathcal{S})$ is the sub-$\bigcup$-semilattice of $\mathcal{P}(G_1)$ generated by $\mathcal{S}$.

In particular, the elements of $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ are arbitrary joins of elements of $\mathcal{S}$.
Condition SB3 crucially guarantees that $\mathcal{S}$ can be traced back from $\mathcal{Q}(\mathcal{G}, \mathcal{S})$:

Proposition 3.9. *For every SP-groupoid $\mathcal{G}$,*

1. $\mathcal{S} = \mathcal{I}(\mathcal{Q}(\mathcal{G}, \mathcal{S}))$.
2. $\mathcal{Q}(\mathcal{G}, \mathcal{S})_e = \{u[U] \mid U \in \Omega(G_0)\}$.
3. $P \in \mathcal{Q}(\mathcal{G}, \mathcal{S})_e$ *is prime iff* $P = u[G_0 \setminus \overline{p}]$ *for some* $p \in G_0$.

Proof. 1. $\mathcal{S} \subseteq \mathcal{I}(\mathcal{Q}(\mathcal{S}, \mathcal{G}))$ immediately follows from $\mathcal{S} \subseteq \mathcal{S}(\mathcal{G}) \subseteq \mathcal{I}(\mathcal{P}(G_1))$ (cf. Proposition 3.2.1). Let $T \in \mathcal{I}(\mathcal{Q}(\mathcal{S}, \mathcal{G}))$, so $T = \bigcup\{S_i\}_{i\in I}$ for some collection $\{S_i\}_{i\in I} \subseteq \mathcal{S}$. Then for every $i, j \in I$, $S_i \cdot S_j^* \subseteq T \cdot T^* \subseteq E$ and $S_i^* \cdot S_j \subseteq T^* \cdot T \subseteq E$, hence by SB3, $T = \bigcup_{i\in I} S_i \in \mathcal{S}$.
2. By SB2, if $U \in \Omega(G_0)$, then $u[U] \in \mathcal{S}$ and clearly $u[U] \subseteq u[G_0]$ so $u[U] \in \mathcal{Q}(\mathcal{G}, \mathcal{S})_e$. Conversely, let $H \in \mathcal{Q}(\mathcal{G}, \mathcal{S})_e$; then $H \subseteq u[G_0]$ and $H = \bigcup_{i\in I} S_i$ for some $\{S_i \mid i \in I\} \subseteq \mathcal{S}$. Let $s_i : U_i \to G_1$ be the corresponding local bisections. Then $S_i \subseteq u[G_0]$ implies that $s_i(p) = u(p)$ for every $p \in U_i$, therefore $H = u[U]$ for $U = \bigcup\{U_i \mid i \in I\}$.
3. The prime elements of $\Omega(G_0)$ are exactly the complements of irreducible closed sets, and by assumption G_0 is sober. □

Example 3.10. 1. Let $\mathcal{G} = (X, X \times G)$ be as in Example 3.3.1, and let $\mathcal{S}$ be the selection base associated with all local bisections (the locally constant maps $U \to G$ s.t. $U \subseteq X$ is an open set). $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ coincides with the product topology on $G_1 = G \times X$ and the resulting topological groupoid is étale.
2. On the other hand, let $R \subseteq X \times X$ be as in Example 3.3.2, and let $\mathcal{S}$ be the selection base associated with all local bisections. The observations in 3.3.2 imply, by the results in [13] and the discussion in Section 7, that the groupoid quantale $\mathcal{Q}(R, \mathcal{S})$ is not an inverse quantal frame.

4. SGF-quantales and their set groupoids

Definition 4.1. An *SGF-quantale* is a unital involutive quantale $\mathcal{Q}$ satisfying the following extra axioms:

SGF1. $\mathcal{Q}$ is $\bigvee$-generated by $\mathcal{I}(\mathcal{Q})$.
SGF2. $f = ff^*f$ for every $f \in \mathcal{I}(\mathcal{Q})$.
SGF3. For any $f, g \in \mathcal{I}(\mathcal{Q})$ and $h \in \mathcal{Q}_e$ if $f \leq h \cdot 1 \vee g$ then $f \leq h \cdot f \vee g$.

Clearly, the first two axioms imply that every SGF-quantale is SG. Let us motivate the axioms by showing that every groupoid quantale is SGF:

Proposition 4.2. *For every SP-groupoid $\mathcal{G}$ and every selection base $\mathcal{S}$ of $\mathcal{G}$, $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ is an SGF-quantale.*

Proof. SGF1 readily follows from $\mathcal{S} \subseteq \mathcal{I}(\mathcal{Q}(\mathcal{S}, \mathcal{G}))$. SGF2 follows from Fact 2.7. For SGF3, let $F, G \in \mathcal{I}(\mathcal{Q}(\mathcal{G}, \mathcal{S}))$ and $H \in \mathcal{Q}(\mathcal{G}, \mathcal{S})_e$. Proposition 3.9 implies that F, G are bisection images: let them correspond respectively to the local bisections f and g. From the same proposition it follows that H can be identified, via the structure map u, with some open subset $h \in \Omega(G_0)$. Assume that $F \subseteq H \cdot 1 \cup G$. This implies that for every $x \in dom(f)$, either $x \in h$, hence $f(x) \in H \cdot F$, or $x \in dom(g)$, which implies, since $d \circ f = id = d \circ g$, that $f(x) = g(x)$. □

The remainder of this section is aimed at constructing the set groupoid associated with any SGF-quantale. This construction is based on the incidence relation, whose definition (Definition 4.5) and properties are given in the following subsection.

4.1. The incidence relation on SGF-quantales

Let $\mathcal{Q}$ be an SGF-quantale. For every $f \in \mathcal{I}(\mathcal{Q})$ let $d(f) = ff^*$ and $r(f) = f^*f$. The following lemma lists some straightforward but useful formal properties of these abbreviations:

Lemma 4.3. *Let $\mathcal{Q}$ be an SGF-quantale, $f, f', g \in \mathcal{I}(\mathcal{Q})$ and $h, k \in \mathcal{Q}_e$. Then:*

1. $d(hf) = hd(f)$ *and* $r(fk) = r(f)k$.
2. *If* $f \leq g$, *then* $d(f) \leq d(g)$ *and* $r(f) \leq r(g)$.
3. $d(ff') = d(f')^{f^*}$ *and* $r(ff') = r(f)^{f'}$.
4. $r(f) = d(f)^f$ *and* $d(f) = r(f)^{f^*}$.

Let $\mathcal{P}_e$ be the set of the *prime* elements of $\mathcal{Q}_e$ (cf. [2]) i.e. those non-top elements $p \in \mathcal{Q}_e$ s.t. for every $h, k \in \mathcal{Q}_e$, if $h \wedge k \leq p$, then $h \leq p$ or $k \leq p$. Let

$$\mathcal{I} = \{(p, f) \in \mathcal{P}_e \times \mathcal{I}(\mathcal{Q}) \mid p \in \mathcal{P}_e,\ d(f) \not\leq p\}.$$

For every $p \in \mathcal{P}_e$, $p \leq e$ and $p \neq e$ imply that $d(e) = e \not\leq p$, hence $(p, e) \in \mathcal{I}$. Moreover, if $f \leq g$ then $d(f) \leq d(g)$ so $(p, f) \in \mathcal{I}$ implies that $(p, g) \in \mathcal{I}$. For every $h \in \mathcal{Q}_e$, $\mathcal{Q}_h = \{k \in \mathcal{Q} \mid k \leq h\}$ is a subframe of $\mathcal{Q}_e$.

Lemma 4.4. *For every* $f \in \mathcal{I}(\mathcal{Q})$,

1. *the assignment* $h \mapsto h^f = f^*hf$ *defines a frame isomorphism* $()^f : \mathcal{Q}_{d(f)} \to \mathcal{Q}_{r(f)}$, *the inverse of which is defined by* $k \mapsto k^{f^*} = fkf^*$.
2. *The prime elements of* $\mathcal{Q}_{d(f)}$ *correspond bijectively to the prime elements of* $\mathcal{Q}_{r(f)}$ *via* $()^f$.

Definition 4.5. The *incidence relation* $\sim$ on $\mathcal{I}$ is defined by setting

$$(p, f) \sim (q, g) \text{ iff } p = q \text{ and } h \not\leq p \text{ and } hf \leq pf \vee g \text{ for some } h \leq d(f) \wedge d(g).$$

We will also alternatively write $f \sim_p g$ (read: f and g are incident in p) in place of $(p, f) \sim (q, g)$.

Remark. Let us interpret the incidence relation if $\mathcal{Q} = \mathcal{Q}(\mathcal{G}, \mathcal{S})$ for some SP-groupoid $\mathcal{G}$ and some selection base $\mathcal{S}$: in this case, by Proposition 3.9, $\mathcal{Q}_e$ can be identified via u with $\Omega(G_0)$, $\mathcal{P}_e$ can be identified with the collection $\{\overline{p}^c \mid p \in G_0\}$ of the complements of the closures $\overline{p}$ of points $p \in G_0$ and $\mathcal{I}(\mathcal{Q}) = \mathcal{S}$. For all $F, G \in \mathcal{I}(\mathcal{Q})$, let f, g be their associated local bisections: then $F \sim_{\overline{p}^c} G$ iff there exists an open subset H of G_0 s.t. $H \cap \overline{p} \neq \varnothing$ (i.e., since p is dense in $\overline{p}$, $p \in H$), s.t. f and g are both defined over H and coincide over $H \cap \overline{p}$. Moreover, if G_0 is T_1, then $\mathcal{P}_e$ corresponds to the collection of the complements of points of G_0 and $F \sim_{\{p\}^c} G$ iff $f(p) = g(p)$.
Notice also that the relation $f \sim_p g$ may be defined by saying that there exist some $f' \leq f$ and $g' \leq g$ s.t. $d(f') = d(g') \not\leq p$ and $f' \leq pf' \vee g'$.

Proposition 4.6. 1. *The relation $\sim$ is an equivalence relation.*
2. *If $f \sim_p g$ and $g \leq g'$ then $f \sim_p g'$.*

Proof. 1. Reflexivity is obvious. Symmetry: $hf \leq pf \vee g$ implies $h = h \wedge d(f) = hd(f) \leq pd(f) \vee gf^*$ hence $h \leq p \vee fg^*$ and so $hg \leq pg \vee fg^*g \leq pg \vee f$. Transitivity: If $h_1 f \leq pf \vee g$ and $h_2 g \leq pg \vee l$ then setting $h = h_2 h_1$ we get $h \not\leq p$, because p is prime, $h \leq d(f) \wedge d(l)$ and

$$hf = h_2h_1f \leq h_2pf \vee h_2g \leq h_2pf \vee pg \vee l \leq p(f \vee g) \vee l \leq p \cdot 1 \vee l.$$

Hence, by SGF3, $hf \leq phf \vee l \leq pf \vee l$.
2. Straightforward. □

Lemma 4.7. *For every SGF-quantale $\mathcal{Q}$, let $(p, f) \in \mathcal{I}$. Then:*

1. *there exists a unique* $q \in \mathcal{P}_e$, *denoted* $q = f[p]$, *s.t.* $r(f) \not\leq q$ *and* $pf = fq$.
2. *For every* $h \in \mathcal{Q}_e$, *if* $h \not\leq p$, *then* $h^f \not\leq f[p]$.
3. *For every* $h \in \mathcal{Q}_e$, *if* $h \not\leq p$, *then* $d(hf) \not\leq p$ *and* $r(hf) \not\leq f[p]$.
4. *If* $f \sim_p g$ *then* $f[p] = g[p]$.
5. $(f[p], f^*) \in \mathcal{I}$ *and* $f^*[f[p]] = p$.

6. *If* $(f[p], g) \in \mathcal{I}$, *then* $(p, fg) \in \mathcal{I}$ *and* $fg[p] = g[f[p]]$.
7. $ff^* \sim_p e$ *and* $f^*f \sim_{f[p]} e$.

Proof. 1. From the basic theory of locales, we recall that for every $0 \neq h \in \mathcal{Q}_e$ p' is a prime element of $\mathcal{Q}_h$ iff $p' = hp$ for a unique $p \in \mathcal{P}_e$ s.t. $h \nleq p$. By Lemma 4.4.2, p'^f is a prime of $\mathcal{Q}_{r(f)}$, hence $p'^f = r(f)q$ for a unique $q \in \mathcal{P}_e$ s.t. $r(f) \nleq q$. Therefore, by Lemma 2.3.1, $pf = fp^f = fp'^f = fr(f)q = fq$.
2. Let $q = f[p]$. Then $pf = fq$, so $h^f \leq q$ implies, by Lemma 2.3.1, that $hf = fh^f \leq fq = pf$, hence $hd(f) \leq pd(f) \leq p$, and since p is prime and $d(f) \nleq p$, then $h \leq p$.
3. Recall that $d(hf) = hd(f)$. Since p is prime and $d(f) \nleq p$, then $h \nleq p$ implies that $d(hf) = hd(f) \nleq p$. Let $k = h^f$ (so, by Lemma 2.3.1, $hf = fk$); by item 2 we get that $k \nleq f[p]$, and since $f[p]$ is prime and $r(f) \nleq f[p]$, then $r(hf) = r(fk) = r(f)k \nleq f[p]$.
4. Assume that $f \sim_p g$. Then there exists some $h \in \mathcal{Q}_e$ s.t. $h \nleq p$, $h \leq d(f) \wedge d(g)$ and $hg \leq pg \vee f$. Let $q = f[p]$ and $q' = g[p]$, so $pf = fq$ and $pg = gq'$, and let us show that $q = q'$. Our assumption implies that there exists some $h \in \mathcal{Q}_e$ s.t. $h \nleq p$ and $hgq \leq pgq \vee fq = pgq \vee pf \leq p \cdot 1 \vee pgq$, hence by SGF3, $hgq \leq p \cdot hgq \vee pgq \leq pg = gq'$. This implies that $r(hg)q = r(hgq) \leq r(gq') = r(g)q' \leq q'$. Since $q' = g[p]$ and $h \nleq p$, by item 3 we get that $r(hg) \nleq q'$. Hence, since q' is prime, $r(hg)q \leq q'$ implies that $q \leq q'$. The proof that $q' \leq q$ is obtained symmetrically, from $g \sim_p f$.
5. By item 2, $d(f) \nleq p$ implies that $d(f^*) = r(f) = d(f)^f \nleq f[p]$, which proves that $(f[p], f^*) \in \mathcal{I}$. Let $q = f[p]$. In order to show that $f^*[q] = p$, by the uniqueness of $f^*[q]$ it is enough to show that $qf^* = f^*p$, which readily follows from $pf = fq$.
6. Let $q = f[p]$. Then by item 5, $p = f^*[q]$, hence $d(g) \nleq q$ implies by item 2 that $d(fg) = d(g)^{f^*} \nleq p$, which proves that $(p, fg) \in \mathcal{I}$. Let $q' = g[q]$, so $qg = gq'$; to finish the proof it is enough to show that $pfg = fgq'$: since $pf = fq$, then $pfg = fqg = fgq'$.
7. By assumption $d(f) \nleq p$, so take $h = d(f)$: clearly $h \leq d(ff^*) \wedge d(e)$ and $hff^* \leq e = pff^* \vee e$. The second relation follows from item 5 and the first relation in this item. □

Proposition 4.8. *For every SGF-quantale* $\mathcal{Q}$, *let* $f, f'g, g' \in \mathcal{I}(\mathcal{Q})$ *and* $p \in \mathcal{P}_e$.

1. *If* $f \sim_p g$ *and* $f' \sim_{f[p]} g'$ *then* $ff' \sim_p gg'$.
2. $f \sim_p g$ *iff* $f^* \sim_{f[p]} g^*$.

Proof. 1. Let $q = f[p] = g[p]$ and $q' = f'[q] = g'[q]$. Hence $pf = fq$, $pg = gq$, $qf' = f'q'$ and $qg' = g'q'$. By assumptions there exist some $h, h' \in \mathcal{Q}_e$ s.t. $h \nleq p$, $h' \nleq q$, $h \leq d(f) \wedge d(g)$, $h' \leq d(f') \wedge d(g')$, $hf \leq pf \vee g$ and $h'f' \leq qf' \vee g'$. By Lemma 4.7.5, $p = f^*[q] = g^*[q]$, hence $h' \nleq q$ implies, by Lemma 4.7.2, that $h'^{f^*} \nleq p$ and $h'^{g^*} \nleq p$, and so $hh'^{f^*}h'^{g^*} \nleq p$. Let $k = hh'^{f^*}h'^{g^*}$: to finish the proof it is enough to show that $k \leq d(ff') \wedge d(gg')$ and $kff' \leq pff' \vee gg'$. $h' \leq d(f')$ implies that $k \leq h'^{f^*} \leq d(f')^{f^*} = d(ff')$, and analogously $k \leq d(gg')$, from which the first inequality follows. For the second inequality,

$$kff' \leq hh'^{f^*}ff' = hfh'f' \leq pfqf' \vee pfg' \vee gqf' \vee gg'.$$

Since $gqf' = pgf'$, then $kff' \leq p(fqf' \vee fg' \vee gf') \vee gg' \leq p \cdot 1 \vee gg'$. By SGF3, we get that $kff' \leq pkff' \vee gg' \leq pff' \vee gg'$.
2. Since $p = f^*[f[p]]$ it is enough to show the left-to-right direction. So let $q = f[p]$. By Lemma 4.7.5, $qf^* = f^*p$. By assumption, there exists some $h \in \mathcal{Q}_e$ s.t. $h \nleq p$, $h \leq d(f) \wedge d(g)$ and $hf \leq pf \vee g$. Let $k = h^f$: then, by Lemma 4.7.2, $k \nleq q$; moreover, $h \leq d(f)$ implies that $h^f \leq d(f)^f = r(f) = d(f^*)$ and likewise $k \leq d(g^*)$. Finally, $kf^* = f^*h \leq f^*p \vee g^* = qf^* \vee g^*$. □

4.2. The set groupoid of an SGF-quantale

Definition 4.9. For every SGF-quantale $\mathcal{Q}$, its associated set groupoid $\mathcal{G}(\mathcal{Q})$ is defined as follows: $G_0 = \mathcal{P}_e$ and $G_1 = \mathcal{I}/\sim$, moreover, denoting the elements of G_1 by $[p, f]$, the structure maps of $\mathcal{G}(\mathcal{Q})$ are given by the following assignments:

$$d([p, f]) = p, \qquad r([p, f]) = f[p], \qquad u(p) = [p, e],$$
$$[p, f][q, g] = [p, fg] \quad \text{only if} \quad q = f[p]$$
$$[p, f]^{-1} = [f[p], f^*].$$

Lemma 4.10. *The structure maps above are indeed well defined.*

Proof. If $(p, f) \sim (p'f')$ then $p = p'$, so d is well defined. Moreover, by Lemma 4.7.4, $f[p] = f'[p']$, so r is well defined. Also by Lemma 4.7.4, it is straightforward to see that if $(p, f) \sim (p', f')$ and $(q, g) \sim (q', g')$ then $[p, f][q, g]$ is defined iff $q = f[p]$ iff $q' = f[p']$ iff $[p', f'][q', g']$ is defined; Proposition 4.8.1 exactly says that the product is well defined. Likewise, Proposition 4.8.2 exactly says that the inverse is well defined. □

Proposition 4.11. *For every SGF-quantale* $\mathcal{Q}$, $\mathcal{G}(\mathcal{Q})$ *is a set groupoid.*

Proof. G2: Recall that for every $p \in \mathcal{P}_e$, $(p, e) \in \mathcal{I}$; then $e[p] = p$ hence $d(u(p)) = d([p, e]) = p = e[p] = r([p, e]) = r(u(p))$.

G3: The associativity of the product readily follows from the definitions using Lemma 4.7.6. If $q = f[p]$, then, by Lemma 4.7.6, $r([p, f][q, g]) = r([p, fg]) = fg[p] = g[f[p]] = g[q] = r([q, g])$.
G4: immediate from the definitions.
G5: $d([p, f]^{-1}) = d([f[p], f^*]) = f[p] = r([p, f])$; by Lemma 4.7.5, $r([p, f]^{-1}) = r([f[p], f^*]) = f^*[f[p]] = p = d([p, f])$. By Lemma 4.7.7, $[p, f][p, f]^{-1} = [p, f][f[p], f^*] = [p, ff^*] = [p, e] = u(p) = u(d([p, f]))$. Likewise, $p = f^*[f[p]]$ implies that the product $[p, f]^{-1}[p, f]$ is well defined and by 4.7.7 $[p, f]^{-1}[p, f] = [f[p], f^*f] = [f[p], e] = u(r([p, f]))$. □

5. Spatial SGF-quantales and SP-groupoids

The last ingredient needed in $\mathcal{G}(\mathcal{Q})$ is a topology on G_0. For this, we need a condition on $\mathcal{Q}$ which guarantees $\mathcal{Q}_e$ to be a spatial frame. The notion of spatial SGF-quantales that we are going to introduce in this section generalizes spatial locales, i.e. the locales that are meet-generated by their prime elements.

Definition 5.1. For every SGF-quantale $\mathcal{Q}$ and every $[p, f] \in \mathcal{I}/\sim$, let

$$\mathcal{I}_{[p,f]} = \{g \in \mathcal{I}(\mathcal{Q}) \mid d(g) \leq p \text{ or } (p, g) \not\sim (p, f)\} \quad \text{and} \quad I_{[p,f]} = \bigvee \mathcal{I}_{[p,f]}.$$

$\mathcal{Q}$ is *spatial* if:

SPQ1. for every $(p, f) \in \mathcal{I}, f \nleq I_{[p,f]}$.
SPQ2. For every $a \in \mathcal{Q}, a = \bigwedge\{I_{[p,f]} \mid a \leq I_{[p,f]}\}$.

It immediately follows from the definition that $p \in \mathcal{I}_{[p,f]}$, hence $p \leq I_{[p,f]}$ for every $(p, f) \in \mathcal{I}$. It is also immediate to see that $(p, f) \sim (p', f')$ implies that $\mathcal{I}_{[p,f]} = \mathcal{I}_{[p',f']}$, and that if $g \nleq I_{[p,f]}$ then $g \sim_p f$.

Lemma 5.2. *For every SGF-quantale $\mathcal{Q}$ s.t. SPQ1 holds and every $g \in \mathcal{I}(\mathcal{Q})$, $g \leq I_{[p,f]}$ iff $g \in \mathcal{I}_{[p,f]}$.*

Proof. The right-to-left direction is clear. Conversely, if $g \leq I_{[p,f]}$ and $g \sim_p f$, then $I_{[p,f]} = I_{[p,g]}$ so $g \leq I_{[p,g]}$, i.e. $g \wedge I_{[p,g]} = g$, contradicting SPQ1. □

An immediate consequence of this lemma is that if $g \sim_p f$ then $g \nleq I_{[p,f]}$ (so indeed these two conditions are equivalent). Let us verify that the axioms for spatial quantales are sound:

Proposition 5.3. *For every SP-groupoid $\mathcal{G}$ and every selection base $\mathcal{S}$ of $\mathcal{G}$, the SGF-quantale $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ is spatial.*

Proof. Recall that $\mathcal{I}(\mathcal{Q}(\mathcal{G}, \mathcal{S})) = \mathcal{S}$ and the prime elements of $\mathcal{Q}(\mathcal{G}, \mathcal{S})_e$ are exactly those $P = u[G_0 \setminus \overline{p}]$ for $p \in G_0$ (cf. Proposition 3.9). For every $F \in \mathcal{S}$, let $f : U_f \to G_1$ be its corresponding local bisection; in particular for every $H \in \mathcal{Q}(\mathcal{G}, \mathcal{S})_e$, its corresponding local bisection is the restriction of the structure map u to some open subset of G_0 that we denote H as well. Then HF is the image of $f_{|H}$, wherever defined. Moreover, $\mathcal{I}_{[P,F]}$ (resp. $I_{[P,F]}$) is (the union of) the collection of all the $G \in \mathcal{S}$ corresponding to local bisections $g : U_g \to G_1$ s.t. either $U_g \cap \overline{p} = \varnothing$ (i.e. $p \notin U_g$) or $HG \nsubseteq PG \cup F$ for every open set H s.t. $p \in H \subseteq U_f \cap U_g$.

SPQ1: Let $P = u[G_0 \setminus \overline{p}]$; it is enough to show that $f(p) \notin I_{[P,F]}$. Suppose that $f(p) \in I_{[P,F]}$; then there exists some g such that $G \not\sim_P F$ and $g(p) = f(p)$. By SB4, $p \in H \cap C \subseteq \{q \in G_0 \mid f(q) = g(q)\}$ for some H open and C closed subsets of G_0. Then $H \cap \overline{p} \subseteq H \cap C \subseteq \{q \in G_0 \mid f(q) = g(q)\}$. This means that $g_{|H}$ coincides with f outside of P. In other words, $HG \subseteq PG \cup F$, contradicting the hypothesis that $G \not\sim_P F$.
SPQ2: Let $A \in \mathcal{Q}(\mathcal{G}, \mathcal{S})$ and let $G \in \mathcal{S}$ s.t. $G \nsubseteq A$. Then $g(p) \notin A$ for some $p \in U_g$. Let $P = u[G_0 \setminus \overline{p}]$, and let us show that if $F \in \mathcal{S}$ and $F \subseteq A$, then $F \in \mathcal{I}_{[P,G]}$: indeed, if $F \subseteq A$ and $p \in U_f$ then $f(p) \neq \mathcal{G}(p)$, since by assumption $g(p) \notin A$, therefore $F \not\sim_P G$. By SGF1, this shows that $A \subseteq I_{[P,G]}$. Since by Lemma 5.2, $G \nsubseteq \mathcal{I}_{[P,G]}$, then $G \nsubseteq \bigwedge\{I_{[Q,G']} \mid A \leq I_{[Q,G']}\}$, which concludes the proof of the non-trivial inclusion. □

Lemma 5.4. *If $\mathcal{Q}$ is spatial then for every $g \in \mathcal{I}(\mathcal{Q})$,*

$$g = \bigwedge\{I_{[p,f]} \mid d(g) \leq p \text{ or } (p, g) \not\sim (p, f)\}.$$

Proof. Let $g \in \mathcal{I}(\mathcal{Q})$. By SPQ2 and Lemma 5.2, $g = \bigwedge\{I_{[p,f]} \mid g \leq I_{[p,f]}\} = \bigwedge\{I_{[p,f]} \mid g \in \mathcal{I}_{[p,f]}\} = \bigwedge\{I_{[p,f]} \mid d(g) \leq p \text{ or } (p, g) \not\sim (p, f)\}$. □

Proposition 5.5. *If $\mathcal{Q}$ is spatial then $\mathcal{Q}_e$ is a spatial frame.*

Proof. Let $h \in \mathcal{Q}_e$ and let us show that $h = \bigwedge\{p \in \mathcal{P}_e \mid h \leq p\}$. Since $\mathcal{Q}$ is spatial, then by Lemma 5.4 $h = \bigwedge\{I_{[q,g]} \mid h \leq q \text{ or } (q, g) \not\sim (q, h)\}$.
Claim. If $h \nleq q$, then for every $g \in \mathcal{I}(\mathcal{Q})$ s.t. $(q, g) \in \mathcal{I}$, if $(q, g) \not\sim (q, h)$ then $(q, g) \not\sim (q, e)$.
From the claim it follows that $e \leq I_{[q,g]}$ for every $(q, g) \in \mathcal{I}$ s.t. $h \nleq q$ and $(q, g) \not\sim (q, h)$. Hence,

$$h = h \wedge e = \bigwedge\{I_{[q,g]} \wedge e \mid h \leq q \text{ or } (q, g) \not\sim (q, h)\} = \bigwedge\{I_{[q,g]} \wedge e \mid h \leq q\}.$$

Since for every $(q, g) \in \mathcal{I}$ s.t. $h \leq q$ there exists some $p = q$ s.t. $h \leq p$ and $p \leq I_{[q,g]}$, we can conclude that

$$h \leq \bigwedge\{p \in \mathcal{P}_e \mid h \leq p\} \leq \bigwedge\{I_{[q,g]} \wedge e \mid h \leq q\} = h.$$

To finish the proof, we need to prove the claim: if $h \not\leq q$ and $g \sim_q e$ then there exists some $k \in \mathcal{Q}_e$ s.t. $k \not\leq q$, $k \leq d(g) \wedge e = d(g)$ and $kg \leq qg \vee e$. Let $h' = hk$: then $h' \not\leq q$, $h' \leq d(g) \wedge h$ and $h'g \leq hqg \vee h \leq qg \vee h$; this shows that $g \sim_q h$. □

Proposition 5.6. *For every spatial SGF-quantale $\mathcal{Q}$,*

1. *every element $f \in \mathcal{I}(\mathcal{Q})$ corresponds to a local bisection of $\mathcal{G}(\mathcal{Q})$.*
2. *$\mathcal{G}(\mathcal{Q})$ is an SP-groupoid.*

Proof. 1. By Proposition 5.5, every $h \in \mathcal{Q}_e$ can be identified with the open set $U_h = \{p' \in \mathcal{P}_e \mid h \not\leq p'\}$ (cf. [2]). Then for every $f \in \mathcal{I}(\mathcal{Q})$, the map $s_f : U_{d(f)} \to \mathcal{I}/\sim$ defined by $s_f(p') = [p', f]$ is a local bisection of $\mathcal{G}(\mathcal{Q})$: indeed, $d \circ s_f = id$ and it readily follows from Lemmas 4.7.2 and 4.7.5 that $r \circ s_f$ is open and its inverse is $r \circ s_{f^*}$ which is also open.
2. If $[p, f] \in \mathcal{I}$, then $[p, f]$ belongs to the bisection image corresponding to the local bisection s_f defined above. □

6. Spatial SGF-quantales are GQs

6.1. The canonical map

Definition 6.1. For every SGF-quantale $\mathcal{Q}$, $\alpha : \mathcal{Q} \to \mathcal{P}(\mathcal{I}/\sim)$ is defined by

$$\alpha(a) = \{[p, f] \mid a \not\leq I_{[p,f]}\}.$$

Theorem 6.2. *For every SGF-quantale $\mathcal{Q}$,*

1. *$\alpha(\bigvee_i a_i) = \bigcup_i \alpha(a_i)$ for any family $\{a_i | i \in I\}$ of elements of $\mathcal{Q}$.*
2. *if $\mathcal{Q}$ is spatial, then α is an embedding.*
3. *$\alpha(ab) = \alpha(a)\alpha(b)$ for any $a, b \in \mathcal{Q}$.*
4. *$\alpha(a^*) = \alpha(a)^*$ for any $a \in \mathcal{Q}$.*
5. *$\alpha(1) = G_1$ and $\alpha(e) = u(G_0)$.*

So α is a strong and strictly unital morphism of unital involutive quantales.

Proof. 1. $[p, f] \in \bigcup_{i\in I} \alpha(a_i)$ iff $a_i \not\leq I_{[p,f]}$ for some $i \in I$ iff $\bigvee_{i\in I} a_i \not\leq I_{[p,f]}$ iff $[p, f] \in \alpha(\bigvee_{i\in I} a_i)$.
2. If $a \not\leq b$, by SGF1 and SPQ2, $f \not\leq I_{[q,g]}$ for some $f \in \mathcal{I}(\mathcal{Q})$ s.t. $f \leq a$ and some $(q, g) \in \mathcal{I}$ s.t. $b \leq I_{[q,g]}$. Then $a \not\leq I_{[q,g]}$, i.e. $[q, g] \in \alpha(a)$, and $[q, g] \notin \alpha(b)$.
3. If $a \not\leq I_{[p,f]}$ and $b \not\leq I_{[q,g]}$ then by SGF1, $g_1 \not\leq I_{[p,f]}$ and $g_2 \not\leq I_{[q,g]}$ for some $g_1, g_2 \in \mathcal{I}(\mathcal{Q})$ s.t. $g_1 \leq a$ and $g_2 \leq b$. Hence $g_1 \sim_p f$ and $g_2 \sim_q g$ and so if $q = f[p]$ then by Proposition 4.8.1, $g_1g_2 \sim_p fg$ which implies by Lemma 5.2 that $g_1g_2 \not\leq I_{[p,fg]}$, i.e. $[p, fg] \in \alpha(g_1g_2) \subseteq \alpha(ab)$. Conversely, if $ab \notin I_{[p,f]}$, then by SGF1 $g_1g_2 \not\leq I_{[p,f]}$ for some $g_1, g_2 \in \mathcal{I}(\mathcal{Q})$ s.t. $g_1 \leq a$ and $g_2 \leq b$. Hence $g_1g_2 \sim_p f$. In particular, $d(g_1g_2) \not\leq p$, which implies, since $d(g_1g_2) \leq d(g_1)$, that $d(g_1) \not\leq p$. So by Lemma 4.7.1, let $q = f[p]$: then $pg_1 = g_1q$. Let us show that $d(g_2) \not\leq q$: if not, then $qd(g_2) = d(g_2)$ and so $pg_1g_2 = g_1qg_2 = g_1qd(g_2)g_2 = g_1d(g_2)g_2 = g_1g_2$, hence $d(g_1g_2) \leq p$. From $d(g_1) \not\leq p$ and $d(g_2) \not\leq q$ we get $[p, g_1] \in \alpha(g_1) \subseteq \alpha(a)$ and $[q, g_2] \in \alpha(b)$.
4. If $a^* \not\leq I_{[p,f]}$ then by SGF1, $g^* \not\leq I_{[p,f]}$ for some $g \in \mathcal{I}(\mathcal{Q})$ s.t. $g \leq a$. Hence $g^* \sim_p f$, and so, by Proposition 4.8.2, $g \sim_{f[p]} f^*$ which implies by Lemma 5.2 that $g \not\leq I_{[f[p],f^*]}$, i.e. $[p, f]^{-1} \in \alpha(g) \subseteq \alpha(a)$. Hence, $[p, f] = ([p, f]^{-1})^{-1} \in \alpha(a)^*$. Conversely, if $[p, f] \in \alpha(a)^*$ then $a \not\leq I_{[f[p],f^*]}$, then by SGF1 $g \not\leq I_{[f[p],f^*]}$ for some $g \in \mathcal{I}(\mathcal{Q})$ s.t. $g \leq a$. Hence $g^* \leq a^*$ and $g \sim_{f[p]} f^*$, and so, by Proposition 4.8.2, $g^* \sim_p f$ which implies by Lemma 5.2 that $g^* \not\leq I_{[p,f]}$, i.e. $[p, f] \in \alpha(g^*) \subseteq \alpha(a^*)$.
5. Since $f \not\leq I_{[p,f]}$ for every $(p, f) \in \mathcal{I}$, then $\alpha(1) = \{[p, f] \mid \bigvee \mathcal{I}(\mathcal{Q}) \not\leq I_{[p,f]}\} = G_1$. For the second equality, $u(G_0) \subseteq \alpha(e)$ follows from $e \not\leq I_{[p,e]}$ for every p. The converse inclusion follows from the fact that $e \not\leq I_{[p,f]}$ by definition implies that $[p, f] = [p, e]$. □

Proposition 6.3. *For every spatial SGF-quantale $\mathcal{Q}$, $\alpha[\mathcal{I}(\mathcal{Q})]$ is a selection base of $\mathcal{G}(\mathcal{Q})$.*

Proof. We already showed (see proof of Proposition 5.6) that for every $f \in \mathcal{I}(\mathcal{Q})$, $s_f : U_{d(f)} \to \mathcal{P}(\mathcal{I}/\sim)$ defined by $s_f(p') = [p', f]$ is a local bisection of $\mathcal{G}(\mathcal{Q})$. Let us show that $s_f[U_{d(f)}] = \alpha(f)$: $[p, g] \in s_f[U_{d(f)}]$ iff $[p, g] = s_f(p') = [p', f]$ for some $p' \in U_{d(f)}$ iff $g \sim_p f$ iff $f \not\leq I_{[p,g]}$ iff $[p, g] \in \alpha(f)$. This shows that $\alpha[\mathcal{I}(\mathcal{Q})]$ is a collection of bisection images of $\mathcal{G}(\mathcal{Q})$.

SB2: In particular, for every $h \in \mathcal{Q}_e$, $s_h[U_{d(h)}] = \alpha(h)$. Moreover, notice that $h \not\leq p$ implies that $[p, h] = [p, e]$, from which it easily follows that $s_h[U_{d(h)}] = u[U_h]$.

SB1: it readily follows from Theorem 6.2 and Proposition 2.2.3.

SB3: it readily follows from Theorem 6.2.3 and .4, and the fact that if $\{f_i \mid i \in I\} \subseteq \mathcal{I}(\mathcal{Q})$ s.t. $f_if_j^* \leq e$ and $f_i^*f_j \leq e$ then $\bigcup_{i\in I} f_i \in \mathcal{I}(\mathcal{Q})$.

SB4: Let $f, g \in \mathcal{I}(\mathcal{Q})$ and let $p \in \mathcal{P}_e$ be s.t. $s_f(p) = s_g(p)$. So $[p, f] = [p, g]$, i.e. $hg \leq pg \vee f$ for some $h \in \mathcal{Q}_e$ s.t. $h \nleq p$. Then, for every $q \in \mathcal{P}_e$ s.t. $p \leq q$ and $h \nleq q$, we get that $hg \leq pg \vee f \leq qg \vee f$, i.e. $s_f(q) = [q, f] = [q, g] = s_g(q)$. Since h can be identified with an open set of G_0 and p can be identified with the topological closure $\overline{p}$ of its corresponding point of G_0 (which we also denote by p), this is enough to show that $\{p \in G_0 \mid s_f(p) = s_g(p)\}$ is the union of locally closed subsets of G_0.

SB5: Lemma 5.2 readily implies that for every $(p, f) \in \mathcal{I}$, $[p, f] \in \alpha(f)$. □

6.2. The correspondence

Proposition 6.4. *For every spatial SGF-quantale,*

$$\mathcal{Q}(\mathcal{G}(\mathcal{Q}), \alpha[\mathcal{I}(\mathcal{Q})]) \cong \mathcal{Q}.$$

For every SP-groupoid $\mathcal{G}$ and every selection base $\mathcal{S}$,

$$\mathcal{G}(\mathcal{Q}(\mathcal{G}, \mathcal{S})) \cong \mathcal{G}.$$

Proof. Let $\mathcal{Q}$ be a spatial SGF-quantale. Then by Proposition 5.6, $\mathcal{G}(\mathcal{Q})$ is an SP-groupoid, and by Proposition 6.3, $\alpha[\mathcal{I}(\mathcal{Q})]$ is a selection base of $\mathcal{G}(\mathcal{Q})$. Then by definition $\mathcal{Q}(\mathcal{G}(\mathcal{Q}), \alpha[\mathcal{I}(\mathcal{Q})])$ is the sub $\bigcup$-semilattice of $\mathcal{P}(G_1) = \mathcal{P}(\mathcal{I}/\sim)$ generated by $\alpha[\mathcal{I}(\mathcal{Q})]$. Theorem 6.1 guarantees that $\mathcal{Q}$ is isomorphic as a unital involutive quantale to its α-image $\alpha[\mathcal{Q}]$ and hence that $\alpha[\mathcal{Q}]$ is $\bigcup$-generated by $\alpha[\mathcal{I}(\mathcal{Q})]$. Hence by definition $\mathcal{Q}(\mathcal{G}(\mathcal{Q}), \alpha[\mathcal{I}(\mathcal{Q})]) = \alpha[\mathcal{Q}]$.
Let $\mathcal{G}$ be an SP-groupoid and $\mathcal{S}$ be a selection base for $\mathcal{G}$. Then by Propositions 4.2 and 5.3 $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ is a spatial SGF-quantale. Moreover, by Proposition 3.9, $\mathcal{I}(\mathcal{Q}(\mathcal{G}, \mathcal{S})) = \mathcal{S}$, $\mathcal{Q}(\mathcal{G}, \mathcal{S}))_e$ can be identified with the topology $\Omega(G_0)$, and since G_0 is sober, the prime elements of $\mathcal{Q}(\mathcal{G}, \mathcal{S}))_e$ bijectively correspond to the points of G_0 via the assignment $p \mapsto u[G_0 \setminus \overline{p}]$. Since the prime elements of $\mathcal{Q}(\mathcal{G}, \mathcal{S}))_e$ form the space of units of $\mathcal{G}(\mathcal{Q}(\mathcal{G}, \mathcal{S}))$, then the assignment defines the map φ_0. Since $\mathcal{S}$ is a selection base, for every $x \in G_1$ $x = g(p)$ for some local bisection $g : U_g \to G_1$ s.t. its corresponding $G \in \mathcal{S}$ and some $p \in U_g$. Hence $(P, G) \in \mathcal{I}$. Clearly, $[P, G] = [P', G']$ for any local bisection $g' : U_g \to G_1$ s.t. $x = g'(p')$ for some $p' \in U_{g'}$, so the assignment $x \mapsto [P, G]$ defines a map $\varphi_1 : G_1 \to \mathcal{I}/\sim$. The map φ_1 is bijective: indeed, if $(P, G) \in \mathcal{I}$, then $p \in U_g$, so $[P, G] = \varphi_1(g(p))$; moreover, if $\varphi_1(x) = [P, F] = \varphi_1(y)$, then $x = f(p) = y$. The fact that (φ_0, φ_1) is indeed a morphism of groupoids is a standard if tedious verification. □

7. Comparison with the étale localic setting

The aim of this section is showing informally that our bijective correspondence extends, in the spatial setting, the non-functorial duality defined in [13] between localic étale groupoids and inverse quantal frames. In [13] inverse quantal frames are defined as unital involutive quantales $\mathcal{Q}$ which are also frames for the lattice operations, are generated by $\mathcal{I}(\mathcal{Q})$ and have a *support*, i.e. a completely join-preserving map $\varsigma : \mathcal{Q} \to \mathcal{Q}_e$ s.t. $\varsigma(a) \leq aa^*$ and $a \leq \varsigma(a)a$ for every $a \in \mathcal{Q}$.[10] Any such quantale is shown to be isomorphic to one of the form $\mathcal{O}(\mathcal{G})$, for some localic étale groupoid[11] $\mathcal{G} = (G_1, G_0)$. In particular, for any such $\mathcal{G}$, its associated quantale is based on the frame $\mathcal{O}(\mathcal{G})$, on which the noncommutative product is defined by using the product of $\mathcal{G}$ in the natural way. When $\mathcal{G}$ is spatial (i.e. isomorphic to a topological groupoid), the back-and-forth correspondence in [13] can be equivalently described in the following way. Recall that a *G*-set of a topological groupoid is a subset $S \subseteq G_1$ such that the maps $d : S \to G_0$ and $r : S \to G_0$ are both homeomorphisms onto open subsets of G_0. A *G*-set[12] S is therefore the image of a *continuous* local bisection $s : U \to G_1$, for some open set U of G_0. Then the inverse quantal frame associated with any étale topological groupoid (G_1, G_0) can be equivalently described as the sub-$\bigcup$-semilattice of $\mathcal{P}(G_1)$ generated by the *G*-sets of G_1. Conversely, if $\mathcal{Q}$ is an inverse quantal frame corresponding to some spatial étale groupoid (G_1, G_0), then G_0 can be equivalently recovered as the topological space dual to the locale $\mathcal{Q}_e$, and G_1 as the set of *germs* of elements of $\mathcal{I}(\mathcal{Q})$, i.e. as the set of the equivalence classes of the relation $\sim$ on $\mathcal{I}(\mathcal{Q})$ defined as $f \sim g$ if and only if $hf = hg$ on some neighborhood h of a point $p \in G_0$.

To show that the spatial version of the correspondence in [13] is a special case of our construction, we make the following remarks. As we remarked early on, the notion of local bisection introduced in Definition 3.1 does not refer to any topology on G_1. However, if for some selection base $\mathcal{S}$ (Definition 3.5), the quantale $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ as in Definition 3.8 happens to be an inverse quantal frame, then this quantale defines a topology on G_1. To continue the discussion, the following lemma will be useful:

Lemma 7.1. *If $\mathcal{Q}$ is an inverse quantal frame, then for all $f, g \in \mathcal{I}(\mathcal{Q})$ and every $p \in \mathcal{P}_e$, if $f \sim_p g$ then there exists some $k \leq d(f)d(g)$ such that $k \nleq p$ and $kf = kg$.*

10 It readily follows that for every $f \in \mathcal{I}(\mathcal{Q})$, $\varsigma(f) = f^*f$, hence $f = ff^*f$.

11 A *localic groupoid* is a groupoid in the category of locales. Such a groupoid is étale if d is a partial homeomorphism.

12 For the sake of highlighting the difference between the bisection images as defined in Definition 3.1 and those of topological groupoids, in this section we will refer to the latter ones as *G*-sets.

Proof. By assumption, there exists some $h \in \mathcal{Q}_e$ s.t. $h \not\leq p$, $h \leq d(f)d(g)$ and $hf \leq pf \vee g$. Since $\mathcal{Q}$ is distributive, $hf = phf \vee (hf \wedge g)$. Let $kf = hf \wedge g$. Since $h = ph \vee k \not\leq p$, we get also $k \not\leq p$. □

Since $\mathcal{I}(\mathcal{Q}(\mathcal{G}, \mathcal{S})) = \mathcal{S}$ (cf. Proposition 3.9), the Lemma above implies that $\mathcal{S}$ is a base for the topology $\mathcal{Q}(\mathcal{G}, \mathcal{S})$. Then, notice that the elements of $\mathcal{S}$ are images of local bisections that are *continuous* with respect to the given topology of G_0 and the topology $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ on G_1: indeed, let $F \in \mathcal{S}$ and let f be its associated local bisection. To show that f is continuous it is enough to check that for every basic open $G \in \mathcal{S}$, $f^{-1}[G] \in \Omega(G_0)$, i.e. that every $p \in f^{-1}[G]$ has an open neighborhood that is contained in $f^{-1}[G]$. But this is again guaranteed by the Lemma. Therefore, bisection images defined as in Definition 3.1 are G-sets according to the standard definition. Analogously, it can be shown that the structure maps of the groupoid (G_0, G_1) are continuous, i.e. (G_0, G_1) is a topological groupoid. By well known results in groupoid theory (cf. [11], chapter I, Definition 2.6, Lemma 2.7 and Proposition 2.8) this topological groupoid is étale.

Conversely, if $\mathcal{Q}$ is an inverse quantal frame, then it not difficult to see that $\mathcal{Q}$ is also an SGF-quantale. If $\mathcal{Q}$ is also a spatial quantale as in Definition 5.1, then its associated groupoid quantale $\mathcal{G}(\mathcal{Q})$ (cf. Definition 4.9) is defined by taking G_1 as the set of equivalence classes $[p, f]$ with respect to the incidence relation as in Definition 4.5.

Lemma 7.1 shows that the equivalence classes $[p, f]$ coincide with the *germs* of local bisections, as in Definition 3.1, (at p), Also in this case, $\mathcal{Q}$ can be identified with a topology on G_1, via the canonical embedding α (cf. Theorem 6.2), and by Proposition 6.3, $\alpha[\mathcal{I}(\mathcal{Q})]$ is a selection base; then axiom SB3 readily implies that $\alpha[\mathcal{I}(\mathcal{Q})]$ is collection of all G-sets, according to the standard definition, i.e. every $S \in \alpha[\mathcal{I}(\mathcal{Q})]$ is associated with a *continuous* local bisection. Hence the construction of G_1 from $\mathcal{Q}$ in [13] and in this paper coincide. As before, from the same results in [11], the groupoid (G_1, G_0) is étale.

8. Examples

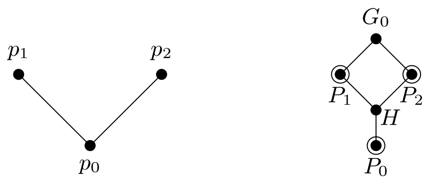
In this section, we present two examples of groupoids arising as the equivalence relations induced by group actions on topological spaces, as described in Example 3.3.2.

8.1. Finite, not T_1 and étale

Consider the finite topological space $X = (G_0, \Omega(G_0))$, defined as follows:

$$G_0 = \{p_0, p_1, p_2\},$$
$$\Omega(G_0) = \{P_0 = \varnothing, H = \{p_0\}, P_1 = \{p_0, p_2\}, P_2 = \{p_0, p_1\}, G_0\}.$$

So the opens are the down-sets of the partial order on the left, and the lattice of the topology is represented on the right:



X is clearly not T_1. The prime elements of $\Omega(G_0)$ are exactly P_0, P_1 and P_2, hence X is sober. The group acting on X is $G = \{\varphi, id_X\}$, where $(\varphi(p_0) = p_0, \varphi(p_1) = p_2, \varphi(p_2) = p_1)$. The equivalence relation induced by the action of G is then

$$R = \{(p_0, p_0), (p_1, p_1), (p_2, p_2), (p_1, p_2), (p_2, p_1)\}.$$

The collection of partial homeomorphisms $X \to X$ consists of the restrictions to the open sets in $\Omega(G_0)$ of the maps φ and id_X. For every $H' \in \Omega(G_0)$, $H'\varphi$ will denote the graph of the restriction of φ to H'. The collection of the graphs of partial homeomorphisms $X \to X$ is

$$\mathcal{S} = \{H' = id_{H'} \mid H' \in \Omega(G_0)\} \cup \{H\varphi, P_1\varphi, P_2\varphi, \varphi\}.$$

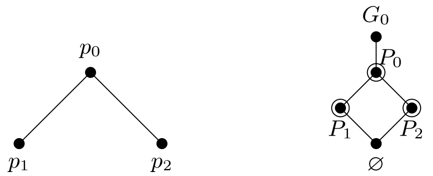
X can be represented as the groupoid $\mathcal{G} = (X, R)$; then $\mathcal{S}$ is the collection of the bisection images of $\mathcal{G}$ and $\mathcal{G}$ is SP. Then $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ is the sub-$\bigcup$-semilattice of $\mathcal{P}(R)$ generated by $\mathcal{S}$. Notice that for any two partial homeomorphisms of X the set over which they coincide is an open set of G_0; this implies that the intersection of the graphs of any two partial homeomorphisms is again a graph of a partial homeomorphism, hence $\mathcal{S}$ is the base of a topology on G_1. So (cf. [9,13], and the discussion in Section 7) $\mathcal{G}$ is étale and $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ is an inverse quantal frame.

8.2. Finite, not T_1 and not étale

Consider the finite topological space $X = (G_0, \Omega(G_0))$, defined as follows:

$$G_0 = \{p_0, p_1, p_2\}, \qquad \Omega(G_0) = \{\varnothing, P_1 = \{p_2\}, P_2 = \{p_1\}, P_0 = \{p_1, p_2\}, G_0\},$$

So the opens are the down-sets of the partial order on the left, and the lattice of the topology is represented on the right:



X is clearly not T_1. The prime elements of $\Omega(G_0)$ are exactly P_0, P_1 and P_2, hence X is sober. The group acting on X is $G = \{\varphi, id_X\}$, where $(\varphi(p_0) = p_0, \varphi(p_1) = p_2, \varphi(p_2) = p_1)$, and the equivalence relation induced by the action of G is

$$R = \{(p_0, p_0), (p_1, p_1), (p_2, p_2), (p_1, p_2), (p_2, p_1)\}.$$

The collection of partial homeomorphisms $X \to X$ consists of the restrictions to the open sets in $\Omega(G_0)$ of the maps φ and id_X. For every $H \in \Omega(G_0)$, $H\varphi$ will denote the graph of the restriction of φ to H. The collection of the graphs of partial homeomorphisms $X \to X$ is

$$\mathcal{S} = \{H = id_H \mid H \in \Omega(G_0)\} \cup \{P_0\varphi, P_1\varphi, P_2\varphi, \varphi\}.$$

X can be represented as the groupoid $\mathcal{G} = (X, R)$; then $\mathcal{S}$ is the collection of the bisection images of $\mathcal{G}$ and $\mathcal{G}$ is SP. Notice that the set over which the graphs of φ and id_X coincide is $\{(p_0, p_0)\}$, which cannot be (nor contain) the graph of any (nonempty) partial homeomorphism since $\{p_0\}$ is closed but not open. Hence, $\mathcal{S}$ is not a topological base. Therefore, (cf. [9,13] and the discussion in Section 7) $\mathcal{G}$ is not étale and $\mathcal{Q}(\mathcal{G}, \mathcal{S})$ is not a distributive lattice.

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