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Bounds for characteristic values of entire matrix pencils ☆

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Abstract

Entire matrix-valued functions of a complex argument (entire matrix pencils) are considered. Upper bounds for sums of characteristic values and a lower bound for the smallest characteristic value are derived in the terms of the coefficients of the Taylor series. These results are new even for polynomial pencils.

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1. Introduction and statement of the main result

Entire matrix-valued functions of a complex argument (entire matrix pencils) arise in various applications, see for instance [1,10,14] and references therein. The spectrum of matrix pencils was investigated in many works. Mainly the spectrum perturbations of polynomial matrix pencils were investigated, cf. [5,7,12,13]. In the very interesting paper [8], upper and lower bounds are derived for the absolute values of characteristic values of polynomial matrix pencils.

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On the other hand, estimates for sums of the roots of entire functions are very important for the theory of these functions and its applications. An essential role here is played by the Hadamard theorem [11, p. 18]. In the present paper we derive upper bounds for sums of characteristic values of entire matrix pencils. In addition, we suggest a lower bound for the smallest characteristic value.

Besides, we generalize the main result from the paper [2] (see also [3, Chapter 19]) on inequalities for sums of zeros of entire scalar functions of finite order. These inequalities supplement the Hadamard theorem. Namely they not only assert the convergence of the series of the zeros, but also give us the estimate for the sums of the zeros. It should be noted that the generalization requires additional mathematical tools.

Let $\mathbf{C}^n$ be a Euclidean space with the Euclidean norm $\|\cdot\|$ and the unit matrix I_n. For an $n \times n$-matrix Q, $\lambda_k(Q)$ $(k = 1, \ldots, n)$ are the eigenvalues with multiplicities taken into account and ordered in the following way: $|\lambda_{k+1}(Q)| \leqslant |\lambda_k(Q)|$.

Let a_k, $k = 1, 2, \ldots$ be $n \times n$-matrices. Our main object in this paper is the entire matrix pencil

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k!)^\gamma} a_k \quad (a_0 = I_n,\ \lambda \in \mathbf{C}) \tag{1.1}$$

with a finite $\gamma > 0$. Put $M_f(r) := \max_{|z|=r} \|f(z)\|$ $(r > 0)$. The limit

$$\rho(f) := \overline{\lim_{r\to\infty}} \frac{\ln \ln M_f(r)}{\ln r}$$

is the order of f. A zero $z_k(f)$ of $\det f(z)$ is called a characteristic value of f. Everywhere in the present paper it is assumed that the set $\{z_k(f)\}$ of all the characteristic values of f is infinite. Note that if f has a finite number l of the characteristic values, we can put $z_k^{-1}(f) = 0$ for $k = l, l+1, \ldots$ and apply our arguments below, where $z_k^{-1}(f)$ means $\frac{1}{z_k(f)}$. The characteristic values of f are enumerated in the non-decreasing way:

$$|z_k(f)| \leqslant |z_{k+1}(f)| \quad (k = 1, 2, \ldots).$$

Assume that the series

$$\theta_f := \sum_{k=1}^{\infty} a_k a_k^* \tag{1.2}$$

converges. Here and below the asterisk denotes the adjointness. So θ_f is an $n \times n$-matrix and by (1.1), and (1.2) we have $\rho(f) \leqslant 1/\gamma$. Put

$$\omega_k(f) = \sqrt{\lambda_k(\theta_f)} \quad \text{for } k = 1, \ldots, n \quad \text{and} \quad \omega_k(f) = 0 \quad \text{for} \quad k \geqslant n+1.$$

Theorem 1.1. *Let condition* (1.2) *hold. Then the characteristic values of the pencil* f *defined by* (1.1) *satisfy the inequalities*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leqslant \sum_{k=1}^{j} \left[\omega_k(f) + \frac{n^\gamma}{(k+1)^\gamma}\right] \quad (j = 1, 2, \ldots).$$

The proof of this theorem is presented in the next section.

Let us assume that under (1.1), there is a constant $d_0 \in (0, 1)$, such that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{\|a_k\|} < 1/d_0 \tag{1.3}$$

and consider the function

$$\tilde{f}(\lambda) = \sum_{k=0}^{\infty} \frac{(d_0\lambda)^k}{(k!)^\gamma} a_k.$$

That is, $\tilde{f}(\lambda) \equiv f(d_0\lambda)$ and $z_k(\tilde{f}) = z_k(f)/d_0$. Clearly, the series

$$\theta_{\tilde{f}} := \sum_{k=1}^{\infty} d_0^{2k} a_k a_k^*$$

converges. Moreover,

$$\omega_k(\tilde{f}) = \sqrt{\lambda_k(\theta_{\tilde{f}})} \quad \text{for } k = 1, \ldots, n \quad \text{and} \quad \omega_k(\tilde{f}) = 0 \quad \text{for } k \geqslant n+1.$$

Theorem 1.1 implies

Corollary 1.2. *Under* (1.1), *let condition* (1.3) *hold. Then*

$$\sum_{k=1}^{j} \frac{1}{d_0^{-1}|z_k(f)|} \leqslant \sum_{k=1}^{j} \omega_k(\tilde{f}) + \frac{n^\gamma}{(k+1)^\gamma} \quad (j = 1, 2, \ldots).$$

2. Proof of Theorem 1.1

Consider the polynomial matrix pencil

$$F(\lambda) = \sum_{k=0}^{m} \frac{\lambda^{m-k}}{(k!)^\gamma} a_k \quad (a_0 = I_n) \tag{2.1}$$

with the characteristic values ordered in the non-increasing way:

$$|z_k(F)| \geqslant |z_{k+1}(F)| \quad (k = 1, \ldots, mn-1).$$

Introduce the block matrix

$$A_m = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\ \frac{1}{2^\gamma} I_n & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{3^\gamma} I_n & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & \frac{1}{m^\gamma} I_n & 0 \end{pmatrix}.$$

Lemma 2.1. *The relation* $\det F(\lambda) = \det(\lambda I_{mn} - A_m)$ *is true.*

Proof. Let z_0 be a characteristic value of F. Then

$$\sum_{k=0}^{m} \frac{z_0^{m-k}}{(k!)^\gamma} a_k h = 0,$$

where $h \in \mathbf{C}^n$ is the eigenvector of $F(z_0)$. Put

$$x_k = \frac{z_0^{m-k}}{(k!)^\gamma} h \quad (k = 1, \ldots, m).$$

Then $z_0 x_k = x_{k-1}/k^\gamma$ $(k = 2, \ldots, m)$ and

$$\sum_{k=0}^{m} \frac{z_0^{m-k}}{(k!)^\gamma} a_k h = \sum_{k=1}^{m} a_k x_k + z_0 x_1 = 0.$$

So $x = (x_1, \ldots, x_m)$ satisfies the equation $A_m x = z_0 x$. If the spectrum of $F(\cdot)$ is simple, the lemma is proved. If det $F(\cdot)$ has non-simple roots, then the required result can be proved by a small perturbation. □

Set

$$\theta_F := \sum_{k=1}^{m} a_k a_k^*$$

and

$$\omega_k(F) = \sqrt{\lambda_k(\theta_F)} \quad \text{for } k = 1, \ldots, n \text{ and } \omega_k(F) = 0 \text{ for } k = n+1, \ldots, mn.$$

Lemma 2.2. *The characteristic values of F satisfy the inequalities*

$$\sum_{k=1}^{j} |z_k(F)| \leqslant \sum_{k=1}^{j} \left[\omega_k(F) + \frac{n^\gamma}{(k+1)^\gamma}\right] \quad (j = 1, \ldots, mn).$$

Proof. By Lemma 2.1,

$$\lambda_k(A_m) = z_k(F) \quad (k = 1, 2, \ldots, nm). \tag{2.2}$$

Take into account that

$$\sum_{k=1}^{j} |\lambda_k(A_m)| \leqslant \sum_{k=1}^{j} s_k(A_m) \quad (j = 1, \ldots, nm), \tag{2.3}$$

where $s_k(A_m)$ are the singular numbers of A_m ordered in the non-increasing way (see, for instance, [4, Section 2.2]). But $A_m = M + C$, where

$$M = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^\gamma} I_n & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{3^\gamma} I_n & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \frac{1}{m^\gamma} I_n & 0 \end{pmatrix}.$$

We have

$$MM^* = \begin{pmatrix} \theta_F & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$CC^* = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & I_n/2^{2\gamma} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 0 & I_n/m^{2\gamma} \end{pmatrix}.$$

We can show that $s_k(M) = \omega_k(F)$. In addition,

$$s_k(C) = \frac{1}{(j+2)^\gamma} \quad (k = jn + l;\ j = 0, \ldots, m-2;\ l = 1, \ldots, n)$$

and

$$s_k(C) = 0 \quad (k = (m-1)n + l;\ l = 1, \ldots, n).$$

Since $j = (k-l)/n$ and $(k-l)/n + 2 \geqslant (k+1)/n$, it follows that,

$$s_k(C) \leqslant \frac{n^\gamma}{(k+1)^\gamma} \quad (k = 1, \ldots, nm).$$

Take into account that

$$\sum_{k=1}^{j} s_k(A_m) = \sum_{k=1}^{j} s_k(M + C) \leqslant \sum_{k=1}^{j} s_k(M) + \sum_{k=1}^{j} s_k(C),$$

cf. [4, Lemma II.4.2]. So

$$\sum_{k=1}^{j} s_k(A_m) \leqslant \sum_{k=1}^{j} \left[\omega_k(F) + n^\gamma (k+1)^{-\gamma}\right] \quad (j = 1, 2, \ldots, mn).$$

Now (2.2) and (2.3) yield the required result. □

Proof of Theorem 1.1. Consider the polynomial pencil

$$f_m(\lambda) = \sum_{k=0}^{m} \frac{\lambda^k}{(k!)^\gamma} a_k.$$

Clearly, $\lambda^m f_m(1/\lambda) = F(\lambda)$. So $z_k(F) = z_k^{-1}(f_m)$. Now Lemma 2.2 yields the inequalities

$$\sum_{k=1}^{j} |z_k(f_m)|^{-1} \leqslant \omega_k(f) + n^\gamma \sum_{k=1}^{j} (k+1)^{-\gamma} \quad (j = 1, \ldots, nm). \tag{2.4}$$

But the characteristic values of entire matrix functions continuously depend on their coefficients. So for any $j = 1, 2, \ldots,$

$$\sum_{k=1}^{j} |z_k(f_m)|^{-1} \to \sum_{k=1}^{j} |z_k(f)|^{-1}$$

as $m \to \infty$. Now (2.4) implies the required result. □

3. Applications of Theorem 1.1

Put

$$\tau_k = \omega_k(f) + \frac{n^\gamma}{(k+1)^\gamma} \quad (k = 1, 2, \ldots).$$

The following result is a consequence of the well-known Lemma II.3.4 from [4] and Theorem 1.1.

Corollary 3.1. *Let $\phi(t)(0 \leqslant t < \infty)$ be a convex scalar-valued function, such that $\phi(0) = 0$. Then under conditions* (1.1) *and* (1.2), *the inequalities*

$$\sum_{k=1}^{j} \phi(|z_k(f)|^{-1}) \leqslant \sum_{k=1}^{j} \phi(\tau_k) \quad (j = 1, 2, \ldots)$$

are valid. In particular, for any $r \geqslant 1$,

$$\sum_{k=1}^{j} |z_k(f)|^{-r} \leqslant \sum_{k=1}^{j} \tau_k^r \quad (j = 2, 3, \ldots)$$

and thus

$$\left[\sum_{k=1}^{j} |z_k(f)|^{-r}\right]^{1/r} \leqslant \left[\sum_{k=1}^{j} \omega_k^r(f)\right]^{1/r} + n^\gamma \left[\sum_{k=1}^{j} \frac{1}{(k+1)^{r\gamma}}\right]^{1/r} \quad (j = 2, 3, \ldots). \tag{3.1}$$

Furthermore, assume that

$$r\gamma > 1, \quad r \geqslant 1. \tag{3.2}$$

Then

$$\sum_{k=1}^{\infty}(k+1)^{-r\gamma} = \zeta(\gamma r) - 1,$$

where $\zeta(\cdot)$ is the Riemann Zeta function. With the notation

$$N_r(f) = \left[\sum_{k=1}^{n} \lambda_k^{r/2}(\theta_f)\right]^{1/r}$$

relation (3.1) yields

Corollary 3.2. *Under conditions* (1.1), (1.2) *and* (3.2), *the inequality*

$$\left(\sum_{k=1}^{\infty} |z_k(f)|^{-r}\right)^{1/r} \leqslant N_r(f) + n^{\gamma}(\zeta(\gamma r) - 1)^{1/r} \tag{3.3}$$

is valid. Moreover, if $\gamma > 1$, *then*

$$\sum_{k=1}^{\infty} |z_k(f)|^{-1} \leqslant N_1(f) + n^{\gamma}(\zeta(\gamma) - 1). \tag{3.4}$$

Consider now a positive scalar-valued function $\Phi(t_1, t_2, \ldots, t_j)$ with an integer j, defined on the domain

$$0 \leqslant t_j \leqslant t_{j-1} \leqslant t_2 \leqslant t_1 < \infty$$

and satisfying

$$\frac{\partial \Phi}{\partial t_1} > \frac{\partial \Phi}{\partial t_2} > \cdots > \frac{\partial \Phi}{\partial t_j} > 0 \quad \text{for } t_1 > t_2 > \cdots > t_j. \tag{3.5}$$

Corollary 3.3. *Under conditions* (1.1), (1.2) *and* (3.5),

$$\Phi(|z_1(f)|^{-1}, |z_2(f)|^{-1}, \ldots, |z_j(f)|^{-1}) \leqslant \Phi(\tau_1, \tau_2, \ldots, \tau_j).$$

Indeed, this result follows from Theorem 1.1 and the well-known Lemma II.3.5 from [4].

In particular, let $\{d_k\}_{k=1}^{\infty}$ be a decreasing sequence of non-negative numbers. Take

$$\Phi(t_1, t_2, \ldots, t_j) = \sum_{k=1}^{j} d_k t_k.$$

Then Corollary 3.3 yields

$$\sum_{k=1}^{j} d_k |z_k(f)|^{-1} \leqslant \sum_{k=1}^{j} \tau_k d_k = \sum_{k=1}^{j} d_k [\omega_k(f) + (k+1)^{-\gamma}].$$

4. Lower bounds for characteristic values

Consider the polynomial F defined by (2.1). If $m < \infty$, we can take $\gamma = 0$. Let h be a normalized eigenvector corresponding to a characteristic value $z(F)$. Then

$$z^m(F)h = -\sum_{k=0}^{m-1} \frac{z(F)^{m-1-k}}{[(k+1)!]^\gamma} a_{k+1} h.$$

Hence,

$$|z(F)|^m \leqslant \sum_{k=0}^{m-1} \frac{\|a_{k+1}\| |z(F)|^{m-1-k}}{[(k+1)!]^\gamma}$$

or

$$1 \leqslant \sum_{k=0}^{m-1} \frac{\|a_{k+1}\|}{[(k+1)!]^\gamma |z(F)|^{1+k}}.$$

Hence it follows that $|z(F)| \leqslant x_0$ where x_0 is the unique positive root of the equation

$$1 = \sum_{k=0}^{m-1} \frac{\|a_{k+1}\|}{[(k+1)!]^\gamma x^{1+k}}.$$

Put

$$p(F) = \sum_{k=0}^{m-1} \frac{\|a_{k+1}\|}{[(k+1)!]^\gamma}$$

and

$$\delta_0(F) := \begin{cases} \sqrt[m-1]{p(F)} & \text{if } p(F) \leqslant 1, \\ p(F) & \text{if } p(F) > 1. \end{cases}$$

Due to Lemma 1.6.1 from [3], $x_0 \leqslant \delta_0(F)$. Moreover, thanks to Corollary 1.6.2 from [3]

$$x_0 \leqslant 2 \sup_{k=1,\ldots,m} \sqrt[k]{\frac{\|a_k\|}{(k!)^\gamma}}.$$

We thus get

Lemma 4.1. *Any characteristic value $z(F)$ of the pencil F defined by* (2.1) *satisfies the inequalities $|z(F)| \leqslant \delta_0(F)$ and*

$$|z(F)| \leqslant 2 \sup_{k=1,\ldots,m} \sqrt[k]{\frac{\|a_k\|}{(k!)^\gamma}}.$$

Again consider the polynomial pencil

$$f_m(\lambda) = \sum_{k=0}^{m} \frac{\lambda^k}{(k!)^\gamma} a_k.$$

Since $z_k(F) = z_k^{-1}(f_m)$, due to the previous lemma we can write

$$2|z(f_m)| \geqslant \inf_{k=1,\ldots,m} \sqrt[k]{\frac{(k!)^\gamma}{\|a_k\|}}.$$

Letting $m \to \infty$, we arrive at

Corollary 4.2. *Any characteristic value $z(f)$ of the pencil f defined by* (1.1) *satisfies the inequality*

$$2|z(f)| \geqslant \inf_{k=1,2,\ldots} \sqrt[k]{\frac{[k!]^\gamma}{\|a_k\|}}.$$

It is well-known that the polynomials of the type (2.1) play an essential role in the theory of discrete-time systems, cf. [6,9]. If any characteristic value of F satisfies the inequality $|z(F)| < 1$, then the corresponding discrete-time system is stable. Thus Lemma 4.1 gives us a stability criterion.

References

[1] F.R. Gantmacher, Theory of Matrices, Chelsea, New York, 1959.
[2] M.I. Gil', Inequalities for zeros of entire functions, Journal of Inequalities 6 (2001) 463–471.
[3] M.I. Gil', Operator Functions and Localization of Spectra, Lectures Notes in Mathematics, Springer Verlag, Berlin, 2003.
[4] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Trans. of Math. Monographs, Amer. Math. Soc., RI, 1969.
[5] I.C. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Academic Press, New York, 1982.
[6] M. Gossel, Nonlinear Discrete-time Control Systems, Lectures Notes in Control and Information Sciences, 1982.
[7] J.M. Gracia, F.E. Velasco, Stability of invariant subspaces of regular matrix pencils, Linear Algebra Appl. 221 (1995) 219–226.
[8] N.J. Higham, F. Tisseur, Bounds for the eigenvalues of matrix polynomials, Linear Algebra Appl. 358 (1–3) (2003) 5–22.
[9] O. Katsuhiko, Discrete-time Control Systems, Prentice Hall, New Jersey, 1987.
[10] V. Kolmanovskii, A. Myshkis, Applied Theory of Functional Differential Equations, Kluwer Academic Publishers, London, 1992.
[11] B.Ya. Levin, Lectures on Entire Functions, Trans. of Math. Monographs, Amer. Math. Soc., RI, 1996.

[12] L. Rodman, An Introduction to Operator Polynomials, Birkhäuser Verlag, Basel, 1989.
[13] G.W. Stewart, Sun Ji-guang, Matrix Perturbation Theory, Academic Press, New York, 1990.
[14] Q.-G. Wang, T.H. Lee, K.K. Tan, Finite Spectrum Assignment for Time-delay Systems, Springer, London, 1999.