Bounds for characteristic values of entire matrix pencils

M.I. Gil’

Department of Mathematics, Ben Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel

Received 9 February 2004; accepted 30 April 2004

Submitted by L. Rodman

Abstract

Entire matrix-valued functions of a complex argument (entire matrix pencils) are considered. Upper bounds for sums of characteristic values and a lower bound for the smallest characteristic value are derived in terms of the coefficients of the Taylor series. These results are new even for polynomial pencils.

© 2004 Elsevier Inc. All rights reserved.

AMS classification: 15A22; 47A56

Keywords: Entire matrix pencils; Characteristic values

1. Introduction and statement of the main result

Entire matrix-valued functions of a complex argument (entire matrix pencils) arise in various applications, see for instance [1,10,14] and references therein. The spectrum of matrix pencils was investigated in many works. Mainly the spectrum perturbations of polynomial matrix pencils were investigated, cf. [5,7,12,13]. In the very interesting paper [8], upper and lower bounds are derived for the absolute values of characteristic values of polynomial matrix pencils.

This research was supported by the Kamea Fund of the Israel No. 8567901.

E-mail address: gilmi@cs.bgu.ac.il

0024-3795/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
On the other hand, estimates for sums of the roots of entire functions are very important for the theory of these functions and its applications. An essential role here is played by the Hadamard theorem [11, p. 18]. In the present paper we derive upper bounds for sums of characteristic values of entire matrix pencils. In addition, we suggest a lower bound for the smallest characteristic value.

Besides, we generalize the main result from the paper [2] (see also [3, Chapter 19]) on inequalities for sums of zeros of entire scalar functions of finite order. These inequalities supplement the Hadamard theorem. Namely they not only assert the convergence of the series of the zeros, but also give us the estimate for the sums of the zeros. It should be noted that the generalization requires additional mathematical tools.

Let $C^n$ be a Euclidean space with the Euclidean norm $\| \cdot \|$ and the unit matrix $I_n$.

For an $n \times n$-matrix $Q$, $\lambda_k(Q)$ ($k = 1, \ldots, n$) are the eigenvalues with multiplicities taken into account and ordered in the following way:

$$|\lambda_k(Q) + 1| \leq |\lambda_k(Q)|.$$ (1.1)

with a finite $\gamma > 0$. Put

$$M_f(r) := \max_{|z|=r} \| f(z) \|$$ (r > 0).

The limit

$$\rho(f) := \lim_{r \to \infty} \frac{\ln \ln M_f(r)}{\ln r}$$

is the order of $f$. A zero $z_k(f)$ of $\det f(z)$ is called a characteristic value of $f$.

Everywhere in the present paper it is assumed that the set $\{z_k(f)\}$ of all the characteristic values of $f$ is infinite. Note that if $f$ has a finite number $l$ of the characteristic values, we can put $z_{k+l}(f) = 0$ for $k = l, l+1, \ldots$ and apply our arguments below, where $z_{k-l}(f)$ means $1/z_l(f)$. The characteristic values of $f$ are enumerated in the non-decreasing way:

$$|z_k(f)| \leq |z_{k+1}(f)|$$ (k = 1, 2, ...).

Assume that the series

$$\theta_f := \sum_{k=1}^{\infty} a_k a_k^*$$ (1.2)

converges. Here and below the asterisk denotes the adjointness. So $\theta_f$ is an $n \times n$-matrix and by (1.1), and (1.2) we have $\rho(f) \leq 1/\gamma$. Put

$$o_k(f) = \sqrt{\lambda_k(\theta_f)}$$ for $k = 1, \ldots, n$ and $o_k(f) = 0$ for $k \geq n + 1$.

**Theorem 1.1.** Let condition (1.2) hold. Then the characteristic values of the pencil $f$ defined by (1.1) satisfy the inequalities

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leq \sum_{k=1}^{j} \left[ o_k(f) + \frac{n^\gamma}{(k+1)^\gamma} \right]$$ (j = 1, 2, ...).
The proof of this theorem is presented in the next section.
Let us assume that under (1.1), there is a constant \( d_0 \in (0, 1) \), such that
\[
\lim_{k \to \infty} \sqrt[k]{\|a_k\|} < 1/d_0
\]
and consider the function
\[
\tilde{f}(\lambda) = \sum_{k=0}^{\infty} \frac{(d_0\lambda)^k}{(k!)^{\gamma}} a_k.
\]
That is, \( f(\lambda) \equiv f(d_0\lambda) \) and \( z_k(\tilde{f}) = z_k(f)/d_0 \). Clearly, the series
\[
\theta_j \triangleq \sum_{k=1}^{\infty} d_0^{2k} a_k a_k^*
\]
converges. Moreover,
\[
\omega_k(\tilde{f}) = \sqrt{\lambda_k(\theta_j)} \quad \text{for} \quad k = 1, \ldots, n \quad \text{and} \quad \omega_k(\tilde{f}) = 0 \quad \text{for} \quad k \geq n + 1.
\]
Theorem 1.1 implies

Corollary 1.2. Under (1.1), let condition (1.3) hold. Then
\[
\sum_{k=1}^{j} \frac{1}{d_0^{k}} |z_k(f)| \leq \sum_{k=1}^{j} \omega_k(\tilde{f}) + \frac{n^{\gamma}}{(k+1)^{\gamma}} \quad (j = 1, 2, \ldots).
\]

2. Proof of Theorem 1.1

Consider the polynomial matrix pencil
\[
F(\lambda) = \sum_{k=0}^{m} \lambda^{m-k} a_k \quad (a_0 = I_n)
\]
with the characteristic values ordered in the non-increasing way:
\[
|z_k(F)| \geq |z_{k+1}(F)| \quad (k = 1, \ldots, mn - 1).
\]

Introduce the block matrix
\[
A_m = \begin{pmatrix}
-a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\
\frac{1}{m^{\gamma}} I_n & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{m^{\gamma}} I_n & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \frac{1}{m^{\gamma}} I_n & 0
\end{pmatrix}.
\]

Lemma 2.1. The relation \( \det F(\lambda) = \det(\lambda I_{mn} - A_m) \) is true.
Proof. Let $z_0$ be a characteristic value of $F$. Then
\[
\sum_{k=0}^{m} \frac{z_0^{m-k}}{(k!)^\gamma} a_k h = 0,
\]
where $h \in \mathbb{C}^n$ is the eigenvector of $F(z_0)$. Put
\[
x_k = \frac{z_0^{m-k}}{(k!)^\gamma} h \quad (k = 1, \ldots, m).
\]
Then $z_0 x_k = x_{k-1}/k^\gamma$ $(k = 2, \ldots, m)$ and
\[
\sum_{k=0}^{m} \frac{z_0^{m-k}}{(k!)^\gamma} a_k h = \sum_{k=1}^{m} a_k x_k + z_0 x_1 = 0.
\]
So $x = (x_1, \ldots, x_m)$ satisfies the equation $A_m x = z_0 x$. If the spectrum of $F(\cdot)$ is simple, the lemma is proved. If $\det F(\cdot)$ has non-simple roots, then the required result can be proved by a small perturbation. □

Set
\[
\theta_F := \sum_{k=1}^{m} a_k a_k^* \quad \text{and} \quad \omega_k(F) = \sqrt{\lambda_k(\theta_F)} \quad \text{for} \quad k = 1, \ldots, n \quad \text{and} \quad \omega_k(F) = 0 \quad \text{for} \quad k = n+1, \ldots, mn.
\]

Lemma 2.2. The characteristic values of $F$ satisfy the inequalities
\[
\sum_{k=1}^{j} |z_k(F)| \leq \sum_{k=1}^{j} \left[ \omega_k(F) + \frac{n^\gamma}{(k+1)^\gamma} \right] \quad (j = 1, \ldots, mn).
\]

Proof. By Lemma 2.1,
\[
\lambda_k(A_m) = z_k(F) \quad (k = 1, 2, \ldots, nm). \tag{2.2}
\]
Take into account that
\[
\sum_{k=1}^{j} |\lambda_k(A_m)| \leq \sum_{k=1}^{j} s_k(A_m) \quad (j = 1, \ldots, nm), \tag{2.3}
\]
where $s_k(A_m)$ are the singular numbers of $A_m$ ordered in the non-increasing way (see, for instance, [4, Section 2.2]). But $A_m = M + C$, where
\[
M = \begin{pmatrix}
-a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\
0 & 0 & \cdots & 0 & 0 \\
. & . & \cdots & . & . \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
and
\[
C = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
\frac{1}{m} I_n & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{m} I_n & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{1}{m} I_n & 0
\end{bmatrix}.
\]

We have
\[
M M^* = \begin{bmatrix}
\theta F & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]
and
\[
C C^* = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & I_n/2^{2^\gamma} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & I_n/m^{2^\gamma}
\end{bmatrix}.
\]

We can show that \(s_k(M) = \omega_k(F)\). In addition,
\[
s_k(C) = \frac{1}{(j+2)^{2^\gamma}} \quad (k = jn + l; j = 0, \ldots, m-2; l = 1, \ldots, n)
\]
and
\[
s_k(C) = 0 \quad (k = (m-1)n + l; l = 1, \ldots, n).
\]

Since \(j = (k-l)/n\) and \((k-l)/n + 2 \geq (k+1)/n\), it follows that,
\[
s_k(C) \leq \frac{n^{2^\gamma}}{(k+1)^{2^\gamma}} \quad (k = 1, \ldots, mn).
\]

Take into account that
\[
\sum_{k=1}^{j} s_k(A_m) = \sum_{k=1}^{j} s_k(M + C) \leq \sum_{k=1}^{j} s_k(M) + \sum_{k=1}^{j} s_k(C),
\]
cf. [4, Lemma II.4.2]. So
\[
\sum_{k=1}^{j} s_k(A_m) \leq \sum_{k=1}^{j} \left[ \omega_k(F) + n^{2^\gamma}(k+1)^{-2^\gamma} \right] \quad (j = 1, 2, \ldots, mn).
\]

Now (2.2) and (2.3) yield the required result. □

**Proof of Theorem 1.1.** Consider the polynomial pencil
\[
f_m(\lambda) = \sum_{k=0}^{m} \frac{\lambda^k}{(k!)^{2^\gamma}} a_k.
\]
Clearly, $\lambda^m f_m(1/\lambda) = F(\lambda)$. So $z_k(F) = z_k^{-1}(f_m)$. Now Lemma 2.2 yields the inequalities

$$\sum_{k=1}^{j} \left| z_k(f_m) \right|^{-1} \leq \omega_k(f) + n^\gamma \sum_{k=1}^{j} (k + 1)^{-\gamma} \quad (j = 1, \ldots, nm). \quad (2.4)$$

But the characteristic values of entire matrix functions continuously depend on their coefficients. So for any $j = 1, 2, \ldots,$

$$\sum_{k=1}^{j} \left| z_k(f_m) \right|^{-1} \rightarrow \sum_{k=1}^{j} \left| z_k(f) \right|^{-1}$$

as $m \rightarrow \infty$. Now (2.4) implies the required result. \[\square\]

### 3. Applications of Theorem 1.1

Put

$$\tau_k = \omega_k(f) + \frac{n^\gamma}{(k + 1)^\gamma} \quad (k = 1, 2, \ldots).$$

The following result is a consequence of the well-known Lemma II.3.4 from [4] and Theorem 1.1.

**Corollary 3.1.** Let $\phi(t)(0 \leq t < \infty)$ be a convex scalar-valued function, such that $\phi(0) = 0$. Then under conditions (1.1) and (1.2), the inequalities

$$\sum_{k=1}^{j} \phi(\left| z_k(f) \right|^{-1}) \leq \sum_{k=1}^{j} \phi(\tau_k) \quad (j = 1, 2, \ldots)$$

are valid. In particular, for any $r \geq 1$,

$$\sum_{k=1}^{j} \left| z_k(f) \right|^{-r} \leq \sum_{k=1}^{j} \tau_k^r \quad (j = 2, 3, \ldots)$$

and thus

$$\left[ \sum_{k=1}^{j} \left| z_k(f) \right|^{-r} \right]^{1/r} \leq \left[ \sum_{k=1}^{j} \omega_k^r(f) \right]^{1/r} + n^\gamma \left[ \sum_{k=1}^{j} \frac{1}{(k + 1)^\gamma} \right]^{1/r} \quad (j = 2, 3, \ldots). \quad (3.1)$$
Furthermore, assume that
\[ r^\gamma > 1, \quad r \geq 1. \] (3.2)
Then
\[ \sum_{k=1}^{\infty} (k+1)^{-r^\gamma} = \zeta(\gamma r) - 1, \]
where \( \zeta(\cdot) \) is the Riemann Zeta function. With the notation
\[ N_r(f) = \left[ \sum_{k=1}^{n} \lambda_k^r \frac{\tau_f^2}{2} \right]^{1/r} \]
relation (3.1) yields

**Corollary 3.2.** Under conditions (1.1), (1.2) and (3.2), the inequality
\[ \left( \sum_{k=1}^{\infty} |z_k(f)|^{-r^\gamma} \right)^{1/r} \leq N_r(f) + n^\gamma (\zeta(\gamma r) - 1)^{1/r} \] (3.3)
is valid. Moreover, if \( \gamma > 1 \), then
\[ \sum_{k=1}^{\infty} |z_k(f)|^{-1} \leq N_1(f) + n^\gamma (\zeta(\gamma) - 1). \] (3.4)

Consider now a positive scalar-valued function \( \Phi(t_1, t_2, \ldots, t_j) \) with an integer \( j \), defined on the domain
\[ 0 \leq t_j \leq t_{j-1} \leq t_2 \leq t_1 < \infty \]
and satisfying
\[ \frac{\partial \Phi}{\partial t_1} > \frac{\partial \Phi}{\partial t_2} > \cdots > \frac{\partial \Phi}{\partial t_j} > 0 \] for \( t_1 > t_2 > \cdots > t_j \). (3.5)

**Corollary 3.3.** Under conditions (1.1), (1.2) and (3.5),
\[ \Phi(|z_1(f)|^{-1}, |z_2(f)|^{-1}, \ldots, |z_j(f)|^{-1}) \leq \Phi(\tau_1, \tau_2, \ldots, \tau_j). \]
Indeed, this result follows from Theorem 1.1 and the well-known Lemma II.3.5 from [4].
In particular, let \( \{d_k\}_{k=1}^{\infty} \) be a decreasing sequence of non-negative numbers. Take
\[ \Phi(t_1, t_2, \ldots, t_j) = \sum_{k=1}^{j} d_k t_k. \]
Then Corollary 3.3 yields
\[ \sum_{k=1}^{j} d_k |z_k(f)|^{-1} \leq \sum_{k=1}^{j} \tau_k d_k = \sum_{k=1}^{j} d_k [\omega_k(f) + (k + 1)^{-\gamma}]. \]

4. Lower bounds for characteristic values

Consider the polynomial \( F \) defined by (2.1). If \( m < \infty \), we can take \( \gamma = 0 \). Let \( h \) be a normalized eigenvector corresponding to a characteristic value \( z(F) \). Then
\[ z^m(F)h = -\sum_{k=0}^{m-1} z(F)^{m-1-k} \frac{[(k + 1)!]^{-\gamma} a_{k+1}}{(k + 1)!} h. \]
Hence,
\[ |z(F)|^m \leq \sum_{k=0}^{m-1} \frac{\|a_{k+1}\| |z(F)|^{m-1-k}}{[(k + 1)!]^{1+\gamma}} \]
or
\[ 1 \leq \sum_{k=0}^{m-1} \frac{\|a_{k+1}\|}{[(k + 1)!]^{1+\gamma}|z(F)|^{1+\gamma}}. \]
Hence it follows that \( |z(F)| \leq x_0 \) where \( x_0 \) is the unique positive root of the equation
\[ 1 = \sum_{k=0}^{m-1} \frac{\|a_{k+1}\|}{[(k + 1)!]^{1+\gamma} x^{1+\gamma}}. \]
Put
\[ p(F) = \sum_{k=0}^{m-1} \frac{\|a_{k+1}\|}{[(k + 1)!]^{1+\gamma}} \]
and
\[ \delta_0(F) := \begin{cases} \sqrt[m-1]{p(F)} & \text{if } p(F) \leq 1, \\ p(F) & \text{if } p(F) > 1. \end{cases} \]
Due to Lemma 1.6.1 from [3], \( x_0 \leq \delta_0(F) \). Moreover, thanks to Corollary 1.6.2 from [3]
\[ x_0 \leq 2 \sup_{k=1, \ldots, m} \sqrt[k]{\|a_k\|}. \]
We thus get

**Lemma 4.1.** Any characteristic value \( z(F) \) of the pencil \( F \) defined by (2.1) satisfies the inequalities \( |z(F)| \leq \delta_0(F) \) and
\[ |z(F)| \leq 2 \sup_{k=1,\ldots,m} \frac{i \|a_k\|}{(k!)^{\gamma}} \]

Again consider the polynomial pencil
\[ f_m(\lambda) = \sum_{k=0}^{m} \frac{\lambda^k}{(k!)^{\gamma}} a_k. \]

Since \( z_k(F) = z_k^{-1}(f_m) \), due to the previous lemma we can write
\[ 2|z(f_m)| \geq \inf_{k=1,\ldots,m} \frac{i \sqrt{k!}}{\|a_k\|}. \]

Letting \( m \to \infty \), we arrive at

**Corollary 4.2.** Any characteristic value \( z(f) \) of the pencil \( f \) defined by (1.1) satisfies the inequality
\[ 2|z(f)| \geq \inf_{k=1,2,\ldots} \frac{i \sqrt{k!}}{\|a_k\|}. \]

It is well-known that the polynomials of the type (2.1) play an essential role in the theory of discrete-time systems, cf. [6,9]. If any characteristic value of \( F \) satisfies the inequality \(|z(F)| < 1\), then the corresponding discrete-time system is stable. Thus Lemma 4.1 gives us a stability criterion.

**References**