A Source of Counterexamples in Operator Theory and How to Construct Them

Boris Mirman
Digital Equipment Corporation
Andover, Massachusetts 01810

Submitted by Chandler Davis

ABSTRACT

It is known that the function \( f(t) = |t| \) fails to satisfy an "operator Lipschitz condition," in the sense that the best bound upon \( |A - B| ||A - B|| \) goes to infinity with the dimensionality of the (finite-dimensional Hilbert) space where \( A \) and \( B \) act. Two new ways of supplying the counterexamples are given here, to exemplify an approach that is believed to have wider applicability.

1. THE METHOD

Counterexamples shed light on many problems, emphasizing the constraints and the importance of hypotheses. Every mathematician constructs counterexamples while trying to prove some conjectured statement. Construction of a counterexample in operator theory turns often on the solution of an operator equation. Trying to determine conditions for solvability of this equation, we consider first the corresponding numerical equation and determine the requirements for existence of its solution. Then we consider how these requirements should be changed in the operator case. It may happen that in the operator case, a "weak" requirement (on the spectra of operators) has to be satisfied for the existence of a solution, rather than a "strong" requirement (on the operators themselves). If so, we can use the facts that (1) spectra of real and imaginary parts of Volterra operators have some special relation (see [4]), and (2) usually Volterra operators have very unstable spectra, i.e., a small perturbation of some specially constructed Volterra operator may satisfy the required spectral conditions. This perturbation will allow us to solve the operator equation, and thereby the counterexample will be constructed.
2. THE COUNTEREXAMPLES

Let us consider operators in n-dimensional Hilbert space $H_n$. Let $\sigma(T)$ denote the spectrum of operator $T$, and $L_\alpha$ the Lipschitz $\alpha$-class of functions, i.e., $f \in L_\alpha$ means that the ratio

$$\frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\alpha}$$

is bounded by some constant. And let $(x, y)$ or $y^*x$ denote the scalar product of $x, y \in H_n$.

1.1

We will illustrate the method described in Section 1 on the following problem posed by M. Reed and B. Simon in [1]: whether the ratio

$$\frac{|||A| - |B||}{||A - B||}$$

is bounded by some constant which does not depend on dim $H$. Here $|A| = (A*A)^{1/2}$ is the "absolute value" of an operator $A$, and the problem may be interpreted as whether the function $f(t) = |t|$ as a function of operators has the Lipschitz property. A negative answer was given by T. Kato in 1973 [2]. This answer was based on an example given by A. McIntosh in 1971 [3].

REMARK. An estimate of the ratio

$$\frac{||f(A) - f(B)||}{||A - B||}$$

for self-adjoint finite-dimensional operators $A$ and $B$ and Lipschitz functions $f \in L_1$ was given by Yu. B. Farforovskaya [5]. This ratio is limited by a constant independent of dim $H$ only if $f$ is smooth enough, in particular, if the derivative $f' \in L_\alpha$. This result was found by M. Sh. Birman and M. Z. Solomyak [6]. Farforovskaya showed in [5] that this ratio is bounded by a constant which essentially depends on dim $H$ and goes to infinity as
For the specific case of \( f(t) = |t| \) and arbitrary operators \( A, B \), we have Kato and McIntosh's solution with similar result.

However, the second edition of [1], issued in 1980, did not mention this solution. I will give here another solution of this problem just to illustrate a convenient way to construct counterexamples in operator theory. Note that this solution does not use self-adjoint \( A \) and \( B \).

Let \( T > 0 \) and \( C = T - (S*S)^{1/2} \). Then \( |T| = T \) and

\[
\frac{||T| - |S||}{||T - S||} = \frac{||C||}{||S| - S + C||},
\]

so the desired counterexample will be constructed if we find, for any \( 0 < \epsilon < 1 \), operators \( S \) and \( C \) such that

\[
||S - S + C|| < \epsilon, \quad (1)
\]

\[
||C|| > 1 - \epsilon, \quad (2)
\]

\[
C > 0. \quad (3)
\]

For this purpose, let us determine the conditions required for existence of a solution of the following equation with respect to \( S \):

\[
|S| - S = W. \quad (4)
\]

The corresponding numerical equation,

\[
|s| - s = w,
\]

has a nonzero solution if and only if

\[
\text{Re } w > 0.
\]

We ask whether a corresponding condition exists in the operator case. If so, is it the strong condition

\[
W + W^* > 0,
\]
the weak condition

\[ \text{Re } \sigma(W) > 0, \tag{5} \]

or something else? It turns out (see Appendix 1 for proof) that the weak condition is sufficient.

Now suppose for a moment that we have a nilpotent operator \( N \) with the following properties for some small positive \( \epsilon_1 \):

\[
\text{Re } N < \epsilon_1, \tag{6}
\]

\[
\|N\| = 1, \tag{7}
\]

\[
\|\text{Im } N\| < \epsilon_1. \tag{8}
\]

This will give us the desired example. Indeed, let

\[
W = N + \epsilon_1 I, \tag{4}
\]

\[
C = \epsilon_1 I - \text{Re } N. \tag{5}
\]

Since \( \sigma(W) = \{\epsilon_1\} \), we have \( \text{Re } \sigma(W) > 0 \). This is the condition (5) for equation (4) to have a solution \( S \). The properties (6)–(8) with \( \epsilon_1 = \epsilon / 3 \) lead to the conditions (1)–(3) for these \( C \) and \( S \).

It remains to construct such a nilpotent operator \( N \) with arbitrarily small imaginary part and with "almost" negative definite and not small real part. This is done in Appendix 2. This completes the construction of our counterexample.

Operators with related properties were constructed by W. Kahan [7] and A. Pokrzywa [8].

1.2

Let us consider now Kato and McIntosh’s example. That is, we seek a construction for any \( n \) of self-adjoint operators \( X \) and \( Y \) such that

\[ X > 0, \tag{9} \]

\[ \|\text{Re } XY\| \leq 1, \tag{10} \]

\[ \|\text{Im } XY\| \geq n. \tag{11} \]
Let us determine the conditions required for existence of a solution of the operator equation

\[ XY = W. \] (12)

The corresponding numerical equation

\[ xy = w \]

has, obviously, a real solution \( x, y \) if and only if

\[ \text{Im} \ w = 0. \]

Again: which of the following conditions corresponds to the latter in the operator case? Is it the strong condition

\[ W = W^*, \]

the weak condition

\[ \text{Im} \ \sigma(W) = 0, \]

or neither of them? It turns out (see Appendix 3) that the weak condition is sufficient.

Now suppose for a moment that we have a nilpotent operator \( N \) such that

\[ \| \text{Re} \ N \| < 1 \] (13)

and

\[ \| \text{Im} \ N \| > n. \] (14)

This will give the solution. Indeed, let \( E \) be the positive operator which, in the Jordan basis for \( N \), has the form \( \text{diag}(\epsilon_1, \ldots, \epsilon_n) \), for suitable small distinct numbers \( \epsilon_j > 0 \). Then \( W = N + E \) is as described in Appendix 3, so Equation (12) has self-adjoint solution \( X, Y \) with the properties (9)–(11).

It remains to construct a nilpotent operator \( N \) with the properties (13), (14). This is done in analogy to Appendix 2. The construction of our Kato-McIntosh example is complete.
APPENDIX 1

**Proposition.** A solution $S$ of the equation

$$|S| - S = W \quad (15)$$

exists if

$$\text{Re } \sigma(W) > 0. \quad (16)$$

**Proof.** If the equation

$$RW + W^*R = W^*W \quad (15^*)$$

has a solution $R > 0$, then by a simple calculation $S = R - W$ is a solution of (15); so we study $(15^*)$. We need the following

**Lemma.** Let $G = (g_{jk})_{j,k=1}^n$ be a positive definite matrix, $\text{Re } \epsilon_j \geq \epsilon_0 > 0$ $(j = 1, \ldots, n)$. Then the matrix

$$A = \left( \frac{g_{jk}}{\epsilon_j + \epsilon_k} \right)_{j,k=1}^n$$

is positive definite also, and $\|A\| \leq \|G\|/(2\epsilon_0)$.

Indeed, let $D = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$. It is obvious that $A = A^*$ and $DA + AD^* = G$. If $Ax = \alpha x$, $x \neq 0$, then $2a \text{Re}(Dx, x) = (Gx, x)$. Hence, $0 < a \leq \|G\|/(2\epsilon_0)$. The lemma is proved.

Now consider first the case of diagonalizable $W$, i.e.,

$$W = \sum_{j=1}^n w_j x^{(j)} y^{(j)*}, \quad (17)$$

where $\text{Re } w_j \geq \epsilon_0 > 0$, and $\{x^{(j)}, y^{(j)}\}_{j=1}^n$ is a biorthonormal system in $H_n$. It is easy to check that the operator

$$R = \sum_{j,k=1}^n \frac{w_j \bar{w}_k}{w_j + w_k} y^{(k)} x^{(k)*} x^{(j)} y^{(j)*}$$

satisfies the equation $RW + W^*R = W^*W$. If

$$z = \sum_{j=1}^{n} z_j x^{(j)}$$

then

$$z^*Rz = \sum_{j,k=1}^{n} \frac{g_{jk}}{w_j + \bar{w}_k} z_j \bar{z}_k,$$

where $g_{jk} = w_j \bar{w}_k x^{(k)^*} x^{(j)}$, i.e., $G = (g_{jk})_{j,k=1}^{n}$ is the Gram matrix of the vectors $(w_j x^{(j)})_{j=1}^{n}$. Hence $G > 0$, and, by the Lemma, $R > 0$. This means that in case $W$ has the form (17), $S = R - W$ is a solution of Equation (15).

Now the Lemma gives the inequality

$$\|R\| \leq \frac{\|W\|^2}{2\epsilon_0}.$$

Therefore if Equation (15*) is to be solved for $W$ not satisfying (17), we can use a limit of the solutions obtained for diagonalizable approximation to $W$. The Proposition is proved.

**Remark.** P. Rosenthal [11] noted that the solvability of Equation (15) follows from Lyapunov's theorem (even in the infinite-dimensional case). This appendix is given just for completeness and because of the simplicity of this direct proof.

**APPENDIX 2**

Let $N$ be the following operator in $L_2(0, 1)$:

$$Nf(x) = 2i \int_{0}^{x} f(t) \sum_{j} \lambda_j e^{(2j-1)\pi i (x-t)} dt,$$
Then we know from [4, III.10] that

(i) $N$ is a compact quasinilpotent operator;
(ii) $\sigma(\text{Im } N) = \{\lambda_j\} \cup \{0\}$;
(iii) $\sigma(\text{Re } N) = \{\mu_k = (2/\pi)\sum_j \lambda_j/(2k-2j+1)\}_{k=0}^{\pm 1, \pm 2} \cup \{0\}$.

**Proposition.** Let $\lambda_j = j^\delta/(2j-1)$, where $0 < \delta < 1$, $j = 1, 2, \ldots$. Then, for any integer $k$,

$$\mu_k = \frac{2}{\pi} \sum_j \frac{j^\delta}{(2j-1)(2k-2j+1)} \leq \frac{1}{\pi}.$$

**Proof.** For $k \leq 0$, obviously, $\mu_k < 0$. Let $k > 0$. Then

$$S_{k, l} = \sum_{j=1}^{kl} \frac{j^\delta}{(2j-1)(2j-2k+1)}$$

$$= \sum_{m=1}^{l} \frac{1}{2k} \sum_{j=(m-1)k+1}^{mk} \left( \frac{j^\delta}{2j-2k-1} - \frac{j^\delta}{2j-1} \right)$$

$$= \frac{1}{2k} \sum_{j=1}^{k} \frac{(j+k)^\delta - j^\delta - (k+1-j)^\delta}{2j-1}$$

$$- \frac{1}{2k} \sum_{j=(l-1)k+1}^{lk} \frac{j^\delta}{2j-1} + \frac{1}{2k} \sum_{m=2}^{l-1} \sum_{j=(m-1)k+1}^{mk} \frac{(j+k)^\delta - j^\delta}{2j-1}$$

$$> - \frac{1}{2k} \frac{1}{2k} \sum_{j=(l-1)k+1}^{lk} \frac{j^\delta}{2j-1}.$$
Hence

\[ \lim_{l \to \infty} S_{k, l} \geq \frac{1}{2} \]

and

\[ \mu_k = -\frac{2}{\pi} \lim_{l \to \infty} S_{k, l} \leq \frac{1}{\pi}. \]

If \( \delta < 1 \) is close enough to 1, then obviously

\[ \mu_0 = -\frac{2}{\pi} \sum_{j=1}^{\infty} \frac{f^j}{(2j-1)^2} < -\frac{1}{\epsilon} \]

for any given \( \epsilon > 0 \). Therefore the operator

\[ N_0 \circ = -\frac{2i}{\mu_0} \int_0^1 \sum_{j=1}^{\infty} \frac{j^\delta}{2j-1} e^{(2j-1)\pi(i(x-1))} dt \]

has the following properties:

(i) \( \| \text{Im } N_0 \| < \epsilon \),
(ii) \( \text{Re } N_0 < \epsilon / \pi \),
(iii) \( \| \text{Re } N_0 \| = 1 \),

i.e., \( N_0 \) is a compact quasinilpotent operator with arbitrarily small imaginary part and "almost" negative definite real part.

A finite-dimensional operator with the same properties exists due to the following statement: Any quasinilpotent compact operator can be uniformly approximated by finite-dimensional nilpotent operators. This statement follows from [9, p. 916]. I thank B. S. Mityagin [10] and the referee for clarifying this point.

APPENDIX 3

**Proposition.** Self-adjoint operators \( X > 0 \) and \( Y \) such that

\[ XY = W \]
exist if

\[ W = \sum_{j=1}^{n} w_j x^{(j)} y^{(j)*}, \]

where

\[ \text{Im} w_j = 0, \quad j = 1, \ldots, n, \]

and \( \{x^{(j)}, y^{(j)}\} \) is a biorthonormal system in \( H_n \).

**Proof.** Let \( x^{(j)} = X y^{(j)}, \quad j = 1, \ldots, n \). Then:

(i) \( X > 0 \). Indeed, for any nonzero \( u \in H_n \),

\[ u = \sum_{j=1}^{n} x^{(j)*} u y^{(j)}, \]

\[ X u = \sum_{j=1}^{n} x^{(j)*} u x^{(j)}, \]

and

\[ u^* X u = \sum_{j=1}^{n} |x^{(j)*} u|^2 > 0. \]

(ii) \( Y = X^{-1} W = Y^* \). Indeed,

\[ X^{-1} WX = \sum_{j=1}^{n} w_j x^{(j)} (X y^{(j)})^* = W^*; \]

hence \( X^{-1} W = W^* X^{-1} \). or \( Y = Y^* \). The proposition is proved.

**REFERENCES**


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