ADVANCES IN MATHEMATICS **71**, 92–112 (1988)

Gorenstein Spaces*

YVES FÉLIX†

Institut de Mathématiques, Université de Louvain-La-Neuve, 2, Chemin du Cyclotron, 1348 Louvain-La-Neuve, Belgique

STEPHEN HALPERIN

Department of Mathematics, Scarborough College, University of Toronto, Scarborough, Canada M1C 1A4

AND

JEAN-CLAUDE THOMAS‡

U.E.R. de Mathématiques Pures et Appliquées, Université de Lille I, 59655 Villeneuve d'Ascq, France

INTRODUCTION

Let X be a finite n-dimensional subcomplex of $\mathbb{R}^{n+k}$. Its *Spivak fibre*, F_X, is the homotopy fibre of the inclusion of the boundary of a regular neighbourhood of X, and is, up to suspension, a homotopy invariant of X. It is introduced in [17] by Spivak, who shows that F_X is a homotopy sphere if and only if X is a Poincaré complex.

This paper starts with the observation (1.1, 1.2) that when X is 1-connected, the reduced homology of F_X with coefficients in any field $\Bbbk$ is given by

$$\tilde{H}_*(F_X; \Bbbk) = \operatorname{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X; \Bbbk)),$$

where $C^*(X; \Bbbk)$ is a left module over itself by multiplication, and Ext is the "differential" Ext of Eilenberg and Moore [16]. Now the right hand side of this equation is a well defined homotopy invariant for any pointed space, X, and we think of it as the reduced homology of a "virtual Spivak fibre," even when geometrically no such fibre exists.

* Part of this research was carried out at the University of Toronto, and was supported by the second named author's NSERC grant.
† Chercheur qualifié FNRS.
‡ Partially supported by a NATO grant.

0001-8708/88 $7.50

The study of this invariant is further motivated by the study of the invariant $\mathrm{Ext}_R(\Bbbk, R)$ associated with a local commutative ring R (cf. [4]).

Our first general result (Proposition 1.6) asserts that if $X = Y \cup e_n$ and e_n is homologically non-trivial in X (coefficients in $\Bbbk$) then $\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X;\Bbbk))$ is non-zero. By contrast (Proposition 1.7) if $H^*(X;\Bbbk)$ is an infinite tensor product, then $\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X;\Bbbk))$ is identically zero.

Next, in Section 2, we show that for this invariant, Eckmann–Hilton duality reduces to the identity

$$\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X;\Bbbk)) \cong \mathrm{Ext}_{C_*(\Omega X;\Bbbk)}(\Bbbk, C_*(\Omega X;\Bbbk))$$

(Proposition 2.1). It follows (from a result of [8]) that if $\mathrm{Ext}_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q})) \neq 0$ then the radical of the homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ is finite dimensional (Theorem 2.2). In particular, the radical for any space of the form $X = Y \cup e_n$ (e_n rationally homologically non-trivial) is finite dimensional.

In Section 3 we turn to duality theorems. Here we have a local analogue (Theorem 3.1) of Spivak's theorem: If $\mathbb{Z}_p$ is the prime field of characteristic p (p prime or zero) then $H^*(X;\mathbb{Z}_p)$ satisfies Poincaré duality if and only if $(F_X)_{(p)} \simeq S^{k-1}_{(p)}$.

The first step in the proof is, of course, the observation that $(F_X)_{(p)} \simeq S^{k-1}_{(p)}$ is equivalent to

$$\dim \mathrm{Ext}_{C^*(X;\mathbb{Z}_p)}(\mathbb{Z}_p, C^*(X;\mathbb{Z}_p)) = 1;$$

one then shows via differential homological algebra that this is equivalent to Poincaré duality in $H^*(H;\mathbb{Z}_p)$.

Again, this condition makes sense (and can hold) for more general spaces and, again, it has an analogue in local rings: a local ring A is called a *Gorenstein ring* if $\dim \mathrm{Ext}_A(\Bbbk, A) = 1$. We therefore make the general

DEFINITION. A simply connected space, X, is *Gorenstein at* $\Bbbk$ if $\dim \mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X;\Bbbk)) = 1$.

When $H^*(X;\Bbbk)$ is finite dimensional we have that X is Gorenstein at $\Bbbk$ if and only if $H^*(X;\Bbbk)$ satisfies Poincaré duality. There are, however, other examples: any finite Postnikov tower is Gorenstein over $\mathbb{Q}$ (Proposition 3.4).

In [10] Gottlieb shows that for a fibration $F \to E \to B$ of finite complexes, the Spivak fibres satisfy $F_E \simeq F_F * F_B$; thus E is a Poincaré complex if and only if F and B are. In Section 4 (Theorem 4.3) we prove a $\mathbb{Q}$-local analogue of this result:

THEOREM. *If $F \to E \to B$ is a fibration of simply connected spaces in which $H^*(F; \mathbb{Q})$ is finite dimensional and $H^*(B; \mathbb{Q})$ has finite type then*

$$\begin{aligned}&\operatorname{Ext}_{C^*(E;\mathbb{Q})}(\mathbb{Q}, C^*(E; \mathbb{Q}))\\ &\qquad = \operatorname{Ext}_{C^*(B;\mathbb{Q})}(\mathbb{Q}, C^*(B; \mathbb{Q})) \otimes \operatorname{Ext}_{C^*(F;\mathbb{Q})}(\mathbb{Q}, C^*(F;\mathbb{Q})).\end{aligned}$$

In particular, E is Gorenstein at $\mathbb{Q}$ if and only if both F and B are.

This theorem also has an analogue for local rings, due to Avramov, Foxby, and Halperin [3].

Lastly, in Section 5 we define the *formal dimension* of any space X at $\mathbb{k}$ and obtain some basic properties. (When dim $H^*(X; \mathbb{k})$ is finite this is just the largest n for which $H^n(X; \mathbb{k}) \neq 0$.)

We have indicated the analogy between Poincaré duality spaces (or Gorenstein spaces) and Gorenstein rings. We mention another in passing: a discrete group G is said to be a weak duality group [5] if $\operatorname{Ext}_{\mathbb{k}[G]}(\mathbb{k}, \mathbb{k}[G])$ has dimension one.

The results of this paper depend heavily on techniques from the differential homological algebra invented by Eilenberg and Moore [16]. For the convenience of the reader we have collected the conventions, definitions, and facts we use into an appendix. Another exposition can be found in [2]. In the appendix we have put hardly any proofs, since the results there are (a) not new and (b) rather easy exercises. For the occasional notation/convention from rational homotopy theory proper the reader is referred to [7, Sect. 1] or [12].

1. *The Spivak Fibre*

Let N be a regular neighbourhood of an n-dimensional simply connected finite sub complex, $X \subset \mathbb{R}^{n+k} k > n+1$, and let F be the homotopy fibre of the inclusion $\partial N \subsetneq N$; it is the *Spivak fibre* for X.

Let $C^*(X; \mathbb{k})$ denote the DGA (differential graded algebra) of singular cochains on a space X and let $C_*(X; \mathbb{k})$ be the singular chains: it is a $C^*(X, \mathbb{k})$-module under cap product. The first result of this sectin reads.

PROPOSITION 1.1. *The reduced cohomology of F is given by*

$$\tilde{H}^*(F; \mathbb{k}) = \operatorname{Tor}^{C^*(X;\mathbb{k})}(\mathbb{k}, s^{-n-k+1} C_*(X; \mathbb{k})).$$

Proof. The Eilenberg–Moore theorem [6] gives $H^*(F; \mathbb{k}) = \operatorname{Tor}^{C^*(N;\mathbb{k})}(\mathbb{k}, C^*(\partial N; \mathbb{k}))$. Now consider the short exact sequence of $C^*(N; \mathbb{k})$-modules: $C^*(N, \partial N; \mathbb{k}) \rightarrowtail C^*(N; \mathbb{k}) \twoheadrightarrow C^*(\partial N; \mathbb{k})$. It leads to a long exact sequence of the modules $\operatorname{Tor}^{C^*(N;\mathbb{k})}(\mathbb{k}, -)$.

Since $\mathrm{Tor}^{C^*(N;\Bbbk)}(\Bbbk, C^*(N;\Bbbk)) = \Bbbk$, we deduce that $\tilde{H}^*(F;\Bbbk) = \mathrm{Tor}^{C^*(N;\Bbbk)}(\Bbbk, sC^*(N, \partial N;\Bbbk))$. Let z be a representative of the fundamental class for the pair $(N, \partial N)$. Then (Poincaré duality) capping with z defines an equivalence $C^*(N, \partial N;\Bbbk) \to s^{-n-k}C_*(N, \Bbbk)$ of $C^*(N;\Bbbk)$-modules.

Finally, the homotopy equivalence $X \xrightarrow{\simeq} N$ gives an equivalence $C_*(X;\Bbbk) \to C_*(N;\Bbbk)$ of $C^*(N;\Bbbk)$-modules. Thus $\mathrm{Tor}^{C^*(N;\Bbbk)}(\Bbbk, C_*(N;\Bbbk))$ and $\mathrm{Tor}^{C^*(X;\Bbbk)}(\Bbbk, C_*(X;\Bbbk))$ are identified by the natural isomorphisms of $\mathrm{Tor}^{C^*(N;\Bbbk)}(\Bbbk, C_*(X;\Bbbk))$ into each. ∎

COROLLARY 1.2. *The reduced homology of F is given by*

$$\tilde{H}_*(F;\Bbbk) = \mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, s^{n+k-1}C^*(X;\Bbbk)).$$

Proof. Note that $H_*(F;\Bbbk)$ has finite type and hence coincides with its double dual. Since $C^*(X;\Bbbk) = C_*(X;\Bbbk)^\vee$ the corollary follows from $\mathrm{Ext}_R(k, M^\vee) = \mathrm{Tor}^R(k, M)^\vee$. ∎

Now for general 1-connected spaces, X, the object $\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X;\Bbbk))$ is a good homotopy invariant, which may be thought of as the reduced homology of a "*virtual Spivak fibre*." In general, of course, it may be non-zero in infinitely many positive and negative degrees. Nonetheless, it turns out that it has important implications for more classical invariants.

We begin with some remarks:

Remarks 1.3. (1) Let $R \to S$ be a quism of augmented $\Bbbk$-DGAs. Then we can identify $\mathrm{Ext}_R(\Bbbk, R)$ with $\mathrm{Ext}_S(\Bbbk, S)$ via

$$\mathrm{Ext}_R(\Bbbk, R) \xrightarrow{\cong} \mathrm{Ext}_R(\Bbbk, S) \xleftarrow{\cong} \mathrm{Ext}_S(\Bbbk, S).$$

(2) For an augmented $\Bbbk$-DGA, R, and an R module M there is an Eilenberg–Moore spectral sequence *converging* from $\mathrm{Tor}^{H(R)}(\Bbbk, H(M))$ to $\mathrm{Tor}^R(\Bbbk, M)$. The analogous spectral sequence for Ext, however, may *fail* to converge for general modules.

In the important special case that $M \simeq N^\vee$, however, we can use the identification $\mathrm{Ext}_R(\Bbbk, M) = \mathrm{Tor}^R(\Bbbk, N)^\vee$ to deduce convergence via duality.

This arises in two ways when $M = R$. If $R = C^*(X;\Bbbk)$ then $R \cong C_*(X;\Bbbk)^\vee$ as R-modules. Alternatively, if each $H^p(R)$ is finite dimensional then $R \to (R^\vee)^\vee$ is an equivalence of R-modules. In either case we get a convergent Eilenberg–Moore spectral sequence from $\mathrm{Ext}_{H(R)}(\Bbbk, H(R))$ to $\mathrm{Ext}_R(\Bbbk, R)$. ∎

Now consider an augmented $\Bbbk$-DGA, $R = R^*$. A natural map

$$\mathrm{Ext}_R(\Bbbk, R) \to H(R), \tag{1.4}$$

compatible with the identifications of Remark 1.3(1) is defined as follows: Choose an R-semifree resolution $\rho: P \xrightarrow{\simeq} \Bbbk$ and let $z \in P$ be a cycle representing 1. Define a chain map $\mathrm{Hom}_R(P, R) \to R$ by $f \mapsto f(z)$, and pass to homology to get (1.4). (This definition is independent of the choice of P and z.)

The following lemma extends the classical identification $\mathrm{socle}(A) = \mathrm{Ext}_A^0(\Bbbk, A)$ for commutative rings to our case.

LEMMA 1.5. *Let $R \xrightarrow{\varepsilon} \Bbbk$ be an augmented $\Bbbk$-DGA. If $\alpha \in R$ is represented by a cycle u satisfying* $(\ker \varepsilon) \cdot z = 0$, *then α is in the image of the map* (1.4).

Proof. The map $\omega: \lambda \mapsto \lambda u$ is a cocycle in $\mathrm{Hom}_R(\Bbbk, R)$ and so $\omega \circ \rho: P \to R$ is a cocycle in $\mathrm{Hom}_R(P, R)$ mapping under (1.4) to α. ∎

The lemma has the following geometric consequence:

PROPOSITION 1.6. *Let $X = Y \bigcup_f e_n$ be obtained from a 1-connected space Y by attaching an n-cell, e_n $(n \geqslant 2)$, and let $\alpha \in H^n(X; \Bbbk)$ be the characteristic class for e_n.*

Then α is in the image of (1.5). *In particular, if $\alpha \neq 0$ then* $\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X; \Bbbk))^n \neq 0$.

Proof. For simplicity, denote $C^*(\ ;\Bbbk)$ simply by $C^*(\ )$. We may suppose f embeds S^{n-1} in Y. Then restriction defines a quism

$$C^*(X) \xrightarrow{\simeq} C^*(Y) \times_{C^*(S^{n-1})} C^*(e_n).$$

Compose the surjections $C^*(Y)$, $C^*(e_n) \to C^*(S^{n-1})$ with a surjective quism $C^*(S^{n-1}) \to A$ with $A^{\geqslant n} = 0$, and choose a quism $H(S^{n-1}) = H^*(S^{n-1}; \Bbbk) \to A$. This gives the diagram

$$\begin{array}{ccc} & & C^*(Y), C^*(e_n) \\ & & \downarrow \\ H^*(S^{n-1}) & \xrightarrow{\simeq} & A \end{array}$$

and we denote the fibred products $H(S^{n-1}) \times_A C^*(Y)$ and $H(S^{n-1}) \times_A C^*(e_n)$ respectively by $\bar{C}(Y)$ and $\bar{C}(e_n)$. We have then a quism

$$\bar{C}(Y) \times_{H(S^{n-1})} \bar{C}(e_n) \xrightarrow{\simeq} C^*(Y) \times_{C^*(S^{n-1})} C^*(e_n).$$

Next, divide $\bar{C}(e_n)$ by an acyclic ideal consisting of all the elements of degree $> n$, and a complement of the cocycles in degree n. Inside this quotient DGA, B, we find a sub-DGA $C = \{1, a, da\}$, with a restricting to the fundamental class of S^{n-1}. This gives quisms

$$\bar{C}(Y) \times_{H(S^{n-1})} \bar{C}(e_n) \xrightarrow{\simeq} \bar{C}(Y) \times_{H(S^{n-1})} B \xleftarrow{\simeq} \bar{C}(Y) \times_{H(S^{n-1})} C.$$

Put $C(Y) \times_{H(S^{n-1})} C = R$. The element $(0, da) \in R$ and annihilates R^+. By Lemma 1.6 its cohomology class in $H(R)$ is in the image of (1.5). But clearly the isomorphism $H(R) \cong H(X)$ defined by the equivalences above carries this class to α. ∎

The proposition above gives examples in which our "generalized" Spivak fibre is non-trivial. By contrast we have the

PROPOSITION 1.7. *Suppose X is the infinite product of 1-connected non-trivial spaces X_i (with connexity increasing to ∞ and finite betti numbers over $\Bbbk$). Then*

$$\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X;\Bbbk)) = 0.$$

Proof. In view of the convergent Eilenberg–Moore spectral sequence (Remark 1.3(2)) it is sufficient to show that $\mathrm{Ext}_H(\Bbbk, H) = 0$, when $H^*(X;\Bbbk)$ is an infinite tensor product.

Let $\mathrm{Ext}^i_H(\Bbbk, H)$ denote classical cohomological degree, and note that $\mathrm{Ext}^0_H(\Bbbk, H) = \mathrm{socle}(H) = 0$. Write (for any n) $H = H(1) \otimes \cdots \otimes H(n)$ where the subalgebras $H(j)$ are themselves infinite tensor products. Then $\mathrm{Ext}_H(\Bbbk, H) = \otimes_{j=1}^n \mathrm{Ext}_{H(j)}(\Bbbk, H(j))$ and so $\mathrm{Ext}^i_H(\Bbbk, H) = 0$, $i < n$, and this for any n. ∎

2. *Loop Spaces*

Let X be a 1-connected space with finite betti numbers over $\Bbbk$. Loop space multiplication in ΩX makes $C_*(\Omega X;\Bbbk)$ into a DGA. Thus we obtain the homotopy invariant $\mathrm{Ext}_{C_*(\Omega X;\Bbbk)}(\Bbbk, C_*(\Omega X;\Bbbk))$ in analogy with the "generalized Spivak fibre" of the previous section. In fact we have the

THEOREM 2.1. *If X is as above, there is an isomorphism*

$$\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X;\Bbbk)) \cong \mathrm{Ext}_{C_*(\Omega X;\Bbbk)}(\Bbbk, C_*(\Omega X;\Bbbk)).$$

Proof. Let R be a DGA equivalent to $C^*(X;\Bbbk)$ but satisfying: $R^0 = \Bbbk$, $R^1 = 0$ and each R^i is of finite dimension. Denote by ΩR the dual of the (reduced) bar construction on R (cf., e.g., [14]) and note that according to Adams [1], $C_*(\Omega K;\Bbbk)$ is equivalent to ΩR.

It is thus sufficient to establish the formula of the proposition with $C_*(\Omega X;\Bbbk)$ and $C^*(X;\Bbbk)$ replaced, respectively, by ΩR and by R.

Consider the acylcic bar construction, $R \otimes BR$, on the DGA, R. It is a left R-module and a right BR-commodule and the projection $R \otimes BR \to^{\rho} \Bbbk$ (resp. the inclusion $\Bbbk \to^{\eta} R \otimes BR$) are equivalences respectively of R-modules and of BR comodules. In particular, $R \otimes BR \to^{\rho} \Bbbk$ is an R-semifree resolution, and so

$$\mathrm{Ext}_R(\Bbbk, R) = H(\mathrm{Hom}_R(R \otimes BR, R)).$$

Denote by $\mathrm{End}_{R\otimes BR}(R\otimes BR)$ the space of $\Bbbk$-linear self-maps of $R\otimes BR$ which are maps both of R-modules and BR-comodules. Since the differential, d, in $R\otimes BR$ respects both structures it follows that the differential $\delta=[d, -]$ in $\mathrm{End}(R\otimes BR)$ restricts to a differential in $\mathrm{End}_{R\otimes BR}(R\otimes BR)$.

On the other hand, an isomorphism $\mathrm{Hom}_R(R\otimes BR, R)\xrightarrow{\cong}\mathrm{End}_{R\otimes BR}(R\otimes BR)$ is given by $f\mapsto (f\otimes \mathrm{id})\circ(\mathrm{id}\otimes\Delta)$, Δ the diagonal in BR. Since Δ respects the differentials, this is an isomorphism of differential spaces.

Finally, note that $(R\otimes BR)^\vee=\Omega R\otimes R^\vee$ is a left ΩR-module and a right $R^\vee$-comodule. Moreover, dualizing provides an isomorphism

$$\vee : (R\otimes BR, d)\xrightarrow{\cong}(\Omega R\otimes R^\vee, d^\vee),$$

and so we get isomorphisms of differential vector spaces:

$$\begin{aligned}\mathrm{Hom}_R(R\otimes BR, R)&\xrightarrow{\cong}(\mathrm{End}_{R\otimes BR}(R\otimes BR), [d,\ \ ])\\ &\xrightarrow[\vee]{\cong}(\mathrm{End}_{\Omega R\otimes R^\vee}(\Omega R\otimes R^\vee), [d^\vee,\ \ ])\\ &\xrightarrow{\cong}\mathrm{Hom}_{\Omega R}(\Omega R\otimes R^\vee, \Omega R).\end{aligned}$$

The filtration of $\Omega R\otimes R^\vee$ by the submodules $\Omega R\otimes R^\vee_{\leqslant i}$ exhibits $\Omega R\otimes R^\vee$ as an ΩR-semifree module; since $\eta^\vee$: $\Omega R\otimes R^\vee\to\Bbbk$ is an ΩR-equivalence, it is an ΩR-semifree resolution. Passing to homology now yields the desired result. ∎

As an application, consider the case $\Bbbk=\mathbb{Q}$. Then ([15]) $H_*(\Omega X;\mathbb{Q})$ is the universal enveloping algebra UL_X on the graded Lie algebra $L_X=\pi_*(\Omega X)\otimes\mathbb{Q}$. Let R (the *radical*) be the sum of the solvable ideals in L_X.

In [8] it is shown that if $\mathrm{Ext}_{UL_X}(\mathbb{Q}, UL_X)\neq 0$ then R is finite dimensional. On the other hand, because UL_X has finite type, the Eilenberg–Moore spectral sequence from $\mathrm{Ext}_{UL_X}(\mathbb{Q}, UL_X)$ to $\mathrm{Ext}_{C_*(\Omega X:\mathbb{Q})}(\mathbb{Q}, C_*(\Omega X:\mathbb{Q}))$ is convergent (Remark 1.3(2)). There follows from Theorem 2.1

THEOREM 2.2. *Suppose X is simply connected with finite rational betti numbers, and assume*

$$\mathrm{Ext}^{C^*(X:\mathbb{Q})}(\mathbb{Q}, C^*(X:\mathbb{Q}))\neq 0.$$

Then the radical of L_X is finite dimensional.

COROLLARY 2.3. *If $X=Y\cup e_n$ and e_n represents a non-trivial class in $H_n(X:\mathbb{Q})$ then the radical of L_X is finite dimensional.*

Proof. Apply Proposition 1.6. ∎

3. *Gorenstein Spaces*

Recall ([4, 9]) that a commutative (ungraded) ring R augmented to $\Bbbk$ is called a *Gorenstein ring* if $\mathrm{Ext}_R(\Bbbk, R)$ is concentrated in a single degree and has $\Bbbk$-dimension one. Using differential Ext we extend this to arbitrary (not necessarily commutative) augmented $\Bbbk$-DGAs $R \to \Bbbk$ and make the

DEFINITION. (1) An augmented $\Bbbk$-DGA, R, is *Gorenstein* if $\mathrm{Ext}_R(\Bbbk, R)$ is concentrated in a single degree and has $\Bbbk$-dimension one.

(2) *A* pointed topological space, X, is *Gorenstein over* $\Bbbk$ if the DGA $C^*(X; \Bbbk)$ is Gorenstein.

Gorenstein rings were introduced because of certain duality properties. Analogously, our introduction of Gorenstein spaces is motivated by the fact that they generalize the notion of Poincaré duality space. Indeed, let $\mathbb{Z}_p$ denote the prime field of characteristic p (p prime or zero). Then we have

THEOREM 3.1. *Suppose X is a simply connected CW complex with finite Lusternik–Schnirelmann category and finite betti numbers over $\mathbb{Z}_p$. Then the following conditions are equivalent*:

(i) *X is Gorenstein over $\mathbb{Z}_p$.*

(ii) *$H^*(X{:}\ \mathbb{Z}_p)$ is Gorenstein.*

(iii) *$H^*(X{:}\ \mathbb{Z}_p)$ is finite dimensional and satisfies Poincaré duality.*

If moreover, X is finite then these conditions are equivalent to

(iv) *the Spivak fibre, F_X, for X localizes to a sphere*: $(F_X)_{(p)} = S^{k-1}_{(p)}$.

Note that this contains the p-local analogue of a result of Spivak [17].

Before proving 3.1 we make various elementary observations which are obvious given the convergence of the Eilenberg–Moore spectral sequence (Remark 1.3(2)).

PROPOSITION 3.2. (i) *If $R \xrightarrow{\simeq} S$ is a quism of augmented $\Bbbk$-DGAs then R is Gorenstein if and only if S is.*

(ii) *If R is an augmented $\Bbbk$-DGA for which each R^i (or R_i) is finite dimensional, then*

$$H(R) \text{ Gorenstein} \Rightarrow R \text{ Gorenstein}.$$

(iii) *If X is a (pointed) topological space then*

$$H^*(X; \Bbbk) \text{ Gorenstein} \Rightarrow X \text{ Gorenstein over } \Bbbk.$$

(iv) *If X is a simply connected topological space with finite betti numbers over* $\Bbbk$ *then*

$$H_*(\Omega X: \Bbbk) \text{ Gorenstein} \Rightarrow X \text{ Gorenstein over } \Bbbk. \quad \blacksquare$$

The following examples show that several of the implications in Proposition 3.2 cannot be reversed.

EXAMPLES 3.3. (1) If X is a finite simply connected complex then $H^*(X: \mathbb{Z}_p)$ is Gorenstein if and only if X is Gorenstein (Theorem 3.1). For infinite X, one implication is always true (Proposition 3.2(iii)) but the other may fail:

Let X be the space with rational minimal model of the form $\Lambda(u_1, u_2, u_3, v)$, $|u_i| = 3$, $du_i = 0$, $|v| = 8$, $dv = u_1 u_2 u_3$. Then X is Gorenstein (Proposition 3.4). But the cocycles $u_1 u_2 v^k$ represent infinitely many linearly independent cohomology classes $\alpha_k \in H^*(X: \mathbb{Q})$ satisfying $\alpha_k \cdot H^+(X: \mathbb{Q}) = 0$. Thus $H^*(X; \mathbb{Q})$ is not Gorenstein (Lemma 1.6).

(2) It is possible for $C_*(\Omega X; \Bbbk)$ to be Gorenstein even if $H_*(\Omega X; \Bbbk)$ is not. In fact let $X = (S^3 \times S^3 \times S^3) \# (S^3 \times S^3 \times S^3)$. Then X is a closed manifold and so X is Gorenstein over $\Bbbk$ for all $\Bbbk$. Thus (equivalently by Theorem 2.1) $C_*(\Omega X; \Bbbk)$ is Gorenstein for all $\Bbbk$, but $H_*(\Omega X; \mathbb{Q})$ is not Gorenstein.

Indeed it is immediate from [13] that if $Y = (S^3 \times S^3 \times S^3) \vee (S^3 \times S^3 \times S^3) \vee S^8$ then $\pi_*(\Omega X) \otimes \mathbb{Q} \cong \pi_*(\Omega Y) \otimes \mathbb{Q}$ as graded Lie algebras. Hence $H_*(\Omega X; \mathbb{Q}) \cong H_*(\Omega Y; \mathbb{Q})$. But Y is not Gorenstein over $\mathbb{Q}$ (Theorem 3.1) and so $H_*(\Omega Y; \mathbb{Q})$ is not Gorenstein either (Proposition 3.2(iv)).

One class of examples with $H_*(\Omega X; \mathbb{Q})$ Gorenstein is provided by the

PROPOSITION 3.4. *Let X be 1-connected and suppose $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional. Then $H_*(\Omega X: \mathbb{Q})$ is Gorenstein and so X is Gorenstein over $\mathbb{Q}$.*

Proof. The second assertion follows from the first, via Proposition 3.2(iv). For the first write $L = \pi_*(\Omega X) \otimes \mathbb{Q}$—it is a finite dimensional graded Lie algebra whose universal enveloping algebra, UL, is isomorphic with $H_*(\Omega X; \mathbb{Q})$.

Fix an element $\alpha \in L$ of maximum degree. Since α is central in L the E_2-term of the Hochschild Serre spectral sequence converging to $\mathrm{Ext}_{\mathrm{UL}}(\mathbb{Q}, \mathrm{UL})$ has the form $\mathrm{Ext}_{\mathrm{UL}/\alpha}(\mathbb{Q}, \mathrm{UL}/\alpha) \otimes \mathrm{Ext}_{U(\alpha)}(\mathbb{Q}, U(\alpha))$.

A short calculation shows that $\mathrm{Ext}_{U(\alpha)}(\mathbb{Q}, U(\alpha))$ is one dimensional, and the proposition follows by induction on dim L. $\blacksquare$

On the other hand, it is not clear how to extend the Proposition to finite fields, given

EXAMPLE 3.5. For $n \geqslant 3$, $K(\mathbb{Z}, n)$ is not Gorenstein over $\mathbb{Z}_p$, p finite. In fact, since its mod p cohomology is an infinite tensor product, it follows exactly as in Proposition 1.8 that $\operatorname{Ext}_{C^*(K(\mathbb{Z},n);\mathbb{Z}_p)}(\mathbb{Z}_p, C^*(K(\mathbb{Z}, n); \mathbb{Z}_p)) = 0$.

We come now to the proof of Theorem 3.1. Since X has finite LS category, the DGA $C^*(X; \mathbb{Z}_p)$ has finite right M-cat, as is shown in [14]. Thus the equivalence of (i)–(iii) is simply a translation of Theorem 3.6 below.

The last assertion in Theorem 3.1 then follows from Corollary 1.2.

THEOREM 3.6. *Let $R = R^*$ be a DGA such that $H^0(R) = \Bbbk$, $H^1(R) = 0$, each $H^i(R)$ is finite dimensional and $H(R)$ is graded commutative. Assume also that right M-*cat(R) *is finite. Then the following are equivalent:*

(i) *R is Gorenstein.*

(ii) *$H(R)$ is Gorenstein.*

(iii) *$H(R)$ is finite dimensional and satisfies Poincaré duality.*

Proof. We may replace R by a free model, and so reduce to the case $R^0 = \Bbbk$, $R^1 = 0$ and each R^i is finite dimensional. That (ii) $\Rightarrow$ (i) follows from Proposition 3.2(ii).

To prove (i) $\Rightarrow$ (iii) we note that $\operatorname{Ext}_R(\Bbbk, R) = \operatorname{Tor}^R(\Bbbk, R^\vee)^\vee$, and so if (i) holds $\dim_\Bbbk \operatorname{Tor}^R(\Bbbk, R^\vee) = 1$. By Theorem A.12, $R^\vee$ admits a minimal R-semifree resolution, P. Since $\operatorname{Tor}^R(\Bbbk, R^\vee)$ then coincides with $\Bbbk \otimes_R P$, it follows that P has a single generator z; i.e., $P = R \cdot z$.

This shows that $H(R^\vee)$ is concentrated in degrees $\geqslant \deg z$. Since it is also concentrated in degrees $\leqslant 0$ it is finite dimensional. Moreover, the induced isomorphism $H(R) \xrightarrow{\cong} H(R^\vee)$ of $H(R)$-modules is precisely Poincaré duality.

Finally, if (iii) holds then $H(R^\vee)$ is a free $H(R)$-module on one generator, z, and so $\operatorname{Tor}^{H(R)}(\Bbbk, H(R^\vee)) = \operatorname{Tor}^{H(R)}(\Bbbk, H(R) \cdot z)$ has dimension one. Thus $H(R)$ is Gorenstein and (iii) $\Rightarrow$ (ii). ∎

4. *Fibrations*

In this section we work exclusively with vector spaces over $\mathbb{Q}$. We consider fibrations $F \to E \to B$ such that

(4.1) B is simply connected and $H^*(B; \mathbb{Q})$ has finite type

(4.2) $H^*(F; \mathbb{Q})$ is finite dimensional and $H^1(F; \mathbb{Q}) = 0$.

Our main result is the

THEOREM 4.3. *Under the hypotheses* (4.1) *and* (4.2),

$$\begin{aligned}&\operatorname{Ext}_{C_*(E;\mathbb{Q})}(\mathbb{Q}, C^*(E;\mathbb{Q}))\\ &\quad\cong \operatorname{Ext}_{C^*(B;\mathbb{Q})}(\mathbb{Q}, C^*(B;\mathbb{Q}))\otimes \operatorname{Ext}_{C^*(F;\mathbb{Q})}(\mathbb{Q}, C^*(F;\mathbb{Q})).\end{aligned}$$

In particular, E is Gorenstein over $\mathbb{Q}$ if and only if B is Gorenstein over $\mathbb{Q}$ and $H^*(F;\mathbb{Q})$ satisfies Poincaré duality.

REMARKS AND EXAMPLES 4.4. (1) When F, E, B are simply connected finite complexes the theorem asserts that the $\mathbb{Q}$-localization of the Spivak fibre of E is the join of the $\mathbb{Q}$-localization of the Spivak fibres of F and B. This is the $\mathbb{Q}$-local analogue of a result of Gottlieb [10].

(2) The analogue of Theorem 4.3 for local rings (arbitrary characteristic) is a result of Avramov, Foxby, and Halperin [3].

(3) In the situation of a trivial fibration $E\simeq F\times B$ of simply connected spaces with finite betti numbers, Theorem 4.3 holds in arbitrary characteristic and without the finiteness assumption on F, as follows from [6; Prop. 16.3].

(4) In the fibration $\Omega(S^3\vee S^3)\to pt\to S^3\vee S^3$, the fibre is an infinite product. Thus although the total space is Gorenstein over any field neither the fibre (Proposition 1.8) nor the base (Theorem 3.1) is.

(5) Let F be the three connected cover of $(S^3\times S^3)\#(S^3\times S^3)$. In the fibration $(\mathbb{C}P^\infty)^4\to F\to(S^3\times S^3)\#(S^3\times S^3)$ both the fibre (Proposition 3.2(iii)) and the base (Theorem 3.1) are Gorenstein over any field.

On the other hand, a standard calculation identifies F (up to rational homotopy type) as an infinite wedge of spheres. Hence F is not Gorenstein over $\mathbb{Q}$ (Proposition 1.6).

Before starting the proof of Theorem 4.3 we deduce one corollary:

COROLLARY 4.5. *Suppose X is a finite* 1*-connected CW complex. The following are then equivalent.*

(1) *The rational cohomology of the Spivak fibre, F_X, is finite dimensional.*

(2) *F_X, localized at* 0, *is a sphere $S^n_{(0)}$.*

(3) *$H^*(X;\mathbb{Q})$ satisfies Poincaré duality.*

Proof. In view of Sections 1 and 3 we have only to check that (1) implies that X is Gorenstein over $\mathbb{Q}$. Write $X\simeq N$ where $N\subset\mathbb{R}^{n+k}$ is a manifold with boundary ∂N. Since $H^*(\partial N;\mathbb{Q})$ satisfies Poincaré duality, we can apply Theorem 4.3 to the fibration $F_X\to\partial N\to X$. ∎

The first step in the proof of Theorem 4.3 is to represent $\pi: E \to B$ by a KS-extension $R \to S = R \otimes \Lambda X$ as in [12, p. 254]. Then $S/\!/R = \mathbb{Q} \otimes_R S$ is the quotient model $(\Lambda X, \bar{d})$ and it is a Sullivan model for F by [12, Chap. 20]. We may suppose that R and S have finite type, that $R^0 = S^0 = \mathbb{Q}$, and that $R^1 = 0$.

The Sullivan model of a space and the DGA $C^*(\ ;\mathbb{Q})$ are naturally connected by quisms [12, Chap. 14]. Thus (Remark 1.3(1)) we may replace $C^*(\ ;\mathbb{Q})$ by the CGDA R (resp. S, $S/\!/R$) in the case of B (resp. E, F). Since these DGAs have finite type, the theorem is equivalent to

$$\operatorname{Tor}^S(S^\vee, \mathbb{Q}) \cong \operatorname{Tor}^R(R^\vee, \mathbb{Q}) \otimes \operatorname{Tor}^{S/\!/R}(S/\!/R^\vee, \mathbb{Q}). \tag{4.6}$$

We now outline the proof. The left hand side of (4.6) is the homology $H(S^\vee \otimes_S \bar{S})$ for any acyclic closure, $\bar{S}$, of S. Consider the map

$$\alpha: \operatorname{Hom}_R(S, R) \otimes_R \operatorname{Hom}(R, \mathbb{Q}) \to \operatorname{Hom}(S, \mathbb{Q})$$

given by $\alpha: f \otimes g \to g \circ f$. Note that because S and R are graded commutative the potentially different structures of R-module (or S-module) all coincide. The first step in the proof is

LEMMA 4.7. *α is an equivalence of S modules. In particular, since $\bar{S}$ is S-semifree $\alpha \otimes_S \bar{S}$ is an equivalence, and*

$$H([\operatorname{Hom}_R(S, R) \otimes_S \bar{S}] \otimes_R R^\vee) \cong H(S^\vee \otimes_S \bar{S}) = \operatorname{Tor}^S(S^\vee, \mathbb{Q}).$$

For the next step we need an acyclic closure $\bar{S}$ in a certain form. Choose first a Sullivan model $R \to \bar{R} = R \otimes \Lambda X_R \xrightarrow{\simeq} \mathbb{Q}$ for the augmentation $R \to \mathbb{Q}$. Then the augmentation $S \otimes_R \bar{R} = S \otimes \Lambda X_R \to \mathbb{Q}$ has a Sullivan model of the form $S \otimes \Lambda X_R \to (S \otimes \Lambda X_R) \otimes \Lambda Y \xrightarrow{\simeq} \mathbb{Q}$ and $\bar{S} = S \otimes \Lambda X_R \otimes \Lambda Y$ is an acyclic closure for S. Our next step is (with $\bar{S}$ as constructed above) the

LEMMA 4.8. (1) *There is a flat equivalence of $\bar{R}$-modules*:

$$(\bar{R} \otimes H(\operatorname{Hom}_R(S, R) \otimes_R \bar{R}), d_{\bar{R}} \otimes \mathrm{id}) \xrightarrow{\simeq} \operatorname{Hom}_R(S, R) \otimes_R \bar{R}.$$

(2) *There is a flat equivalence of $\bar{R}$-modules*

$$(\bar{R} \otimes H(\operatorname{Hom}_R(S, R) \otimes_S \bar{S}), d_{\bar{R}} \otimes \mathrm{id}) \xrightarrow{\simeq} \operatorname{Hom}_R(S, R) \otimes_S \bar{S}.$$

The last step is (with $\bar{S}$ as constructed above)

LEMMA 4.9. $H(\operatorname{Hom}_R(S, R) \otimes_S \bar{S}) = \operatorname{Tor}^{S/\!/R}((S/\!/R)^\vee, \mathbb{Q})$.

Indeed, given these three Lemmas we deduce (4.6) as follows:

$$\begin{aligned}\mathrm{Tor}^S(S^\vee, \mathbb{Q}) &\overset{(4.7)}{=} H([\mathrm{Hom}_R(S, R)\otimes_S \bar{S}]\otimes_R R^\vee)\\ &\overset{4.8(2)}{=} H(\bar{R}\otimes H(\mathrm{Hom}_R(S, R)\otimes_S \bar{S})\otimes_R R^\vee)\\ &= H(\bar{R}\otimes_R R^\vee)\otimes H(\mathrm{Hom}_R(S, R)\otimes_S \bar{S})\\ &\overset{(4.9)}{=} \mathrm{Tor}^R(R^\vee, \mathbb{Q})\otimes \mathrm{Tor}^{S/\!/R}((S/\!/R)^\vee, \mathbb{Q}).\end{aligned}$$

It remains to prove the three lemmas.

We begin by observing that since S is concentrated in degrees $\geqslant 0$, it follows that S admits a minimal R-semifree resolution $\phi\colon P \xrightarrow{\simeq} S$. Since S itself is R-semifree we obtain an R-module equivalence $\psi\colon S \xrightarrow{\simeq} P$ such that $\phi\psi \sim_R \mathrm{id}_P$ and $\psi\phi \sim_R \mathrm{id}_S$.

It follows that for each R-module, M, the morphisms $\phi^*\colon \mathrm{Hom}_R(P, M) \leftarrow \mathrm{Hom}_R(S, M)$ and $\psi^*\colon \mathrm{Hom}_R(S, M) \leftarrow \mathrm{Hom}_R(P, M)$ also satisfy $\phi^*\psi^* \sim_R \mathrm{id}$ and $\psi^*\phi^* \sim_R \mathrm{id}$. In particular, they are R-flat equivalences.

On the other hand, because P is minimal its underlying $R_\#$-module, $P_\#$ satisfies $P_\# \cong R_\# \otimes \mathrm{Tor}^R(P, \mathbb{Q})$. But $\mathrm{Tor}^R(P, \mathbb{Q}) = \mathrm{Tor}^R(S, \mathbb{Q}) = H^*(F; \mathbb{Q})$ and so

$$P_\# \cong R_\# \otimes H^*(F; \mathbb{Q}). \tag{4.10}$$

Choose n so that $H^i(F; \mathbb{Q})$ is zero for $i \notin [0, n]$. Because $R^1 = 0$ we have

$$d_P\colon H^i(F; \mathbb{Q}) \to R_\# \otimes H^{<i}(F; \mathbb{Q}).$$

Thus filtering $\mathrm{Hom}_R(P, R)$ by the submodules of functions vanishing on $H^{\leqslant i}(F; \mathbb{Q})$, we exhibit $\mathrm{Hom}_R(P, R)$ as R-semifree. Thus we obtain

LEMMA 4.11. (i) *ϕ^* and ψ^* are R-flat equivalences.*
(ii) *$\mathrm{Hom}_R(S, R)$ is R-flat.*

We now prove our three lemmas.

Proof of 4.7. With the aid of the R-flat equivalence ϕ^* of Lemma 4.11, 4.7 is easily reduced to showing that

$$\alpha_P\colon \mathrm{Hom}_R(P, R)\otimes_R \mathrm{Hom}(R, \mathbb{Q}) \to \mathrm{Hom}(P, \mathbb{Q})$$

induces an isomorphism of homology.

But it is immediate from (4.10) and the finite dimensionality of $H^*(F; \mathbb{Q})$ that α_P, itself, is an isomorphism. ∎

Proof of 4.8. (1) Since $\mathrm{Hom}_R(S, R)$ is R-flat, $\mathrm{Hom}_R(S, R)\otimes_R \bar{R}$ is $\bar{R}$-flat. Thus there is an $\bar{R}$-flat equivalence $Q \xrightarrow{\simeq} \mathrm{Hom}_R(S, R)\otimes_R \bar{R}$ where Q is $\bar{R}$-semifree. Since $\bar{R}$ is acyclic, we may suppose that $Q \cong (\bar{R}, d_{\bar{R}})\otimes(H(Q), 0)$ as $\bar{R}$-modules. Moreover, the equivalence $Q \xrightarrow{\simeq} \mathrm{Hom}_R(S, R)\otimes_R \bar{R}$ identifies $H(Q)$ and $H(\mathrm{Hom}_R(S, R)\otimes_R \bar{R})$. ▮

Proof of 4.8. (2) As in 4.8(1) it suffices to prove that $\mathrm{Hom}_R(S, R)\otimes_S \bar{S}$ is $\bar{R}$-flat. Recall that $\bar{S}$ is a KS-extension of the CGDA $T = S\otimes_R \bar{R} = S\otimes \Lambda X_R$. Thus $\bar{S}$ is, in particular, T-flat. Since $\mathrm{Hom}_R(S, R)\otimes_S \bar{S} = (\mathrm{Hom}_R(S, R)\otimes_S T)\otimes_T \bar{S}$, and since $\bar{S}$ is T-flat we need only show that $\mathrm{Hom}_R(S, R)\otimes_S T$ is $\bar{R}$-flat. But $\mathrm{Hom}_R(S, R)\otimes_S T = \mathrm{Hom}_R(S, R)\otimes_S S\otimes_R \bar{R} = \mathrm{Hom}_R(S, R)\otimes_R \bar{R}$, which is $\bar{R}$-flat by 4.8(1). ▮

Proof of 4.9. Put $M = \mathrm{Hom}_R(S, R)$, and let $W \xrightarrow{\omega} M$ be an S-semifree resolution. Since S is R-semifree, this is also an R-semifree resolution, and since M is R-flat (Lemma 4.11) it follows that ω is an R-flat equivalence. In particular it follows that

$$\omega\otimes \mathrm{id}\colon W\otimes_R \mathbb{Q} \to M\otimes_R \mathbb{Q}$$

is an $S/\!/R$-semifree resolution.

Now we have

$$\begin{aligned} H(M\otimes_S \bar{S}) &= \mathrm{Tor}^S(M, \mathbb{Q}) = H(W\otimes_S \mathbb{Q}) \\ &= H(W\otimes_R \mathbb{Q}\otimes_{S/\!/R} \mathbb{Q}) \\ &= \mathrm{Tor}^{S/\!/R}(M\otimes_R \mathbb{Q}, \mathbb{Q}). \end{aligned}$$

Thus it remains to construct an equivalence $M\otimes_R \mathbb{Q} \simeq (S/\!/R)^\vee$ of $S/\!/R$-modules.

Consider the morphism $\mathrm{Hom}_R(S, R) \to \mathrm{Hom}(S/\!/R, \mathbb{Q})$ of S-modules obtained by composing $f\colon S \to R$ with the augmentation of R. It factors to yield an $S/\!/R$-morphism

$$\mathrm{Hom}_R(S, R)\otimes_R \mathbb{Q} \to \mathrm{Hom}(S/\!/R, \mathbb{Q}), \tag{4.12}$$

and we have to show this induces an isomorphism of homology. But we may use the equivalence $\phi\colon P \xrightarrow{\simeq} S$ (cf. Lemma 4.11) to replace (4.12) by $\mathrm{Hom}_R(P, R)\otimes_R \mathbb{Q} \to \mathrm{Hom}(P\otimes_R \mathbb{Q}, \mathbb{Q})$, and this map is an isomorphism. ▮

5. *Formal Dimension*

Definition. Let X be a pointed topological space. Its *formal dimension over* $\Bbbk$, $\mathrm{fd}(X; \Bbbk)$ is defined by

$$\mathrm{fd}(X; \Bbbk) = \sup\{r \in \mathbb{Z} \mid [\mathrm{Ext}_{C^*(X;\Bbbk)}(k, C^*(X; \Bbbk))]^r \neq 0\}.$$

If $\mathrm{Ext}_{C^*(X;\Bbbk)}(\Bbbk, C^*(X; \Bbbk)) = 0$ we put $\mathrm{fd}(X; \Bbbk) = -\infty$.

PROPOSITION 5.1. *If $H^*(X; \Bbbk)$ is finite dimensional and simply connected, then*

$$\mathrm{fd}(X; \Bbbk) = \sup\{r \mid H^r(X; \Bbbk) \neq 0\}.$$

Proof. The DGA $C^*(X; \Bbbk)$ is equivalent through quisms to a finite dimensional DGA, R, concentrated in degrees $i \in [0, n]$, where $n = \sup\{r \mid H^r(X; \Bbbk) \neq 0\}$. Since an R-semifree resolution, P, of $\Bbbk$ will be concentrated in degrees $\geqslant 0$, the complex $\mathrm{Hom}_R(P, R)$ is concentrated in degrees $\leqslant n$.

The same is therefore true for $\mathrm{Ext}_R(\Bbbk, R)$. Moreover, Lemma 1.6 shows that $[\mathrm{Ext}_R(\Bbbk, R)]^n \to H^n(R)$ is surjective, and so $\mathrm{fd}(X; \Bbbk) = n$. ∎

PROPOSITION 5.2. *If $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional and X is simply connected then*

$$\mathrm{fd}(X; \mathbb{Q}) = \sum_{|x_i| \text{ odd}} |X_i| - \sum_{|x_i| \text{ even}} (|x_i| - 1),$$

where x_i is a basis of $\pi^*(X) \otimes \mathbb{Q}$.

Proof. This follows from the proof of Proposition 3.4 via the fact that if (α) is a 1-dimensional Lie algebra then $\mathrm{Ext}_{U(\alpha)}(\mathbb{Q}, U(\alpha))$ is concentrated in the degree $-|\alpha|$ (if $|\alpha|$ is odd) and $|\alpha| + 1$ if $|\alpha|$ is even. ∎

From Theorem 4.3 there follows

PROPOSITION 5.3. *Let $F \to E \to B$ be a fibration satisfying* (4.1) *and* (4.2). *Then*

$$\mathrm{fd}(E; \mathbb{Q}) = \mathrm{fd}(F; \mathbb{Q}) + \mathrm{fd}(B; \mathbb{Q})$$

(*note that the right hand side of* 5.3 *is well defined because of Proposition* 5.1.)

As an immediate consequence of Proposition 1.6 we have

PROPOSITION 5.4. *Suppose* $X = Y \bigcup_f e_n$, *where* e_n $(n \geqslant 2)$ *carries a non-trivial class in* $H_n(X; \Bbbk)$. *Then* $\mathrm{fd}(X; \Bbbk) \geqslant n$.

5.5. EXAMPLES. (1) $\mathrm{fd}(\mathbb{C}P^\infty; \Bbbk) = -1$ for all $\Bbbk$.

(2) $\mathrm{fd}(BU(n); \Bbbk) = -n^2$ for all $\Bbbk$.

APPENDIX: DIFFERENTIAL HOMOLOGICAL ALGEBRA

A.1. *Conventions*

All vector spaces are defined over a fixed field $\Bbbk$ and the unadorned $\otimes$ and Hom mean with respect to $\Bbbk$. Gradations are written either as superscripts or as subscripts, with the convention $M^k = M_{-k}$, and we say M has *finite type* if each M^k is finite dimensional. We denote $\mathrm{Hom}(M, \Bbbk)$ by $M^\vee$: $(M^\vee)_p = \mathrm{Hom}(M^p, \Bbbk)$. If x is an object in a graded space, $|x|$ is the absolute value of its degree.

Differential graded algebras (DGAs) R are assumed to be either of the form R^* $(R^{<0} = 0)$ or R_* $(R_{<0} = 0)$ with differential d of upper (resp. lower) degree 1 (resp. -1). Following Moore, we denote the underlying graded algebra by $R_\#$. A CGDA is a DGA whose elements satisfy $ab = (-1)^{|a||b|} ba$ and $a^2 = 0$ if $|a|$ is odd.

If R is a DGA a (left) *R-module* M is a $\mathbb{Z}$-graded $R_\#$-module, $M_\#$, together with a differential in $M_\#$ of upper (resp. lower) degree 1 (resp. -1) satisfying $d(r \cdot m) = dr \cdot m + (-1)^{|r|} r \cdot dm$. In particular, $M^\vee = \mathrm{Hom}(M; \Bbbk)$ is a right module via $(f \cdot r)(m) = f(rm)$, and $(df)(m) = (-1)^{|f|+1} f(dm)$.

A *morphism* of R-modules $f\colon M \to N$ is a $\Bbbk$ linear map of some degree k such that $f(rm) = (-1)^{|f||r|} r \cdot f(m)$ and $f(dm) = (-1)^{|f|} df(m)$. Two morphisms f and g are *R-homotopic* via a *homotopy*, h, if $f - g = dh - (-1)^{|h|} hd$ where $h\colon M \to N$ satisfies $h(rm) = (-1)^{|h||r|} rh(m)$. In this case we write $f \sim_R g$, which implies $H(f) = H(g)$.

A DGA morphism inducing a homology isomorphism is called a *quism and an* R-module morphism of degree zero inducing a homology isomorphism is called an *equivalence of R-modules*. In either case we indicate this property by $\xrightarrow{\simeq}$.

A.2. *Differential* Tor *and* Ext

In this section we set out the basic definitions and results we need from differential homological algebra. The proofs are all simple exercises using standard techniques, and are omitted.

Following [2] we call an R-module *R-free* it is free as an $R_{\#}$-module on a basis of cycles, and we call it *R-semifree* if it is the increasing union of submodules $0 = F_{-1} \subset F_0 \cdots$ such that each F_i/F_{i-1} is *R-free.*

Suppose $R^0 = \Bbbk$ and R is augmented to $\Bbbk$. Then we say an R-semifree-module, P, is *minimal* if the differential in $\Bbbk \otimes_R P$ is zero.

LEMMA A.3. *Let R be a DGA and suppose M is an R-module.*

(i) *There exists an equivalence $P \xrightarrow{\simeq} M$ from an R-semifree-module P.*

(ii) *If $R = R^*$ with $R^0 = \Bbbk$ and $R^1 = 0$ and if $M = M^{\geqslant r}$ (some $r \in \mathbb{Z}$) then P may be chosen minimal.*

(iii) *If $R = R_*$ with $R_0 = \Bbbk$ and if $M = M_{\geqslant r}$ (some $r \in \mathbb{Z}$) then P may be chosen minimal.*

(iv) *Given a diagram*

$$\begin{tikzcd} & N \arrow[d, "g"] \\ P \arrow[r, "f"'] & M \end{tikzcd}$$

of R-modules in which P is R-semifree and g is an equivalence, there exists a morphism $F\colon P \to N$ such that $g \circ F \sim_R f$. If g is surjective we may choose F so that $g \circ F = f$. In any case, F is unique up to homotopy.

DEFINITION. An equivalence $P \xrightarrow{\simeq} M$ of R-modules is called an *R-semifree resolution* (resp. a minimal *R-semifree resolution*) if P is R-semifree (resp. minimal and R-semifree).

We are now in a position to define Moore's functors [16] Tor and Ext in the differential category. We use a slightly different approach (cf. [2]).

DEFINITION. If M and Q are right R-modules and N is a left R-module, and if $P \to M$ is any R-semifree resolution, then

$$\mathrm{Tor}^R(M, N) = H(P \otimes_R N)$$

and

$$\mathrm{Ext}_R(M, Q) = H(\mathrm{Hom}_R(P, Q)).$$

It follows from Lemma A.3 that this definition is independent of the choice of P.

The next result is classical, cf. [16, Theorem 2.3], where it is proved via spectral sequences. With the approach described above it has a simple, elementary proof as well.

PROPOSITION A.4. *Suppose* M, N *(resp.* M', N'*) are* R*-modules (resp.* R'*-modules) and that* $f: R \to R'$, $\alpha: M \to M'$, $\beta: N \to N'$ *are morphisms* (M' *and* N'*) regarded as modules over* R *via* f*. Then*

(i) *There is a natural map*

$$\mathrm{Tor}^f(\alpha, \beta): \mathrm{Tor}^R(M, N) \to \mathrm{Tor}^{R'}(M', N').$$

(ii) *If* f *is a quism and* α, β *are equivalences then* $\mathrm{Tor}^f(\alpha, \beta)$ *is an isomorphism.*

Remark A.5. The analogue of A.4 for Ext (for arbitrary modules over arbitrary DGAs) is true, and has the same elementary proof.

PROPOSITION A.6. *Let* M *be an* R*-module. Then*

(i) $\mathrm{Tor}^R(\Bbbk, M)^\vee \cong \mathrm{Ext}_R(\Bbbk, M^\vee)$.

(ii) *If* R *is a* CGDA *and* N *is another* R*-module then*

$$\mathrm{Tor}^R(M, N) = \mathrm{Tor}^R(N, M).$$

Finally we need the concept of *flatness.*

DEFINITION. An R-module equivalence $\phi: M \to N$ is called *flat* if $H(\phi \otimes_R -): H(M \otimes_R -) \to H(N \otimes_R -)$ is always an isomorphism.

An R-module, M is *flat* if it admits an R-semifree resolution $P \to^\pi M$ which is a flat equivalence.

Let M be a right R-module and let $P \xrightarrow[\simeq]{\pi} M$ be an R-semifree resolution. The map

$$H(\pi \otimes -): \mathrm{Tor}^R(M, -) \to H(M \otimes_R -)$$

is independent of the choice of resolution, by Lemma A.3(iv).

PROPOSITION A.7. *The following are equivalent*:

(i) M *is* R*-flat.*

(ii) *Every* R*-semifree resolution of* M *is a flat equivalence.*

(iii) *The map* $\mathrm{Tor}^R(M, -) \to H(M \otimes_R -)$ *is always an isomorphism.*

(iv) $M \otimes_R -$ *preserves equivalences.*

In particular an R*-semifree-module is* R*-flat.*

PROPOSITION A.8. *If* M *is the increasing union of* R*-submodules* $0 = F_{-1} \subset F_0 \subset \cdots$ *such that each* F_{i+1}/F_i *is* R*-flat and* $R_\#$*-flat, then* M *is* R*-flat.*

A.9. *Free DGA Models*

Let $\phi: R \to S$ be a DGA morphism in which $R^0 = S^0 = k$ (or $R_0 = S_0 = k$). As observed in [14] we may factor ϕ in the form $R \to R \amalg T(V) \to^{\pi} S$ in which π is a quism, $(R \amalg T(V))_{\#}$ is the coproduct of $R_{\#}$ and a tensor algebra $T(V)$, and V is the increasing union of subspaces $F_i(V)$, $i \geqslant 0$, such that $d: F_i(V) \to R \amalg T(F_{i-1}(V))$. In particular, $R \amalg T(V) \to^{\pi} S$ is an R-semifree resolution of S.

An analogous construction obtains for CGDA morphisms $\phi: R \to S$, which we factor as $R \to R \otimes \Lambda V \to^{\pi} S$, ΛV denoting the free graded commutative algebra on V. The analogous properties obtain; in particular, π is an R-semifree resolution of S.

When ϕ is an augmentation $R \to \Bbbk$ the DGA model $R \amalg T(V) \to^{\simeq} \Bbbk$ (or $R \otimes \Lambda V \to^{\simeq} \Bbbk$) is called an *acyclic closure* for R. When ϕ is the inclusion $\Bbbk \to R$ the DGA model $T(V) \to^{\simeq} \mathbb{R}$ (resp. $\Lambda V \to^{\simeq} R$) is a *free model* (resp. a *Sullivan model*) for R.

A.10. *Finite Category*

Let R be a DGA such that $R^0 = k$ (or $R_0 = \Bbbk$), and let $T(V) \to^{\simeq} R$ be a free model; we may assume that either $V = V^{>0}$ or $V = V_{>0}$. Fix an integer $m \geqslant 1$ and let

$$\begin{tikzcd} T(V) \arrow[r, "i"] \arrow[dr, "\pi"'] & T(V) \amalg T(W) \arrow[d, "\rho", "\simeq"'] \\ & T(V)/T^{>m}(V) \end{tikzcd}$$

be a free model of the projection π of $T(V)$ onto the $(m+1)$st power, $T^{>m}(V)$, of its maximal ideal.

Recall ([14]) that right (resp. left) M-cat$(R) \leqslant m$ if and only if there is a retraction $r: T(V) \amalg T(W) \to T(V)$ for i which is a morphism of right (resp. left) $T(V)$-modules. For such DGAs we can extend Lemma A.3(ii), which guaranteed minimal R-semifree resolutions for R-modules with $M^i = 0$, $i \leqslant k$, some $k \in \mathbb{Z}$. For convenience we restrict to DGAs for which

$$R = R^*, \qquad R^0 = \Bbbk, \qquad R^1 = 0. \tag{A.11}$$

THEOREM A.12. *Suppose R satisfies* (A.11) *and that right* M-cat$(R) \leqslant m$. *Let M be a left R-module for which* $\operatorname{Tor}^R(\Bbbk\, M)^i = 0$, $i \leqslant k$ *(some $k \in \mathbb{Z}$). Then M admits a minimal R-semifree resolution.*

Remark A.13. The theorem remains true if left and right are respectively everywhere replaced by right and left.

LEMMA A.14. *Under the hypotheses of* A.12, *if* $\operatorname{Tor}^R(\Bbbk, M) = 0$ *then* $H(M) = 0$.

Proof. We may suppose $R=T(V)$ with $V=V^{\geqslant 2}$, and that M is R-semifree. Let $T(V)\to^{i} T(V)\amalg T(W)$ be a model for $T(V)\to T(V)/T^{>m}(V)$ and let $T(V)\amalg T(W)\to^{r} T(V)$ be a right $T(V)$-linear retraction for i. This gives a $\Bbbk$-linear map $r\otimes_{T(V)} M$: $[T(V)\amalg T(W)]\otimes_{T(V)} M\to M$ and $(r\otimes_{T(V)} M)\ (1\otimes u)=u$, $u\in M$.

It is thus sufficient to show that $H([T(V)\amalg T(W)]\otimes_{T(V)} M)=0$. Denote $T(V)/T^{>m}(V)$ by S. Then, because M is $T(V)$-semifree,

$$H([T(V)\amalg T(W)]\otimes_{T(V)} M)=H(T(V)/T^{>m}V\otimes_{T(V)} M)=H(S\otimes_{T(V)} M).$$

On the other hand, since M is $T(V)$-semifree,

$$0=\mathrm{Tor}^{T(V)}(\Bbbk, M)=H(\Bbbk\otimes_{T(V)} M)=H(\Bbbk\otimes_S(S\otimes_{T(V)} M)).$$

It follows that $\Bbbk\otimes_S(S\otimes_{T(V)} M)$ has a basis of the form $\bar{x}_i$, $\bar{d}x_i$, and this is represented by a subset of the form $\{x_i, dx_i\}\subset S\otimes_{T(V)} M$. Because the maximal ideal $\bar{S}\subset S$ satisfies $\bar{S}\cdot\ \cdots\ \cdot\bar{S}=0$ $(m+1)$ factors, it follows that $\{x_i, dx_i\}$ is an $S_{\#}$-basis for $(S\otimes_{T(V)} M)_{\#}$ and hence $0=H(S\otimes_{T(V)} M)=H([T(V)\amalg T(W)]\otimes_{T(V)} M)$, as desired. ∎

Proof of Theorem A.12. We may suppose M is R-semifree. Let $\bar{d}$ denote the differential in $\Bbbk\otimes_R M$. We may suppose that $M_{\#}$ is $R_{\#}$-free on a basis x_j, y_j and h_α such that in $\Bbbk\otimes_R M$, $\bar{d}x_j=y_j$, $\bar{d}h_\alpha=0$, and such that the basis of cycles of each $F_i(M)/F_{i-1}(M)$ is a subset of the x_j, y_j, h_α.

By hypothesis the h_α have degree $\geqslant k+1$. Let $N_{\#}$ be the $R_{\#}$-module on the subset of the basis x_j, y_j, h_α of elements of degree $\leqslant k$, together with the x_j of degree $k+1$. It follows easily from (A.11) that $N_{\#}$ is stable under d, and so defines a submodule $N\subset M$; moreover, N is R-semifree. Finally $\Bbbk\otimes_R N$ consists of only elements x_j, $\bar{d}x_j$ and so $\mathrm{Tor}^R(\Bbbk, N)=H(\Bbbk\otimes_R N)=0$.

Now Lemma A.14 implies that $H(N)=0$ and $M\to M/N$ is an equivalence. Since $(M/N)^i=0$, $i\leqslant k$, it admits a minimal R-semifree resolution by Lemma A.3(ii). ∎

As an easy consequence of Lemma A.14 we also have

THEOREM A.15. *Suppose R satisfies* (A.11) *and right M-*cat(R) *is finite. Then a morphism ϕ: $M\to N$ of left R-modules is an equivalence if and only if* $\mathrm{Tor}^R(\Bbbk, \phi)$ *is an isomorphism.*

REFERENCES

1. J. F. ADAMS, On the cobar construction, *Proc. Nat. Acad. Sci.* **42** (1956), 409–412.
2. L. AVRAMOV AND S. HALPERIN, Through the looking glass: A dictionary between rational homotopy theory and local algebra, *in* "Algebra, Algebraic Topology and Their Interactions," pp. 3–27, Lecture Notes in Mathematics, Vol. 1183, Springer-Verlag, Berlin, 1986.

3. L. Avramov, H. B. Foxby, and S. Halperin, Descent and ascent of local properties along homomorphisms of finite flat dimension, *J. Pure Appl. Algebra* **38** (1985), 167–185.
4. H. Bass, On the uniquity of Gorenstein rings, *Math. Z.* **82** (1963), 8–28.
5. R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré duality, *Invent. Math.* **20** (1973), 103–124.
6. S. Eilenberg and J. C. Moore, Homology and fibrations, *Comment. Math. Helv.* **40** (1966), 199–236.
7. Y. Félix and S. Halperin, Rational L. S. category and its applications, *Trans. Amer. Math. Soc.* **273** (1982), 1–37.
8. Y. Félix, S. Halperin, C. Jacobsson, C. Löfwall, and J. C. Thomas, The radical of the homotopy Lie algebra, *Amer. J. Math.* **109** (1987).
9. D. Gorenstein, An arithmetic theory of adjoint plane curves, *Trans. Amer. Math. Soc.* **72** (1952), 414–436.
10. D. Gothlieb, Poincaré duality and fibrations, *Proc. Amer. Math. Soc.* **76** (1979), 148–150.
11. S. Halperin, Finiteness in the minimal models of Sullivan, *Trans. Amer. Math. Soc.* **230** (1977), 173–199.
12. S. Halperin, Lectures on minimal models. *Mém. Soc. Math. France* (*N.S.*) No. 9/10 (1983).
13. S. Halperin and J. M. Lemaire, Suites inertes dans des algèbres de Lie graduées, *Math. Scand.* **61** (1987), 39–67.
14. S. Halperin and J. M. Lemaire, Notions of category in differential algebra, preprint.
15. J. Milnor and J. C. Moore, On the structure of Hopf algebras, *Ann. of Math. (2)* **81** (1965), 211–264.
16. J. C. Moore, Algèbre homologique et homologie des espaces classifiants, *in* "Séminaire Cartan 1959/1960," exposé 7.
17. M. Spivak, Spaces satisfying Poincaré duality, *Topology* **6** (1967), 77–102.
18. D. Sullivan, Infinitesimal computations in topology, *Inst. Hautes Etudes Sci. Publ. Math.* (1978), 269–331.