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Some results on $(a:b)$-choosability

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ABSTRACT

A solution to a problem of Erdős, Rubin and Taylor is obtained by showing that if a graph G is $(a:b)$-choosable, and $c/d > a/b$, then G is not necessarily $(c:d)$-choosable. Applying probabilistic methods, an upper bound for the kth choice number of a graph is given. We also prove that a directed graph with maximum outdegree d and no odd directed cycle is $(k(d+1):k)$-choosable for every $k \geq 1$. Other results presented in this article are related to the strong choice number of graphs (a generalization of the strong chromatic number). We conclude with complexity analysis of some decision problems related to graph choosability.

1. Introduction

The paper is based on the first author's master's thesis under the supervision of the second author [16]. Many of the recent results on choosability can be found in the survey papers [3,20,19] and their many references.

All graphs considered are finite, undirected and simple. A graph $G = (V, E)$ is $(a:b)$-*choosable* if for every family of sets $\{S(v) : v \in V\}$, where $|S(v)| = a$ for all $v \in V$, there exist subsets $C(v) \subseteq S(v)$, where $|C(v)| = b$ for every $v \in V$, and $C(u) \cap C(v) = \emptyset$ whenever $u, v \in V$ are adjacent. The kth *choice number* of G, denoted by $\mathrm{ch}_k(G)$, is the smallest integer n for which G is $(n:k)$-choosable. A graph $G = (V, E)$ is k-*choosable* if it is $(k:1)$-choosable. The *choice number* of G, denoted by $\mathrm{ch}(G)$, is equal to $\mathrm{ch}_1(G)$.

The concept of $(a:b)$-choosability was defined and studied by Erdős, Rubin and Taylor [11]. In the present paper we present some new results related to that topic. Part of our work generalizes previous results from [2,1,5,11]. We list our results in this introduction section. The detailed proofs are given separately in later sections.

The following theorem examines the behavior of $\mathrm{ch}_k(G)$ when k is large.

Theorem 1.1. *Let G be a graph. For every $\epsilon > 0$ there exists an integer k_0 such that for every $k \geq k_0$, $\mathrm{ch}_k(G) \leq k(\chi(G) + \epsilon)$.*

The following question is stated in [11]:

If G is $(a:b)$-choosable, and $\frac{c}{d} > \frac{a}{b}$, does it imply that G is $(c:d)$-choosable?

The following is a negative answer to this question:

Corollary 1.2. *If $l > m \geq 3$, then there exists a graph G which is $(a:b)$-choosable but not $(c:d)$-choosable, where $\frac{c}{d} = l$ and $\frac{a}{b} = m$.*

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Let K_{m*r} denote the complete r-partite graph with m vertices in each vertex class, and let $K_{m_1,\ldots,m_r}$ denote the complete r-partite graph with m_i vertices in the ith vertex class. It is shown in [1] that there exist two positive constants c_1 and c_2 such that $c_1 r \log m \le \mathrm{ch}(K_{m*r}) \le c_2 r \log m$, for every $m \ge 2$ and $r \ge 2$. The following theorem generalizes the upper bound.

Theorem 1.3. *If $r \ge 1$ and $m_i \ge 2$ for every $1 \le i \le r$, then*

$$\mathrm{ch}_k(K_{m_1,\ldots,m_r}) \le 948r\left(k + \log\frac{m_1 + \cdots + m_r}{r}\right).$$

Logarithms are in the natural base e. Following are two applications of the above.

Corollary 1.4. *For every graph G and $k \ge 1$*

$$\mathrm{ch}_k(G) \le 948\chi(G)\left(k + \log\left(\frac{|V|}{\chi(G)} + 1\right)\right).$$

The second corollary generalizes a result from [1] regarding the choice numbers of random graphs. We refer to the standard model $G_{n,p}$ (see, e.g., [8]), a graph on n vertices, every pair of which is expected to be the endvertices of an edge, randomly and independently, with probability p.

Corollary 1.5. *For every two constants $k \ge 1$ and $0 < p < 1$, the probability that $\mathrm{ch}_k(G_{n,p}) \le 475\log(1/(1-p))n\frac{\log\log n}{\log n}$ tends to 1 as n tends to infinity.*

A theorem stated in [5] reveals the connection between the choice number of a graph and its orientations. We present here a generalization of this theorem for a specific case:

Theorem 1.6. *Let $D = (V, E)$ be a digraph and k a positive integer. For each $v \in V$, let $S(v)$ be a set of size $k(d_D^+(v) + 1)$, where $d_D^+(v)$ is the outdegree of v. If D contains no odd directed (simple) cycle, then there exist subsets $C(v) \subseteq S(v)$, where $|C(v)| = k$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$. The subsets $C(v)$ can be found in polynomial time with respect to $|V|$ and k.*

Corollary 1.7. *Let G be an undirected graph. If G has an orientation D which contains no odd directed (simple) cycle and maximum outdegree d, then G is $(k(d+1) : k)$-choosable for every $k \ge 1$.*

Corollary 1.8. *An even cycle is $(2k : k)$-choosable for every $k \ge 1$.*

The last corollary enables us to generalize a variant of Brooks' Theorem which appears in [11].

Corollary 1.9. *If a connected graph G is not K_n, and not an odd cycle, then $\mathrm{ch}_k(G) \le k\Delta(G)$ for every $k \ge 1$, where $\Delta(G)$ is the maximum degree of G.*

For a graph $G = (V, E)$, define $M(G) = \max(|E(H)|/|V(H)|)$, where $H = (V(H), E(H))$ ranges over all subgraphs of G. The following two corollaries are generalizations of results which appear in [5].

Corollary 1.10. *Every bipartite graph G is $(k(\lceil M(G)\rceil + 1) : k)$-choosable for every $k \ge 1$.*

Corollary 1.11. *Every bipartite planar graph G is $(3k : k)$-choosable for every $k \ge 1$.*

Following are some more applications:

Corollary 1.12. *If every induced subgraph of a graph G has a vertex of degree at most d, then G is $(k(d+1) : k)$-choosable for every $k \ge 1$.*

Corollary 1.13. *If G is a triangulated (chordal) graph, then $\mathrm{ch}_k(G) = k\chi(G) = k\omega(G)$ for every $k \ge 1$, where $\omega(G)$ is the clique number of G.*

The list-chromatic conjecture asserts that for every graph G, $\mathrm{ch}(L(G)) = \chi(L(G))$, where $L(G)$ denotes the line graph of G. The list-chromatic conjecture is easy to verify for trees, graphs of degree at most 2, and $K_{2,m}$. It has also been proven true for snarks [18], $K_{3,3}$ [9], $K_{4,4}$, $K_{6,6}$ [5], and 2-connected regular planar graphs [10]. Galvin proved the list-chromatic conjecture for all bipartite multigraphs [13]. The following corollary shows that the assertion of the list-chromatic conjecture is true for a graph whose 2-connected components are at most triangles:

Corollary 1.14. *If a graph G contains no simple circuit of size 4 or more then $\mathrm{ch}(L(G)) = \chi(L(G))$.*

The *core* of a graph G is the graph obtained from G by successively deleting vertices of degree 1 until there are no such vertices left. The graph $\Theta_{a,b,c}$ consists of three paths of lengths a, b, and c, which share a pair of endvertices and they are otherwise vertex disjoint. The following theorem from [11] gives a complete characterization of 2-choosable graphs:

Theorem 1.15. *A connected graph G is 2-choosable if, and only if, the core of G belongs to $\{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\}$.*

The following is asked in [11]:

If G is $(a : b)$-choosable, does it follow that G is $(am : bm)$-choosable?

The first instant of that question is proved in [16], where it is shown that if a graph G is 2-choosable, then G is also $(4 : 2)$-choosable. Tuza and Voigt later generalized this result by showing that every 2-choosable graph is $(2m : m)$-choosable [21].

In the other direction we obtain:

Theorem 1.16. *Suppose that k and m are positive integers and that k is odd. If a graph G is $(2mk : mk)$-choosable, then G is also $2m$-choosable.*

A graph $G = (V, E)$ is *f-choosable* for a function $f : V \mapsto N$ if for every family of sets $\{S(v) : v \in V\}$, where $|S(v)| = f(v)$ for all $v \in V$, there is a proper vertex-coloring of G assigning to each vertex $v \in V$ a color from $S(v)$. It is shown in [11] that the following problem is Π_2^p-complete: (for terminology see [14])

BIPARTITE GRAPH $(2, 3)$-**CHOOSABILITY** (**BG** $(2, 3)$-**CH**)

INSTANCE: A bipartite graph $G = (V, E)$ and a function $f : V \mapsto \{2, 3\}$.

QUESTION: Is G f-choosable?

We consider the following decision problem:

BIPARTITE GRAPH k-**CHOOSABILITY** (**BG** k-**CH**)

INSTANCE: A bipartite graph $G = (V, E)$.

QUESTION: Is G k-choosable?

If follows from Theorem 1.15 that this problem is solvable in polynomial time for $k = 2$.

Theorem 1.17. ***BIPARTITE GRAPH** k-**CHOOSABILITY** is Π_2^p-complete for every constant $k \geq 3$.*

Results concerning the complexity of planar graph choosability are proved in [17].

A graph $G = (V, E)$ is *strongly k-colorable* if every graph obtained from G by appending a union of vertex disjoint cliques of size at most k (on the vertex set V) is k-colorable. An analogous definition of *strongly k-choosable* is made by replacing colorability with choosability. The *strong chromatic number* of a graph G, denoted by $s\chi(G)$, is the minimum k such that G is strongly k-colorable. Define $s\chi(d) = \max(s\chi(G))$, where G ranges over all graphs with maximum degree at most d. (The definition of strong colorability given in [2] is slightly different. It is claimed there that if G is strongly k-colorable, then it is also strongly $(k + 1)$-colorable. However, it is not known how to prove this if the original definition given in [2] is used).

Theorem 1.18. *If G is strongly k-colorable, then it is strongly $(k + 1)$-colorable as well.*

We give a weaker version of this theorem for choosability.

Theorem 1.19. *If G is strongly k-choosable, then it is also strongly km-choosable for any integer m.*

Theorem 1.20. *Let $G = (V, E)$ be a graph, and suppose that km divides $|V|$. If the choice number of any graph obtained from G by appending a union of vertex disjoint k-cliques (on the vertex set V) is k, then the choice number of any graph obtained from G by appending a union of vertex disjoint km-cliques is km.*

Corollary 1.21. *Let n and k be positive integers, and let G be a $(3k + 1)$-regular graph on $3kn$ vertices. Assume that G has a decomposition into a Hamiltonian circuit and n pairwise vertex disjoint $3k$-cliques. Then $\mathrm{ch}(G) = 3k$.*

Alon [2] proved that there is a constant c such that for every d, $3\lfloor d/2 \rfloor < s\chi(d) \leq cd$. The following theorem improves the lower bound.

Theorem 1.22. *For every $d \geq 1$, $s\chi(d) \geq 2d$.*

2. A solution to a problem of Erdős, Rubin and Taylor

In this section we prove an upper bound for the kth choice number of a graph when k is large and apply this bound to settle a problem raised in [11].

Proof of Theorem 1.1. Let $G = (V, E)$ be a graph and $\epsilon > 0$. Let r stand for the chromatic number of G and let $\{V_1, \ldots, V_r\}$ be a partition of V into stable sets. Assign a set $S(v)$ of $\lfloor k(\chi(G) + \epsilon)\rfloor$ distinct colors to every $v \in V$. Let $S = \cup_{v \in V} S(v)$ be the set of all colors. Define $R = \{1, 2, \ldots, r\}$ and let $f : S \mapsto R$ be a random function, obtained by randomly selecting, the value of $f(c)$, independently for each color $c \in S$, according to a uniform distribution on R. The colors c for which $f(c) = i$ will be used to color the vertices in V_i. To complete the proof, it thus suffices to show that the probability of the following event is positive: For every i, $1 \le i \le r$, and for every vertex $v \in V_i$ there are at least k colors $c \in S(v)$ for which $f(c) = i$.

For a fixed vertex v, included in a set V_i, define $X = |S(v) \cap f^{-1}(i)|$. The probability that there are less than k colors $c \in S(v)$ for which $f(c) = i$ is equal to $Pr(X < k)$. Since X is a random variable with distribution $B(\lfloor k(r + \epsilon)\rfloor, 1/r)$, Chebyshev's inequality (see, e.g., [4]) implies

$$Pr(X < k) \le Pr\left(\left|X - \frac{\lfloor k(r+\epsilon)\rfloor}{r}\right| \ge \frac{\lfloor k\epsilon\rfloor}{r}\right) \le \frac{\lfloor k(r+\epsilon)\rfloor \frac{1}{r}\left(1 - \frac{1}{r}\right)}{\left(\frac{\lfloor k\epsilon\rfloor}{r}\right)^2} = O\left(\frac{1}{k}\right).$$

It follows that there is an integer k_0 such that for every $k \ge k_0$, $P(X < k) < 1/|V|$. There are $|V|$ vertices from which v is selected (and i is determined) and hence, the probability that for some i and some $v \in V_i$ there are less than k colors $c \in S(v)$ for which $f(c) = i$ is smaller than 1. □

Proof of Corollary 1.2. Suppose that $l > m \ge 3$, and let G be a graph such that $\mathrm{ch}(G) = l + 1$ and $\chi(G) = m - 1$ (it is proven in [22] that for every $l \ge m \ge 2$ there is a graph G, where $\mathrm{ch}(G) = l$ and $\chi(G) = m$. Take for example the disjoint union of K_m and $K_{n,n}$ for an appropriate value of n). By Theorem 1.1, for $\epsilon = 1$ there exists an integer k such that G is $(k(\chi(G) + 1) : k)$-choosable. Hence G is $(km : k)$-choosable but not $(l : 1)$-choosable, as claimed. □

Note that it is not true that for every graph G there exists an integer k_0 such that $\mathrm{ch}_k(G) \le k\chi(G)$ for every $k \ge k_0$. For example the chromatic number of $G = K_{3,3}$ is 2, but that graph is not 2-choosable and therefore, by Theorem 1.16, it is not $(2k : k)$-choosable for any odd k. Thus $\mathrm{ch}_k(G) > k\chi(G)$ for every odd k.

3. An upper bound for the *k*th choice number

In this section we establish an upper bound for $\mathrm{ch}_k(K_{m_1,\ldots,m_r})$, and use it to prove two consequences. The following lemma appears in [4].

Lemma 3.1. *If X is a random variable with distribution $B(n, p)$, $0 < p \le 1$, and $k < pn$ then*

$$Pr(X < k) < \mathrm{e}^{-\frac{(np-k)^2}{2pn}}.$$

In the rest of this section we denote $t = \frac{m_1+\cdots+m_r}{r}$, $t_1 = \frac{m_1+\cdots+m_{r/2}}{r/2}$, and $t_2 = \frac{m_{r/2+1}+\cdots+m_r}{r/2}$. Notice that $t = (t_1 + t_2)/2$, and therefore $\log t_1 t_2 \le 2 \log t$.

Lemma 3.2. *If $1 \le r \le t$, $k \ge 1$, and $m_i \ge 2$ for every i, $1 \le i \le r$, then $\mathrm{ch}_k(K_{m_1,\ldots,m_r}) \le 4r(k + \log t)$.*

Proof. Let $V_1, V_2, \ldots, V_r$ be the stable sets of $K = K_{m_1,\ldots,m_r}$, where $|V_i| = m_i$ for all i, and let $V = V_1 \cup \cdots \cup V_r$ be the set of all vertices of K. For each $v \in V$, let $S(v)$ be a set of $\lfloor 4r(k + \log t)\rfloor$ distinct colors. Define $R = \{1, 2, \ldots, r\}$ and let $f : S \mapsto R$ be a random function, obtained by choosing, for each color $c \in S$, randomly and independently, the value of $f(c)$ according to a uniform distribution on R. The colors c for which $f(c) = i$ will be the ones to be used for coloring the vertices in V_i. To complete the proof it thus suffices to show that with positive probability for every i, $1 \le i \le r$, and every vertex $v \in V_i$ there are at least k colors $c \in S(v)$ so that $f(c) = i$.

Fix an i and a vertex $v \in V_i$, and define $X = |S(v) \cap f^{-1}(i)|$. The probability that there are less than k colors $c \in S(v)$ so that $f(c) = i$ is equal to $Pr(X < k)$. Since X is a random variable with distribution $B(\lfloor 4r(k + \log t)\rfloor, 1/r)$, by Lemma 3.1

$$Pr(X < k) < \mathrm{e}^{-\frac{(E(X)-k)^2}{2E(X)}} \le \mathrm{e}^{-\frac{(4(k+\log t)-1-k)^2}{8(k+\log t)}} < \mathrm{e}^{-\frac{16(k+\log t)^2-8(k+1)(k+\log t)}{8(k+\log t)}} \le \mathrm{e}^{-2\log t} = \frac{1}{t^2} \le \frac{1}{rt},$$

where the last inequality follows the fact that $r \le t$. There are rt possible ways to choose i, $1 \le i \le r$ and $v \in V_i$, and hence, the probability that for some i and some $v \in V_i$ there are less than k colors $c \in S(v)$ so that $f(c) = i$ is smaller than 1, completing the proof. □

Lemma 3.3. *Suppose that r is even, $r > t$, $k \ge 1$, $d \ge 244$, and $m_i \ge 2$ for every i, $1 \le i \le r$. Denote by K_1 and K_2 the two multipartite graphs $K_{m_1,\ldots,m_{r/2}}$ and $K_{m_{r/2+1},\ldots,m_r}$, respectively. If $\mathrm{ch}_k(K_j) \le d\left(1 - \frac{1}{5r^{1/3}}\right)\frac{r}{2}(k + \log t_j)$ for all $j \in \{1, 2\}$, then $\mathrm{ch}_k(K_{m_1,\ldots,m_r}) \le dr(k + \log t)$.*

Proof. Let $V_1, V_2, \dots, V_r$ be the stable sets of $K = K_{m_1,\dots,m_r}$, where $|V_i| = m_i$ for all i, and let $V = V_1 \cup \cdots \cup V_r$ be the vertex set of K. For each $v \in V$, let $S(v)$ be a set of $\lfloor dr(k + \log t)\rfloor$ distinct colors. Define $R = \{1, 2, \dots, r\}$, and let $S = \cup_{v\in V} S(v)$ be the set of all colors. Define $R_1 = \{1, 2, \dots, r/2\}$ and $R_2 = \{r/2 + 1, \dots, r\}$. Let $f : S \mapsto \{1, 2\}$ be a random function obtained by choosing, for each $c \in S$ randomly and independently, $f(c) \in \{1, 2\}$ where for all $j \in \{1, 2\}$

$$Pr(f(c) = j) = \frac{k + \log t_j}{2k + \log t_1 t_2}.$$

The colors c for which $f(c) = 1$ will be used to color the vertices in $\cup_{i\in R_1} V_i$, whereas the colors c for which $f(c) = 2$ will be used to color the vertices in $\cup_{i\in R_2} V_i$.

For every vertex $v \in V$, define $C(v) = S(v) \cap f^{-1}(\{1\})$ if v belongs to $\cup_{i\in R_1} V_i$, and $C(v) = S(v) \cap f^{-1}(\{2\})$ if v belongs to $\cup_{i\in R_2} V_i$. Because of the assumptions of the lemma, it remains to show that with positive probability,

$$|C(v)| \ge d\left(1 - \frac{1}{5r^{1/3}}\right)\frac{r}{2}(k + \log t_j) \tag{1}$$

for all $j \in \{1, 2\}$ and $v \in \cup_{i\in R_j} V_i$.

Fix a $j \in \{1, 2\}$ and a vertex $v \in \cup_{i\in R_j} V_i$, and define $X = |C(v)|$. The expectation of X is

$$\lfloor dr(k + \log t)\rfloor \frac{k + \log t_j}{2k + \log t_1 t_2} \ge (dr(k + \log t) - 1)\frac{k + \log t_j}{2k + 2\log t} \ge d\frac{r}{2}(k + \log t_j) - 1 = T.$$

If follows from Lemma 3.1 and the inequality $E(X) \ge T$ that

$$Pr(X < T - T^{2/3}) < e^{-\frac{(E(X)-T+T^{2/3})^2}{2E(X)}} \le e^{-\frac{1}{2}T^{1/3}} \le e^{-\frac{1}{2}\left(d\frac{r}{2}\right)^{1/3}}.$$

Since $|\cup_{i\in R_j} V_i| \le rt < r^2$, the probability that $|C(v)| < T - T^{2/3}$ holds for some $v \in \cup_{i\in R_j} V_i$ is at most

$$r^2 \cdot e^{-\frac{1}{2}\left(d\frac{r}{2}\right)^{1/3}} < 1/2,$$

where the last inequality follows the fact that $d \ge 244$. One can easily verify that

$$T - T^{2/3} = T\left(1 - \frac{1}{T^{1/3}}\right) \ge d\frac{r}{2}(k + \log t_j)\left(1 - \frac{1}{5r^{1/3}}\right),$$

and therefore, with positive probability (1) holds for all $j \in \{1, 2\}$ and $v \in \cup_{i\in R_j} V_i$. $\square$

Proof of Theorem 1.3. Define for every r that is a power of 2

$$f(r) = \prod_{j=0}^{\log_2 r}\left(1 - \frac{1}{5\cdot 2^{j/3}}\right) \Big/ \prod_{j=0}^{2}\left(1 - \frac{1}{5\cdot 2^{j/3}}\right).$$

We claim that for every r that is a power of 2

$$\mathrm{ch}_k(K_{m_1,\dots,m_r}) \le \frac{244r(k + \log t)}{f(r)}. \tag{2}$$

The proof is by induction on r.

Case 1: $r \le t$.

The result follows from Lemma 3.2 since

$$\frac{244}{f(r)} \ge 244\prod_{j=1}^{2}\left(1 - \frac{1}{5\cdot 2^{j/3}}\right) > 4.$$

Case 2: $r > t$.

Notice that $t \ge 2$, and therefore $r \ge 4$. By the induction hypothesis

$$\mathrm{ch}_k(K_{m_1,\dots,m_{r/2}}) \le \frac{244\left(1 - \frac{1}{5r^{1/3}}\right)\frac{r}{2}(k + \log t_1)}{f(r)}$$

and

$$\mathrm{ch}_k(K_{m_{r/2+1},\dots,m_r}) \le \frac{244\left(1 - \frac{1}{5r^{1/3}}\right)\frac{r}{2}(k + \log t_2)}{f(r)}.$$

Since $r \ge 4$, we have $244/f(r) \ge 244$ and it follows from Lemma 3.3 that (2) holds, as claimed.

It is easy to check that

$$\prod_{j=3}^{\log_2 r}\left(1-\frac{1}{5\cdot 2^{j/3}}\right)\geq 1-\sum_{j=3}^{\log_2 r}\frac{1}{5\cdot 2^{j/3}}\geq 1-\frac{1}{10(1-2^{-1/3})},$$

and therefore $244/f(r)\leq 474$. It follows from (2) that for every r that is a power of 2

$$\mathrm{ch}_k(K_{m_1,\ldots,m_r})\leq 474r(k+\log t). \tag{3}$$

Returning to the general case, assume that $r\geq 1$. Choose an integer r' which is a power of 2 and $r\leq r'<2r$. By applying (3), we get

$$\begin{aligned}\mathrm{ch}_k(K_{m_1,\ldots,m_r}) &\leq \mathrm{ch}_k(K_{m_1,\ldots,m_r,\underbrace{2,\ldots,2}_{r'-r}})\\ &\leq 474r'\left(k+\log\frac{m_1+\cdots+m_r+2(r'-r)}{r'}\right)\leq 948r\left(k+\log\frac{m_1+\cdots+m_r}{r}\right),\end{aligned}$$

completing the proof. □

Define $K=K_{m,\underbrace{s,\ldots,s}_{r}}$, where $m\geq 2$ and $s\geq 2$. Every induced subgraph of K has a vertex of degree at most rs, and therefore by Corollary 1.12 $\mathrm{ch}_k(K)\leq k(rs+1)$ for all $k\geq 1$. Note that this upper bound for $\mathrm{ch}_k(K)$ does not depend of m, which means that a good lower bound for $\mathrm{ch}_k(K_{m_1,\ldots,m_r})$ has a more complicated form than the upper bound given in Theorem 1.3.

Proof of Corollary 1.4. Let $G=(V,E)$ be a graph and $k\geq 1$. Define $r=\chi(G)$, and let $V=V_1\cup\cdots\cup V_r$ be a partition of the vertices, such that each V_i is a stable set. Define $m_i=|V_i|$ for all i, $1\leq i\leq r$. By Theorem 1.1

$$\mathrm{ch}_k(G)\leq \mathrm{ch}_k(K_{m_1+1,\ldots,m_r+1})\leq 948r\left(k+\log\frac{m_1+\cdots+m_r+r}{r}\right)=948\chi(G)\left(k+\log\left(\frac{|V|}{\chi(G)}+1\right)\right),$$

as claimed. □

Proof of Corollary 1.5. As proven by Bollobás in [7], for a fixed probability p, $0<p<1$, almost surely (i.e., with probability that tends to 1 as n tends to infinity), the random graph $G_{n,p}$ has chromatic number

$$\left(\frac{1}{2}+o(1)\right)\log(1/(1-p))\frac{n}{\log n}.$$

By Corollary 1.4, for every $\epsilon>0$ almost surely

$$\mathrm{ch}_k(G_{n,p})\leq 948\left(\frac{1}{2}+\epsilon\right)\log(1/(1-p))\frac{n}{\log n}\left(k+\log\left(\frac{3\log n}{\log(1/(1-p))}+1\right)\right).$$

The result follows since k and p are constants. □

Note that in the proof of the last corollary we have not used any knowledge concerning independent sets of $G_{n,p}$, as was done in [1] for the proof of the special case.

4. Choice numbers and orientations

Let $D=(V,E)$ be a digraph. We denote the set of out-neighbors of v in D by $N_D^+(v)$. A set of vertices $K\subseteq V$ is called a *kernel* of D if K is an independent set and $N_D^+(v)\cap K\neq\emptyset$ for every vertex $v\notin K$. Richardson's theorem (see, e.g., [6]) states that any digraph with no odd directed cycle has a kernel.

Proof of Theorem 1.6. Let $D=(V,E)$ be a digraph which contains no odd directed (simple) cycle and $k\geq 1$. For each $v\in V$, let $S(v)$ be a set of size $k(d_D^+(v)+1)$. We claim that the following algorithm finds subsets $C(v)\subseteq S(v)$, where $|C(v)|=k$ for all $v\in V$, and $C(u)\cap C(v)=\emptyset$ for every two adjacent vertices $u,v\in V$.

(1) $S\leftarrow\cup_{v\in V}S(v)$, $W\leftarrow V$ and for every $v\in V$, $C(v)\leftarrow\emptyset$.
(2) Choose a color $c\in S\cap\cup_{v\in W}S(v)$ and put $S\leftarrow S-\{c\}$.
(3) Let K be a kernel of the subgraph of D induced by the vertex set $\{v\in W:c\in S(v)\}$.
(4) $C(v)\leftarrow C(v)\cup\{c\}$ for all $v\in K$.
(5) $W\leftarrow W-\{v\in K:|C(v)|=k\}$.
(6) If $W=\emptyset$, stop. If not, go to step 2.

During the algorithm, W is equal to $\{v \in V : |C(v)| < k\}$, and S is the set of remaining colors. We first prove that in step 2, $S \cap \cup_{v \in W} S(v) \neq \emptyset$. When the algorithm reaches step 2, it is obvious that $W \neq \emptyset$. Suppose that $w \in W$ in this step, and therefore $|C(w)| < k$. It follows easily from the definition of a kernel that every color from $S(w)$, which has been previously chosen in step 2, belongs either to $C(w)$ or to $\cup_{v \in N_D^+(w)} C(v)$. Since

$$|C(w)| + \left| \bigcup_{v \in N_D^+(w)} C(v) \right| < k + k \cdot d_D^+(w) = |S(w)|,$$

not all the colors of $S(w)$ have been used. This means that $S \cap S(w) \neq \emptyset$, as needed. It follows easily that the algorithm always terminates.

Upon termination of the algorithm, $|C(v)| = k$ for all $v \in V$. In step 4 the same color is assigned to the vertices of a kernel which is an independent set, and therefore $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$. This proves the correctness of the algorithm.

In step 4, the operation $C(v) \leftarrow C(v) \cup \{c\}$ is performed for at least one vertex. Upon termination $| \cup_{v \in V} C(v)| \leq k|V|$, which means that the algorithm performs at most $k|V|$ iterations. There is a polynomial time algorithm for finding a kernel in a digraph with no odd directed cycle. Thus, the algorithm is of polynomial time complexity in $|V|$ and k, completing the proof. □

Proof of Corollary 1.7. This is an immediate consequence of Theorem 1.6, since $k(d_D^+(v) + 1) \leq k(d + 1)$ for every $v \in V$. □

Proof of Corollary 1.8. The result follows from 1.7 by taking the cyclic orientation of the even cycle. □

The proof of Corollary 1.9 is similar to the proof of the special case which appears in [11]. A graph $G = (V, E)$ is *k-degree-choosable* if for every family of sets $\{S(v) : v \in V\}$, where $|S(v)| = kd(v)$ for all $v \in V$, there are subsets $C(v) \subseteq S(v)$, where $|C(v)| = k$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$.

Lemma 4.1. *If a graph $G = (V, E)$ is connected, and G has a connected induced subgraph $H = (V', E')$ which is k-degree-choosable, then G is k-degree-choosable.*

Proof. For each $v \in V$, let $S(v)$ be a set of size $kd(v)$. The proof is by induction on $|V|$. In case $|V| = |V'|$ there is nothing to prove. Assuming that $|V| > |V'|$, let v be a vertex of G which is at a maximal distance from H. This guarantees that $G - v$ is connected. Choose any subset $C(v) \subseteq S(v)$ such that $|C(v)| = k$, and remove the colors of $C(v)$ from all the vertices adjacent to v. The choice can be completed by applying the induction hypothesis on $G - v$. □

Lemma 4.2. *If $c \geq 2$, then $\Theta_{a,b,c}$ is k-degree-choosable for every $k \geq 1$.*

Proof. Suppose that $\Theta_{a,b,c}$ has vertex set $V = \{u, v, x_1, \ldots, x_{a-1}, y_1, \ldots, y_{b-1}, z_1, \ldots, z_{c-1}\}$ and contains the three paths $u - x_1 - \cdots - x_{a-1} - v$, $u - y_1 - \cdots - y_{b-1} - v$, and $u - z_1 - \cdots - z_{c-1} - v$. For each $w \in V$, let $S(w)$ be a set of size $kd(w)$. For the vertex u we choose a subset $C(u) \subseteq S(u) - S(z_1)$ of size k. For each vertex according to the sequence $x_1, \ldots, x_{a-1}, y_1, \ldots, y_{b-1}, v, z_{c-1}, \ldots, z_1$ we choose a subset of k colors that were not chosen in adjacent earlier vertices. □

For the proof of Corollary 1.9, we shall need the following lemma which appears in [11].

Lemma 4.3. *If there is no vertex which disconnects G, then G is an odd cycle, or $G = K_n$, or G contains, as a vertex induced subgraph, an even cycle without chord or with only one chord.*

Proof of Corollary 1.9. Suppose that a connected graph G is not K_n, and not an odd cycle. If G is not a regular graph, then every induced subgraph of G has a vertex of degree at most $\Delta(G) - 1$, and by Corollary 1.12 $\mathrm{ch}_k(G) \leq k\Delta(G)$ for all $k \geq 1$. If G is a regular graph, then there is a part of G not disconnected by a vertex, which is neither an odd cycle nor a complete graph. It follows from Lemma 4.3 that G contains, as a vertex induced subgraph, an even cycle or a particular kind of $\Theta_{a,b,c}$ graph. We know from Corollary 1.8 and Lemma 4.2 that both an even cycle and $\Theta_{a,b,c}$ are k-degree-choosable for every $k \geq 1$. The result follows from Lemma 4.1. □

Proof of Corollary 1.10. It is proved in [5] that a graph $G = (V, E)$ has an orientation D in which every outdegree is at most d if and only if $M(G) \leq d$. Therefore, there is an orientation D of G in which the maximum outdegree is at most $\lceil M(G) \rceil$. Since D contains no odd directed cycles, the result follows from Corollary 1.7. □

Proof of Corollary 1.11. $M(G) \leq 2$, since any planar bipartite (simple) graph on r vertices contains at most $2r - 2$ edges. The result follows from Corollary 1.10. □

Proof of Corollary 1.12. We claim that if every induced subgraph of a graph $G = (V, E)$ has a vertex of degree at most d, then G has an acyclic orientation in which the maximum outdegree is d. The proof is by induction on $|V|$. If $|V| = 1$, the result is trivial. If $|V| > 1$, let v be a vertex of G with degree at most d. By the induction hypothesis, $G - v$ has an acyclic orientation in which the maximum outdegree is d. We complete this orientation of $G - v$ by orienting every edge incident to v from v to its appropriate neighbor and obtain the desired orientation of G, as claimed. The result follows from Corollary 1.7. □

An undirected graph G is called *triangulated* if G does not contain an induced subgraph isomorphic to C_n for $n \geq 4$. Being triangulated is a hereditary property inherited by all the induced subgraphs of G. A vertex v of G is called *simplicial* if its adjacency set $Adj(v)$ induces a complete subgraph of G. It is proved in [15] that every triangulated graph has a simplicial vertex.

Proof of Corollary 1.13. Suppose that G is a triangulated graph, and let H be an induced subgraph of G. Since H is triangulated, it has a simplicial vertex v. The set of vertices $\{v\} \cup Adj_H(v)$ induces a complete subgraph of H, and therefore v has degree at most $\omega(G) - 1$ in H. It follows from Corollary 1.12 that $\mathrm{ch}_k(G) \leq k\omega(G)$ for every $k \geq 1$. For every graph G and $k \geq 1$, $\mathrm{ch}_k(G) \geq k\omega(G)$ and hence $\mathrm{ch}_k(G) = k\omega(G)$ for every $k \geq 1$. Since G is triangulated, it is also perfect, which means that $\chi(G) = \omega(G)$, as needed. □

Proof of Corollary 1.14. It is easy to see that $L(G)$ is triangulated if and only if G contains no C_n for every $n \geq 4$. The result follows from Corollary 1.13. □

The validity of the list-chromatic conjecture for graphs of class 2 with maximum degree 3 (and in particular for snarks) follows easily from Corollary 1.9. Suppose that G is a graph of class 2 with $\Delta(G) = 3$. Let C be a connected component of $L(G)$. If C is not a complete graph, and not an odd cycle, then $\mathrm{ch}(C) \leq \Delta(C) \leq \Delta(L(G)) \leq 4$. If C is a complete graph or an odd cycle, then it is easy to see that $\Delta(C) \leq 2$, and therefore by Corollary 1.12 $\mathrm{ch}(C) \leq \Delta(C) + 1 \leq 3$. It follows that $\mathrm{ch}(L(G)) \leq 4$. Since G is a graph of class 2, $\mathrm{ch}(L(G)) \geq \chi(L(G)) = \Delta(G) + 1 = 4$, and hence, $\mathrm{ch}(L(G)) = \chi(L(G)) = 4$.

5. Properties of $(2k : k)$-choosable graphs

In this section we establish an upper bound for the choice number of $(2k : k)$-choosable graphs.

Proof of Theorem 1.16. Suppose that $G = (V, E)$ is $(2mk : mk)$-choosable for k odd. We prove that G is $2m$-choosable as well. For each $v \in V$, let $S(v)$ be a set of size $2m$. With every color c we associate a set $F(c)$ of size k, such that $F(c) \cap F(d) = \emptyset$ if $c \neq d$. For every $v \in V$, we define $T(v) = \cup_{c \in S(v)} F(c)$. Since G is $(2mk : mk)$-choosable, there are subsets $C(v) \subseteq T(v)$, where $|C(v)| = mk$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$.

Fix a vertex $v \in V$. Since k is odd, there is a color $c \in S(v)$ for which $|C(v) \cap F(c)| > k/2$, so we define $f(v) = c$. In case u and v are adjacent vertices for which $c \in S(u) \cap S(v)$, it is not possible that both $|C(u) \cap F(c)|$ and $|C(v) \cap F(c)|$ are greater than $k/2$. This proves that f is a proper vertex-coloring of G assigning to each vertex $v \in V$ a color in $S(v)$. □

6. The complexity of graph choosability

Let $G = (V, E)$ be a graph. We denote by G' the graph obtained from G by adding a new vertex to G, and joining it to every vertex in V. Consider the following decision problem:

GRAPH k-COLORABILITY

INSTANCE: A graph $G = (V, E)$.

QUESTION: Is G k-colorable?

The standard technique to show a polynomial transformation from GRAPH k-COLORABILITY to GRAPH $(k + 1)$-COLORABILITY is to use the fact that $\chi(G') = \chi(G) + 1$ for every graph G. However, it is not true that $\mathrm{ch}(G') = \mathrm{ch}(G) + 1$ for every graph G. To see that, we first prove that $K'_{2,4}$ is 3-choosable.

Suppose that $K'_{2,4}$ has vertex set $V = \{v, x_1, x_2, y_1, y_2, y_3, y_4\}$, and contains exactly the edges $\{x_i, y_j\}$, $\{v, x_i\}$, and $\{v, y_j\}$. For each $w \in V$, let $S(w)$ be a set of size 3.

Case 1: All the sets are the same.

A choice can be made since $K'_{2,4}$ is 3-colorable.

Case 2: There is a set $S(x_i)$ which is not equal to $S(v)$.

Without loss of generality, suppose that $S(v) \neq S(x_1)$. For the vertex v, choose a color $c \in S(v) - S(x_1)$, and remove c from the sets of the other vertices. We can assume that every set $S(y_j)$ is of size 2 now.

Suppose first that $S(x_1)$ and $S(x_2)$ are disjoint. The number of different sets consisting of one color from each of the $S(x_i)$ is at least 6, and therefore we can choose colors $c_i \in S(x_i)$, such that $\{c_1, c_2\}$ does not appear as a set of $S(y_j)$. We complete the choice by choosing for every vertex y_j a color from $S(y_j) - \{c_1, c_2\}$. Suppose next that $d \in S(x_1) \cap S(x_2)$. For every vertex x_i we choose d, and for every vertex y_j we choose a color from $S(y_j) - \{d\}$.

Case 3: There is a set $S(y_j)$ which is not equal to $S(v)$.

Without loss of generality, suppose that $S(v) \neq S(y_1)$. For the vertex v, choose a color $c \in S(v) - S(y_1)$, and remove c from the sets of the other vertices. Suppose first that $S(x_1)$ and $S(x_2)$ are disjoint. The number of different sets consisting of one color from each of the $S(x_i)$ is at least 4, and since $|S(y_1)| = 3$ we can choose colors $c_i \in S(x_i)$, such that $S(y_j) - \{c_1, c_2\} \neq \emptyset$ for every vertex y_j. We can complete the choice as in case 2. In case $S(x_1)$ and $S(x_2)$ are not disjoint, we proceed as in case 2.

This completes the proof that $K'_{2,4}$ is 3-choosable. It follows from Theorem 1.15 and Corollary 1.12 that $\mathrm{ch}(K_{2,4}) = 3$, and therefore $\mathrm{ch}(K'_{2,4}) = \mathrm{ch}(K_{2,4}) = 3$. The following lemma exhibits a construction which increases the choice number of a graph by exactly 1.

Lemma 6.1. *Let $G = (V, E)$ be a graph. If H is the disjoint union of $|V| + 1$ copies of G, then* $\mathrm{ch}(H') = \mathrm{ch}(G) + 1$.

Proof. Let H be the disjoint union of the graphs $\{G_i : 1 \leq i \leq |V| + 1\}$, where each G_i is a copy of G. Suppose that H' is obtained from H by joining the new vertex v to all the vertices of H.

We claim that if G is k-choosable, then H' is $(k+1)$-choosable. For each $w \in V(H')$, let $S(w)$ be a set of size $k+1$. Choose a color $c \in S(v)$, and remove c from the sets of the other vertices. We can complete the choice since G is k-choosable.

We now prove that if H' is k-choosable, then G is $(k-1)$-choosable. By what precedes, $\mathrm{ch}(H') \leq \mathrm{ch}(G) + 1 \leq |V(G)| + 1$. Hence, we can assume that $k \leq |V(G)| + 1$. For each $w \in V(G)$, let $L(w)$ be a list of $k - 1$ colors. Without loss of generality, let $C = \{1, 2, \ldots, k\}$ be a set of k colors disjoint from $\cup_{w \in V(G)} L(w)$. As noted, $k \leq |V(G)| + 1$. Define a list-assignment S of H' by putting $S(v) := C$, and if w is in the ith copy with $S(w) := L(w) \cup \{i\}$, where i is modulo k (so that $1 \leq i \leq k$). Now, if f is an S-list-coloring of H', then its restriction to $G_{f(v)}$ is an L-list-coloring of G, as desired.

By Corollary 1.9, $\mathrm{ch}(G) + 1 \leq |V|$ if G is not a complete graph and the proof still goes through if $|V| + 1$ is replaced by $|V|$ in the statement of the lemma. $\square$

Lemma 6.2. ***BIPARTITE GRAPH* 3-*CHOOSABILITY*** *is Π_2^p-complete.*

Proof. It is easy to see that **BG** 3-**CH** $\in \Pi_2^p$. We transform **BG** (2, 3)-**CH** to **BG** 3-**CH**. Let $G = (V, E)$ and $f : V \mapsto \{2, 3\}$ be an instance of **BG** (2, 3)-**CH**. We shall construct a bipartite graph H'' such that H'' is 3-choosable if and only if G is f-choosable.

Let H be the disjoint union of the graphs $\{G_{i,j} : 1 \leq i, j \leq 3\}$, where each $G_{i,j}$ is a copy of G. Let (X, Y) be a bipartition of the bipartite graph H. The graph H'' is obtained from H by adding two new vertices u and v, joining u to every vertex $w \in X$ for which $f(w) = 2$, and joining v to every vertex $w \in Y$ for which $f(w) = 2$.

Since H is bipartite, H'' is also a bipartite graph. It is easy to see that if G is f-choosable, then H'' is 3-choosable. We now prove that if H'' is 3-choosable, then G is f-choosable. For every $w \in V$, let $S(w)$ be a list of $f(w)$ colors, such that $S(w) \cap \{1, 2, 3\} = \emptyset$. For every i and j, $1 \leq i, j \leq 3$, the vertices of the graph $G_{i,j}$ are assigned the list of colors $S(w)$, with the vertices for which f is equal to 2 receiving another color as follows: the vertices which belong to X are assigned the list $S(w) \cup \{i\}$, whereas the vertices which belong to Y are assigned the list $S(w) \cup \{j\}$. The vertices u and v are both assigned the list $\{1, 2, 3\}$. Let f be a proper vertex-coloring of H'' assigning to each vertex a color from its list. Denote $f(u) = i$ and $f(v) = j$, then f restricted to $G_{i,j}$ is a proper vertex-coloring of G assigning to each vertex $w \in V$ a color in $S(w)$. $\square$

Proof of Theorem 1.17. The proof is by induction on k. For $k = 3$, the result follows from Lemma 6.2. Assuming that the result is true for k, $k \geq 3$, we prove it is true for $k+1$. It is easy to see that **BG** $(k+1)$-**CH** $\in \Pi_2^p$. We transform **BG** k-**CH** to **BG** $(k+1)$-**CH**. Let $G = (V, E)$ be an instance of **BG** k-**CH**. We shall construct a bipartite graph W such that W is $(k+1)$-choosable if and only if G is k-choosable.

Let H be the disjoint union of the graphs $\{G_{i,j} : 1 \leq i, j \leq k + 1\}$, where each $G_{i,j}$ is a copy of G. Let (X, Y) be a bipartition of the bipartite graph H. The graph W is obtained from H by adding two new vertices u and v, joining u to every vertex of X, and joining v to every vertex of Y.

It is easy to see that if G is k-choosable, then W is $(k + 1)$-choosable. In a way similar to the proof of Lemma 6.2, we can prove that if W is $(k + 1)$-choosable, then G is k-choosable. $\square$

7. The strong choice number

Let $G = (V, E)$ be a graph, and let $V_1, \ldots, V_r$ be pairwise disjoint subsets of V. We denote by $[G, V_1, \ldots, V_r]$ the graph obtained from G by appending the union of cliques induced by each V_i, $1 \leq i \leq r$.

Suppose that $G = (V, E)$ is a graph with maximum degree at most 1. We claim that G is strongly k-choosable for every $k \geq 2$. To see that, let $V_1, \ldots, V_r$ be pairwise disjoint subsets of V, each of size at most k. The graph $[G, V_1, \ldots, V_r]$ has maximum degree at most k, and therefore by Corollary 1.9 it is k-choosable.

Proof of Theorem 1.18. Let $G = (V, E)$ be a strongly k-colorable graph. Let $V_1, \ldots, V_r$ be pairwise disjoint subsets of V, each of size at most $k + 1$. Without loss of generality, we can assume that $V_1, \ldots, V_m$ are subsets of size exactly $k + 1$, and $V_{m+1}, \ldots, V_r$ are subsets of size less than $k + 1$. Let H be the graph $[G, V_1, \ldots, V_r]$. To complete the proof, it suffices to show that H is $(k+1)$-colorable. For every i, $1 \leq i \leq m$, we define $W_i = V_i - \{c\}$ for an arbitrary element $c \in V_i$, whereas for every j, $m + 1 \leq j \leq r$, we define $W_i = V_i$. Since $[G, W_1, \ldots, W_r]$ is k-colorable, there exists an independent set S of H which is composed of exactly one vertex from each V_i, $1 \leq i \leq m$. For every i, $1 \leq i \leq m$, we define $W'_i = V_i - S$, whereas for every j, $m + 1 \leq j \leq r$, we define $W'_i = V_i$. Since $[G, W'_1, \ldots, W'_r]$ is k-colorable, we can obtain a proper $(k + 1)$-vertex-coloring of H by using k colors for $V - S$ and another color for S. $\square$

Lemma 7.1. *Suppose that $k, l \geq 1$. If $\mathcal{F}$ is a family of $k+l$ sets of size $k+l$, then it is possible to partition $\mathcal{F}$ into a family $\mathcal{F}_1$ of k sets and a family $\mathcal{F}_2$ of l sets, to choose for each set $S \in \mathcal{F}_1$ a subset $S' \subseteq S$ of size k, and to choose for each set $T \in \mathcal{F}_2$ a subset $T' \subseteq T$ of size l, so that $S' \cap T' = \emptyset$ for every $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$.*

Proof. Suppose that $\mathcal{F} = \{C_1, \dots, C_{k+l}\}$, and define $C = \cup_{i=1}^{k+l} C_i$. For every partition π of C into two subsets R and L, we denote $\mathcal{R}(\pi) = \{V \in \mathcal{F} : |V \cap R| > k\}$, $\mathcal{L}(\pi) = \{V \in \mathcal{F} : |V \cap L| > l\}$, and $\mathcal{M}(\pi) = \{V \in \mathcal{F} : |V \cap R| = k$ and $|V \cap L| = l\}$. We now start with the partition of C into two subsets $R = C$ and $L = \emptyset$, and start moving one element at a time from R to L until we obtain a partition π_1 of C into two subsets R and L and a partition π_2 into two subsets $R' = R - \{c\}$ and $L' = L \cup \{c\}$, such that $|\mathcal{R}(\pi_1)| > k$ and $|\mathcal{R}(\pi_2)| \leq k$. Thus, $\mathcal{L}(\pi_2) \subseteq \mathcal{L}(\pi_1) \cup \mathcal{M}(\pi_1)$, and therefore $|\mathcal{L}(\pi_2)| < l$. We now partition $\mathcal{M}(\pi_2)$ into two subsets $\mathcal{M}_1$ and $\mathcal{M}_2$, such that $\mathcal{F}_1 = \mathcal{R}(\pi_2) \cup \mathcal{M}_1$ has size k and $\mathcal{F}_2 = \mathcal{L}(\pi_2) \cup \mathcal{M}_2$ has size l. For every set $S \in \mathcal{F}_1$ we choose a subset $S' \subseteq S \cap R'$ of size k, whereas for every $T \in \mathcal{F}_2$ we choose a subset $T' \subseteq T \cap L'$ of size l. Since R' and L' are disjoint, we have $S' \cap T' = \emptyset$ for every $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$. □

Lemma 7.2. *Suppose that $k, m \geq 1$. If $\mathcal{F}$ is a family of km sets of size km, then it is possible to partition $\mathcal{F}$ into m subsets $\mathcal{F}_1, \dots, \mathcal{F}_m$, each of size k, and to choose for each set $S \in \mathcal{F}$ a subset $S' \subseteq S$ of size k, so that $S' \cap T' = \emptyset$ for every $i \neq j$, $S \in \mathcal{F}_i$ and $T \in \mathcal{F}_j$.*

Proof. By induction on m. For $m = 1$ the result is trivial. Assuming that the result is true for m, $m \geq 1$, we prove that it is true for $m+1$. Let $\mathcal{F}$ be a family of $k(m+1)$ sets of size $k(m+1)$. By Lemma 7.1, it is possible to partition $\mathcal{F}$ into a family $\mathcal{F}_1$ of k sets and a family $\mathcal{F}_2$ of km sets, to choose for each $S \in \mathcal{F}_1$ a subset $S' \subseteq S$ of size k, and to choose for each set $T \in \mathcal{F}_2$ a subset $T' \subseteq T$ of size km, so that $S' \cap T' = \emptyset$ for every $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$. The proof is completed by applying the induction hypothesis on $\mathcal{F}_2$. □

Proof of Theorem 1.19. Let $G = (V, E)$ be a strongly k-choosable graph. Let $V_1, \dots, V_r$ be pairwise disjoint subsets of V, each of size at most km. Let H be the graph $[G, V_1, \dots, V_r]$. To complete the proof, it suffices to show that H is km-choosable. For each $v \in V$, let $S(v)$ be a set of size km. By Lemma 7.2, for every i, $1 \leq i \leq r$, is it possible to partition V_i into the m subsets $V_{i,1}, \dots, V_{i,m}$, each of size at most k, and to choose for each vertex $v \in V_i$ a subset $C(v) \subseteq S(v)$ of size k, so that $C(u) \cap C(v) = \emptyset$ for every $p \neq q$, $u \in V_{i,p}$ and $v \in V_{i,q}$. Since the graph $[G, V_{1,1}, \dots, V_{r,m}]$ is k-choosable, we can obtain a proper vertex-coloring of H assigning to each vertex a color from its set. □

Proof of Theorem 1.20. Apply Lemma 7.2 as in the proof of Theorem 1.19. □

Proof of Corollary 1.21. It is proved in [12] that if G is a 4-regular graph on $3n$ vertices and G has a decomposition into a Hamiltonian circuit and n pairwise vertex disjoint triangles, then $\mathrm{ch}(G) = 3$. The result follows from Theorem 1.20. □

Proof of Theorem 1.22. Since $s\chi(1) = 2$, we can assume that $d > 1$. Suppose first that d is even, and denote $d = 2r$. Construct a graph G with $12r-3$ vertices, partitioned into 8 classes, as follows. Let these classes be $A, B_1, B_2, C_1, C_2, D_1, D_2, E$, where $|A| = |D_1| = |D_2| = 2r$, $|B_1| = |B_2| = r$, $|C_1| = |C_2| = r-1$, and $|E| = 2r-1$. Each vertex in A is joined by edges to each member of B_1 and each member of B_2. Each member of D_1 is adjacent to each member of D_2. Consider the following partition of the vertex set of G into three classes of cardinality $4r-1$ each:

$$V_1 = B_1 \cup C_1 \cup D_1, \qquad V_2 = B_2 \cup C_2 \cup D_2, \qquad V_3 = A \cup E.$$

We claim that $H = [G, V_1, V_2, V_3]$ is not $(4r-1)$-colorable. In a proper $(4r-1)$-vertex-coloring of H, every color used for coloring the vertices of A must appear on a vertex of $C_1 \cup D_1$ and on a vertex of $C_2 \cup D_2$. Since $|C_1 \cup C_2| < |A|$, there is a color used for coloring the vertices of A which appears on both D_1 and D_2. But this is impossible as each vertex in D_1 is adjacent to each member of D_2. Thus $s\chi(G) > 4r-1$ and as the maximum degree in G is $2r$, this shows that $s\chi(2r) \geq 4r$.

Next suppose that d is odd, and denote $d = 2r+1$. Construct a graph G with $12r+3$ vertices, partitioned into 8 classes, as follows. Let these classes be named as before, where $|A| = |D_1| = |D_2| = 2r+1$, $|B_1| = r+1$, $|C_1| = r-1$, $|B_2| = |C_2| = r$, and $|E| = 2r$. In the same manner we can prove that $[G, V_1, V_2, V_3]$ is not $(4r+1)$-colorable. Thus $s\chi(G) > 4r+1$ and as the maximum degree in G is $2r+1$, this shows that $s\chi(2r+1) \geq 4r+2$, completing the proof. □

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