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The alternating groups and K3 surfaces

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Abstract

In this note, we consider all possible extensions G of a non-trivial perfect group H acting faithfully on a $K3$ surface X. The pair (X, G) is proved to be uniquely determined by G if the transcendental value of G is maximum. In particular, we have $G/H \le (\mathbf{Z}/(2))^{\oplus 2}$, if H is the alternating group A_5 and normal in G.

MSC: 14J28; 14J50; 14L30

0. Introduction

We work over the complex numbers field $\mathbf{C}$. A *K3* surface X is a simply connected projective surface with a nowhere vanishing holomorphic 2-form ω_X. In this note, we will consider finite groups in $\mathrm{Aut}(X)$. An element $h \in \mathrm{Aut}(X)$ is *symplectic* if h acts trivially on the 2-form ω_X. A group $G_N \subseteq \mathrm{Aut}(X)$ is *symplectic* if every element of G_N is symplectic.

According to Nikulin [13], Mukai [11] and Xiao [24], there are exactly 80 abstract finite groups which can act symplectically on $K3$ surfaces. Among these 80, there are exactly four perfect groups (G is perfect if the commutator subgroup $[G, G] = G$): $A_5, L_2(7), A_6, M_{20} = C_2^4 : A_5$ (the Mathieu group of degree 20), where the first three are also the only non-abelian simple groups which can act on a $K3$ surface symplectically, and the last is the symplectic finite group with the largest order 960.

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The common thing shared by the three bigger perfect groups $G_N = L_2(7)$, A_6 and M_{20}, is that they all can be extended to a bigger group $G = G_N \cdot \mu_4$ acting faithfully on a $K3$ surface X. Moreover, the pair (X, G) turns out to be unique in each case, [9,18,6].

So one would expect that A_5, being a smaller one, should be extendable to a bigger group $G = A_5 \cdot \mu_I$ for some $I \geq 3$. However, our result below shows that this is not the case. Indeed, only $I = 1$, or 2 is possible.

Theorem A. *Suppose that a finite group G acts faithfully on a $K3$ surface. Suppose further that G contains A_5 as a normal subgroup. Then G equals one of the following four groups, each realizable (see Example 1.10):*

$$A_5,\ S_5,\ A_5 \times \mu_2,\ S_5 \times \mu_2.$$

To be precise, as in 1.0 below, for every finite group G acting on a $K3$ surface X, the symplectic elements of G (i.e., those h acting trivially on the non-zero 2-form ω_X) form a normal subgroup G_N such that $G/G_N \cong \mu_I$ (the cyclic group of order I in $\mathbf{C}^*$). Namely, we have $G = G_N \cdot \mu_I$ (see *Notation* below). The natural number $I = I(G)$ is determined by the action of G on X and called the *transcendental value* of (the action of) G.

It is proved in [9,18] and [6] that for the three bigger perfect groups G_N above, there is an extension $G = G_N \cdot \mu_I$ such that the transcendental value $I = I(G)$ equals 4. However, for the smaller perfect (and also simple) group A_5, we have:

Theorem B. *Suppose that a finite group G acts faithfully on a $K3$ surface. Suppose further that G contains A_5 as a normal subgroup. Then the transcendental value $I(G)$ equals* 1 *or* 2 *(both attainable as shown in Example 1.10).*

A bit more surprise comes from the next result: the existence of action by a perfect group (together with the transcendental value being 4) will guarantee the existence of action by a quite large group G as well as the uniqueness of the pair (X, G).

Theorem C. *Suppose that a finite group G acts faithfully on a $K3$ surface X. Suppose further that G contains a non-trivial perfect group H as a subgroup (not necessarily normal). Then we have:*

(1) *The transcendental value $I(G) \leq 4$.*
(2) *If $I(G) = 4$, then $G = L_2(7) \cdot \mu_4$, $A_6 \cdot \mu_4$ or $M_{20} \cdot \mu_4$, and the pair (X, G) is unique, up to isomorphisms, in all three cases.*

Remark D. (1) The three subgroups $L_2(7)$, A_6 and M_{20} of G in Theorem C are all equal to G_N in the notation of 1.0, and are the only perfect groups among the 11 maximum symplectic $K3$ groups [11]. So the maximality of the transcendental value $I(G)$ in the situation of Theorem C guarantees the maximality of the symplectic part G_N of G. This also shows the importance of studying non-symplectic K3 groups.

(2) Regarding Theorems B and C, the readers may wonder whether the action of $\widetilde{A}_6 = A_6 : \mu_4$ on a $K3$ surface X induces an action of $H \cdot \mu_4$ on X with $H = A_5$ a smaller perfect (indeed simple) group. To elaborate, the unique group structure of $\widetilde{A}_6$ (and also the unique pair $(X, \widetilde{A}_6)$) is described in [6,7]. In particular, the natural conjugation

map $\widetilde{A}_6 \to \mathrm{Aut}(A_6)$ ($x \mapsto c_x$; see *Notation* below) has the Mathieu group M_{10} as its image; therefore, the conjugation μ_4 action switches the two different conjugacy classes of order 3 in A_6 [4, Ch 10, Section 1.5]. On the other hand, for $\widetilde{A}_6$ to contain an $A_5 \cdot \mu_4$, the conjugation μ_4 action should stabilize at least one A_5 in A_6 and also preserve the unique conjugacy class of order 3 in this A_5, which is impossible.

(3) The same construction in [18, Appendix] shows that there is a smooth non-isotrivial family of $K3$ surfaces $f : \mathcal{X} \to \mathbf{P}^1$ such that all fibres admit A_6 actions and infinitely many of them are algebraic K3 surfaces. So, the symplectic part alone cannot determine the surface uniquely, and the study of transcendental value is needed.

The main tools of the paper are the Lefschetz fixed point formula (both the topological version and vector bundle version due to Atiyah–Segal–Singer [1,2]), the representation theory on the $K3$ lattice and the study in [26] on automorphism groups of Niemeier lattices (in connection with Golay binary or ternary codes) where the latter is much inspired by Conway–Sloane [4], Kondo [8] and Mukai [12].

The reduction to the study of automorphisms of Niemeier lattices was pioneered by Nikulin (see e.g., [15, end of section 1.14]) and further developed by Kondo (see e.g. [8]).

We believe that the way of combining different very powerful machinaries to deduce results as done in the paper should be applicable to the study of other problems. Our humble paper also demonstrates the powerfulness and depth of these algebraic results in the study of geometry. The information we compute in Proposition 1.4 (and its generalization in the future) should be of independent interest and use in understanding the geometry of $K3$ surfaces.

Note. "**Maple**" was used in solving the linear equations in the crucial Proposition 1.4. We refer to Shimada [19–21] for more computations in Algebraic Geometry.

Notation. 1. When we write $G = G_N.\mu_I$ we mean that G acts on a $K3$ surface X satisfying the situation in 1.0 below.

2. S_n is the symmetric group in n letters, A_n ($n \geq 3$) the alternating group in n letters and $\mu_I = \langle \exp(2\pi\sqrt{-1})/I \rangle$ the multiplicative group of order I in $\mathbf{C}^*$.
3. For a group G, we write $G = A.B$ if A is normal in G so that $G/A = B$. We write $G = A : B$ if we assume further that A is normal in G and B is a subgroup of G such that the composition $B \to G \to G/A = B$ is the identity (we say then that G is a *semi-direct product* of A and B).
4. For groups $H \leq G$ and $x \in G$ we denote by $c_x : H \to G$ ($h \mapsto c_x(h) = x^{-1}hx$) the *conjugation* map.
5. For a $K3$ surface X, we let S_X and T_X be the Neron–Severi and transcendental lattices. So the $K3$ lattice $H^2(X, \mathbf{Z})$ contains $S_X \oplus T_X$ as a sublattice of finite index.

1. Preparations and examples

1.0. In this section, we will prepare some basic results to be used later. Let X be a $K3$ surface with a non-zero 2-form ω_X and let $G \subseteq \mathrm{Aut}(X)$ be a finite group of automorphisms. For every $h \in G$, we have $h^*\omega_X = \alpha(h)\omega_X$ for some scalar $\alpha(h) \in \mathbf{C}^*$. Clearly, $\alpha : G \to \mathbf{C}^*$ is a homomorphism. A fact in basic group theory says that $\alpha(G)$ is a finite cyclic group, so $\alpha(G) = \mu_I = \langle \exp(2\pi\sqrt{-1}/I)\rangle$ for some $I \geq 1$. This natural number $I = I(G)$ is called the *transcendental* value of G. It is known that $I = I(G)$ for some G if and only if that the Euler function $\varphi(I) \leq 21$ and $I \neq 60$ [10].

Set $G_N = \mathrm{Ker}(\alpha)$. Then we have the *basic exact sequence* below:

$$1 \longrightarrow G_N \longrightarrow G \xrightarrow{\alpha} \mu_I \longrightarrow 1.$$

For the G in the basic exact sequence, we write $G = G_N.\mu_I$, though there is no guarantee that $G = G_N : \mu_I$ (a semi-direct product).

Fact 1.0A. If G is a finite perfect group, i.e., the commutator group $[G, G] = G$ (especially if G is a non-abelian simple group like A_5), then $G = G_N$.

Fact 1.0B. G_N acts trivially on the transcendental lattice T_X (Lefschetz theorem on (1, 1)-classes).

Fact 1.0C. If a subgroup $H \leq G_N$ fixes a point P, then $H < SL(T_{X,P}) \cong SL_2(\mathbf{C})$ [11, (1.5)]. The finite subgroups of $SL_2(\mathbf{C})$ are listed up in [11, (1.6)]. These are cyclic C_n, binary dihedral (or quaternion) Q_{4n} ($n \geq 2$), binary tetrahedral T_{24}, binary octahedral O_{48} and binary icosahedral I_{120}.

Lemma 1.1. *Suppose that $G := A_5 \cdot \mu_I$ (with $G_N = A_5$) acts faithfully on a $K3$ surface X.*

(1) *The Picard number $\rho(X) \geq 19$, and $I = 1, 2, 3, 4, 6$. Moreover, $\rho(X) = 20$ if $I \geq 3$.*
(2) *We have $G = A_5 : \mu_I$, i.e., a semi-product of a normal subgroup A_5 and a subgroup μ_I of G. Moreover, $G = A_5 \times \mu_I$ if $I = 3$.*

Proof. (1) In the notation of [24, the list], $\rho(X) = \mathrm{rank}\, S_X \geq c + 1 = 19$. Also the Euler function $\varphi(I)$ divides $\mathrm{rank}\, T_X = 22 - \rho(X)$ by [13, Theorem 0.1]. So (1) follows.
(2) Let $g \in G$ such that $\alpha(g)$ is a generator of μ_I. Since $\mathrm{Aut}(A_5) = S_5 > A_5$ and the conjugation homomorphism $A_5 \to \mathrm{Aut}(A_5)$ ($x \mapsto c_x$) is an isomorphism onto A_5, the conjugation map c_g equals $c_{(12)a}$ or c_a on A_5 for some $a \in A$. Replacing g by ga^{-1}, we may assume that $c_g = c_{(12)}$ or c_{id}. Thus g^2 commutes with every element in A_5. If $2|I$, then $g^I \in \mathrm{Ker}(\alpha) = A_5$ is in the centre of A_5 (which is trivial) and hence $\mathrm{ord}(g) = I$; thus $G = A_5 : \mu_I$. If $I = 3$, then $\gcd(3, \mathrm{ord}(g)/3) = 1$ as proved in [5] or [16, Proposition 5.1]; so replacing g by g^ℓ with $\ell = \mathrm{ord}(g)/3$ (or $2\mathrm{ord}(g)/3$), we have $G = A_5 \times \langle g\rangle = A_5 \times \mu_3$. □

The third result below [13, Section 5] is crucial in classifying symplectic groups in [11]. The second uses the fact $A_5 \subset \mathrm{Aut}(X)$ in an essential way.

Lemma 1.2. (1) *Let h be a non-symplectic involution on a $K3$ surface X. Then X^h is a disjoint union of s smooth curves C_i with $0 \leq s \leq 10$. To be precise, X^h (if not empty) is either a disjoint union of a genus ≥ 2 curve C and a few $\mathbf{P}^1$'s, or a disjoint union of a few elliptic curves and $\mathbf{P}^1$'s, or a disjoint union of a few $\mathbf{P}^1$'s.*
(2) *For h in* (1), *suppose further that $A_5 \subseteq \mathrm{Aut}(X)$. Then $\chi_{\mathrm{top}}(X^h) \leq 18$.*

(3) *If δ is a non-trivial symplectic automorphism of finite order on a $K3$ surface X, then* $\operatorname{ord}(\delta) \le 8$ *and X^δ is a finite set. To be precise, if* $\operatorname{ord}(\delta) = 2, 3, 4, 5, 6, 7, 8$, *then* $|X^\delta| = 8, 6, 4, 4, 2, 3, 2$, *respectively; see* [13, *Section 5*] *for the proof. In particular, if* $A_5 \subseteq \operatorname{Aut}(X)$ *then* $\sum_{\delta \in A_5} \chi_{\text{top}}(X^\delta) = 360$ *(see 1.0A).*

Proof. (1) Locally, at a point $P \in X^h$, we have $h|P : (x, y) \to (x, -y)$ for some coordinates around P, because h is non-symplectic. Thus around P, our $X^h = \{y = 0\}$ which is smooth. For the range of s, see [14] or [25]. If X^h contains a genus ≥ 2 curve C, then the big and nefness of C and the Hodge index theorem show that the other $s - 1$ curves are negative definite, whence are $\mathbf{P}^1$'s. So (1) is true.
(2) Let $X^h = \coprod_{i=1}^s C_i$ be as in (1). Then $\chi_{\text{top}}(X^h) = \sum_{i=1}^s (2 - 2g(C_i)) \le 2s \le 20$. If (2) is false, then $s = 10$ and $C_i \cong \mathbf{P}^1$. Thus, by [17, Theorem 4], X equals X_4: the unique $K3$ surface of Picard number $\rho(X) = 20$ and $|\text{Pic } X| = -4$. Now $A_5 \subset \operatorname{Aut}(X_4)$, where the latter is given in [23]. This is impossible by the simplicity of A_5 and the precise description of $\operatorname{Aut}(X_4)$ there (see the proof of [6, Proposition 4.1(3)]). □

For an automorphism h on a smooth algebraic surface Y, we split the pointwise fixed locus as the disjoint union of the 1-dimensional part and the isolated part: $Y^h = Y^h_{1-\dim} \coprod Y^h_{\text{isol}}$. The proof of (1) below is similar to that for (1) in 1.2.

Fact 1.3. (1) $Y^h_{1-\dim}$ (if not empty) is a disjoint union of smooth curves.
(2) The Euler number $\chi_{\text{top}}(Y^h_{1-\dim}) = \sum_C (2 - 2g(C)) = 2n_h$ for some integer n_h, where C runs in the set $Y^h_{1-\dim}$ of curves.
(3) The Euler number $\chi_{\text{top}}(Y^h) = m_h + 2n_h$, where $m_h = |Y^h_{\text{isol}}|$.

The results of [5] below follow from the application of the Lefschetz fixed point formula to the trivial vector bundle in Atiyah–Segal–Singer [1,2, pages 542 and 567]. The results themselves should be very useful and informative for other studies in the future.

Important Proposition 1.4. *Let X be a $K3$ surface and let $h \in \operatorname{Aut}(X)$ be of order I such that $h^*\omega_X = \eta_I \omega_X$ for some primitive Ith root η_I of 1.*

(1) *Suppose that $I = 3$. Then $m_h = 3 + n_h$ and hence $\chi_{\text{top}}(X^h) = 3(1 + n_h)$. Moreover, $-3 \le n_h \le 6$.*
(2) *Suppose that $I = 4$. Then $m_h = 4 + 2n_h$ and hence $\chi_{\text{top}}(X^h) = 4(1 + n_h)$. Moreover, $-2 \le n_h \le 4$.*
(3) *Suppose that $I = 3$, or 4. If $\delta \in \operatorname{Aut}(X)$ is symplectic of order 5 and commutes with h. Then $|X^{h\delta}| = 4$.*
(4) *Suppose that $I = 4$. If $\delta \in \operatorname{Aut}(X)$ is symplectic of order 3 and commutes with h then $6 \ge |X^{h^2\delta}| \ge |X^{h\delta}| \in \{2, 4, 6\}$.*

Proof. (1) The first part is proved in [17, Lemma 2.3]. Note that $h^*|T_X$ can be diagonalized as $\operatorname{diag}[\eta_3, \eta_3^2]^{\oplus s}$ $(s \ge 1)$ by [13, Theorem 0.1]. So as in 1.7 below, $\chi_{\text{top}}(X^h) = 2 + \operatorname{Tr}(h^*|T_X) + \operatorname{Tr}(h^*|S_X) \le 2 - s + \operatorname{rank} S_X \le 21$, whence $n_h \le 6$. Also $m_h \ge 0$ implies that $n_h \ge -3$.
(2) As in [17, Lemma 2.3], we calculate the holomorphic Lefschetz number $L(h)$ in two ways as in [1,2, pages 542 and 567], where $X^h_{\text{isol}} = \{P_j | 1 \le j \le m_h\}$ (so

$h^*|T_{P_j} = (\eta_4^{-1}, \eta_4^2)$ up to switching the coordinates of the tangent plane at P_j), $X^h_{1-\dim} = \{C_k\}$, $gC_k = g(C_k)$ the genus, and η_4^{-1} the eigenvalue of the action h_* on the normal bundle of C_k (in the first equation below we used Serre duality, while the last is from the first two with $x = \eta_4$):

$$L(h) = \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(h^*|H^i(X, \mathcal{O}_X)) = 1 + \eta_4^{-1},$$

$$L(h) = \sum_{j=1}^{m_h} a(P_j) + \sum_k b(C_k),$$

$$a(P_j) = 1/\det(1 - h^*|T_{P_j}) = 1/(1 - \eta_4^{-1})(1 - \eta_4^2),$$

$$b(C_k) = (1 - gC_k)/(1 - \eta_4) - \eta_4 C_k^2/(1 - \eta_4)^2 = (1 - gC_k)(1 + \eta_4)/(1 - \eta_4)^2,$$

$$0 = -(1 + x^{-1}) + m_h/(1 - x^{-1})(1 - x^2) + n_h(1 + x)/(1 - x)^2.$$

Noting that $x = \eta_4$ satisfies $x^2 = -1$ and solving the last equation, we get $m_h = 4 + 2n_h$. The second part of (2) is similar to (1), noting that $h^*|T_X$ can be diagonalized as $\mathrm{diag}[\eta_4, -\eta_4]^{\oplus s}$ $(s \geq 1)$.

(3) & (4). In (4), note that $X^{h^i\delta} = X^{h^i} \cap X^\delta$ $(i = 1, 2)$. So the inequalities there hold and we have only to calculate $|X^{h\delta}|$; see 1.2.

Let $g \in \mathrm{Aut}(X)$ such that $\mathrm{ord}(g) = kI$ and $g^*\omega_X = \eta^k\omega_X$ where $\eta = \eta_{kI}$ is a primitive kIth root of 1. (We set $g = h\delta$ in (3) and (4).) If $k \geq 2$ and $\gcd(k, I) = 1$ (these are true in (3) and (4)), then g^I is of order k and symplectic, so $X^g \subseteq X^{g^I}$ is a finite set by 1.2. Namely, $X^g = X^g_{\mathrm{isol}} = \{P_j | 1 \leq j \leq m_g\}$ say. Let $M_g(i)$ be the set of points P in X^g satisfying $g^*|T_P = (\eta^{-i}, \eta^{k+i})$ (up to switching the coordinates of the tangent plane at P; so $a(P) = 1/(1 - \eta^{-i})(1 - \eta^{k+i})$ in the notation for the formula of $L(g)$). Put $m_g(i) = |M_g(i)|$. Then for $(I, k) = (3, 5)$ (the first case in (3)), we have $X^g = \coprod M_g(i)$ and $m_g = \sum_i m_g(i)$, where $i \in \{1, \ldots, 4, 11, 12\}$; for $(I, k) = (4, 5)$ (the second case in (3)), we have $m_g = \sum_i m_g(i)$, where $i \in \{1, \ldots, 4, 6, 7, 16, 17\}$; for $(I, k) = (4, 3)$ (the case in (4)), we have $m_g = \sum_i m_g(i)$, where $i \in \{1, 2, 4, 10\}$.

As in (2), we have the following, where $x = \eta = \eta_{kI}$ and i runs in the set specified above:

$$0 = -(1 + x^{-k}) + \sum_i \sum_{P \in M_g(i)} a(P)$$

$$= -(1 + x^{-k}) + \sum_i m_g(i)/(1 - x^{-i})(1 - x^{k+i}). \qquad (*)$$

For $(I, k) = (3, 5)$, x satisfies the minimal polynomial $\Phi_g(x) = 1 - x + x^3 - x^4 + x^5 - x^7 + x^8$ and also $x^{15} = 1$, $x^{10} = -1 - x^5$. Substituting these into (*) multiplied by the common denominator (which is not zero), we will get an equation of degree ≤ 7 in x with coefficients linear in $m_g(i)$. The minimality of $\Phi_g(x)$ implies that all 8 coefficients are zero. Solving these 8 linear equations, we obtain, where $m_i = m_g(i)$:

$$m_1 = m_4, \quad m_2 = -1 + m_3, \quad m_{11} = -1 + m_4, \quad m_{12} = m_3. \qquad (**)$$

By 1.2, we have $4 = m_{g^3} \geq m_g = \sum_{i=1}^{4} m_i + \sum_{i=11}^{12} m_i = -2 + 3(m_3 + m_4)$. So $m_3 + m_4 \leq 2$. This together with the condition $m_i \geq 0$ and the relations in (**), imply that $[m_1, m_2, m_3, m_4, m_{11}, m_{12}] = [1, 0, 1, 1, 0, 1]$. In particular, $m_g = 4$.

For $(I, k) = (4, 5)$, x satisfies the minimal polynomial $\Phi_g(x) = 1 - x^2 + x^4 - x^6 + x^8$ and also $x^{20} = 1$, $x^{10} = -1$. As above, solving (*), we obtain, where $m_i = m_g(i)$:

$$\begin{aligned} &m_1 = -3 + 2m_3 - 3m_4 + 4m_6 - 2m_7, \quad m_2 = -1 + m_3 - 2m_4 + 2m_6, \\ &m_{16} = -5 + 2m_3 - 4m_4 + 5m_6 - 2m_7, \quad m_{17} = 3 + 2m_4 - 2m_6 + m_7. \end{aligned} \tag{***}$$

One can check that the following is the only possibility of m_i satisfying the relations in (***) and that $0 \leq m_i \leq m_g \leq m_{g^4} = 4$ by 1.2; in particular, $m_g = 4$:

$$[m_1, m_2, m_3, m_4, m_6, m_7, m_{16}, m_{17}] = [1, 1, 0, 0, 1, 0, 0, 1].$$

For $(I, k) = (4, 3)$, x satisfies the minimal polynomial $\Phi_g(x) = 1 - x^2 + x^4$ and also $x^{12} = 1$, $x^6 = -1$. As above, solving (*), we obtain, where $m_i = m_g(i)$:

$$m_1 = 3 + 3m_2 - 2m_4, \quad m_{10} = 1 + 2m_2 - m_4. \tag{****}$$

One can check that the following are the only possibilities of m_i satisfying the relations in (****) and $0 \leq m_i \leq m_g \leq m_{g^4} = 6$, 1.2; in particular, $m_g = 2, 4, 6$ (so 1.4 is done):

$$[m_1, m_2, m_4, m_{10}] = [3, 0, 0, 1], \quad [1, 0, 1, 0], \quad [2, 1, 2, 1], \quad [0, 1, 3, 0]. \quad \square$$

The following two results can be found in [13, Theorem 0.1], [10, Lemma (1.1)], or [18, Lemma (2.8)].

Lemma 1.5. *Suppose that X is a $K3$ surface of Picard number $\rho(X) = 20$ and g an order-4 automorphism such that $g^*\omega_X = \eta_4\omega_X$ with a primitive 4th root η_4 of 1. Then we can express the transcendental lattice T_X as $T_X = \mathbf{Z}[t_1, t_2]$ so that $t_2 = g^*(t_1)$, $g^*(t_2) = -t_1$. In particular, the intersection forms $(t_i \cdot t_j) = \mathrm{diag}[2m, 2m]$ for some $m \geq 1$.*

Now we assume that $G = G_N \cdot \mu_I$ (with $I = I(G)$) acts on a $K3$ surface X. When $G_N = A_5$, we will determine the action of G_N on the Neron–Severi lattice S_X of X:

Lemma 1.6. (1) *Suppose that A_5 acts on a $K3$ surface X, and* rank $S_X = 20$ *(this is true if $I \geq 3$ by 1.1). Then we have the irreducible decomposition below (in the notation of Atlas for the characters of A_5), where χ_1 (the trivial character), χ_4 and χ_5 have dimensions 1, 4 and 5, respectively, where χ_i' is a copy of χ_i:*

$$S_X \otimes \mathbf{C} = \chi_1 \oplus \chi_1' \oplus \chi_4 \oplus \chi_4' \oplus \chi_5 \oplus \chi_5'.$$

(2) *For conjugacy class nA (and nB) of order n in A_5 and the characters χ_i of A_5, we have the following Table 1 from [3], where Z is respectively $1A$, $2A$, $3A$, $5A$ or $5B$:*

$$\begin{aligned} &[\chi_1, \chi_2, \chi_3, \chi_4, \chi_5](Z) = [1, 3, 3, 4, 5], \quad [1, -1, -1, 0, 1], \quad [1, 0, 0, 1, -1], \\ &[1, (1-\sqrt{5})/2, (1+\sqrt{5})/2, -1, 0], \quad [1, (1+\sqrt{5})/2, (1-\sqrt{5})/2, -1, 0]. \end{aligned}$$

Proof. Applying the Lefschetz fixed point formula to the action of A_5 on $H^*(X, \mathbf{Z}) = \oplus_{i=0}^{4} H^i(X, \mathbf{Z})$ and noting that $H^2(X, \mathbf{Z})$ contains $S_X \oplus T_X$ as a finite index sublattice, we obtain (see also 1.0A–1.0C and 1.2):

$$2 + \operatorname{rank} T_X + \operatorname{rank}(S_X)^{A_5} = \operatorname{rank} H^*(X, \mathbf{Z})^{A_5} = \frac{1}{|A_5|}\sum_{a\in A_5}\chi_{\rm top}(X^a) = 360/60 = 6.$$

Thus $\operatorname{rank} S_X^{A_5} = 2$. So the irreducible decomposition is of the following form, where a_i are non-negative integers:

$$S(X) \otimes \mathbf{C} = 2\chi_1 \oplus a_2\chi_2 \oplus a_3\chi_3 \oplus a_4\chi_4 \oplus a_5\chi_5.$$

As in 1.7 below, using the topological Lefschetz fixed point formula, the fact that $\operatorname{rank} T(X) = 2$ and 1.0B, we have, for $a \in A_5$, that:

$$\chi_{\rm top}(X^a) = 4 + \operatorname{Tr}(a^*|S(X)).$$

Running a through the five conjugacy classes and calculating both sides, using 1.2 and the character Table 1 in (2), we obtain the following system of equations:

$$20 = 2 + 3(a_2 + a_3) + 4a_4 + 5a_5,$$
$$4 = 2 - (a_2 + a_3) + a_5,$$
$$2 = 2 + a_4 - a_5,$$
$$0 = 2 + \frac{1-\sqrt{5}}{2}a_2 + \frac{1+\sqrt{5}}{2}a_3 - a_4,$$
$$0 = 2 + \frac{1+\sqrt{5}}{2}a_2 + \frac{1-\sqrt{5}}{2}a_3 - a_4.$$

Now, we get the result by solving this system of Diophantine equations. □

1.7. Note that $\operatorname{Aut}(A_5) = S_5$. For a group $G = A_5 \cdot \mu_I$ (and the map α) in 1.0, we have the natural homomorphism below, which is injective (since its restriction on A_5 is an injection onto $A_5 \times (1)$), where $c_x : a \mapsto c_x(a) = x^{-1}ax$ is the conjugation map:

$$G \longrightarrow \operatorname{Aut}(A_5) \times \mu_I = S_5 \times \mu_I,$$
$$x \mapsto (c_x, \alpha(x)).$$

Lemma. *Suppose that $G = A_5{\cdot}\mu_4$ acts on a $K3$ surface X (i.e., $G_N = A_5$ and $I(G) = 4$). Then $G = A_5 : \mu_4$, but $G \ne A_5 \times \mu_4$. Our $G \to S_5 \times \mu_4$ ($x \mapsto (c_x, \alpha(x))$) is an injective homomorphism and the group structure of such G is unique up to isomorphisms.*

Proof. By 1.1, we have $G = A_5 : \mu_4$. Suppose the contrary $G = A_5 \times \mu_4$. Write $\mu_4 = \langle g\rangle$. In the notation of 1.6, the g either stabilizes χ_i or switches χ_i with χ_i' ($i = 4$ or 5; then denoted as $\chi_i \overset{g}{\longleftrightarrow} \chi_i$, and $\operatorname{Tr}(g^*|(\chi_i \oplus \chi_i')) = 0$)). Since G stabilizes an ample line bundle (the pull back of an ample line bundle on X/G) and since G acts on $S_X^{A_5}$ (whose $\mathbf{C}$-extension is $\chi_1 \oplus \chi_1'$), we may assume that $g^*|(\chi_1 \oplus \chi_1') = \operatorname{diag}[1, \pm 1]$ w.r.t. a suitable basis. If χ_i is g-stable then $g^*|\chi_i$ is a scalar ζ_4^c with $\zeta_4 = \exp(2\pi\sqrt{-1}/4)$, by Schur's lemma.

Let $a \in A_5$. Then $(ga)^*|T_X = g^*|T_X$ (see 1.0B) and the latter can be diagonalized as $\mathrm{diag}[\zeta_4, \zeta_4^{-1}]$ by [13, Theorem 0.1] and 1.1. Hence $\mathrm{Tr}(ga)^*|T_X = 0$. By the topological Lefschetz fixed point formula and noting that $H^2(X, \mathbf{Z})$ contains $S_X \oplus T_X$ as a sublattice of finite index, we have $\chi_{\mathrm{top}}(X^{ga}) = \oplus_{i=0}^4 \mathrm{Tr}(ga)^*|H^i(X, \mathbf{Z}) = 2 + \mathrm{Tr}(ga)^*|S_X + \mathrm{Tr}(ga)^*|T_X = 2 + \mathrm{Tr}(ga)^*|S_X$. For $a = 5A$ (an order-5 element) in A_5, by 1.4 and Table 1 in 1.6 (and Schur's lemma), we have: $4 = \chi_{\mathrm{top}}(X^{g5A}) = 2 + \mathrm{Tr}(g^*|\chi_1 \oplus \chi_1') + \mathrm{Tr}(g5A)^*|(\chi_4 \oplus \chi_4) + 0$, so one of the following cases occurs (using Schur's lemma):
Case (i). $g^*|S_X \otimes \mathbf{C} = \mathrm{diag}[1, -1, -I_4, -I_4, ?, ?]$,
Case (ii). $g^*|S_X \otimes \mathbf{C} = \mathrm{diag}[1, 1, \chi_4 \overset{g}{\longleftrightarrow} \chi_4, ?, ?]$,
Case (iii). $g^*|S_X \otimes \mathbf{C} = \mathrm{diag}[1, 1, I_4, -I_4, ?, ?]$,
Case (iv). $g^*|S_X \otimes \mathbf{C} = \mathrm{diag}[1, 1, \zeta_4 I_4, \zeta_4^{-1} I_4, ?, ?]$.

By 1.4, we have (*): $-4 \le \chi_{\mathrm{top}}(X^g) = 4(1 + n_g) = 0 \pmod 4$ with $-2 \le n_g \le 4$. So $\chi_{\mathrm{top}}(X^g) = 4$ in Cases (ii), (iii) and (iv) (using Schur's lemma). Thus $n_g = 0$ and $m_g = 4 + 2n_g = 4$ by 1.4. Now A_5 (commuting with g) acts on the four isolated points P_i in X^g, whence fixing these four points (see 1.8 below). So $A_5 < SL(T_{X,P_1})$, contradicting 1.0C. In Case (i), by the fact (*) above and Schur's lemma, we have $\chi_{\mathrm{top}}(X^g) = 2 + (1 - 1 - 4 - 4 + 5 + 5) = 4$, which will lead to the same contradiction.

By the proof of 1.1 and the result in the above paragraph, we may assume that there is an order-4 element $g \in G$ such that $\alpha(g)$ is the generator of μ_4, so that $G = A_5 : \langle g \rangle = A_5 : \mu_4$ and the conjugation map $c_g = c_{(12)}$ on A_5. Clearly, the group structure of G is unique. The lemma is proved. □

The two results below are either easy or well known and will be frequently used in the arguments of the subsequent sections.

Lemma 1.8. *Let $f : A_5 \to S_r$ $(r \ge 2)$ be a homomorphism.*
(1) *If $r = 2, 3,$ or 4, then f is trivial.*
(2) *If $\mathrm{Im}(f)$ is a transitive subgroup of the full symmetry group S_r in r letters $\{1, 2, \dots, r\}$ (whence $r \ge 5$ by (1)), then $r \| A_5|$ with $|A_5|/r$ equal to the order of the subgroup of A_5 stabilizing the letter 1, so $r \in \{5, 6, 10, 12, 15, 20, 30\}$.*

Lemma 1.9. (1) $\mathrm{Aut}(\mathbf{P}^1)$ *includes A_5 but not S_5* [22, *Theorem 6.17*].
(2) *If $\mathrm{id} \ne f \in \mathrm{Aut}(\mathbf{P}^1)$ is an automorphism of finite order, then f fixes exactly two points of $\mathbf{P}^1$ (by the diagonalization of a lifting of f to $GL_2(\mathbf{C})$).*
(3) *If f_r $(r = 2$ or $3)$ is an order $-r$ automorphism of an elliptic curve E, then either f_r acts freely on E, or the fix locus satisfies $|X^{f_r}| = 4$ (resp. $= 3$) if $r = 2$ (resp. $r = 3$) (by the Hurwitz formula).*

The examples below are to show the existence of the groups in Theorems A and B.

Example 1.10. (1) *$G = G_N \cdot \mu_I = S_5 \times \mu_2$ (with $G_N = S_5$, $I = 2$) acts on a $K3$.*

Let $X = \{\sum_{i=1}^5 X_i = \sum_{i=1}^6 X_i^2 = \sum_{i=1}^5 X_i^3 = 0\} \subset \mathbf{P}^5$. We define the symplectic action of $\sigma \in S_5$ on X (the same as in [11, no 3]) and a non-symplectic involution g on X as follows (see [11, Lemma 2.1]):

$$\sigma : [X_1 : \cdots : X_6] \mapsto [X_{\sigma(1)} : \cdots : X_{\sigma(5)} : (\mathrm{sign}\,\sigma) X_6],$$
$$g : [X_1 : \cdots : X_6] \mapsto [X_1 : \cdots : X_5 : -X_6].$$

Let $G = \langle S_5, g \rangle$. Then $G = S_5 \times \langle g \rangle$ is the required one.

(2) $G = G_N.\mu_I = A_5 : \mu_2 = S_5$ (*with* $G_N = A_5$, $I = 2$) *acts on a* $K3$ *surface*.

Let $X = \{\sum_{i=1}^{6} X_i = \sum_{i=1}^{6} X_i^2 = \sum_{i=1}^{6} X_i^3 = 0\} \subset \mathbf{P}^5$. We define the action of $\sigma \in S_6$ on X (the same as in [11, no 2]):

$$\sigma : [X_1 : \cdots : X_6] \mapsto [X_{\sigma(1)} : \cdots : X_{\sigma(6)}].$$

Since A_6 is perfect, its action on X is symplectic 1.0A. If we let $\widetilde{G} = S_6$, then $\widetilde{G} = \widetilde{G}_N.\mu_2$ with $\widetilde{G}_N = A_6$ and $I = 2$ (see [11, Lemma 2.1]). Now a subgroup $G = S_5$ of $\widetilde{G}$ is the required one.

2. The determination of some topological invariants

Let X be a $K3$ surface with a faithful action by a group of the form $G := A_5 \cdot \mu_4$ as in 1.0. Then $G = A_5 : \mu_4$ and the *unique* group structure of such G is given in 1.7.

We will use the notation in 1.6. Let g be a generator of $\mu_4 < G$. We may also assume the following is true (after a change of g):

Lemma 2.1. (1) *The conjugation action* $c_g(.) = c_{(12)}(.)$ *on* A_5. *So* $\langle g^2 \rangle$ *is in the centre of* G *and* $G \to \mathrm{Aut}(A_5) = S_5$ ($x \mapsto c_x$) *induces an isomorphism* $G/\langle g^2 \rangle \cong S_5$.

(2) $g^*\omega_X = \zeta_4\omega_X$ *with* $\zeta_4 = \exp(2\pi\sqrt{-1}/4)$.

(3) g^2 *is a non-symplectic involution on* X *and commutes with every element in* A_5.

(4) *Set* $\sigma = (12)(34)$ *and* $\tau = (345)$. *Then* g *commutes with every element in* $\langle \sigma, \tau \rangle = S_3$. *So* $G = A_5 : \mu_4 > S_3 \times \mu_4$.

(5) *Set* $\sigma = (12)(34)$, $\gamma = (123)$. *Then* g *normalizes* $\langle \sigma, \gamma \rangle = A_4$. *So* $G = A_5 : \mu_4 > A_4 : \mu_4$. *Set* $\sigma_1 = \sigma$ *and* $\sigma_2 = (13)(24)$ *(all in* A_4*).*

(6) g *stabilizes both* χ_1 *and* χ_1'; *the restrictions* $g^*|\chi_1 = \mathrm{id}$ *and* $g^*|\chi_1' = \pm\mathrm{id}$ *(after a change of basis).*

(7) g *either stabilizes both* χ_4 *and* χ_4' *(so the restrictions of* g^* *on* χ_4 *and* χ_4' *are scalar multiplications), or switches* χ_4 *with* χ_4'.

(8) g *either stabilizes both* χ_5 *and* χ_5' *(so the restrictions of* g^* *on* χ_5 *and* χ_5' *are scalar multiplications), or switches* χ_5 *with* χ_5'.

(9) *Both* $g^2|\chi_i$ *and* $g^2|\chi_i'$ ($i = 4, 5$) *are scalar multiplications.*

Proof. (1) is from the last part of the proof of 1.7. (2) is true because g is a generator of $\mu_4 < G = A_5 : \mu_4$. (3), (4) and (5) follow from (1). (6) is true because $G = A_5 : \langle g \rangle$ stabilizes one ample line bundle (the pull back of an ample line bundle on X/G) and g acts on $S_X^{A_5}$ (defined over $\mathbf{Z}$) whose $\mathbf{C}$-extension is $\chi_1 \oplus \chi_1'$. (7), (8) and (9) are from the form of the decomposition in 1.6 and Schur's lemma. □

In the rest of the section, we will prove the Key result 2.2 below which will be used in the proof of Theorems A, B and C in Section 3 and is the consequence of 2.6–2.9 below. The representation theory (mainly on A_5) is fully applied. We divide into cases according to whether g stabilizes or switches χ_i ($i = 4, 5$).

Key Proposition 2.2. *Suppose that $G = A_5 : \mu_4$ acts on a $K3$ surface X. Then with the notation in 2.1 and 1.4, $(n_g, m_g;\ \chi_{\text{top}}(X^g), \chi_{\text{top}}(X^{g\tau}), \chi_{\text{top}}(X^{g^2\tau}), \chi_{\text{top}}(X^{g^2}))$ is one of the following:*

$$(1, 6; 8, 2, 6, 0), \quad (0, 4; 4, 4, 6, 0), \quad (-1, 2; 0, 6, 6, 0).$$

The result below is used in 2.4 to determine the representation of $S_3 \times \mu_4 < G$ there.

Lemma 2.3. (1) *Suppose that g stabilizes χ_4. Then w.r.t. one and the same basis $\{v_1, \dots, v_4\}$, we have the following matrix representation of $A_4 : \mu_4$ on χ_4:*

$$\sigma_1^* = \text{diag}[1, 1, -1, -1], \qquad \sigma_2^* = [1, -1, 1, -1],$$

$$\gamma^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \end{pmatrix}, \qquad g^* = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & \alpha_4 & 0 \end{pmatrix}.$$

We have exactly the same kind of matrix representation of $A_4 : \mu_4$ w.r.t. one and the same basis $\{v'_1, \dots, v'_4\}$ of χ'_4. But we use β'_i and α'_i for $\gamma^*|\chi'_4$ and $g^*|\chi'_4$ instead.

(2) *Suppose g stabilizes χ_5. Then w.r.t. one and the same basis $\{y_1, \dots, y_5\}$, we have the following matrix representation of $A_4 : \mu_4$ on χ_5, where η_3 is a primitive 3rd root of 1:*

$$\sigma_1^* = \text{diag}[1, 1, 1, -1, -1], \qquad \sigma_2^* = [1, 1, -1, 1, -1],$$

$$\gamma^* = \begin{pmatrix} \eta_3 & 0 & 0 & 0 & 0 \\ 0 & \eta_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 \end{pmatrix}, \qquad g^* = \begin{pmatrix} 0 & a_2 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix}.$$

We have exactly the same kind of matrix representation of $A_4 : \mu_4$ w.r.t. one and the same basis $\{y'_1, \dots, y'_5\}$ of χ'_5. But we use b'_i and a'_i for $\gamma^*|\chi'_5$ and $g^*|\chi'_5$ instead.

Proof. This follows from the character Table 1 in 1.6 and the fact that the conjugation c_g fixes σ_1, and exchanges σ_2 with $\sigma_1\sigma_2$ and γ with γ^{-1}. □

Lemma 2.4. (1) *Suppose that g stabilizes χ_4. Then w.r.t. one and the same basis $\{u_1, \dots, u_4\}$, we have the following matrix representation of $S_3 \times \mu_4$ on χ_4, where η_3 is a primitive 3rd root of 1. Moreover, $d_3 = \pm d_1$ and $(g^2)^*|\chi_4 = d_1^2\,\text{id}$:*

$$\tau^* = [1, 1, \eta_3, \eta_3^2], \qquad g^* = \text{diag}[d_1, -d_3, d_3, d_3],$$

$$\sigma^* = \text{diag}\left[1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

We have exactly the same kind of matrix representation of $S_3 \times \mu_4$ w.r.t. one and the same basis $\{u'_1, \dots, u'_4\}$ of χ'_4. But we use d'_i for $g^*|\chi'_4$ instead.

(2) *Suppose that g stabilizes χ_5. Then w.r.t. one and the same basis $\{x_1, \dots, x_5\}$, we have the following matrix representation of $S_3 \times \mu_4$ on χ_5, where η_3 is a primitive 3rd root*

of 1*. Moreover,* $e_2 = \pm e_1$, $(g^2)^*|\chi_5 = e_1^2\,\mathrm{id}$ *(and* e_1 *equals* a_3 *in* 2.3*):*

$$\tau^* = \mathrm{diag}[1, \eta_3, \eta_3^2, \eta_3, \eta_3^2], \qquad g^* = [e_1, e_2, e_2, -e_2, -e_2],$$
$$\sigma^* = \mathrm{diag}\left[1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

We have exactly the same kind of matrix representation of $S_3 \times \mu_4$ *w.r.t. one and the same basis* $\{x'_1, \ldots, x'_5\}$ *of* χ'_5*. But we use* e'_i *for* $g^*|\chi'_5$*, instead.*

Proof. (1) follows from the character Table 1 in 1.6 and the fact that g commutes with both σ, τ, if we claim only $g^*|\chi_4 = \mathrm{diag}[d_1, d_2, d_3, d_3]$ instead. It suffices to show that $d_2 = -d_3$. On the one hand, over the eigenspace $V_4(\sigma = -1) \subset \chi_4$ of σ corresponding to the eigenvalue -1, we have $g^*|V_4(\sigma = -1) = \mathrm{diag}[d_2, d_3]$. On the other hand, by 2.3, $g^*|V_4(\sigma = -1) = \mathrm{diag}[\sqrt{\alpha_4\alpha_5}, -\sqrt{\alpha_4\alpha_5}]$. Thus $d_2 = -d_3$. Now $d_1 = \pm d_3$ follows from the fact that $(g^2)^*|\chi_i$ is a scalar.

(2) is similar, except the determination of e_i in $g^* = \mathrm{diag}[e_1, e_2, e_2, e_4, e_4]$. Indeed, comparing the diagonalization in 2.3 and here we see also that $\mathrm{diag}[e_2, e_4] = g^*|V_5(\sigma = -1) = \mathrm{diag}[\sqrt{a_4a_5}, -\sqrt{a_4a_5}]$, whence $e_4 = -e_2$. Taking the trace in 2.3 and here, we obtain $a_3 = \mathrm{Tr}(g^*|\chi_5) = e_1$. □

Lemma 2.5. (1) *Suppose that* g *switches* χ_4 *with* χ'_4*. Then w.r.t. one and the same basis* $\{u_1, \ldots, u_8\}$*, we have the following matrix representation of* $S_3 \times \mu_4$ *on* $\chi_4 \oplus \chi'_4$*, where* η_3 *is a primitive* 3*rd root of* 1*. Moreover,* $(g^2)^*|\chi_4 = (d_1d_5)\,\mathrm{id} = (g^2)^*|\chi'_4$*:*

$$\tau^* = [1, 1, \eta_3, \eta_3^2, 1, 1, \eta_3, \eta_3^2],$$
$$\sigma^* = \mathrm{diag}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$
$$g^* = \begin{pmatrix} 0 & 0 & 0 & 0 & d_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_8 \\ d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(2) *Suppose that* g *switches* χ_5 *with* χ'_5*. Then w.r.t. one and the same basis* $\{x_1, \ldots, x_{10}\}$*, we have the following matrix representation of* $S_3 \times \mu_4$ *on* $\chi_5 \oplus \chi'_5$*, where* η_3 *is a primitive* 3*rd root of* 1*. Moreover,* $(g^2)^*|\chi_5 = (e_1e_6)\,\mathrm{id} = (g^2)^*|\chi'_5$*:*

$$\tau^* = [1, \eta_3, \eta_3^2, \eta_3, \eta_3^2, 1, \eta_3, \eta_3^2, \eta_3, \eta_3^2],$$
$$\sigma^* = \mathrm{diag}\left[1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

$$g^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_9 \\ e_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. The proof is similar to 2.4. □

To prove 2.2, we consider first the case where both χ_4 and χ_5 are g-stable:

Lemma 2.6. *Suppose that both χ_4 and χ_5 are g-stable.*

(1) *We have the following, where by $\sum d_1$, etc. we mean $d_1 + d_1'$, etc.:*

$$\chi_{\text{top}}(X^{g^{\pm}}) = 2 + \text{Tr}(g^*|\chi_1 \oplus \chi_1') + \sum(d_1 + d_3 + e_1),$$
$$\chi_{\text{top}}(X^{g^{-1}\tau^{\mp}}) = \chi_{\text{top}}(X^{g\tau^{\pm}}) = 2 + \text{Tr}(g^*|\chi_1 \oplus \chi_1') + \sum(d_1 - 2d_3 + e_1),$$
$$\chi_{\text{top}}(X^{g^2}) = 2 + \sum(4d_1^2 + 5e_1^2),$$
$$\chi_{\text{top}}(X^{g^2\tau^{\pm}}) = 2 + \sum(d_1^2 - e_1^2).$$

(2) *We have $d_1^4 = e_1^4 = (d_1')^4 = (e_1')^4 = 1$ and $d_3 \in \{\pm d_1\}$, $d_3' \in \{\pm d_1'\}$.*
(3) *Among six 4th roots of 1: e_1, e_1', d_i, d_i' ($i = 1, 3$), either all six of them are primitive, or exactly e_1, e_1' are primitive, or exactly the d_i, d_i' ($i = 1, 3$) are primitive 4th roots of 1.*
(4) *2.2 holds.*

Proof. (1) and (2) follow from 2.4. For (3), the formula for $\chi_{\text{top}}(X^{g^2})$ in (1) and its upper bound 18 in 1.2 imply that at least one of the six 4th roots of 1 in (3) is primitive. Now (3) is a consequence of (2) and the description of $\chi_{\text{top}}(X^g)$ and $\chi_{\text{top}}(X^{g\tau})$ in (1) and the difference (i.e., $3\sum d_3 = 3(d_3 + d_3')$) of which must be real numbers (indeed, integers).

To prove (4), we apply (3). If exactly these four: d_i, d_i' ($i = 1, 3$) are primitive 4th roots of 1, then $\chi_{\text{top}}(X^{g^2\tau}) = 2 + (-2) - 2 < 0$, contradicting 1.4. If all these six in (3) are primitive 4th roots of 1, then $\chi_{\text{top}}(X^g)$ and $\chi_{\text{top}}(X^{g\tau})$, given in (1) and being real numbers, must all be equal to $2 + \text{Tr}(g^*|\chi_1 \oplus \chi_1')$; hence they are all equal to 4 — the only possible common value of these two, by 1.4; but then $\chi_{\text{top}}(X^{g^2\tau}) = 2 + (-2) - (-2) = 2 < 4 = \chi_{\text{top}}(X^{g\tau})$, a contradiction to 1.4.

Thus, exactly e_1, e_1' are primitive 4th roots of 1, while $d_i, d_i' \in \{\pm 1\}$ ($i = 1, 3$). So (*): $-2 \le \chi_{\text{top}}(X^g) \le 8$. Also $\chi_{\text{top}}(X^{g^2}) = 2 + 4 \times 2 + 5 \times (-2) = 0$ and $\chi_{\text{top}}(X^{g^2\tau^{\pm}}) = 2 + 2 - (-2) = 6$. Now (1) implies that $\chi_{\text{top}}(X^{g\tau^{\pm}}) + 3\sum d_3 = \chi_{\text{top}}(X^g) = 0 \pmod 4$ by 1.4, and also $\sum d_3 = d_3 + d_3' \in \{0, \pm 2\}$ and $\chi_{\text{top}}(X^{g\tau^{\pm}}) \in \{2, 4, 6\}$ by 1.4. These and (*) above infer that the cases in 2.2 occur. The lemma is proved. □

The first two assertions below are consequences of 2.4 and 2.5 and an argument similar to 2.6.

Lemma 2.7. *Suppose that g switches χ_4 with χ_4' but keeps χ_5 (and χ_5') stable.*
(1) *We have the following, where $\delta \in S_3 = \langle \sigma, \tau \rangle$ and by $\sum e_1$ etc. we mean $e_1 + e_1'$ etc.:*

$$\chi_{\mathrm{top}}(X^{g^{-1}\delta^{-1}}) = \chi_{\mathrm{top}}(X^{g\delta}) = 2 + \mathrm{Tr}(g^*|\chi_1 \oplus \chi_1') + \sum e_1,$$
$$\chi_{\mathrm{top}}(X^{g^2}) = 2 + 8d_1d_5 + 5\sum e_1^2,$$
$$\chi_{\mathrm{top}}(X^{g^2\tau^\pm}) = 2 + 2d_1d_5 - \sum e_1^2.$$

(2) *We have $e_1^4 = (e_1')^4 = (d_1d_5)^2 = 1$. Either $\{e_1, e_1'\} = \{\pm\sqrt{-1}\}$, or $e_1, e_1' \in \{\pm 1\}$.*
(3) *2.2 holds.*

Proof. To prove (3), by (1) $\chi_{\mathrm{top}}(X^g)$ $(= 0 \bmod 4)$ and $\chi_{\mathrm{top}}(X^{g\tau})$ $(\in \{2, 4, 6\})$ are equal (see 1.4). Hence they are all equal to 4. If both e_1, e_1' are in $\{\pm 1\}$, then $\chi_{\mathrm{top}}(X^{g^2\tau}) = 2 + 2d_1d_5 - 2 \le 2 < 4 = \chi(X^{g\tau})$, contradicting 1.4. Thus, $\{e_1, e_1'\} = \{\pm\sqrt{-1}\}$. By 1.4, we have $4 = \chi_{\mathrm{top}}(X^{g\tau}) \le \chi_{\mathrm{top}}(X^{g^2\tau}) = 2 + 2d_1d_5 + 2$, whence the latter equals 6 and $d_1d_5 = 1$. Now $\chi_{\mathrm{top}}(X^{g^2}) = 2 + 8 + 5 \times (-2) = 0$. Therefore, the second case in 2.2 occurs. This proves the lemma. □

Lemma 2.8. *Suppose that χ_4 (and χ_4' are) is g-stable but g switches χ_5 with χ_5'.*
(1) *We have the following, where by $\sum d_1$ etc. we mean $d_1 + d_1'$ etc.:*

$$\chi_{\mathrm{top}}(X^{g^\pm}) = 2 + \mathrm{Tr}(g^*|\chi_1 \oplus \chi_1') + \sum(d_1 + d_3),$$
$$\chi_{\mathrm{top}}(X^{g^{-1}\tau^\mp}) = \chi_{\mathrm{top}}(X^{g\tau^\pm}) = 2 + \mathrm{Tr}(g^*|\chi_1 \oplus \chi_1') + \sum(d_1 - 2d_3),$$
$$\chi_{\mathrm{top}}(X^{g^2}) = 2 + 4\sum d_1^2 + 10e_1e_6,$$
$$\chi_{\mathrm{top}}(X^{g^2\tau^\pm}) = 2 + \sum d_1^2 - 2e_1e_6.$$

(2) *We have $d_1^4 = (d_1')^4 = (e_1e_6)^2 = 1$ and $d_3 \in \{\pm d_1\}$, $d_3' \in \{\pm d_1'\}$.*
(3) *Either the four 4th roots of 1: d_i, d_i' $(i = 1, 3)$ are all in $\{\pm\sqrt{-1}\}$, or these four are all in $\{\pm 1\}$ (so $e_1e_6 = -1$ and $\chi_{\mathrm{top}}(X^{g^2}) = 0$ by 1.2).*
(4) *2.2 holds.*

Proof. (1)–(2) are consequences of 2.5 and 2.6, while the proof of (3)–(4) are similar to the argument for the case of 2.6. Indeed, if the first (resp. second) situation in (3) occurs, then a contradiction (resp. 2.2 holds). This proves the lemma. □

Lemma 2.9. *Suppose that g switches χ_4 with χ_4' and χ_5 with χ_5'. Then 2.2 holds.*

To be precise, we have the following, where δ is in $S_3 = \langle \sigma, \tau \rangle$, where $(d_1d_5)^2 = (e_1e_6)^2 = 1$:

$$\chi_{\mathrm{top}}(X^{g^{-1}\delta^{-1}}) = \chi_{\mathrm{top}}(X^{g\delta}) = 2 + \mathrm{Tr}(g^*|\chi_1 \oplus \chi_1'),$$
$$\chi_{\mathrm{top}}(X^{g^2}) = 2 + 8d_1d_5 + 10e_1e_6,$$
$$\chi_{\mathrm{top}}(X^{g^2\tau^\pm}) = 2 + 2d_1d_5 - 2e_1e_6.$$

Proof. The formulae or equalities are consequences of 2.4 and 2.5. As in 2.7, the formulae in (1) and 1.4 imply that $\chi_{\text{top}}(X^g) = \chi_{\text{top}}(X^{g\tau}) = 4$. The formula for $\chi_{\text{top}}(X^{g^2\tau})$ and its lower bounder $4 = \chi_{\text{top}}(X^{g\tau})$ by 1.4 infer that it equals 6 and $d_1d_5 = 1$, $e_1e_6 = -1$. This proves the lemma. The proof of 2.2 is completed. □

3. The proofs of Theorems A–C

In this section we shall prove Theorems A–C. We first prove the result below which is a consequence of 3.2–3.8 below.

Theorem 3.1. (1) *There is no faithful group action of the form $A_5 \cdot \mu_4$ (see 1.0) on a $K3$ surface.*
(2) *If $A_5 \cdot \mu_I$ acts faithfully on a $K3$ surface, then $I = 1$, or 2.*

(2) follows from (1), 1.1 and [26, Theorem 3.1]. Let us prove 3.1 (1). Suppose the contrary that $G := A_5 \cdot \mu_4$ acts on a $K3$ surface X. Then $G = A_5 : \mu_4$ and the *unique* group structure of such G is given in 1.7. We use the notation in 2.1 and 2.2. First, we need:

Proposition 3.2. *Suppose that $G = A_5 : \mu_4$ acts on a $K3$ surface X. Then with the notation in 2.1, the fixed locus $X^{g^2} = C \coprod_{i=1}^{6} D_i$ is a disjoint union of a genus-7 curve C (hence $C^2 = 12$) and six smooth rational curves. Both C and $\sum_{i=1}^{6} D_i$ are G-stable.*

Proof. We apply 2.2. Then we always have $\chi_{\text{top}}(X^{g^2}) = 0$. Also 1.4 implies that $X^{g^2} \supseteq X^g \neq \emptyset$, so either $X^{g^2} = \coprod_{i=1}^{s} E_i$ with $1 \le s \le 10$ (by 1.2) is a disjoint union of a few smooth elliptic curves E_i (so $X^g_{1-\text{dim}}$ is, if not empty, a disjoint union of some of the E_i's, and hence $n_g = 0$ in the notation of 1.4), or $X^{g^2} = C \coprod_{i=1}^{s} D_i$ is a disjoint union of a smooth curve C and s smooth rational curves D_i with $9 \ge s = g(C) - 1 \ge 1$ (see 1.2).

Consider the case where $X^{g^2} = \coprod_{i=1}^{s} E_i$. Then $n_g = 0$ and $(n_g, m_g; \chi_{\text{top}}(X^g), \chi_{\text{top}}(X^{g\tau}), \chi_{\text{top}}(X^{g^2\tau}), \chi_{\text{top}}(X^{g^2})) = (0, 4; 4, 4, 6, 0)$. Note that $|X^g_{\text{isol}}| = m_g = 4$. We may assume that E_1 contains an isolated g-fixed point. Since the restriction $g|E_1$ is now of order 2, this E_1 contains all four isolated g-fixed points by 1.9. Now g commutes with every element of $\langle \sigma, \tau \rangle = S_3$ as mentioned in 2.1, and hence there is a natural homomorphism $S_3 \to S_4$ (= the full symmetry group of the 4-point set X^g_{isol}). By 1.2 and 1.9, the restriction $\tau | X^g_{\text{isol}} \neq \text{id}$. So the image of this homomorphism equals one of the four 1-point (say P_1) stabilizer subgroups ($\cong S_3$) in S_4. This leads to that $S_3 < SL(T_{X,P_1})$, contradicting 1.0C.

Next we consider the case where $X^{g^2} = C \coprod_{i=1}^{s} D_i$. We claim that $s = 1, 5, 6$. Since g^2 is in the centre of G by 2.1, our G acts on X^{g^2} and hence stabilizes C and permutes D_i's. Note that C and the A_5-orbits of $\{D_1, \ldots, D_s\}$ will give linearly independent classes in $S_X^{A_5} \otimes \mathbf{Q}$. Since the latter is of rank 2 by 1.6, this A_5 acts transitively on the set $\{D_1, \ldots, D_s\}$ and hence the image of the natural homomorphism $A_5 \to S_s$ is a transitive subgroup of S_s. Now the claim follows from 1.8.

We assert that C is not g-fixed. Indeed, let $\delta = (13)(24)$, then $c_\delta(g) = g\sigma$ with $\sigma = (12)(34)$ (because $c_g = c_{(12)}$ on A_5). Hence $X^{g\sigma} = \delta(X^g)$. So $\delta(C)$ is contained in $X^{g\sigma} \subseteq X^{g^2}$ (noting that $(g\sigma)^2 = g^2$), whence it equals the unique curve C of genus ≥ 2

in X^{g^2}. Thus $C = \delta(C)$ is pointwise $g\sigma$-fixed. However, C is also pointwise g-fixed, whence it is pointwise σ-fixed. This contradicts 1.2. So the assertion is proved.

We claim that $s = 1$ is impossible. Consider the case $s = 1$. Then $G = A_5 : \langle g \rangle$ acts on the set $\{C, D_1\}$ and hence stabilizes both C and D_1. If D_1 is pointwise g-fixed, then as above, D_1 would be pointwise ($g\sigma$ and hence) σ-fixed, a contradiction. So the restriction $g|D_1$ is not identity. We consider the natural homomorphism $f : S_5 = A_5 : \langle \overline{g} \rangle = G/\langle g^2 \rangle \to \mathrm{Aut}(D_1)$ (see 2.1), where $\overline{g}$ is the coset in $\langle g \rangle / \langle g^2 \rangle$ containing g. Clearly, the restriction $f|A_5$ is an injection by 1.2. Hence $|\mathrm{Ker}(f)| \le 2$ and $\mathrm{Ker}(f)$ is normal in S_5. So $\mathrm{Ker}(f) = (1)$ and $S_5 \cong f(S_5) < \mathrm{Aut}(\mathbf{P}^1)$, contradicting 1.9.

We still have to rule out the case $s = 5$. Since C is not pointwise g-fixed as proved above, $X^g_{1-\mathrm{dim}}$ is (if not empty) a disjoint union of $n_g/2$ (≥ 0) of D_i's. If $\tau = (345)$ stabilizes some D_j then τ fixes exactly two points on D_j by 1.2 and 1.9. Since $|X^\tau| = 6$, this τ stabilizes at most three D_j's. Thus we may assume that τ permutes D_1, D_2, D_3 while it stabilizes D_4, D_5. Now the commutability of g with τ implies that g stabilizes each D_i ($i = 1, 2, 3$); also none of D_i ($i = 1, 2, 3$) is pointwise g-fixed, for otherwise all these three D_i (forming one τ-orbit) are pointwise g-fixed, whence $n_g \ge 3$, contradicting 2.2. Thus, $m_g = |X^g_{\mathrm{isol}}| \ge \sum_{i=1}^{3} |D_i^g| = 6$. So the first case in 2.2 occurs and $n_g = 1$, $m_g = 6$. Here $n_g = 1$ implies that (after switching D_4 with D_5 if necessary) D_5 is pointwise g-fixed, and D_4 is g-stable but not g-fixed. This leads to $6 = |X^g_{\mathrm{isol}}| \ge \sum_{i=1}^{4} |D_i^g| = 8$, a contradiction. So 3.2 is proved. Indeed, for the last part, note that g^2 is in the centre of G by 2.1 and hence G acts on X^{g^2}. □

We continue the proof of 3.1 (1). In the notation of 3.2, we set $D = \sum_{i=1}^{6} D_i$ and $L_0 := \mathbf{Z}[C, D]$. Then we have:

Lemma 3.3. *Suppose that $G = A_5 : \mu_4$ acts on a $K3$ surface X.*

(1) *L_0 is a sublattice (with intersection form* $\mathrm{diag}[12, -12]$*) of $S_X^{A_5}$ of finite index d_1. In particular, $S_X^G = S_X^{A_5}$, i.e., $g^*|S_X^{A_5} = \mathrm{id}$.*

(2) *If $d_1 > 1$, then $d_1 = 2$ and $S_X^{A_5}$ equals $\mathbf{Z}[u_1, u_2]$ with $u_1 = \frac{1}{2}(C + D)$ and $u_2 = \frac{1}{2}(C - D)$ and with the intersection form $U(6)$, i.e., $u_i^2 = 0$ and $u_1 \cdot u_2 = 6$.*

Proof. (1) Clearly, $S_X^{A_5} \supseteq S_X^G \supseteq L_0$ by 3.2. Now (1) follows from the fact that rank $S_X^{A_5} = 2$ by 1.6.

(2) Suppose that $d_1 > 1$. Let $\theta = \frac{1}{12}(aC + bD)$ be in $S_X^{A_5} \subseteq L_0^\vee = \mathrm{Hom}(L_0, \mathbf{Z}) = \mathbf{Z}[C/12, D/12]$ but not in L_0. Since $-2b/12 = \theta \cdot D_1 \in \mathbf{Z}$, we have $6|b$. This and $(a^2 - b^2)/12 = \theta^2 \in \mathbf{Z}$ imply that 12 divides a^2, whence $6|a$. So modulo L_0, our $\theta = C/2$, or $D/2$ or $(C + D)/2$. Since $\theta^2 \in 2\mathbf{Z}$, we have $\theta = (C + D)/2$ and hence $S_X^{A_5} = \mathbf{Z}[C, (C + D)/2] = \mathbf{Z}[(C + D)/2, (C - D)/2]$. The lemma is proved. □

Set $L = H^0(X, \mathbf{Z})$ which contains $S_X \oplus T_X$ as a sublattice of finite index. Also L^{A_5} contains $S_X^{A_5} \oplus T_X$ as a sublattice of finite index d by 1.0A and 1.0B.

Lemma 3.4. *The quotient $L^{A_5}/(S_X^{A_5} \oplus T_X)$ is 2-elementary of order d and isomorphic to* (0) *($d = 1$),* $\mathbf{Z}/(2)$ *($d = 2$) or* $(\mathbf{Z}/(2))^{\oplus 2}$ *($d = 4$).*

Proof. For a lattice M, we denote by $M^\vee = \mathrm{Hom}(M, \mathbf{Z})$ the dual and $A_M = M^\vee/M$ the discriminant group. Then we have, where ι is the inclusion:

$$S_X^{A_5} \oplus T_X \subseteq L^{A_5} \subseteq (L^{A_5})^\vee \subseteq (S_X^{A_5})^\vee \oplus T_X^\vee,$$
$$\iota : L^{A_5}/(S_X^{A_5} \oplus T_X) \to A_{S_X^{A_5}} \oplus A_{T_X}.$$

Let pr_1 and pr_2 be the projections from $A_{S_X^{A_5}} \oplus A_{T_X}$ to its first and second summands, respectively. Since $S_X^{A_5}$ and T_X are primitive in L^{A_5}, both compositions $pr_i \circ \iota$ are injective. In particular, the quotient group in 3.4 is regarded as a subgroup of a bigger group A_{T_X}, whence it is generated by 2 elements because the same is true for the bigger group (since rank $T_X = 2$ by 1.1). We still have to show that this quotient group is 2-elementary.

Take a coset $\overline{\theta}$ from the quotient group in 3.4. In the notation of 1.5, we write

$$\theta = u + \frac{1}{2m}(at_1 + bt_2) \in (S_X^{A_5})^\vee \oplus T_X^\vee.$$

Regarding $\overline{\theta}$ as an element of $A_{S_X^{A_5}}$ via the injection $pr_1 \circ \iota$, we have by 3.3, modulo $S_X^{A_5} \oplus T_X$, that

$$0 = g^*\theta - \theta = \frac{1}{2m}[a(g^*t_1 - t_1) + b(g^*t_2 - t_2)] = \frac{1}{2m}[-(a+b)t_1 + (a-b)t_2].$$

So $2m$ divides $a+b$, $a-b$ (and hence $2a$ and $2b$) because T_X is primitive in L. Thus m divides a and b and we write $a = ma'$ and $b = mb'$ so that $\theta = u + \frac{1}{2}(a't_1 + b't_1)$. Therefore, modulo T_X, we have $2u = 2\theta \in 2L^{G_N} \subset L^{G_N}$, whence $2u \in L \cap (S_X^{A_5})^\vee = S_X^{A_5}$ (because the latter is primitive in L). So $2\overline{\theta} = 0$. The lemma is proved. □

Lemma 3.5. *One of the following cases occurs.*

(1) *We have $m = 5$. Both the quotients $S_X^{A_5}/L_0$ and $L^{A_5}/(S_X^{A_5} \oplus T_X)$ are isomorphic to $\mathbf{Z}/(2)$. Moreover, the discriminant form of $(L^{A_5})^\vee/L^{A_5} \cong (\mathbf{Z}/(30))^{\oplus 2}$ is given in* [26, *Theorem 2.1*] *(corresponding to the matrix M_1 there) and generated by the cosets $\overline{\varepsilon}_i$ with $\varepsilon_1 = e_1^*$, $\varepsilon_2 = e_2^* + e_3^* + e_4^*$ and the intersection form (note that $\overline{\varepsilon}_i^2$ is in $\mathbf{Q}/2\mathbf{Z}$ while $\overline{\varepsilon}_1.\overline{\varepsilon}_2$ is in $\mathbf{Q}/\mathbf{Z}$):*

$$(\overline{\varepsilon}_i.\overline{\varepsilon}_j) = \begin{pmatrix} -23/30 & -1/5 \\ -1/5 & -35/30 \end{pmatrix}.$$

(2) *We have $m = 10$, $S_X^{A_5}/L_0 \cong \mathbf{Z}/(2)$ and $L^{A_5}/(S_X^{A_5} \oplus T_X) \cong (\mathbf{Z}/(2))^{\oplus 2}$.*

(3) *We have $m = 5$, $L_0 = S_X^{A_5}$ and $L^{A_5}/(S_X^{A_5} \oplus T_X) \cong (\mathbf{Z}/(2))^{\oplus 2}$.*

Proof. In the notation of 3.3 and 3.4, we have $-(12^2)(4m^2) = |L_0||T_X| = d_1^2 d^2 |L^{A_5}|$. On the other hand, $-|L^{A_5}| = 30^2, 3 \times 10^2, 20^2, 3 \times 20^2, 3 \times 40^2$ by the calculation in [26, Theorem 2.1]. Then the lemma follows easily. □

Lemma 3.6. *The case* (3) *in* 3.5 *does not occur.*

Proof. Consider the case (3) in 3.5. Let θ be an element in L^{A_5} but not in the smaller set $S_X^{A_5} \oplus T_X$. We claim that $\theta^2 \in 2\mathbf{Z}$ implies that modulo this smaller set, our θ equals some θ_j below, where $u_1 := C$, $u_2 := D$ and $T_X = \mathbf{Z}[t_1, t_2]$ as in 1.5. Here $\theta_j := \frac{1}{2}(t_1 + t_2) + \frac{1}{2}u_j$.

Indeed, since the quotient group in 3.5(3) is 2-elementary, we can write, modulo the smaller set, that $\theta = \frac{1}{2}(a_1 t_1 + a_2 t_2 + b_1 u_1 + b_2 u_2)$ with a_i, b_j in $\{0, 1\}$ but not all zero. Indeed, $(a_1, a_2) \neq (0, 0) \neq (b_1, b_2)$ because both $S_X^{A_5}$ and T_X are primitive in L. Now modulo $2\mathbf{Z}$, we have the following, so the claim follows:

$$\frac{1}{2}(a_1^2 + a_2^2) + b_1^2 + b_2^2 = \frac{2m}{4}(a_1^2 + a_2^2) + \frac{12}{4}(b_1^2 - b_2^2) = \theta^2 = 0.$$

Since $\theta_1 - \theta_2$ is not in L^{A_5} (not in L at all, by the primitivity of $S_X^{A_5}$ in L), at most one of θ_j is in L^{A_5}. So $L^{A_5}/(S_X^{A_5} \oplus T_X)$ is of order ≤ 2, a contradiction. □

We start anew. By 3.3 and 3.6, the lattice $S_X^{A_5}$ equals $\mathbf{Z}[u_1, u_2]$ with $u_1 = \frac{1}{2}(C + D)$ and $u_2 = \frac{1}{2}(C - D)$, and has the intersection form $U(6)$.

Lemma 3.7. *The case* (2) *in* 3.5 *is impossible.*

Proof. Take θ in L^{A_5} but not in the smaller set $S_X^{A_5} \oplus T_X$. As in 3.6, $\theta^2 \in 2\mathbf{Z}$ implies that modulo the smaller set, our θ is one of the following:

$$\theta^i = \frac{1}{2}t_i + \frac{1}{2}(u_1 + u_2), \qquad \theta_j = \frac{1}{2}(t_1 + t_2) + \frac{1}{2}u_j.$$

Since $\theta^1 - \theta^2$ is not in L^{A_5} (not in L at all), not both θ^i are in L^{A_5}. By the same reasoning not both θ_j are in L^{A_5}. Since $L^{A_5}/(S_X^{A_5} \oplus T_X) \cong (\mathbf{Z}/(2))^{\oplus 2}$ is generated by two elements, one of θ^i $(i = 1, 2)$ and one of θ_j $(j = 1, 2)$ are in L^{A_5}. But $\theta^i.\theta_j = \frac{2m}{4} + \frac{6}{4} = \frac{13}{2}$, which is not an integer. This is a contradiction. □

Lemma 3.8. *Suppose the case* (1) *in* 3.5 *occurs. Then we have:*

(1) *L^{A_5} is generated by S_X, T_X and $\theta = \frac{1}{2}(t_1 + t_2 + u_1 + u_2)$.*

(2) *The discriminant group $A_{L^{A_5}} = (L^{A_5})^\vee / L^{A_5}$ (with the dual $(L^{A_5})^\vee = \mathrm{Hom}(L^{A_5}, \mathbf{Z})$) is generated by the cosets $\overline{\delta}_j$ $(j = 1, 2)$ which (together with the intersection form) is given as follows (where $t_i^*.t_j = \delta_{ij}$, and $u_i^*.u_j = \delta_{ij}$ in Kronecker's symbol):*

$$\delta_1 = t_2^* + u_1^* + 2u_2^* = \frac{1}{10}t_2 + \frac{1}{6}(2u_1 + u_2), \qquad \delta_2 = t_1^* + u_1^* = \frac{1}{10}t_1 + \frac{1}{6}u_2,$$

$$(\overline{\delta}_i.\overline{\delta}_j) = \begin{pmatrix} 23/30 & 1/3 \\ 1/3 & 1/10 \end{pmatrix}.$$

Proof. (1) can be proved as in 3.6, by making use of that $\theta_1^2 \in 2\mathbf{Z}$ for every θ_1 in L^{A_5}.

(2) Since $\delta_i.\theta$, $\delta_i.t_j$ and $\delta_i.u_j$ are all in $\mathbf{Z}$ by a direct calculation, we see that both δ_i are in $(L^{A_5})^\vee$. One checks easily that the subgroup $\langle \overline{\delta}_1, \overline{\delta}_2 \rangle$ of the discriminant group in (2) is isomorphic to $(\mathbf{Z}/(30))^{\oplus 2}$, whence this subgroup is indeed the whole discriminant group in (2) (because the latter is of order 30^2 by 3.5). This proves the lemma. □

Here comes the punch line. By 3.5–3.8, there is an isometry $\varphi : \langle \overline{\varepsilon}_1, \overline{\varepsilon}_2 \rangle \longrightarrow \langle \overline{\delta}_1, \overline{\delta}_2 \rangle$, so for some integers a, b, c, d, we can write $(\varphi(\overline{\varepsilon}_1), \varphi(\overline{\varepsilon}_2)) = (\overline{\delta}_1, \overline{\delta}_2) \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Thus,

$$-23/30 = \varepsilon_1^2 = \varphi(\varepsilon_1)^2 = (a\delta_1 + b\delta_2)^2 = \frac{1}{30}(23a^2 + 3b^2 + 20ab) \pmod{2\mathbf{Z}},$$
$$-23 = 23a^2 + 3b^2 + 20ab \pmod{60\mathbf{Z}}.$$

The congruence above implies that modulo 4, we have $1 = -a^2 - b^2$, which is impossible. This completes the proof of 3.1 (1) and also the whole of 3.1.

We now prove Theorems A–C in the introduction. In Theorem C, we have $H \le G_N$ by 1.0A; so H is either one of A_5, $L_2(7)$, A_6 and $M_{20} = C_2^{\oplus 4} : A_5$, by [24, the list]; if $H = L_2(7)$ then $G_N = H$ by [11] and Theorem C follows from [18, Main Theorem].

Therefore, we may assume that in all three theorems, G is a finite group containing A_5 and acting faithfully on a $K3$ surface X. Write $G = G_N \cdot \mu_I$ as in 1.0. By 1.0A the A_5 in G is contained in G_N. So G_N is either one of A_5, S_5, A_6 and $M_{20} = C_2^{\oplus 4} : A_5$, by [24, the list].

Consider the case $G_N = A_5$. Then $I = 1, 2$, by 1.1, [26, Theorem 3.1] and 3.1. If $I = 1$, then $G = A_5$. If $I = 2$, let $\rho : G \to S_5 \times \mu_2$ $(x \mapsto (c_x, \alpha(x)))$ be the injection as in 1.7 so that $\rho(A_5) = A_5 \times \langle 1 \rangle$; if the projection $pr_1 : S_5 \times \mu_2 \to S_5$ maps $\rho(G)$ to A_5 (resp. to S_5), then $G \cong \rho(G) = A_5 \times \mu_2$ (resp. $G \cong \rho(G) \cong pr_1(\rho(G)) = S_5$, by comparing the orders); see the argument below. Thus Theorems A–C are true.

Consider the case $G_N = S_5$. Let g be in G such that $\alpha(g)$ is a generator of μ_I. Since $\mathrm{Aut}(S_5) = S_5$ and $x \mapsto c_x$ gives rise to an isomorphism $S_5 \to \mathrm{Aut}(S_5)$, we see that the map $G \to \mathrm{Aut}(S_5) = S_5$ $(x \mapsto c_x)$ is surjective, and the conjugation maps $c_g = c_s$ on S_5, for some $s \in S_5$. Replacing g by gs^{-1}, we may assume that g commutes with every element in $G_N = S_5$. So $g^I \in \mathrm{Ker}(\alpha) = G_N$ is in the centre of $G_N = S_5$ (which is (1)), whence $\mathrm{ord}(g) = I$, while $\alpha(g)$ is a generator of μ_I. Thus $G = S_5 \times \mu_I \ge A_5 \times \mu_I$. So $I = 1, 2$ by 1.1, [26, Theorem 3.1] and 3.1. Hence Theorems A–C are true.

Consider the case where $G_N = A_6$ or $G_N = M_{20} = C_2^4 : A_5$. Then G_N does not contain an A_5 as a normal subgroup (otherwise, in the latter case, $M_{20} = C_2^4 \times A_5$, absurd). So A_5 is also not normal in G. Thus Theorems A and B are void this time. Now Theorem C follows from [9] and [6].

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