Global existence and asymptotic behavior of classical solutions of quasilinear hyperbolic systems with linearly degenerate characteristic fields

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Abstract

In this paper, we study the global existence and the asymptotic behavior of classical solution of the Cauchy problem for quasilinear hyperbolic system with constant multiple and linearly degenerate characteristic fields. We prove that the global $C^1$ solution exists uniquely if the BV norm of the initial data is sufficiently small. Based on the existence result on the global classical solution, we show that, when the time $t$ tends to the infinity, the solution approaches a combination of $C^1$ traveling wave solutions. Finally, we give an application to the equation for time-like extremal surfaces in the Minkowski space–time $\mathbb{R}^{1+n}$.

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1. Introduction and main results

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0 ,$$

(1.1)
where \( u = (u_1, \ldots, u_n)^T \) is the unknown vector-valued function of \((t, x)\), \( A(u) = (a_{ij}(u)) \) is an \( n \times n \) matrix.

By hyperbolicity, for any given \( u \) on the domain under consideration, \( A(u) \) has \( n \) real eigenvalues \( \lambda_1(u), \ldots, \lambda_n(u) \) and a complete system of left (respectively right) eigenvectors \( l_1(u), \ldots, l_n(u) \) (respectively \( r_1(u), \ldots, r_n(u) \)). In this paper, we assume that

\[(H_1) \quad (1.1) \text{ is a hyperbolic system with constant multiple characteristic fields.}\]

To simplify the computations, in this paper we suppose that

\[\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u). \quad (1.2)\]

For general situation, the same results can be obtained in a completely similar way. Under the assumption (1.2), when \( p = 1 \), the system (1.1) is strictly hyperbolic; while, when \( p > 1 \), (1.1) is a non-strictly hyperbolic system.

For \( i = 1, \ldots, n \), let

\[l_i(u) = (l_{i1}(u), \ldots, l_{in}(u)) \quad \text{(respectively } r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))^T)\]

be a left (respectively right) eigenvector corresponding to \( \lambda_i(u) \), i.e.,

\[l_i(u)A(u) = \lambda_i(u)l_i(u) \quad \text{(respectively } A(u)r_i(u) = \lambda_i(u)r_i(u)). \quad (1.3)\]

We have

\[\det|l_{ij}(u)| \neq 0 \quad \text{(equivalently, } \det|r_{ij}(u)| \neq 0). \quad (1.4)\]

Without loss of generality, we suppose that on the domain under consideration

\[l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n) \quad (1.5)\]

and

\[r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \ldots, n), \quad (1.6)\]

where \( \delta_{ij} \) stands for the Kronecker’s symbol.

We furthermore suppose that

\[(H_2) \quad A(u) \in C^2.\]

\[(H_3) \quad \text{The system } (1.1) \text{ is linearly degenerate in the sense of Lax.}\]

**Remark 1.1.** Noting (H1) and using (H2), we can prove that all \( \lambda_i(u), r_{ij}(u), \) and \( l_{ij}(u) \) \((i, j = 1, \ldots, n)\) possess the \( C^2 \) regularity (cf. [4]).

**Remark 1.2.** By the definition (cf. [11]), the hypothesis (H3) means that

\[\nabla \lambda_i(u)r_i(u) \equiv 0 \quad (i = 1, \ldots, p)\]
and

$$\nabla \lambda_i(u)r_i(u) \equiv 0 \quad (i = p + 1, \ldots, n). \quad (1.8)$$

For the sake of simplicity, we introduce

$$\Gamma \triangleq \{(j, k) \mid j, k \in \{1, \ldots, n\}, j \neq k, j \text{ or } k \notin \{1, \ldots, p\}\}. \quad (1.9)$$

It is easy to see that

if \((j, k) \in \Gamma\), then \((i, j) \in \Gamma\) or \((i, k) \in \Gamma\) for each \(i \in \{1, \ldots, n\}\). \quad (1.10)

By this notation, the hypothesis \((H_3)\), i.e., \((1.7), (1.8)\) can be replaced by

$$\nabla \lambda_i(u) r_j(u) \equiv 0, \quad \forall (i, j) \notin \Gamma. \quad (1.11)$$

Consider the Cauchy problem for the system \((1.1)\) with the following initial data

$$t = 0: \quad u(0, x) = f(x), \quad (1.12)$$

where \(f(x)\) is a \(C^1\) vector-valued function of \(x\).

For the case that the initial data \(f(x)\) satisfies the following decay property: there exists a constant \(\mu > 0\) such that

$$\varrho \triangleq \sup_{x \in \mathbb{R}} \left\{ \left(1 + |x|\right)^{1+\mu} \left(\|f(x)\| + \|f'(x)\|\right) \right\} < +\infty \quad (1.13)$$

is sufficiently small, by means of the normalized coordinates Li et al. proved that the Cauchy problem \((1.1)\) and \((1.12)\) admits a unique global classical solution, provided that the system \((1.1)\) is weakly linearly degenerate\(^1\) (see [12–15] and [7]). In their works, the condition \(\mu > 0\) is essential. If \(\mu = 0\), a counterexample was constructed by Kong [8] showing that the classical solution may blow up in a finite time, even when the system \((1.1)\) is weakly linearly degenerate.

Our goal of this paper is to prove the global existence and uniqueness of classical solution of the Cauchy problem \((1.1), (1.12)\) and describe the exact time asymptotic behavior of the global classical solution. The main results are the following two theorems.

\(^1\) We consider a strictly hyperbolic system. The \(i\)th characteristic \(\lambda_i(u)\) is called to be weakly linearly degenerate, if, along the \(i\)th characteristic trajectory passing through the origin \(u = 0\), it holds that

$$\lambda_i(u) \equiv \lambda(0).$$

If all characteristics \(\lambda_i(u)\) \((i = 1, \ldots, n)\) are weakly linearly degenerate, then the system is weakly linearly degenerate. For non-strictly hyperbolic system, we have a similar definition.
Theorem 1.1 (Global existence). Under the assumptions (H1)–(H3), there exists a small constant \( \epsilon > 0 \) such that the Cauchy problem (1.1), (1.12) admits a unique global \( C^1 \) solution \( u = u(t, x) \) for all \( t \in \mathbb{R} \), provided that

\[
\int_{-\infty}^{+\infty} |f'(x)| \, dx \leq \epsilon. \tag{1.14}
\]

Remark 1.3. Suppose that the system (1.1) is strictly hyperbolic and linearly degenerate, and the initial data \( f(x) \) is a \( C^2 \) function with compact support. In this case Bressan [1] proved that the Cauchy problem (1.1), (1.12) admits a unique global \( C^2 \) solution, provided that the BV norm of the initial data, i.e., \( BV(f) \) is small enough. Theorem 1.1 generalizes Bressan’s result to the case that (1.1) is a hyperbolic system with constant multiple characteristic fields and the initial data \( f(x) \) does not require to have compact support.

Remark 1.4. Under the assumption that the system (1.1) is strictly hyperbolic and weakly linearly degenerate (cf. [14]), Zhou [17] recently proved that there exists a small constant \( \epsilon > 0 \) such that the Cauchy problem (1.1), (1.12) admits a unique global \( C^1 \) solution, provided that (1.14) and the following condition (1.15) hold

\[
N \overset{\Delta}{=} \int_{-\infty}^{+\infty} |f(x)| \, dx < \frac{\epsilon}{M}, \tag{1.15}
\]

where

\[
M \overset{\Delta}{=} \sup_{x \in \mathbb{R}} |f'(x)| < +\infty. \tag{1.16}
\]

Comparing with Zhou’s result, in Theorem 1.1 we suppose that (1.1) is linearly degenerate instead of weakly linearly degenerate. However, we remove his assumption (1.15).

The following definition comes from Kong [7] (see Definition 4.3 in [7]).

Definition 1.1. If there exists an invertible smooth transformation \( u = u(\tilde{u}) \) \( (u(0) = 0) \) such that in \( \tilde{u} \)-space

\[
\tilde{r}_i \left( \sum_{h=1}^{p} \tilde{u}_h e_h \right) \equiv e_i \quad (i = 1, \ldots, p), \quad \forall |\tilde{u}_h| \text{ small } (h = 1, \ldots, p) \tag{1.17}
\]

and

\[
\tilde{r}_j (\tilde{u}_j e_j) \equiv e_j, \quad \forall |\tilde{u}_j| \text{ small } (j = p + 1, \ldots, n), \tag{1.18}
\]

then the transformation \( u = u(\tilde{u}) \) is called the normalized transformation, and the corresponding unknown variables \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T \) are called the normalized variables or normalized coordinates.
**Remark 1.5.** If the system (1.1) is strictly hyperbolic, then there always exists the normalized transformation (cf. [14]). On the other hand, quasilinear hyperbolic systems of conservation laws with eigenvalues with constant multiplicity always have the normalized coordinates (cf. [13]). More detailed, for strictly hyperbolic system, we do not needed the conservation form, the reason is as follows: by standard differential geometry theory, in a neighborhood of \( u = 0 \) there exists an invertible smooth transformation \( u = u(\tilde{u}) \) \((u(0) = 0)\) such that in \( \tilde{u} \)-space, for each \( i = 1, \ldots, n \), the \( i \)th characteristic trajectory passing through \( \tilde{u} = 0 \) coincides the \( \tilde{u}_i \)-axis at least for \( |\tilde{u}_i| \) small, that is, the variables \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T \) are the normalized coordinates. However, for hyperbolic system, in particular, the hyperbolic system with constant multiple characteristics, we need the conservation form, the reason is that, for a hyperbolic system of conservation laws with \( p \)-multiple eigenvalue \( \lambda(u) \), by Frobenius’ Theorem, the multiple eigenvalue \( \lambda(u) \) satisfies the following completely integrable condition:

\[
[r_i, r_j] \in \text{span}\{r_1(u), \ldots, r_p(u)\}, \quad \forall i, j = 1, \ldots, p,
\]

where \( r_i(u) \) \((i = 1, \ldots, p)\) are the right eigenvectors corresponding to \( \lambda(u) \), \( \text{span}\{r_1(u), \ldots, r_p(u)\} \) stands for the linear space spanned by \( r_1(u), \ldots, r_p(u) \), and \([\cdot, \cdot]\) denotes Poisson’s bracket defined by

\[
[r_i, r_j] = (r_i^T \cdot \nabla) r_j - (r_j^T \cdot \nabla) r_i.
\]

Therefore, in a neighborhood of \( u = 0 \) there exists an invertible smooth transformation \( u = u(\tilde{u}) \) \((u(0) = 0)\) such that the new variables \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T \) are the normalized coordinates.

We finally suppose that

**Assumption (H4)** For the system (1.1), there exist the normalized coordinates.

In the normalized coordinates, it follows from (1.7), (1.8), i.e., (1.11) that

\[
\begin{align*}
\lambda_i \left( \sum_{h=1}^{p} u_h e_h \right) &= \lambda_i(0) \quad (i = 1, \ldots, p), \quad \forall |u_h| \text{ small } (h = 1, \ldots, p); \\
\lambda_i(u_i e_i) &= \lambda_i(0), \quad \forall |u_i| \text{ small } (i = p + 1, \ldots, n).
\end{align*}
\]

**Theorem 1.2 (Asymptotic behavior).** Under the assumptions (H1)–(H4), if

\[
\int_{-\infty}^{+\infty} |f(x)| \, dx < +\infty,
\]

then there exists a unique \( C^1 \) vector-valued function \( \varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))^T \) such that in the normalized coordinates

\[
u(t, x) \rightarrow \sum_{i=1}^{n} \varphi_i(x - \lambda_i(0)t)e_i, \quad \text{as } t \rightarrow +\infty.
\]

(1.20)
Moreover, \( \varphi_i(x) \) \((i = 1, \ldots, n)\) are globally Lipschitz continuous, more precisely, there exists a positive constant \( \kappa_1 \) independent of \( \epsilon, M, N, x_1 \) and \( x_2 \) such that for every \( i \in \{1, \ldots, n\} \), it holds that
\[
|\varphi_i(x_1) - \varphi_i(x_2)| \leq \kappa_1 M |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}.
\] (1.21)

Furthermore, if \( f'(x) \), the derivative of the initial data, is globally \( \rho \)-Hölder \((0 < \rho \leq 1)\) continuous, that is, there exists a positive constant \( \varsigma \) such that
\[
|f'(x_1) - f'(x_2)| \leq \varsigma |x_1 - x_2|^\rho, \quad \forall x_1, x_2 \in \mathbb{R},
\] (1.22)
then \( \varphi'(x) \) is also globally \( \rho \)-Hölder continuous and satisfies that
\[
|\varphi'(x_1) - \varphi'(x_2)| \leq \kappa_2 \varsigma (1 + MN + \epsilon)^\rho |x_1 - x_2|^\rho
+ \kappa_2 M^2 (1 + \epsilon)(1 + MN + \epsilon) |x_1 - x_2|,
\] (1.23)
where \( \kappa_2 \) is a positive constant independent of \( \epsilon, M, N, \varsigma, x_1 \) and \( x_2 \).

Remark 1.6. Theorem 1.2 gives the exact time asymptotic behavior of the global classical solution presented in Theorem 1.1. For the initial data satisfying the decay property (1.13), Kong and Yang [10] proved that, when \( t \) tends to the infinity, the global classical solution approaches a combination of \( C^1 \) traveling wave solutions at algebraic rate \((1 + t)^{-\mu}\). The goal of the present paper is to generalize the result in [10] to the case of the initial data with small BV norm. It is well known that, the BV space is a suitable framework for one-dimensional Cauchy problem for quasilinear hyperbolic systems (see Bressan [2]). Comparing with [10], because of the lack of the decay rate of the initial data, in the present situation there is no any estimate on the convergence rate.

 Remark 1.7. For the quasilinear strictly hyperbolic system with weakly linearly degenerate characteristic fields, based on the global existence result in Zhou [17], we have recently studied the asymptotic behavior of the global classical solution (cf. [5]). Moreover, for the inhomogeneous quasilinear hyperbolic system with weakly linearly degenerate characteristic fields, based on the global existence result in Du [6], we have also established the asymptotic behavior of the global classical solution (cf. [3]), provided that the source term satisfies the Kong’s matching condition (cf. [7]).

The paper is organized as follows. For the sake of completeness, in Section 2 we briefly recall John’s formula on the decomposition of waves with some supplements. Section 3 is devoted to the proof of the global existence of classical solution, i.e., the proof of Theorem 1.1. The key point of the proof is to establish the uniform a priori estimate on the \( C^1 \) norm of the solution on its existence domain. In Section 4, we give some new estimates, which will play an important role in the proof of Theorem 1.2. Using these estimates, we prove Theorem 1.2 in Section 5. An application is given in Section 6.
2. Preliminaries

For the sake of completeness, in this section we briefly recall John’s formula on the decomposition of waves with some supplements, which play an important role in our discussion.

Let

\[ v_i(u) = l_i(u)u \quad (i = 1, \ldots, n), \]  
\[ w_i(u) = l_i(u)u_x \quad (i = 1, \ldots, n). \]  

By (1.5), we have

\[ u = \sum_{k=1}^{n} v_k r_k(u), \]  
\[ u_x = \sum_{k=1}^{n} w_k r_k(u). \]  

Let

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x} \]  

be the directional derivative along the \( i \)th characteristic. We have (see [14,15] or [7])

\[ \frac{dv_i}{dt} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \triangleq F_i(t,x), \]  

where

\[ \beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \]  

Hence,

\[ \beta_{ijk}(u) = 0, \quad \forall (i, k) \notin \Gamma. \]  

By (1.17), (1.18), in the normalized coordinates (if any!),

\[ \beta_{ijk} \left( \sum_{h=1}^{n} u_h e_h \right) \equiv 0, \quad \forall |u_h| \text{ small, } \forall i \in \{1, 2, \ldots, n\}, \forall j, k \in \{1, \ldots, p\}, \]  
\[ \beta_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small, } \forall i \in \{1, 2, \ldots, n\}, \forall j \in \{p+1, \ldots, n\}. \]  

See [7]. Moreover, noting (2.4) and (2.6), we have

\[ \frac{\partial v_i}{\partial t} + \frac{\partial (\lambda_i(u) v_i)}{\partial x} = \sum_{j,k=1}^{n} \tilde{\beta}_{ijk}(u) v_j w_k \triangleq \tilde{F}_i(t,x), \]
equivalently,
\[
d[v_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^n \tilde{\beta}_{ijk}(u)v_j w_k dt \wedge dx = \tilde{F}_i(t, x) dt \wedge dx, \tag{2.12}
\]
where
\[
\tilde{\beta}_{ijk}(u) = \beta_{ijk}(u) + \nabla \lambda_i(u)r_k(u)\delta_{ij}. \tag{2.13}
\]
In the normalized coordinates (if any!),
\[
\tilde{\beta}_{ijk} \left( \sum_{h=1}^p u_h e_h \right) \equiv 0, \quad \forall |u_h| \text{ small}, \forall i \in \{p + 1, \ldots, n\}, \forall j, k \in \{1, \ldots, p\}, \tag{2.14}
\]
\[
\tilde{\beta}_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \forall j \in \{p + 1, \ldots, n\}, j \neq i. \tag{2.15}
\]
By the hypothesis (H3), i.e., (1.11),
\[
\tilde{\beta}_{ijk}(u) \equiv 0, \quad \forall (i, k) \notin \Gamma. \tag{2.16}
\]
On the other hand, we have (cf. [14,15] or [7])
\[
\frac{d w_i}{d t} = \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k \triangleq G_i(t, x), \tag{2.17}
\]
where
\[
\gamma_{ijk}(u) = (\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u)r_j(u) - \nabla \lambda_k(u)r_j(u)\delta_{ik}. \tag{2.18}
\]
It follows from (2.18) that
\[
\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i. \tag{2.19}
\]
Noting the hypothesis (H3), i.e., (1.11), we get
\[
\gamma_{ijk}(u) \equiv 0, \quad \forall (j, k) \notin \Gamma. \tag{2.20}
\]
Similar to (2.11), (2.12), we have
\[
\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = \sum_{j,k=1}^n \tilde{\gamma}_{ijk}(u) w_j w_k \triangleq \tilde{G}_i(t, x), \tag{2.21}
\]
equivalently,
\[
d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^n \tilde{\gamma}_{ijk}(u) w_j w_k dt \wedge dx = \tilde{G}_i(t, x) dt \wedge dx, \tag{2.22}
\]
where
\[
\tilde{\gamma}_{ijk}(u) = \frac{1}{2} (\lambda_j(u) - \lambda_k(u)) l_i(u) \left[ \nabla r_k(u) r_j(u) - \nabla \lambda_j(u) r_k(u) \right].
\]  
(2.23)

Noting the hypothesis (H3), i.e., (1.11) again, we have
\[
\tilde{\gamma}_{ijk}(u) \equiv 0, \quad \forall (j,k) \notin \Gamma.
\]  
(2.24)

3. Global existence of classical solutions—Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. At the same time, we also establish some uniform estimates which will play an important role in the proof of Theorem 1.2. Throughout this section, the influence of the work of Zhou [17] is apparent.

By the existence and uniqueness of local $C^1$ solution to the Cauchy problem, in order to prove Theorem 1.1, it suffices to establish a priori estimate on the $C^0$ norm of $u$ and $\frac{\partial u}{\partial x}$ on the existence domain of $C^1$ solution $u = u(t, x)$.

In this section, without loss of generality, we suppose that
\[
f(0) = 0. \quad \text{(H)}
\]
In fact, by the following transformation
\[
\tilde{u} = u - f(0),
\]
we can always realize the assumption (H). Noting (1.14) and using (H), we have
\[
|f(x)| \leq \epsilon, \quad \forall x \in \mathbb{R}.
\]  
(3.1)

On the other hand, by (1.2), there exist positive constants $\delta_0, \delta_1$ and $\delta$ such that
\[
|\lambda_i(u) - \lambda_j(v)| \geq \delta_0, \quad \forall |u|, |v| \leq \delta, \; \forall (i, j) \in \Gamma,
\]  
(3.2)
and
\[
|\lambda_i(u) - \lambda_i(v)| \leq \delta_1, \quad \forall |u|, |v| \leq \delta \; (i = 1, \ldots, n).
\]  
(3.2a)

For the time being it is supposed that on the existence domain of the $C^1$ solution $u = u(t, x)$ we have
\[
|u(t, x)| \leq K_0 \epsilon,
\]  
(3.3)
where $K_0 > 0$ is a constant independent of $\epsilon$ and $M$. At the end of Lemma 3.1, we shall explain that this hypothesis is reasonable, provided that the assumptions (1.14) and (H) are satisfied. Therefore, taking $\epsilon$ suitably small, we always have
\[
|u| \leq \delta.
\]  
(3.4)

For any fixed $T \geq 0$, we introduce
\begin{align*}
U_\infty(T) &= \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |u(t, x)|, \quad V_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |v(t, x)|, \\
W_\infty(T) &= \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |w(t, x)|, \\
W_1(T) &= \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |w(t, x)| \, dx, \\
\tilde{W}_1(T) &= \max_{(i, j) \in \Gamma} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |w_i(t, x)| \, dt, \\
\bar{W}_1(T) &= \max_{(i, j) \in \Gamma} \sup_{L_j} \int_{L_j} |w_i(t, x)| \, dt,
\end{align*}

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$, $v = (v_1, \ldots, v_n)^T$ and $w = (w_1, \ldots, w_n)^T$ in which $v_i$ and $w_j$ are defined by (2.1) and (2.2) respectively, $\tilde{C}_j$ stands for any given $j$th characteristic on the domain $[0, T] \times \mathbb{R}$, while $L_j$ stands for any given ray with the slope $\lambda_j(0)$ on the region $[0, T] \times \mathbb{R}$.

**Lemma 3.1.** Under the assumptions of Theorem 1.1, there exists a positive constant $K_1$ independent of $\epsilon$, $M$, and $T$ such that

\begin{align*}
W_1(T), \tilde{W}_1(T), \bar{W}_1(T) &\leq K_1 \epsilon, \\
U_\infty(T), V_\infty(T) &\leq K_1 \epsilon, \\
W_\infty(T) &\leq K_1 M.
\end{align*}

**Proof.** Introduce

\begin{align*}
Q_W(T) &= \sum_{(j, k) \in \Gamma} \int_0^T \int_{\mathbb{R}} |w_j(t, x)| |w_k(t, x)| \, dt \, dx.
\end{align*}

By (2.21), it follows from Lemma 3.2 in Zhou [17] that

\begin{align*}
Q_W(T) &\leq c_1 \left( W_1(0) + \int_0^T \int_{\mathbb{R}} |\tilde{G}(t, x)| \, dt \, dx \right)^2,
\end{align*}

where $\tilde{G} = (\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n)$, here and hereafter $c_j$ ($j = 1, 2, \ldots$) will denote a constant independent of $\epsilon$, $M$ and $T$.

Noting (2.24), we have

\begin{align*}
\int_0^T \int_{\mathbb{R}} |\tilde{G}(t, x)| \, dt \, dx &\leq c_2 Q_W(T).
\end{align*}
It follows from (3.13), (3.14) that

$$Q_W(T) \leq c_3(W_1(0) + Q_W(T))^2.$$  \hspace{1cm} (3.15)

Thus, noting (1.14) we have

$$Q_W(T) \leq c_4\epsilon^2.$$  \hspace{1cm} (3.16)

By (2.17) and (2.20), it follows from Lemma 3.1 in Zhou [17] that

$$+\infty \int_{-\infty}^{T} \left| w_i(t,x) \right| dx \leq W_1(0) + \int_0^T \int |G(t,x)| \ dt \ dx \leq W_1(0) + c_5 Q_W(T),$$  \hspace{1cm} (3.17)

where $G(t,x) = (G_1, \ldots, G_n)$. Therefore, noting (1.14) and (3.16), (3.17) we obtain

$$W_1(T) \leq c_6\epsilon.$$  \hspace{1cm} (3.18)

For any fixed $\alpha \in \mathbb{R}$, let $\tilde{C}_j$ be the $j$th characteristic passing through $(0, \alpha)$, and let $P$ be the intersection point of $\tilde{C}_j$ with the line $t = T$. Passing through the point $P$, we draw the $i$th characteristic $\tilde{C}_i$ and denote the intersection point of this characteristic with the $x$-axis by $(0, \beta)$, where $(i, j) \in \Gamma$. For fixing the idea we may suppose that $\alpha < \beta$. Let $\Omega$ be the domain bounded by $\tilde{C}_j$, the $x$-axis and $\tilde{C}_i$. Using Green formula on the region $\Omega$, we obtain from (2.22) that

$$\int_{\tilde{C}_j} w_i(t,x)(\lambda_j(u) - \lambda_i(u)) \ dt = \int_{\alpha}^{\beta} w_i(0,x) \ dx - \iint_{\Omega} \tilde{G}_i(t,x) \ dt \ dx.$$  \hspace{1cm} (3.19)

It follows from (3.2) and (3.14) that

$$\int_{\tilde{C}_j} \left| w_i(t,x) \right| dt \leq \frac{c_7}{\delta_0}(W_1(0) + Q_W(T)).$$  \hspace{1cm} (3.20)

Thus, we have

$$\tilde{W}_1(T) \leq c_8(W_1(0) + Q_W(T)) \leq c_9\epsilon.$$  \hspace{1cm} (3.21)

Similarly, replacing the $j$th characteristic $\tilde{C}_j$ by the ray $L_j$ with the slope $\lambda_j(0)$, we obtain

$$\tilde{W}_1(T) \leq c_{10}(W_1(0) + Q_W(T)) \leq c_{11}\epsilon.$$  \hspace{1cm} (3.22)

By (2.17), we get

$$\left| w_i(t,x) \right|_{C^0} \leq W_\infty(0) + \int_{\tilde{C}_i} \left| G_i(t,x) \right| dx.$$  \hspace{1cm} (3.23)
Noting (2.20) and (3.21), we have
\[
\int_{\tilde{c}_i} \left| \tilde{C}_i \left| G_i(t,x) \right| \right| dx \leq c_{12} W_{\infty}(T) \tilde{W}_1(T) \leq c_{13} \epsilon W_{\infty}(T). \tag{3.24}
\]

Then, noting (1.16) we get
\[
W_{\infty}(T) \leq c_{14} M. \tag{3.25}
\]

Finally, we estimate \( U_{\infty}(T) \) and \( V_{\infty}(T) \).

We first assume that \( V_{\infty}(T) \leq A \epsilon \), where \( A \) is a fixed positive constant which is determined below.

Obviously, for any fixed \( \alpha \in \mathbb{R} \),
\[
v_i(t,x) = \int_{x_i}^x (v_j)_y(t,y) dy + v_i(t,x_i),
\]
where \( x_i \) is the intersection point of the line \( t = t \) and the \( i \)-characteristic through the point \((0, \alpha)\).

Then, noting (2.1)–(2.4) and (3.18) we get
\[
\left| v_i(t,x) \right| \leq c_{15} W_1(t) + \left| v_i(t,x_i) \right| \leq c_{16} \epsilon + \left| v_i(t,x_i) \right|. \tag{3.26}
\]

On the other hand, noting (2.6), we have,
\[
v_i(t,x_i) = v_i(0, \alpha) + \int_0^t \sum_{j,k=1}^n \beta_{ijk}(u)v_j w_k(s, x_i(s, \alpha)) ds. \tag{3.27}
\]

It follows from (2.8) that
\[
\left| v_i(t,x_i) \right| \leq \left| v_i(0, \alpha) \right| + c_{17} n \times A \epsilon \times \tilde{W}_1(t) \leq \left| v_i(0, \alpha) \right| + c_{18} A \epsilon^2. \tag{3.28}
\]

Therefore, it follows from (3.26) and (3.28) that
\[
\left| v_i(t,x) \right| \leq c_{16} \epsilon + \left| v_i(0, \alpha) \right| + c_{18} A \epsilon^2, \quad \forall i \in \{1, \ldots, n\}.
\]

Noting (3.1),
\[
\left| v(t,x) \right| \leq c_{19} \epsilon + c_{18} A \epsilon^2 \leq 2c_{19} \epsilon, \tag{3.29}
\]

provided that \( c_{18} A \epsilon \leq c_{19} \). Taking \( A \geq 4c_{19} \), we have
\[
V_{\infty}(T) \leq 2c_{19} \epsilon \leq \frac{A}{2} \epsilon. \tag{3.30}
\]
By continuous induction, we have

\[ V_\infty(T) \leq \frac{A}{2} \epsilon < +\infty. \]  

(3.31)

Equivalently,

\[ U_\infty(T) \leq c_{20} A \epsilon. \]  

(3.32)

Taking \( K_0 \) suitably large, (3.32) shows that the hypothesis (3.3) is reasonable.

Finally, taking \( K_1 \) suitably large and noting (3.18), (3.21), (3.22), (3.25) and (3.31), (3.32), we obtain (3.10)–(3.12) immediately. Thus, the proof of Lemma 3.1 is completed. \( \square \)

**Proof of Theorem 1.1.** Theorem 1.1 follows from Lemma 3.1 directly. \( \square \)

4. Uniform estimates

In this section, we shall establish some new uniform estimates which play a key role in the proof of Theorem 1.2. In what follows, we assume that the normalized coordinates exist. Without loss of generality, we suppose that \( u = (u_1, \ldots, u_n)^T \) are already the normalized coordinates.

Similar to Section 3, we introduce

\[ U_1(T) = \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |v(t, x)| \, dx, \quad V_1(T) = \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |v(t, x)| \, dx, \]  

(4.1)

\[ \tilde{U}_1(T) = \max_{(i,j) \in \Gamma} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |u_i(t, x)| \, dt, \quad \tilde{V}_1(T) = \max_{(i,j) \in \Gamma} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |v_i(t, x)| \, dt, \]  

(4.2)

\[ \bar{U}_1(T) = \max_{(i,j) \in \Gamma} \sup_{L_j} \int_{L_j} |u_i(t, x)| \, dt, \quad \bar{V}_1(T) = \max_{(i,j) \in \Gamma} \sup_{L_j} \int_{L_j} |v_i(t, x)| \, dt. \]  

(4.3)

**Lemma 4.1.** Under the assumptions of Theorem 1.2, there exists a positive constant \( K_2 \) independent of \( \epsilon, M, N \) and \( T \) such that

\[ U_1(T), \tilde{U}_1(T), \bar{U}_1(T), V_1(T), \tilde{V}_1(T), \bar{V}_1(T) \leq K_2 N, \]  

(4.4)

\[ U_\infty(T), V_\infty(T) \leq K_2 \epsilon. \]  

(4.5)

**Proof.** By (1.19), we observe

\[ \lim_{x \to \pm \infty} f(x) = 0. \]  

(4.6)

By the finiteness of propagation speed of waves, we have

\[ \lim_{x \to \pm \infty} u(t, x) = 0, \quad \forall t \in [0, T]. \]  

(4.7)
Noting
\[ |u(t, x)| \leq \int_{-\infty}^{x} u_y(t, y) \, dy, \]
we get
\[ |u(t, x)| \leq \int_{-\infty}^{+\infty} |u_y(t, y)| \, dy \leq c_1 W_1(t). \] (4.8)

Here and hereafter \( c_j \) \((j = 1, 2, \ldots)\) will denote a constant independent of \( \epsilon, M, N \) and \( T \). Therefore,
\[ U_{\infty}(T) \leq c_1 W_1(T) \leq c_2 \epsilon. \] (4.9)

Equivalently, we have
\[ V_{\infty}(T) \leq c_3 \epsilon. \] (4.10)

We next introduce
\[ Q_V(T) = \sum_{(j, k) \in \Gamma} \int_{0}^{T} \int_{\mathbb{R}} \|v_j(t, x)\| \|w_k(t, x)\| \, dt \, dx. \]

By (2.11) and (2.21), it follows from Lemma 3.2 in Zhou [17] that
\[ Q_V(T) \leq c_4 \left( V_1(0) + \int_{0}^{T} |\tilde{F}(t, x)| \, dt \, dx \right) \left( W_1(0) + \int_{0}^{T} |\tilde{G}(t, x)| \, dt \, dx \right), \] (4.11)

where \( \tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_n) \).

Noting (2.14)–(2.16) and using Hadamard’s formula, we obtain from (2.11) that
\[ \tilde{F}_i(t, x) = \sum_{j, k=1}^{n} \tilde{\beta}_{ijk}(u) v_j w_k \]
\[ = \sum_{(j, k) \in \Gamma} \tilde{\beta}_{ijk}(u) v_j w_k + \sum_{j, k=1, \ldots, p} \left( \tilde{\beta}_{ijk}(u) - \tilde{\beta}_{ijk} \left( \sum_{h=1}^{p} u_h e_h \right) \right) v_j w_k \]
\[ + \sum_{j=k \in \{p+1, \ldots, n\}} \left( \tilde{\beta}_{ijk}(u) - \tilde{\beta}_{ijk}(u_j e_j) \right) v_j w_k \]
\[ = \sum_{(j, k) \in \Gamma} \tilde{\beta}_{ijk}(u) v_j w_k + \sum_{j, k=1}^{p} \sum_{l=p+1}^{n} \Gamma_{ijkl}(u) v_j w_k u_l. \]
\begin{equation}
+ \sum_{j=k=p+1}^{n} \sum_{l \neq j}^{n} \Gamma_{ijkl} v_j w_k u_l, \tag{4.12}
\end{equation}

where \( \Gamma_{ijkl}(u) \) are \( C^0 \) functions of \( u \), which are defined by

\begin{equation}
\begin{aligned}
\Gamma_{ijkl}(u) &= \int_{0}^{1} \frac{\partial \tilde{\beta}_{ijk}(u_1, \ldots, u_p, su_{p+1}, \ldots, su_n)}{\partial u_l} ds, \\
& \quad \text{if } j, k \in \{1, \ldots, p\}, l \in \{p+1, \ldots, n\}, \\
\Gamma_{ijkl}(u) &= \int_{0}^{1} \frac{\partial \tilde{\beta}_{ijk}(su_1, \ldots, su_{k-1}, u_k, su_{k+1}, \ldots, su_n)}{\partial u_l} ds, \\
& \quad \text{if } j = k \in \{p+1, \ldots, n\}, l \neq j.
\end{aligned} \tag{4.13}
\end{equation}

Noting (1.17), (1.18), (2.3) and (4.9), (4.10) and using Hadamard’s formula, we have

\begin{equation}
\sum_{j=p+1}^{n} |u_j| = \sum_{j=p+1}^{n} |u \ast e_j| = \sum_{j=p+1}^{n} \left| \sum_{k=1}^{n} v_k r_k(u) e_j \right| \\
= \sum_{j=p+1}^{n} \left| \sum_{k=1}^{p} v_k \left( r_k(u) - r_k \left( \sum_{h=1}^{p} u_h e_h \right) \right) e_j + \sum_{k=p+1}^{n} v_k r_k(u) e_j \right| \\
\leq c_4 \epsilon \sum_{j=p+1}^{n} |u_j| + c_4 \sum_{j=p+1}^{n} |v_j| \tag{4.14}
\end{equation}

and

\begin{equation}
\sum_{j \neq k}^{n} |u_j| = \sum_{j \neq k}^{n} |u \ast e_j| = \sum_{j \neq k}^{n} \left| \sum_{l=1}^{n} v_l r_l(u) e_j \right| \\
= \sum_{j \neq k}^{n} \left| \sum_{l \neq k} v_l r_l(u) e_j + v_k \left( r_k(u) - r_k(u_k e_k) \right) e_j \right| \\
\leq c_5 \sum_{j \neq k}^{n} |v_j| + c_5 \epsilon \sum_{j \neq k}^{n} |u_j|, \quad \forall k \in \{p+1, \ldots, n\}. \tag{4.15}
\end{equation}

It follows from (4.14), (4.15) that

\begin{equation}
\sum_{j=p+1}^{n} |u_j| \leq c_6 \sum_{j=p+1}^{n} |v_j|, \quad \sum_{j \neq k}^{n} |u_j| \leq c_7 \sum_{j \neq k}^{n} |v_j|, \quad \text{when } k \in \{p+1, \ldots, n\}. \tag{4.16}
\end{equation}

Then, by (4.12) and (4.16), we have

\begin{equation}
|\tilde{F_i}(t, x)| \leq c_8 \sum_{(j,k) \in \Gamma} |v_j w_k|, \quad \forall i \in \{1, \ldots, n\}. \tag{4.17}
\end{equation}
Hence,
\[
\int_{0}^{T} \int_{\mathbb{R}} |\tilde{F}(t, x)| \, dt \, dx \leq c_9 Q_V(T). \tag{4.18}
\]

Noting (1.14), (1.19), (3.14), (3.16) and (4.18), we obtain from (4.11) that
\[
Q_V(T) \leq c_{10} (V_1(0) + Q_V(T))(W_1(0) + Q_W(T)) \\
\leq c_{11} (N + Q_V(T))(\epsilon + \epsilon^2). \tag{4.19}
\]

Therefore,
\[
Q_V(T) \leq c_{12} N \epsilon. \tag{4.20}
\]

Similar to (4.17), by (2.6) we get
\[
|F_i(t, x)| \leq c_{13} \sum_{(j, k) \in \Gamma} |v_j w_k|, \quad \forall i \in \{1, 2, \ldots, n\}. \tag{4.21}
\]

On the other hand, similar to (3.17), by (4.21) we obtain from (2.11) that
\[
\int_{-\infty}^{+\infty} |v_i(t, x)| \, dx \leq V_1(0) + \int_{0}^{T} |F(t, x)| \, dt \, dx \\
\leq V_1(0) + c_{14} Q_V(T) \leq c_{15} N, \tag{4.22}
\]

where \( F(t, x) = (F_1, \ldots, F_n) \). Therefore,
\[
V_1(T) \leq c_{16} N. \tag{4.23}
\]

Similar to (3.19), we obtain from (2.12) that
\[
\int_{\tilde{C}_j} v_i(t, x) (\lambda_j(u) - \lambda_i(u)) \, dt = \int_{\alpha}^{\beta} v_i(0, x) \, dx - \int_{\Omega} \tilde{F}_i(t, x) \, dt \, dx. \tag{4.24}
\]

It follows from (3.2) and (4.17) that
\[
\int_{\tilde{C}_j} |v_i(t, x)| \, dt \leq \frac{c_{17}}{\delta_0} (V_1(0) + Q_V(T)). \tag{4.25}
\]

Thus, by (1.19) and (4.20) we have
\[
\tilde{V}_1(T) \leq c_{18} (V_1(0) + Q_V(T)) \leq c_{19} N. \tag{4.26}
\]
Similarly, replacing the $j$th characteristic $\tilde{C}_j$ by the ray $L_j$ with the slope $\lambda_j(0)$, we get

$$\tilde{V}_1(T) \leq c_{20} \left(V_1(0) + Q_V(T)\right) \leq c_{21} N.$$  \hfill (4.27)

Moreover, noting (3.11) and using (4.23), we have

$$U_1(T) \leq c_{22} N.$$  \hfill (4.28)

Finally, we estimate $\tilde{U}_1(T)$ and $\bar{U}_1(T)$.

Noting (1.17), (1.18) and (1.11a) and using Hadamard’s formula we have

$$u_i = \sum_{k=1}^{n} v_k r_k(u)e_i$$

$$= v_i + \sum_{k=1}^{p} v_k \left(r_k(u) - r_k \left(\sum_{h=1}^{p} u_h e_h\right)\right) e_i + \sum_{k=p+1}^{n} v_k (r_k(u) - r_k (u_k e_k)) e_i$$

$$= v_i + \sum_{k=1}^{p} v_k \sum_{j=p+1}^{n} u_j \int_{0}^{1} \frac{\partial r_k(u_1, \ldots, u_p, s u_{p+1}, \ldots, s u_n)}{\partial u_j} e_i \, ds$$

$$+ \sum_{k=p+1}^{n} v_k \sum_{j \neq k}^{n} u_j \int_{0}^{1} \frac{\partial r_k(s u_1, \ldots, s u_{k-1}, u_k, s u_{k+1}, \ldots, s u_n)}{\partial u_j} e_i \, ds$$

$$= v_i + \sum_{(j,k) \in \Gamma} \Xi_{ijk}(u) u_j v_k,$$ \hfill (4.29)

where $\Xi_{ijk}(u)$ are $C^1$ functions of $u$, which are defined by

$$\Xi_{ijk}(u) = \int_{0}^{1} \frac{\partial r_k(u_1, \ldots, u_p, s u_{p+1}, \ldots, s u_n)}{\partial u_j} e_i \, ds,$$

if $k \in \{1, \ldots, p\}$, $j \in \{p+1, \ldots, n\}$;  \hfill (4.30)

$$\Xi_{ijk}(u) = \int_{0}^{1} \frac{\partial r_k(s u_1, \ldots, s u_{k-1}, u_k, s u_{k+1}, \ldots, s u_n)}{\partial u_j} e_i \, ds,$$

if $k \in \{p+1, \ldots, n\}$, $j \neq k$.

Integrating (4.29) along the $j$th characteristic $\tilde{C}_j: x = x_j(s, \alpha)$ (where $i$, $j$ satisfy $(i, j) \in \Gamma$) and noting (1.10) gives

$$\int_{\tilde{C}_j} \left|u_i(t, x)\right| \, dt \leq \tilde{V}_1(T) + c_{23} \left\{ U_\infty(T) \tilde{V}_1(T) + V_\infty(T) \tilde{U}_1(T) \right\}.$$  \hfill (4.31)
Noting (4.9), (4.10) and (4.26), we obtain from (4.31) that
\[ \tilde{U}_1(T) \leq c_{24}N. \] (4.32)

On the other hand, for arbitrary \( i \) and \( j \) with \( (i, j) \in \Gamma \), integrating (4.29) along the ray \( L_j \) with slope \( \lambda_j(0) \) and noting (1.10), (4.27) and (4.9), (4.10), we have
\[ \bar{U}_1(T) \leq c_{25}N. \] (4.33)

Taking \( K_2 \) suitably large and noting (4.9), (4.10), (4.23), (4.26)–(4.28) and (4.32), (4.33), we obtain (4.4), (4.5) immediately. Thus, the proof of Lemma 4.1 is completed. \( \Box \)

By Lemmas 3.1 and 4.1, we have

**Lemma 4.2.** Under the assumptions of Theorem 1.2, there exists a positive constant \( K_3 \) independent of \( \epsilon, M \) and \( N \) such that
\[ U_1(\infty), \tilde{U}_1(\infty), \bar{U}_1(\infty), V_1(\infty), \tilde{V}_1(\infty), \bar{V}_1(\infty) \leq K_3N, \] (4.34)
\[ W_1(\infty), \tilde{W}_1(\infty), \bar{W}_1(\infty) \leq K_3\epsilon, \] (4.35)
\[ U_{\infty}(\infty), V_{\infty}(\infty) \leq K_3\epsilon, \] (4.36)
\[ W_{\infty}(\infty) \leq K_3M, \] (4.37)

where
\[ V_1(\infty) = \sup_{0 \leq t \leq \infty} \int_{-\infty}^{+\infty} |v(t, x)| \, dx, \]

etc.

**Lemma 4.3.** Under the assumptions of Theorem 1.2, for any \( t \in \mathbb{R}^+ \) and arbitrary \( \alpha, \beta \in \mathbb{R} \), it holds that
\[ |u(t, \alpha + \lambda_i(0)t) - u(t, \beta + \lambda_i(0)t)| \leq c_1M|\alpha - \beta|, \] (4.38)
\[ |u(t, x_i(t, \alpha)) - u(t, x_i(t, \beta))| \leq c_2M|\alpha - \beta|. \] (4.39)

Moreover, for any given \( C^1 \) function \( g(u) \),
\[ |g(u(t, \alpha + \lambda_i(0)t)) - g(u(t, \beta + \lambda_i(0)t))| \leq c_3M|\alpha - \beta| \] (4.40)
and
\[ |g(u(t, x_i(t, \alpha))) - g(u(t, x_i(t, \beta)))| \leq c_4M|\alpha - \beta|, \] (4.41)

where \( x = x_i(t, \cdot) \) stands for the \( i \)th characteristic passing through the point \((0, \cdot)\), here and hereafter \( c_i \) (\( i = 1, 2, \ldots \)) stand for some positive constants independent of \( \epsilon, M, N, t, \alpha \) and \( \beta \).
Proof. For fixing the idea we suppose that $\alpha \leq \beta$. Since the solution $u = u(t, x)$ is classical, i.e., $u \in C^1([0, +\infty) \times \mathbb{R})$, noting Lemma 4.2 and using Taylor’s formula, we can easily get (4.38) and (4.40).

We next prove (4.39).

Using (2.4) and noting (4.5) and (4.37), we have

$$
|u(t, x_i(t, \alpha)) - u(t, x_i(t, \beta))| \leq \sup_{x \in \mathbb{R}} \left| u_x(t, x) \right| \times \sup_{\xi \in \mathbb{R}} \left| \frac{\partial x_i(t, \xi)}{\partial \xi} \right| \times |\alpha - \beta|
$$

$$
\leq c_5 W_\infty(t) \times \sup_{\xi \in \mathbb{R}} \left| \frac{\partial x_i(t, \xi)}{\partial \xi} \right| \times |\alpha - \beta|
$$

$$
\leq c_6 M|\alpha - \beta| \times \sup_{\xi \in \mathbb{R}} \left| \frac{\partial x_i(t, \xi)}{\partial \xi} \right|. \quad (4.42)
$$

In what follows, we estimate $|\frac{\partial x_i(t, \xi)}{\partial \xi}|$.

Noting

$$
\frac{\partial x_i(t, \xi)}{\partial t} = \lambda_i(u)(t, x_i(t, \xi)), \quad (4.43)
$$

we have

$$
\frac{\partial}{\partial t} \left( \frac{\partial x_i(t, \xi)}{\partial \xi} \right) = \nabla \lambda_i(u) u_x \frac{\partial x_i(t, \xi)}{\partial \xi}. \quad (4.44)
$$

Noticing $x_i(0, \xi) = \xi$ gives

$$
\frac{\partial x_i(0, \xi)}{\partial \xi} = 1. \quad (4.45)
$$

Then it follows from (4.44), (4.45) that

$$
\frac{\partial x_i(t, \xi)}{\partial \xi} = \exp \left\{ \int_0^t \left( \nabla \lambda_i(u) u_x \right) \left( s, x_i(s, \xi) \right) \, ds \right\}. \quad (4.46)
$$

Noting (1.11), we have

$$
\nabla \lambda_i(u) u_x = \nabla \lambda_i(u) \sum_{j=1}^n w_j r_j(u) = \sum_{(i, j) \in \Gamma} \left[ \nabla \lambda_i(u) r_j(u) \right] w_j. \quad (4.47)
$$

Noting (4.35), we obtain from (4.47) that

$$
\int_0^t \left| \left( \nabla \lambda_i(u) u_x \right) \left( s, x_i(s, \xi) \right) \right| \, ds \leq c_7 \epsilon. \quad (4.48)
$$
where the constants $c_7$ are independent of not only $\epsilon, M, N, t$ but also $\alpha, \beta$ and $\xi$. Combining (4.46) and (4.48) gives

$$
\sup_{(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}} \left\{ \left| \frac{\partial x_i(t, \xi)}{\partial \xi} \right| \right\} \leq e^{c_7 \epsilon}.
$$

Substituting (4.49) into (4.42) yields (4.39) immediately. Finally, noting (4.39) we get (4.41) by Taylor’s formula. Thus, the proof of Lemma 4.3 is completed. □

For any fixed $T \geq 0$ and for arbitrary $\alpha, \beta \in \mathbb{R}$, we introduce

$$
U_{\alpha}^\beta(T) = \max_{(i,j) \in \Gamma} \int_0^T |u_j(s, \alpha + \lambda_i(0)s) - u_j(s, \beta + \lambda_i(0)s)| \, ds,
$$

$$
V_{\alpha}^\beta(T) = \max_{(i,j) \in \Gamma} \int_0^T |v_j(s, \alpha + \lambda_i(0)s) - v_j(s, \beta + \lambda_i(0)s)| \, ds,
$$

$$
W_{\alpha}^\beta(T) = \max_{(i,j) \in \Gamma} \int_0^T |w_j(s, \alpha + \lambda_i(0)s) - w_j(s, \beta + \lambda_i(0)s)| \, ds,
$$

$$
\tilde{U}_{\alpha}^\beta(T) = \max_{(i,j) \in \Gamma} \int_0^T |u_j(s, x_i(s, \alpha)) - u_j(s, x_i(s, \beta))| \, ds,
$$

$$
\tilde{V}_{\alpha}^\beta(T) = \max_{(i,j) \in \Gamma} \int_0^T |v_j(s, x_i(s, \alpha)) - v_j(s, x_i(s, \beta))| \, ds,
$$

$$
\tilde{W}_{\alpha}^\beta(T) = \max_{(i,j) \in \Gamma} \int_0^T |w_j(s, x_i(s, \alpha)) - w_j(s, x_i(s, \beta))| \, ds,
$$

where $x = x_i(s, \cdot)$, as before, stands for the $i$th characteristic passing through the point $(0, \cdot)$.

**Lemma 4.4.** Under the assumptions of Theorem 1.2, there exists a positive constant $K_4$ independent of $\epsilon, M, N, T, \alpha$ and $\beta$ such that

$$
\tilde{U}_{\alpha}^\beta(T) \leq K_4(MN + \epsilon)|\alpha - \beta|,
$$

$$
\tilde{V}_{\alpha}^\beta(T) \leq K_4(MN + \epsilon)|\alpha - \beta|,
$$

and

$$
\tilde{W}_{\alpha}^\beta(T) \leq K_4M(1 + \epsilon)|\alpha - \beta|.
$$
Proof. We first prove (4.57).

Let \( \tilde{C}_i(\alpha) \) and \( \tilde{C}_i(\beta) \) be the \( i \)th characteristics passing through the points \( P_1: (0, \alpha) \) and \( P_2: (0, \beta) \), respectively. For the sake of simplicity, we assume that \( \alpha < \beta \). Denote the intersection point of \( \tilde{C}_i(\alpha) \) (respectively \( \tilde{C}_i(\beta) \)) with the straight line \( t = T \) by \( P_4 \): \( (T, x_i(T, \alpha)) \) (respectively \( P_3 \): \( (T, x_i(T, \beta)) \)). Let \( \tilde{\Omega} \) be the region bounded by the curves \( \tilde{C}_i(\alpha), \tilde{C}_i(\beta), t = 0 \) and \( t = T \), i.e., the curved-quadrilateral \( P_1P_2P_3P_4 \). It follows from (2.12) that

\[
\begin{aligned}
&\frac{d}{dt} \left[ \xi(t)v_j(dx - \lambda_j(u)dt) \right] = \xi(t) \tilde{F}_j(t, x) dt dx, \quad \text{a.e.,}
&\xi(t) = \text{sign}\left[ (v_j(t, x_i(t, \alpha)) - v_j(t, x_i(t, \beta))) (\lambda_i(u)(t, x_i(t, \beta)) - \lambda_j(u)(t, x_i(t, \alpha))) \right].
\end{aligned}
\]

By Green formula, we have

\[
\begin{aligned}
\int_{\tilde{\Omega}} \int \xi(s) \tilde{F}_j(s, x) ds dx
&= \int_\alpha^\beta \xi(0)v_j(0, x) dx + \int_0^T \xi(s) \left[ v_j(\lambda_i(u) - \lambda_j(u)) \right] (s, x_i(s, \beta)) ds

& \quad - \int_\alpha^\beta \xi(T)v_j(T, x_i(T, \gamma)) d\gamma - \int_0^T \xi(s) \left[ v_j(\lambda_i(u) - \lambda_j(u)) \right] (s, x_i(s, \alpha)) ds,
\end{aligned}
\]

where \((i, j) \in \Gamma\), that is,

\[
\begin{aligned}
&\int_0^T \left| (v_j(s, x_i(s, \alpha)) - v_j(s, x_i(s, \beta))) (\lambda_i(u)(s, x_i(s, \beta)) - \lambda_j(u)(s, x_i(s, \alpha))) \right| ds

&= \int_0^T \xi(s)v_j(s, x_i(s, \beta)) \left[ \lambda_j(u)(s, x_i(s, \alpha)) - \lambda_j(u)(s, x_i(s, \beta)) \right] ds

& \quad - \int_0^T \xi(s)v_j(s, x_i(s, \alpha)) \left[ \lambda_i(u)(s, x_i(s, \alpha)) - \lambda_j(u)(s, x_i(s, \beta)) \right] ds

& \quad + \int_\alpha^\beta \left[ \xi(0)v_j(0, \gamma) - \xi(T)v_j(T, x_i(T, \gamma)) \right] d\gamma - \int_{\tilde{\Omega}} \int \xi(s) \tilde{F}_j(s, x) ds dx. \quad (4.59)
\end{aligned}
\]

When \((i, j) \in \Gamma\), noting (3.2), (4.41) and using (3.10), (3.12) and (4.4), (4.5), we obtain from (4.59) that
\[
\int_0^T \left| v_j(s, x_i(s, \alpha)) - v_j(s, x_i(s, \beta)) \right| ds \leq \frac{1}{\delta_0} \left\{ 2V_\infty(T) + 2c_4 M \tilde{V}_1(T) \right\} |\alpha - \beta| + \int_\Omega |\tilde{F}_j(s, x)| \, ds \, dx \\
\leq c_8 \left\{ (MN + \epsilon)|\alpha - \beta| + \int_\Omega |\tilde{F}_j(s, x)| \, ds \, dx \right\}. \tag{4.60}
\]

On the other hand, noting (4.17) and using (3.10), (3.12) and (4.4), (4.5), we have
\[
\int_\Omega |\tilde{F}_j(s, x)| \, ds \, dx \leq c_9 \sum_{(k,l) \in \Gamma} \int_\Omega |v_k w_l| \, ds \, dx \\
= c_9 \sum_{(k,l) \in \Gamma} \int_0^\beta d\gamma \int_0^T |v_k w_l|(s, x_i(s, \gamma)) \, ds \\
\leq c_{10} \left\{ V_\infty(T) \tilde{W}_1(T) + W_\infty(T) \tilde{V}_1(T) \right\} |\alpha - \beta| \\
\leq c_{11} \{K_2 \epsilon \times K_1 \epsilon + K_1 M \times K_2 N\}|\alpha - \beta| \\
\leq c_{12} (MN + \epsilon^2)|\alpha - \beta|. \tag{4.61}
\]

Substituting (4.61) into (4.60) gives
\[
\int_0^T \left| v_j(s, x_i(s, \alpha)) - v_j(s, x_i(s, \beta)) \right| ds \leq c_{13} (MN + \epsilon)|\alpha - \beta|. \tag{4.62}
\]

This proves (4.57).

We next prove (4.58).

It follows from (2.22) that
\[
d\left[ \xi(t) w_j(dx - \lambda_j(u) \, dt) \right] = \xi(t) \tilde{G}_j(t, x) \, dt \, dx, \quad \text{a.e.,}
\]
where
\[
\xi(t) = \text{sign}\left[ \left( w_j(t, x_i(t, \alpha)) - w_j(t, x_i(t, \beta)) \right) \left( \lambda_i(u)(t, x_i(t, \beta)) - \lambda_j(u)(t, x_i(t, \alpha)) \right) \right].
\]

Similar to (4.59), by Green formula we have
\[
\int_0^T \left| (w_j(s, x_i(s, \alpha)) - w_j(s, x_i(s, \beta))) (\lambda_i(u)(s, x_i(s, \beta)) - \lambda_j(u)(s, x_i(s, \alpha))) \right| ds
\]
\[
\begin{align*}
&= \int_0^T \xi(s) w_j(s, x_i(s, \beta)) [\lambda_j(u)(s, x_i(s, \alpha)) - \lambda_j(u)(s, x_i(s, \beta))] \, ds \\
&\quad - \int_0^T \xi(s) w_j(s, x_i(s, \alpha)) [\lambda_i(u)(s, x_i(s, \alpha)) - \lambda_i(u)(s, x_i(s, \beta))] \, ds \\
&\quad + \int_0^\beta \left[ \xi(0) w_j(0, \gamma) - \xi(T) w_j(T, x_i(T, \gamma)) \right] \, d\gamma - \int_\tilde{\Omega} \xi(s) \tilde{G}_j(s, x) \, ds \, dx. \tag{4.63}
\end{align*}
\]

When \((i, j) \in \Gamma\), noting (3.2), (4.41) and using (3.10), (3.12), (4.4), (4.5), we obtain from (4.63) that

\[
\begin{align*}
&\int_0^T \left| w_j(s, x_i(s, \alpha)) - w_j(s, x_i(s, \beta)) \right| \, ds \\
&\leq \frac{1}{\delta_0} \left\{ 2W_\infty(T) + c_2 M \tilde{W}_1(T) \right\} |\alpha - \beta| + \int_\tilde{\Omega} \left| \tilde{G}_j(s, x) \right| \, ds \, dx \\
&\leq c_{14} \left\{ M(1 + \epsilon) |\alpha - \beta| + \int_\tilde{\Omega} \left| \tilde{G}_j(s, x) \right| \, ds \, dx \right\}. \tag{4.64}
\end{align*}
\]

On the other hand, noting (1.10) and (2.24) and using (3.10), (3.12) and (4.4), (4.5), we obtain from (4.64) that

\[
\begin{align*}
&\int_\tilde{\Omega} \left| \tilde{G}_j(s, x) \right| \, ds \, dx \leq c_{15} \sum_{(k, l) \in \Gamma} \int_\tilde{\Omega} \left| w_k w_l \right| \, ds \, dx \\
&\quad = c_{15} \sum_{(k, l) \in \Gamma} \int_0^\beta d\gamma \int_0^T \left| w_k w_l \right|(s, x_i(s, \gamma)) \, ds \\
&\quad \leq c_{16} W_\infty(T) \tilde{W}_1(T) |\alpha - \beta| \\
&\quad \leq c_{16} K_1 M \times K_1 \epsilon |\alpha - \beta| \\
&\quad \leq c_{17} M \epsilon |\alpha - \beta|. \tag{4.65}
\end{align*}
\]

Substituting (4.65) into (4.64) gives

\[
\begin{align*}
&\int_0^T \left| w_j(s, x_i(s, \alpha)) - w_j(s, x_i(s, \beta)) \right| \, ds \leq c_{18} M(1 + \epsilon) |\alpha - \beta|. \tag{4.66}
\end{align*}
\]

This proves (4.58).
We finally prove (4.56).
Noting (4.29), we have
\[
\xi(t)u_j = \xi(t)v_j + \xi(t) \sum_{(k,l) \in \Gamma} \Xi_{jkl}(u)u_k v_l, \quad \text{a.e.,}
\]
where \( \Xi_{jkl}(u) \) is defined by (4.30) and
\[
\xi(t) = \text{sign}\left( [u_j](\alpha) - [u_j](\beta) \right).
\]
For arbitrary \( i,j \) with \( (i,j) \in \Gamma \), integrating (4.67) from 0 to \( T \) along the characteristic: \( x = x_i(s,\alpha) \) and \( x = x_i(s,\beta) \), respectively, and then subtracting the last integral from the first, we obtain
\[
\int_0^T \left| [u_j](\alpha) - [u_j](\beta) \right| ds = \int_0^T \tilde{\xi}(s) \left\{ [v_j](\alpha) - [v_j](\beta) \right\} ds
\]
\[
+ \sum_{(k,l) \in \Gamma} \int_0^T \tilde{\xi}(s) \left\{ \left[ \Xi_{jkl}(u)u_k v_l \right](\alpha) - \left[ \Xi_{jkl}(u)u_k v_l \right](\beta) \right\} ds
\]
\[
= \int_0^T \tilde{\xi}(s) \left\{ [v_j](\alpha) - [v_j](\beta) \right\} ds
\]
\[
+ \sum_{(k,l) \in \Gamma} \int_0^T \tilde{\xi}(s) \left\{ \left[ \Xi_{jkl}(u) \right](\alpha) - \left[ \Xi_{jkl}(u) \right](\beta) \right\} \left[ u_k(\alpha)[v_l](\alpha) \right] ds
\]
\[
+ \sum_{(k,l) \in \Gamma} \int_0^T \tilde{\xi}(s) \left\{ \left[ \Xi_{jkl}(u) \right](\beta)\left[ u_k(\alpha) - u_k(\beta) \right] \right\} \left[ v_l(\alpha) \right] ds
\]
\[
+ \sum_{(k,l) \in \Gamma} \int_0^T \tilde{\xi}(s) \left\{ \left[ \Xi_{jkl}(u) \right](\beta)\left[ u_k(\beta) \right] \right\} \left[ v_l(\alpha) - [v_l](\beta) \right] ds,
\]
where \([\cdot](\alpha)\) stands for \([\cdot](s,x_i(s,\alpha))\), while \([\cdot](\beta)\) stands for \([\cdot](s,x_i(s,\beta))\). Thus, noting (3.2), (3.4) and using (2.1), (4.39) and (4.41), we obtain from (4.68) that
\[
\int_0^T \left| [u_j](\alpha) - [u_j](\beta) \right| ds \leq \tilde{V}_\alpha^\beta(T) + c_{19} \left\{ c_4 M|\alpha - \beta| [U_\infty(T)\tilde{V}_1(T) + V_\infty(T)\tilde{U}_1(T)]
\]
\[
+ c_2 M|\alpha - \beta| \tilde{V}_1(T) + \tilde{U}_\alpha^\beta(T)V_\infty(T)
\]
\[
+ \tilde{V}_\alpha^\beta(T)U_\infty(T) + c_2 M|\alpha - \beta| \tilde{U}_1(T) \right\}.
\]
Then, using (3.10), (3.12), (4.4), (4.5) and (4.62), we have

$$\int_0^T \left| [u_j](\alpha) - [u_j](\beta) \right| ds \leq c_{13} (MN + \epsilon) |\alpha - \beta| + c_{20} \left\{ 2c_4 M |\alpha - \beta| \times K_2 \epsilon \times K_2 N 
+ 2c_2 M |\alpha - \beta| \times K_2 N + K_2 \epsilon \tilde{U}_\alpha^\beta (T) 
+ K_2 \epsilon \times c_{13} (MN + \epsilon) |\alpha - \beta| \right\}$$

$$\leq c_{21} (MN + \epsilon) |\alpha - \beta| + c_{22} K_2 \epsilon \tilde{U}_\alpha^\beta (T). \quad (4.70)$$

It follows from (4.70) that

$$\tilde{U}_\alpha^\beta (T) \leq c_{21} (MN + \epsilon) |\alpha - \beta| + c_{22} K_2 \epsilon \tilde{U}_\alpha^\beta (T). \quad (4.71)$$

(4.56) comes from (4.71) directly. Thus, the proof of Lemma 4.4 is completed. \qed

Similarly, we can prove the following lemma.

**Lemma 4.5.** Under the assumptions of Theorem 1.2, there exists a positive constant $K_5$ independent of $\epsilon$, $M$, $N$, $T$, $\alpha$ and $\beta$ such that

$$U_\alpha^\beta (T) \leq K_5 (MN + \epsilon) |\alpha - \beta|, \quad (4.72)$$

$$V_\alpha^\beta (T) \leq K_5 (MN + \epsilon) |\alpha - \beta| \quad (4.73)$$

and

$$W_\alpha^\beta (T) \leq K_5 (M + \epsilon) |\alpha - \beta|. \quad (4.74)$$

**Proof.** Noting (4.40) instead of (4.41), in a manner completely similar to the proof of Lemma 4.4, we can prove Lemma 4.5. Here we omit the details. \qed

Combining Lemmas 4.4, 4.5 gives

**Lemma 4.6.** Under the assumptions of Theorem 1.2, there exists a positive constant $K_6$ independent of $\epsilon$, $M$, $N$, $\alpha$ and $\beta$ such that

$$U_\alpha^\beta (\infty), \tilde{U}_\alpha^\beta (\infty), V_\alpha^\beta (\infty), \tilde{V}_\alpha^\beta (\infty) \leq K_6 (MN + \epsilon) |\alpha - \beta| \quad (4.75)$$

and

$$W_\alpha^\beta (\infty), \tilde{W}_\alpha^\beta (\infty) \leq K_6 (M + \epsilon) |\alpha - \beta|, \quad (4.76)$$

where
\( U_\alpha^\beta (\infty) = \max_{(i,j) \in \Gamma} \int_0^\infty |u_j(s, \alpha + \lambda_i(0)s) - u_j(s, \beta + \lambda_i(0)s)| \, ds, \) \hspace{1cm} (4.77)

eq \quad \text{etc.}

We finally estimate the difference of \( w_i \) on two differential \( i \)th characteristic at the same time. For arbitrary \( \alpha, \beta \in \mathbb{R} \), we introduce

\[
\begin{align*}
W^*_{\alpha, \beta} (\infty) &= \sum_{i=1}^p \sup_{t \in [0, \infty)} |w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta))|; \\
W^i_{\alpha, \beta} (\infty) &= \sup_{t \in [0, \infty)} |w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta))|, \quad \text{for } i \in \{p+1, \ldots, n\},
\end{align*}
\]

where \( x = x_i(t, \cdot) \) stands for the \( i \)th characteristic passing through the point \((0, \cdot)\).

\textbf{Lemma 4.7.} For any given \( i \in \{1, \ldots, n\} \) and for any fixed \( \alpha \in \mathbb{R} \), the limit

\[
\lim_{t \to +\infty} w_i(t, x_i(t, \alpha))
\]

exists, denoted it by \( \Psi_i(\alpha) \), that is,

\[
\lim_{t \to +\infty} w_i(t, x_i(t, \alpha)) = \Psi_i(\alpha), \quad \forall \alpha \in \mathbb{R},
\]

where \( x = x_i(t, \alpha) \) stands for the \( i \)th characteristic passing through the point \((0, \alpha)\). Moreover, \( \Psi_i(\alpha) \) is a continuous function of \( \alpha \in \mathbb{R} \) and satisfies that there exists a positive constant \( K_7 \) independent of \( \epsilon, M, N, \) and \( \alpha \) such that

\[
|\Psi_i(\alpha)| \leq (1 + K_7 \epsilon) M, \quad \forall \alpha \in \mathbb{R}.
\]

Furthermore, there exists a positive constant \( K_8 \) independent of \( \epsilon, M, N, \alpha, \) and \( \beta \) such that

\[
\begin{align*}
W^*_{\alpha, \beta} (\infty) &\leq (1 + K_8 \epsilon) \sum_{i=1}^p |w_i(0, \alpha) - w_i(0, \beta)| + K_8 M^2 (1 + \epsilon) |\alpha - \beta|; \\
W^i_{\alpha, \beta} (\infty) &\leq (1 + K_8 \epsilon) |w_i(0, \alpha) - w_i(0, \beta)| + K_8 M^2 (1 + \epsilon) |\alpha - \beta|, \quad \forall i \in \{p+1, \ldots, n\}.
\end{align*}
\]

In particular, if \((1.22)\) is satisfied, then there exists a positive constant \( K_9 \) independent of \( \epsilon, M, N, \alpha, \) and \( \beta \) such that

\[
|\Psi_i(\alpha) - \Psi_i(\beta)| \leq K_9 |\alpha - \beta|^p + K_9 M^2 (1 + \epsilon) |\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R},
\]

where \( 0 < \rho \leq 1 \).
Remark 4.1. (4.81) indicates that \( u(t, x) \) is a \( C^{1,\rho} \) continuous vector-valued function with respect to \( x \), while (4.82) implies that \( \Psi_i(\alpha) \) is a globally \( \rho \)-Hölder continuous function of \( \alpha \in \mathbb{R} \), provided that (1.22) holds.

Proof. It follows from (2.17) and (2.20) that

\[
w_i(t, x_i(t, \alpha)) = w_i(0, \alpha) + \int_0^t G_i(s, x_i(s, \alpha)) \, ds
\]

\[
= w_i(0, \alpha) + \sum_{(j,k) \in \Gamma} \int_0^t \gamma_{ijk}(u) w_j w_k(s, x_i(s, \alpha)) \, ds.
\]

(4.83)

Then, Lemma 4.2 implies that the integrals in the right-hand side of (4.83) converge absolutely when \( t \) tends to \( +\infty \). Then, the right-hand side of (4.83) converges when \( t \) tends to \( +\infty \). We denote the limit by \( \Psi_i(\alpha) \). That is,

\[
\lim_{t \to \infty} w_i(t, x_i(t, \alpha)) = \Psi_i(\alpha).
\]

(4.84)

It follows from (3.10), (3.12) and (4.83) that

\[
|w_i(t, x_i(t, \alpha))| \leq |w_i(0, \alpha)| + c_{23} W_\infty(t) \tilde{W}_1(t)
\]

\[
\leq (1 + K_5\epsilon) M.
\]

(4.85)

(4.80) follows from (4.85) directly.

In what follows, we calculate \( w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta)) \):

\[
w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta))
\]

\[
= w_i(0, \alpha) - w_i(0, \beta) + \int_0^t [G_i(s, x_i(s, \alpha)) - G_i(s, x_i(s, \beta))] \, ds
\]

\[
= w_i(0, \alpha) - w_i(0, \beta) + \sum_{(j,k) \in \Gamma} \int_0^t [\gamma_{ijk}(u) w_j w_k(\alpha) - \gamma_{ijk}(u) w_j w_k(\beta)] \, ds,
\]

(4.86)

where \([\cdot](\ast)\) stands for \([\cdot](s, x_i(s, \ast))\). Noting Lemmas 4.2, 4.3, 4.6 and making use of the method of (4.68), we get

\[
|w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta))|
\]

\[
\leq |w_i(0, \alpha) - w_i(0, \beta)| + c_4 M |\alpha - \beta| W_\infty(t) \tilde{W}_1(t)
\]

\[
+ c_{24} W_\infty(t) \tilde{W}_\alpha^\beta(t) + c_{24} \tilde{W}_1(t) \sup_{(i,k) \in \Gamma} \sup_{t \in \mathbb{R}^+} |w_k(t, x_i(t, \alpha)) - w_k(t, x_i(t, \beta))|
\]

\[
\leq |w_i(0, \alpha) - w_i(0, \beta)| + c_4 M |\alpha - \beta| K_1 M K_1 \epsilon + c_{24} K_1 M K_3 M (1 + \epsilon) |\alpha - \beta|
\]
\[ + c_{24} K_1 \epsilon \sum_{(i,k) \notin \Gamma} \sup_{t \in [0, +\infty)} |w_k(t, x_i(t, \alpha)) - w_k(t, x_i(t, \beta))| \]
\[ \leq |w_i(0, \alpha) - w_i(0, \beta)| + c_{25} M^2 (1 + \epsilon) |\alpha - \beta| \]
\[ + c_{24} K_1 \epsilon \sum_{(i,k) \notin \Gamma} \sup_{t \in [0, +\infty)} |w_k(t, x_i(t, \alpha)) - w_k(t, x_i(t, \beta))|. \]  \hspace{1cm} (4.87)

We divide into two cases to discuss.

**Case I:** \( i \in \{1, \ldots, p\} \).
When \( i \in \{1, \ldots, p\} \) and \((i,k) \notin \Gamma\), we have
\[ k \in \{1, \ldots, p\}. \]

By the definition of \( i \)th characteristic \( x_i(t, \alpha) \),
\[ x_1(t, \alpha) = x_2(t, \alpha) = \cdots = x_p(t, \alpha) \triangleq x(t, \alpha), \quad \forall t \in [0, +\infty), \forall \alpha \in \mathbb{R}. \]

Then,
\[ |w_i(t, x(t, \alpha)) - w_i(t, x(t, \beta))| \]
\[ \leq |w_i(0, \alpha) - w_i(0, \beta)| + c_{25} M^2 (1 + \epsilon) |\alpha - \beta| \]
\[ + c_{24} K_1 \epsilon \sum_{k=1}^{p} \sup_{t \in [0, +\infty)} |w_k(t, x(t, \alpha)) - w_k(t, x(t, \beta))|. \]  \hspace{1cm} (4.88)

Summing (4.88) with respect to \( i = 1, \ldots, p \) gives
\[ \sum_{k=1}^{p} \sup_{t \in [0, +\infty)} |w_k(t, x(t, \alpha)) - w_k(t, x(t, \beta))| \]
\[ \leq (1 + K_8 \epsilon) \sum_{k=1}^{p} |w_k(0, \alpha) - w_k(0, \beta)| + K_8 M^2 (1 + \epsilon) |\alpha - \beta|. \]  \hspace{1cm} (4.89)

**Case II:** \( i \in \{p + 1, \ldots, n\} \).
When \( i \in \{p + 1, \ldots, n\} \) and \((i,k) \notin \Gamma\), we have
\[ k = i. \]

It follows from (4.87) that
\[ \sup_{t \in [0, +\infty)} |w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta))| \]
\[ \leq (1 + K_8 \epsilon) |w_i(0, \alpha) - w_i(0, \beta)| + K_8 M^2 (1 + \epsilon) |\alpha - \beta|. \]  \hspace{1cm} (4.90)

Then, (4.81) follows from (4.89) and (4.90) directly. Because \( w_i(0, x) \) is continuous, it follows from (4.81) that \( \Psi_i(\alpha) \in C^0(\mathbb{R}) \).
If (1.22) holds, we see that $w_i(0, x)$ is globally $\rho$-Hölder continuous. (4.82) follows from (4.81) easily. Thus, the proof of Lemma 4.7 is completed. □

5. Asymptotic behavior of global classical solution—Proof of Theorem 1.2

This section is devoted to the study of the asymptotic behavior of the global classical solution of the Cauchy problem (1.1), (1.12) and gives the proof of Theorem 1.2.

In this section, we assume that there exists a normalized transformation. Without loss of generality, we assume that $u = (u_1, \ldots, u_n)^T$ are already the normalized coordinates.

Let

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \lambda_i(0) \frac{\partial}{\partial x}. \tag{5.1}$$

Noting (1.1) and (2.4), we have

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \lambda_i(0) \frac{\partial u}{\partial x} = -A(u) \frac{\partial u}{\partial x} + \lambda_i(0) \frac{\partial u}{\partial x} = \sum_{j=1}^{n} (\lambda_i(0) - \lambda_j(u)) w_j r_j(u). \tag{5.2}$$

Therefore,

$$\frac{Du_i}{Dt} = \frac{Du}{Dt} e_i = \sum_{j=1}^{n} (\lambda_i(0) - \lambda_j(u)) w_j r_j(u) e_i. \tag{5.3}$$

Noting (1.11) and using Hadamard’s formula, we have

$$\frac{Du_i}{Dt} = \frac{1}{p+1} \sum_{j=p+1}^{n} (\lambda_i(0) - \lambda_j(u)) w_j r_j(u) e_i + \frac{1}{p} \sum_{j=1}^{p} \left( \lambda_i \left( \sum_{h=1}^{p} u_h e_h \right) - \lambda(u) \right) w_j r_j(u) e_i,$$

when $i \in \{1, \ldots, p\}$;

$$\frac{Du_i}{Dt} = \sum_{j \neq i} (\lambda_i(0) - \lambda_j(u)) w_j r_j(u) e_i + \left( \lambda_i(0) - \lambda_i(u) \right) w_i r_i(u) e_i \tag{5.4}$$

$$= \sum_{j \neq i} (\lambda_i(0) - \lambda_j(u)) w_j r_j(u) e_i + \left( \lambda_i(u_i e_i) - \lambda_i(u) \right) w_i r_i(u) e_i,$$

when $i \in \{p+1, \ldots, n\}$.

Thus, noting (1.10) and (1.11a) we have

$$\frac{D}{Dt} = \sum_{(i,j) \in \Gamma} B_{ij}(u) w_j + \Gamma_i(u) u_j \left[ \sum_{k \neq i} w_k r_k(u) \right]. \tag{5.5}$$
where

\[ B_{ij}(u) = (\lambda_i(0) - \lambda_j(u))r_j(u)e_i, \quad (5.6) \]

\[
\left\{
\begin{array}{l}
\Gamma_{ij}(u) = -e_i \int_0^1 \frac{\partial \lambda_i(u_1, \ldots, u_p, su_{p+1}, \ldots, su_n)}{\partial u_j} \, ds, \quad \text{if } i \in \{1, \ldots, p\}; \\
\Gamma_{ij}(u) = -e_i \int_0^1 \frac{\partial \lambda_i(su_1, \ldots, su_{i-1}, u_i, su_{i+1}, \ldots, su_n)}{\partial u_j} \, ds, \quad \text{if } i \in \{p+1, \ldots, n\}.
\end{array}
\right. \quad (5.7)
\]

For any fixed \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\), define

\[ \alpha = x - \lambda_i(0)t. \quad (5.8) \]

It follows from (5.5) that

\[ u_i(t, x) = u_i(t, \alpha + \lambda_i(0)t) = u_i(0, \alpha) \]

\[ + \sum_{(i, j) \in \Gamma} \int_0^t \left\{ B_{ij}(u)w_j + \Gamma_{ij}(u)u_j \left[ \sum_{(i,k) \notin \Gamma} w_k r_k(u) \right] \right\} (s, \alpha + \lambda_i(0)s) \, ds. \quad (5.9) \]

Noting (4.34)–(4.37), we observe that the integral in the right-hand side of (5.9) converges absolutely when \(t\) tends to \(+\infty\). Therefore, when \(t\) tends to \(+\infty\), the limit of the right-hand side of (5.9) exists, we denote it by \(\Phi_i(\alpha)\). That is,

\[ u_i(t, \alpha + \lambda_i(0)t) \rightarrow \Phi_i(\alpha), \quad \text{as } t \rightarrow +\infty. \quad (5.10) \]

Moreover, noting (4.36) we obtain that there exists a positive constant \(K_{10}\) independent of \(\epsilon, M, N\) and \(\alpha\) such that

\[ |\Phi_i(\alpha)| \leq K_{10}\epsilon. \quad (5.11) \]

Thus, we have proved the following lemma.

**Lemma 5.1.** For every \(i \in \{1, \ldots, n\}\) and any given \(\alpha \in \mathbb{R}\), the limit

\[ \lim_{t \rightarrow +\infty} u_i(t, \alpha + \lambda_i(0)t) = \Phi_i(\alpha) \quad (5.12) \]

exists; moreover, the limit function \(\Phi_i(\alpha)\) satisfies the estimate (5.11).

In what follows, we shall investigate the regularity of the limit function \(\Phi_i(\alpha)\). First, we prove that \(\Phi_i(\alpha)\) is a globally Lipschitz continuous function of \(\alpha\).
For any fixed $(t, \alpha + \lambda_i(0)t)$, there exists a unique $\theta_i(t, \alpha) \in \mathbb{R}$ such that

$$\theta_i(t, \alpha) + \int_0^t \lambda_i(u(s, x_i(s, \theta_i(t, \alpha)))) ds = \alpha + \lambda_i(0)t, \quad (5.13)$$

namely,

$$\theta_i(t, \alpha) = \alpha + \int_0^t \left[ \lambda_i(0) - \lambda_i(u(s, x_i(s, \theta_i(t, \alpha)))) \right] ds, \quad (5.14)$$

where $x = x_i(s, \theta_i(t, \alpha))$ stands for the $i$th characteristic passing through the point $(0, \theta_i(t, \alpha))$, which is defined by

$$\frac{dx_i(s, \theta_i(t, \alpha))}{ds} = \lambda_i(u(s, x_i(s, \theta_i(t, \alpha)))), \quad x_i(0, \theta_i(t, \alpha)) = \theta_i(t, \alpha). \quad (5.15)$$

**Lemma 5.2.** Under the assumptions of Theorem 1.2, for any given $\alpha \in \mathbb{R}$ there exists a unique $\vartheta_i(\alpha)$ such that $\theta_i(t, \alpha)$ converges to $\vartheta_i(\alpha)$ when $t$ tends to $\infty$; moreover, $\vartheta_i(\alpha)$ satisfies

$$|\vartheta_i(\alpha) - \alpha| \leq K_{11} N, \quad (5.16)$$

and it is a globally Lip-continuous function of $\alpha$, more precisely, the following estimate holds

$$|\vartheta_i(\alpha) - \vartheta_i(\beta)| \leq \left[ 1 + K_{12}(MN + \epsilon) \right] |\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}, \quad (5.17)$$

where $K_{11}$ is a positive constant independent of $\epsilon$, $M$, $N$ and $\alpha$, while $K_{12}$ is another positive constant independent of $\epsilon$, $M$, $N$, $\alpha$ and $\beta$.

**Proof.** By (1.11a), we have

$$\begin{cases}
\lambda_i(0) - \lambda_i(u) = \lambda_i \left( \sum_{h=1}^p u_h e_h \right) - \lambda_i(u) = \sum_{j=p+1}^n \Lambda_{ij}(u) u_j, & \text{if } i \in \{1, \ldots, p\}; \\
\lambda_i(0) - \lambda_i(u) = \lambda_i(u_i e_i) - \lambda_i(u) = \sum_{j \neq i} \Lambda_{ij}(u) u_j, & \text{if } i \in \{p+1, \ldots, n\},
\end{cases} \quad (5.18)$$

where

$$\begin{align*}
\Lambda_{ij}(u) &= - \int_0^1 \frac{\partial \lambda_i(u_1, \ldots, u_p, su_{p+1}, \ldots, su_n)}{\partial u_j} ds, & \text{if } i \in \{1, \ldots, p\}, j \in \{p+1, \ldots, n\}; \\
\Lambda_{ij}(u) &= - \int_0^1 \frac{\partial \lambda_i(su_1, \ldots, su_{i-1}, u_i, su_{i+1}, \ldots, su_n)}{\partial u_j} ds, & \text{if } i \in \{p+1, \ldots, n\}, j \neq i.
\end{align*} \quad (5.19)$$
Then, it follows from (5.14) that
\[
\theta_i(t, \alpha) = \alpha + \int_0^t \left( \lambda_i(0) - \lambda_i(u) \right) (s, x_i(s, \theta_i(t, \alpha))) \, ds
\]
\[= \alpha + \sum_{(i,j) \in \Gamma} \int_0^t \left( A_{ij}(u) u_j \right) (s, x_i(s, \theta_i(t, \alpha))) \, ds. \tag{5.20}
\]

Noting (4.34), we observe that the integral in the right-hand side of (5.20) converges absolutely when \( t \) tends to \( +\infty \). This implies that \( \theta_i(t, \alpha) \) converges when \( t \) tends to \( +\infty \). We denote the limit by \( \vartheta_i(\alpha) \). That is,
\[
\vartheta_i(\alpha) = \lim_{t \to \infty} \theta_i(t, \alpha). \tag{5.21}
\]

Therefore,
\[
\vartheta_i(\alpha) - \vartheta_i(\beta) = \alpha - \beta + \lim_{t \to +\infty} \sum_{(i,j) \in \Gamma} \int_0^t \left[ \left[ A_{ij}(u) u_j \right] (\alpha) - \left[ A_{ij}(u) u_j \right] (\beta) \right] \, ds
\]
\[= (\alpha - \beta) + \sum_{(i,j) \in \Gamma} \lim_{t \to +\infty} \int_0^t \left\{ \left[ A_{ij}(u) \right] (\alpha) - \left[ A_{ij}(u) \right] (\beta) \right\} [u_j](\alpha) \, ds
\]
\[+ \int_0^t \left[ A_{ij}(u) \right] (\beta) \left[ [u_j](\alpha) - [u_j](\beta) \right] \, ds \right\}, \tag{5.22}
\]
where \([-\cdot](\alpha)\) stands for \([-\cdot](s, x_i(s, \theta_i(t, \alpha)))\), etc. Thus, using (4.40) and noting (4.41), (4.43), (4.59), we obtain from (5.22) that
\[
|\vartheta_i(\alpha) - \vartheta_i(\beta)| \leq |\alpha - \beta| + c_{26} \left\{ c_{24} M |\alpha - \beta| \times \tilde{U}_1(\infty) + \tilde{U}_2(\infty) \right\}
\]
\[\leq |\alpha - \beta| + c_{26} \left\{ c_{24} M |\alpha - \beta| \times K_2 N + K_7 (MN + \epsilon) |\alpha - \beta| \right\}
\]
\[\leq \left[ 1 + c_{27} (MN + \epsilon) \right] |\alpha - \beta|, \tag{5.23}
\]
here and hereafter, as before, \( c_j \ (j = 26, 27, \ldots) \) stand for some positive constants independent of \( \epsilon, M, N, \alpha \) and \( \beta \). This proves (5.17).

On the other hand, noting (4.36) and using (4.34), we obtain form (5.20) that
\[
|\theta_i(t, \alpha) - \alpha| \leq c_{28} \tilde{U}_1(t) \leq c_{28} K_2 N, \quad \forall t \geq 0, \ \forall \alpha \in \mathbb{R}. \tag{5.24}
\]
Letting \( t \to \infty \) in (5.24) gives
\[
|\vartheta_i(\alpha) - \alpha| \leq c_{28} K_2 N, \quad \forall \alpha \in \mathbb{R}. \tag{5.25}
\]
Taking $K_{11} \geq c_{28} K_2$, we obtain (5.16) from (5.25) immediately. Thus, the proof of Lemma 5.2 is completed. □

**Lemma 5.3.** For every $i \in \{1, \ldots, n\}$, there exists a positive constant $K_{13}$ independent of $\epsilon, M, N, \alpha$ and $\beta$ such that

$$
|\Phi_i(\alpha) - \Phi_i(\beta)| \leq K_{13} (M + M^2 N + M \epsilon)|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}.
$$

(5.26)

**Proof.** By (5.13) and (5.15), for any $t \in \mathbb{R}^+$ and any $\alpha \in \mathbb{R}$ it holds that

$$
u_i(t, \alpha + \lambda_i(0)t) = u_i(t, x_i(t, \theta_i(t, \alpha))),
$$

(5.27)

where, as before, $x = x_i(s, \theta_i(t, \alpha))$ stands for the $i$th characteristic passing through the point $(0, \theta_i(t, \alpha))$. Noting Lemma 5.2 and using (5.12), we have

$$
\Phi_i(\alpha) - \Phi_i(\beta) = \lim_{t \to \infty} u_i(t, \alpha + \lambda_i(0)t) - \lim_{t \to \infty} u_i(t, \beta + \lambda_i(0)t)
$$

$$
= \lim_{t \to \infty} u_i(t, x_i(t, \theta_i(t, \alpha))) - \lim_{t \to \infty} u_i(t, x_i(t, \theta_i(t, \beta)))
$$

$$
= \lim_{t \to \infty} \{u_i(t, x_i(t, \partial_i(\alpha))) - u_i(t, x_i(t, \partial_i(\beta)))\}.
$$

(5.28)

Then, using Taylor’s formula and noting (2.4), (4.5), (4.37), (4.49) and (5.17), we obtain

$$
|\Phi_i(\alpha) - \Phi_i(\beta)| \leq \sup_{(t, x) \in \mathbb{R}^+ \times \mathbb{R}} \left| \frac{\partial u_i}{\partial x} (t, x) \right| \times \sup_{(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}} \left| \frac{\partial x_i}{\partial \xi} (t, \xi) \right| \times |\partial_i(\alpha) - \partial_i(\beta)|
$$

$$
\leq c_{29} W_\infty(\infty) \times e^{c_7 \epsilon} \times |\partial_i(\alpha) - \partial_i(\beta)|
$$

$$
\leq c_{30} (M + M^2 N + M \epsilon)|\alpha - \beta|,
$$

(5.29)

here and hereafter, as before, $c_j$ ($j = 29, 30, \ldots$) stand for some positive constants independent of $\epsilon, M, N, \alpha$ and $\beta$. (5.29) is nothing but the desired estimate (5.26). Thus, the proof of Lemma 5.3 is completed. □

**Lemma 5.4.** Suppose that the limit

$$
\lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t)
$$

exists, then

$$
\frac{d\Phi_i(\alpha)}{d\alpha} = \lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t).
$$

(5.30)
Proof. By the definition,

\[
\frac{d\Phi_i(\alpha)}{d\alpha} = \lim_{\Delta\alpha \to 0} \frac{\Phi_i(\alpha + \Delta\alpha) - \Phi_i(\alpha)}{\Delta\alpha}
\]

\[
= \lim_{\Delta\alpha \to 0} \lim_{t \to +\infty} \frac{u_i(t, \alpha + \lambda_i(0)t) - u_i(t, \alpha + \lambda_i(0)t)}{\Delta\alpha}
\]

\[
= \lim_{t \to +\infty} \frac{u_i(t, \alpha + \lambda_i(0)t) - u_i(t, \alpha + \lambda_i(0)t)}{\Delta\alpha}
\]

\[
= \lim_{t \to +\infty} u_x(t, \alpha + \lambda_i(0)t)e_i
\]

\[
= \lim_{t \to +\infty} \sum_{j=1}^{n} w_j(t, \alpha + \lambda_i(0)t)r_j(\alpha + \lambda_i(0)t)e_j
\]

\[
= \lim_{t \to +\infty} \left\{ \sum_{j=1}^{p} w_j \left( r_j(u) - r_j \left( \sum_{h=1}^{p} u_h e_h \right) \right) e_i \right\}(t, \alpha + \lambda_i(0)t)
\]

\[
+ \left\{ \sum_{j=p+1}^{n} w_j(\alpha + \lambda_i(0)t)e_i + w_i \right\}(t, \alpha + \lambda_i(0)t)
\]

\[
= \lim_{t \to +\infty} \left\{ \sum_{(j,k) \in \Gamma} O_{ijk}(u)w_j u_k + w_i \right\}(t, \alpha + \lambda_i(0)t),
\]

where

\[
O_{ijk}(u) = \begin{cases}
\int_{0}^{1} \frac{\partial r_j(u_1, \ldots, u_p, s u_{p+1}, \ldots, s u_n)}{\partial u_k} e_i ds, & \text{if } j \in \{1, \ldots, p\}, k \in \{p+1, \ldots, n\}; \\
\int_{0}^{1} \frac{\partial r_j(s u_1, \ldots, s u_{j-1}, u_j, s u_{j+1}, \ldots, s u_n)}{\partial u_k} e_i ds, & \text{if } j \in \{p+1, \ldots, n\}, k \neq j.
\end{cases}
\]

By (4.34) and (4.35), we find

\[
u_j(t, \alpha + \lambda_i(0)t) \to 0 \quad ((i, j) \in \Gamma), \quad \text{as } t \to \infty,
\]

and

\[
w_j(t, \alpha + \lambda_i(0)t) \to 0 \quad ((i, j) \in \Gamma), \quad \text{as } t \to \infty.
\]

Then, noting Lemma 4.1 and using (5.33), (5.34), we obtain (5.30) from (5.31) immediately. This proves Lemma 5.4. \(\square\)

In what follows, we prove the existence of \(\lim_{t \to +\infty} u_i(t, \alpha + \lambda_i(0)t)\).
Lemma 5.5. For every $i \in \{1, \ldots, n\}$, the limit $\lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t)$ exists, and

$$\lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t) = \Psi_i(\vartheta_i(\alpha)) \in C^0(\mathbb{R}).$$

(5.35)

Moreover, if (1.22) is satisfied, then the limit function $\Psi_i(\vartheta_i(\alpha))$ is globally $\rho$-Hölder continuous, and satisfies that, for all $\alpha, \beta \in \mathbb{R}$,

$$\left| \Psi_i(\vartheta_i(\alpha)) - \Psi_i(\vartheta_i(\beta)) \right| \leq K_{14}\varsigma(1 + MN + \epsilon)^\rho |\alpha - \beta|^\rho + K_{14}M^2(1 + \epsilon)(1 + MN + \epsilon)|\alpha - \beta|. \quad (5.36)$$

Proof. It follows from (5.13) that

$$w_i(t, \alpha + \lambda_i(0)t) = w_i(t, x_i(t, \theta_i(t, \alpha))). \quad (5.37)$$

Then noting Lemma 5.2, we have

$$\lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t) = \lim_{t \to +\infty} w_i(t, x_i(t, \theta_i(t, \alpha))) = \lim_{t \to +\infty} w_i(t, x_i(t, \vartheta_i(\alpha))), \quad (5.38)$$

and then by (4.79),

$$\lim_{t \to +\infty} w_i(t, \alpha + \lambda_i(0)t) = \lim_{t \to +\infty} w_i(t, x_i(t, \vartheta_i(\alpha))) = \Psi_i(\vartheta_i(\alpha)). \quad (5.39)$$

Since $\Psi_i(\cdot)$ and $\vartheta_i(\ast)$ are continuous with respect to $\cdot$ and $\ast$ respectively, $\Psi_i(\vartheta_i(\alpha))$ is a continuous function of $\alpha \in \mathbb{R}$. This proves (5.35).

Moreover, if (1.22) is satisfied, then using (4.82) and (5.17), we obtain (5.36) immediately. (5.36) shows that the limit function $\Psi_i(\vartheta_i(\alpha))$ is globally $\rho$-Hölder continuous. Thus, the proof of Lemma 5.5 is completed. □

Combining Lemmas 5.4 and 5.5 gives

Lemma 5.6. For every $i \in \{1, \ldots, n\}$, it holds that

$$\frac{d\Phi_i(\alpha)}{d\alpha} = \Psi_i(\vartheta_i(\alpha)) \in C^0(\mathbb{R}).$$

(5.40)

Moreover, if (1.22) is satisfied, then the following estimate hold

$$\left| \frac{d\Phi_i(\alpha)}{d\alpha} - \frac{d\Phi_i(\beta)}{d\alpha} \right| \leq K_{15}\varsigma(1 + MN + \epsilon)^\rho |\alpha - \beta|^\rho + K_{15}M^2(1 + \epsilon)(1 + MN + \epsilon)|\alpha - \beta|, \quad (5.41)$$

where $K_{15}$ is a positive constant independent of $\epsilon$, $M$, $N$, $\varsigma$, $\alpha$ and $\beta$.  

Proof of Theorem 1.2. Taking
\[ \phi_i(x - \lambda_i(0)t) = \Phi_i(x - \lambda_i(0)t) \quad (i = 1, \ldots, n), \] (5.42)
and noting Lemmas 5.1, 5.3 and 5.6, we get the conclusion of Theorem 1.2 immediately. Thus, the proof of Theorem 1.2 is completed. □

6. An application to the equation for time-like extremal surfaces in the Minkowski space–time

Let \((t, x, y)\) be points in the \((1 + 2)\)-dimensional Minkowski space. A time-like surface takes the form
\[ y = \phi(t, x). \] (6.1)

This surface is called to be extremal surface if \(\phi\) is the critical point of the area functional
\[ I_1 = \int\int \sqrt{1 + \phi_x^2 - \phi_t^2} \, dx \, dt. \] (6.2)
The corresponding Euler–Lagrange equation is
\[ \left( \frac{\phi_t}{\sqrt{1 + \phi_x^2 - \phi_t^2}} \right)_t - \left( \frac{\phi_x}{\sqrt{1 + \phi_x^2 - \phi_t^2}} \right)_x = 0. \] (6.3)

More generally, we consider a vector function \(\phi = (\phi_1, \ldots, \phi_n)^T\) which is the critical point of the area functional
\[ I_n = \int\int \sqrt{1 + |\phi_x|^2 - |\phi_t|^2} \, dx \, dt \] (6.4)
where \((\cdot, \cdot)\) stands for the inner product. The Euler–Lagrange equation is
\[ \left( \frac{\phi_t + |\phi_x|^2 \phi_t - (\phi_t, \phi_x) \phi_x}{\sqrt{1 + |\phi_x|^2 - |\phi_t|^2} \phi_x} \right)_t - \left( \frac{\phi_x + (\phi_t, \phi_x) \phi_t - |\phi_t|^2 \phi_x}{\sqrt{1 + |\phi_x|^2 - |\phi_t|^2} \phi_x} \right)_x = 0. \] (6.5)

Remark 6.1. When \(n = 1\), Eq. (6.5) is nothing but Eq. (6.3).

(6.5) is the equation for time-like extremal surfaces in the Minkowski space \(\mathbb{R}^{2+n}\). The extremal surfaces in the Minkowski space are \(C^2\) surfaces with vanishing mean curvature.

Let
\[ u = \phi_x, \quad v = \phi_t, \] (6.6)
where $u = (u_1, \ldots, u_n)^T$ and $v = (v_1, \ldots, v_n)^T$. Then (6.5) can be equivalently rewritten as

$$
\begin{cases}
    u_t - v_x = 0, \\
    \left( \frac{v + |u|^2 v - \langle u, v \rangle u}{\sqrt{1 + |u|^2 - |v|^2 - |v|^2|u|^2 + \langle u, v \rangle^2}} \right)_t \\
    - \left( \frac{u + \langle u, v \rangle v - |v|^2 u}{\sqrt{1 + |u|^2 - |v|^2 - |v|^2|u|^2 + \langle u, v \rangle^2}} \right)_x = 0
\end{cases}
$$

(6.7)

for classical solutions.

Introduce

$$
\Delta(u, v) = 1 + |u|^2 - |v|^2 - |v|^2|u|^2 + \langle u, v \rangle^2.
$$

(6.8)

Recently, the following theorem was proved by Kong, Sun and Zhou [9].

**Theorem A.** Suppose $n \geq 2$ and suppose furthermore that on the domain under consideration

$$
\Delta(u, v) > 0
$$

(6.9)

holds, then (6.7) is a non-strictly hyperbolic system with two $n$-constant multiple eigenvalues; moreover, the characteristic propagation speeds are bounded (not large than the light speed), and all characteristic fields are linearly degenerate in the sense of Lax (see [11]) and the system (6.7) is rich in the sense of Serre (see [16]).

Consider the Cauchy problem for Eq. (6.5) with the initial data

$$
\phi(0, x) = f(x), \quad \phi_t(0, x) = g(x),
$$

(6.10)

where $f$ is a given $C^2$ vector-valued function and $g$ is a given $C^1$ vector-valued function.

Using Theorem A and Theorems 1.1, 1.2, we can easily prove the following two theorems.

**Theorem 6.1.** Suppose that there exists a point $x_0 \in \mathbb{R}$ such that

$$
\Delta(f'(x_0), g(x_0)) > 0.
$$

(6.11)

Then there exists a small constant $\epsilon > 0$ such that the Cauchy problem (6.5), (6.10) admits a unique global $C^2$ solution $\phi = \phi(t, x)$ for all $t \in \mathbb{R}$, provided that

$$
\int_{-\infty}^{+\infty} \left| f''(x) \right| + \left| g'(x) \right| \, dx \leq \epsilon.
$$

(6.12)

Moreover, the solution $\phi = \phi(t, x)$ satisfies

$$
\Delta(\phi_x(t, x), \phi_t(t, x)) > 0, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}.
$$
Remark 6.2. Theorem 6.1 implies that, if the surface is time-like at the initial point \((0, x_0)\), then it is time-like globally, provided that \(\epsilon > 0\) is small enough.

On the other hand, let

\[
M \triangleq \sup_{x \in \mathbb{R}} |f''(x)| + |g'(x)| < +\infty.
\]

(6.13)

Theorem 6.2. Under the assumptions of Theorem 6.1, assume furthermore that

\[
N \triangleq \int_{-\infty}^{+\infty} \left[ |f'(x)| + |g(x)| \right] dx < +\infty.
\]

(6.14)

Then there exists a unique \(C^1\) vector-valued function \(\varphi(x) = (\varphi_1(x), \ldots, \varphi_{2n}(x))^T\) such that in the normalized coordinates\(^2\)

\[
(\phi^T_x(t, x), \phi^T_t(t, x)) \rightarrow \sum_{i=1}^{2n} \varphi_i(x - \lambda_i(0)t)e_i, \quad \text{as } t \rightarrow +\infty.
\]

(6.15)

Moreover, \(\varphi_i(x) (i = 1, \ldots, 2n)\) are globally Lipschitz continuous, more precisely, there exists a positive constant \(\kappa_1\) independent of \(\epsilon, M, N, x_1\) and \(x_2\) such that for every \(i \in \{1, \ldots, 2n\}\), it holds that

\[
|\varphi_i(x_1) - \varphi_i(x_2)| \leq \kappa_1 M |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}.
\]

(6.16)

Furthermore, if \(f''(x)\) and \(g'(x)\) are globally \(\rho\)-Hölder \((0 < \rho \leq 1)\) continuous, that is, there exists a positive constant \(\varsigma\) such that

\[
|f''(x_1) - f''(x_2)| + |g'(x_1) - g'(x_2)| \leq \varsigma |x_1 - x_2|^\rho, \quad \forall x_1, x_2 \in \mathbb{R},
\]

then \(\varphi'(x)\) is also globally \(\rho\)-Hölder continuous and satisfies that

\[
|\varphi'(x_1) - \varphi'(x_2)| \leq \kappa_2 \varsigma (1 + MN + \epsilon)^\rho |x_1 - x_2|^\rho
\]

\[+ \kappa_2 M^2 (1 + \epsilon)(1 + MN + \epsilon)|x_1 - x_2|,
\]

(6.18)

where \(\kappa_2\) is a positive constant independent of \(\epsilon, M, N, \varsigma, x_1\) and \(x_2\).

Remark 6.3. Theorem 6.2 gives the asymptotic behavior of the tangent plane of the extremal surface.

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References