On the helical pipe flow with a pressure-dependent viscosity

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Abstract We address the flow of incompressible fluid with a pressure-dependent viscosity through a pipe with helical shape. The viscosity-pressure relation is defined by the Barus law. The thickness of the pipe and the helix step are assumed to be of the same order and considered as the small parameter. After transforming the starting problem, we compute the asymptotic solution using curvilinear coordinates and standard perturbation technique. The solution is provided in the explicit form clearly showing the influence of viscosity-pressure dependence and pipe’s geometry on the effective flow.

Keywords pressure-dependent viscosity, Barus law, helical pipe flow, curvilinear coordinates, asymptotic approximation

Curved-pipe flows have been the subject of many theoretical investigations since they naturally appear in numerous engineering and industrial applications. Among all, one type of curved pipes has been of particular interest and that is the pipe with helical shape. Such pipes can be found in many applications, namely air conditioners, refrigeration systems, central heating radiators, chemical reactors etc. In view of that, in this paper we consider the fluid flow through a thin (or long) pipe with a helical shape. To be more precise, introducing the small parameter $\varepsilon$, we assume that the pipe’s central curve is parameterized by $x_1 \mapsto x_1 t + a \cos(x_1/\varepsilon)\hat{i} + a \sin(x_1/\varepsilon)\hat{k}$, $x_1 \in [0, \ell]$. In other words, the distance between two coils of the helix (helix step) and the thickness of the pipe are assumed to be small (of order $O(\varepsilon)$), while the diameter of the helix is of order $O(1)$. We are strongly motivated to take such assumptions by the fact that they can be retrieved from numerous realistic coiled pipes appearing in the applications.

In addition, we would like to study the case of incompressible fluid with a pressure-dependent viscosity. Such fluids have relevance from both theoretical and practical point of view, as it has been confirmed in many engineering papers.\textsuperscript{1–6} The most common relation for the viscosity-pressure dependence has been the exponential law proposed by Barus in 1893. Here $\mu_0$ represents viscosity at atmospheric pressure while $\alpha > 0$ is the pressure-viscosity coefficient. Barus law (1) has been extensively used by the engineers and, thus, it is going to be the subject of this investigation.

It is not reasonable to expect that we are going to be able to solve the governing 3D boundary-

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value problem and to find the exact solution. It is, of course, due to the complex geometry and governing equations. Therefore, we employ the asymptotic analysis (with respect to $\varepsilon$) and try to build the asymptotic approximation of the flow. The key idea is to transform the starting problem into the system with negligible nonlinear perturbation and then to write the obtained problem in the curvilinear coordinates attached to the helix. Using two-scale asymptotic expansion, we obtain the asymptotic solution in the explicit form and that we find as our main contribution. We compute the higher-order terms in the velocity and pressure expansion explicitly acknowledging the effects of viscosity-pressure dependence and the specific pipe’s geometry. By taking those effects into account, we believe that the obtained result could be relevant with regards to numerical simulations.

There exist numerous papers\textsuperscript{7–11} on helical pipe flow of classical, Newtonian fluid ($\mu = \text{const.}$). In the case of pressure-dependent viscosity, analytical solutions have been reported only for unidirectional and plane-parallel flows and in the case of the linear law $\mu(p) = \alpha p$. We refer the reader to Refs.\textsuperscript{12–14}. However, recently, straight-pipe flow has been successfully resolved in the case of exponential law (1).\textsuperscript{15} Our approach, presented here, combines the methods from Ref.\textsuperscript{16}, enabling us to handle the specific helical geometry with the asymptotic technique proposed in Ref.\textsuperscript{15}.

In this part we formally describe the pipe’s geometry and introduce the governing equations with the boundary conditions. Let $(i, j, k)$ denotes the standard Cartesian basis. The pipe’s center curve is given by the helix with parametrization $r_\varepsilon(x_1) = x_1i + a\cos(x_1/\varepsilon)j + a\sin(x_1/\varepsilon)k$, $x_1 \in [0, \ell]$ ($a = \text{const.} > 0$). The Frenet basis attached to this helix is computed as

$$t_\varepsilon(x_1) = \left| \frac{dx_\varepsilon}{dx_1} \right|^{-1} \frac{dx_\varepsilon}{dx_1} = \left( \varepsilon i - a \sin(x_1/\varepsilon) j + a \cos(x_1/\varepsilon) k \right) (\varepsilon^2 + \varepsilon^2)^{-1/2},$$

$$n_\varepsilon(x_1) = \left| \frac{dt_\varepsilon}{dx_1} \right|^{-1} \frac{dt_\varepsilon}{dx_1} = -\cos(x_1/\varepsilon) j - \sin(x_1/\varepsilon) k,$$

$$b_\varepsilon(x_1) = t_\varepsilon \times n_\varepsilon = (ai + \varepsilon \sin(x_1/\varepsilon) j - \varepsilon \cos(x_1/\varepsilon) k) (\varepsilon^2 + \varepsilon^2)^{-1/2}.$$

For small parameter $\varepsilon > 0$ and a unit circle $B = B(0, 1) \subset \mathbb{R}^2$ we first introduce a thin straight pipe with circular cross section $T_\varepsilon = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < \ell, x' = (x_2, x_3) \in \varepsilon B \}$. Then we define the mapping $\Phi_\varepsilon : T_\varepsilon \rightarrow \mathbb{R}^3$ by $\Phi_\varepsilon(x) = r_\varepsilon(x_1) + x_2 n_\varepsilon(x_1) + x_3 b_\varepsilon(x_1)$. Finally, we define $\Omega_\varepsilon = \Phi_\varepsilon(T_\varepsilon)$ representing our 3D domain occupied by the fluid. The pipe’s lateral boundary is denoted by $\Gamma_\varepsilon = \Phi_\varepsilon([0, \ell] \times \varepsilon \partial B)$, while the pipe’s ends are given by $\Sigma_\varepsilon^i = \Phi_\varepsilon([i] \times \varepsilon B)$, $i = 0, \ell$ (see Fig.\textsuperscript{1}).

In the case of incompressible fluid with a viscosity depending on pressure, the stress tensor is taken to be $T = -p I + 2\mu(p) D(u)$, where $D(u) = ([\nabla u] + [\nabla u]^T)/2$. Assuming that the Reynolds number is not too large, we assume that the flow in $\Omega_\varepsilon$ is governed by the following system for the velocity $u_\varepsilon$ and pressure $p_\varepsilon$

$$-\text{div} \left[ 2\mu(p_\varepsilon) D(u_\varepsilon) \right] + \nabla p_\varepsilon = \mathbf{0} \text{ in } \Omega_\varepsilon, \quad (2)$$

$$\text{div} u_\varepsilon = 0 \text{ in } \Omega_\varepsilon. \quad (3)$$

As indicated before, we use the well-known Barus law to describe the viscosity-pressure depen-
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\[ \mu(\rho) = \mu_0 e^{\alpha \rho}, \quad \mu_0, \alpha = \text{const.} > 0. \] (4)

The choice of boundary conditions highly depends on the devices to be considered. Here we want to study the real-life situation in which the flow is governed by the pressure drop prescribed between pipe’s ends. Thus, we impose \( \mathbf{u} = 0 \) on \( \Gamma_0 \) and \( \mathbf{u} \times t = 0, p = p_i \) on \( \Sigma_i, i = 0, \ell \), for given pressures \( p_i, i = 0, \ell \) \((p_0 > p_\ell)\). The well-posedness of the above problem has been recently established (see Appendix in Ref. 17).

**Transformed problem** In view of Eq. (4), the momentum equation (2) reads

\[- \text{div}(2\mu_0 e^{\alpha \rho} \mathbf{D}(\mathbf{u})) + \nabla p = -\mu_0 e^{\alpha \rho} \Delta \mathbf{u} - 2\mu_0 \alpha e^{\alpha \rho} \mathbf{D}(\mathbf{u}) \nabla p + \nabla p = 0.\]

Dividing it by \( \mu_0 e^{\alpha \rho} \) we obtain

\[-\Delta \mathbf{u} + (1/\mu_0) e^{-\alpha \rho} \nabla p = 2\alpha \mathbf{D}(\mathbf{u}) \nabla p.\] (5)

We introduce a new function, denoted by \( q \), such that

\[(1/\mu_0) e^{-\alpha \rho} \nabla p = \nabla q.\] (6)

From Eq. (6) we deduce

\[q = [1/(\mu_0 \alpha)](e^{-\alpha q_0} - e^{-\alpha q}).\] (7)

Taking into account that \( \nabla p = \mu_0 e^{\alpha \rho} \nabla q = [\mu_0/(e^{-\alpha q_0} - \alpha \mu_0 q)] \nabla q \), Eq. (5) transforms into

\[-\Delta \mathbf{u} + \nabla q = [2\mu_0 \alpha/(e^{-\alpha q_0} - \alpha \mu_0 q)] \mathbf{D}(\mathbf{u}) \nabla q.\] As a result, we arrive at the following system satisfied by velocity \( \mathbf{u} \) and the new function \( q \)

\[-\Delta \mathbf{u} + \nabla q = [2\mu_0 \alpha/(e^{-\alpha q_0} - \alpha \mu_0 q)] \mathbf{D}(\mathbf{u}) \nabla q \text{ in } \Omega,\] (8)

\[\text{div} \mathbf{u} = 0 \text{ in } \Omega.\] (9)
Due to the specific pipe’s geometry, the pair of basis vectors (x) in the curvilinear coordinates

\[ u_e = 0 \text{ on } \Gamma_e, \]

\[ u_e \times t_e = 0, \quad q_e = (e^{-\alpha q_0} - e^{-\alpha q_0})/(\alpha \mu_0) \text{ on } \Sigma_i^e, \quad i = 0, \ell. \]

Following Ref. 15, the idea is to construct the asymptotic approximation satisfying the transformed problem (8)–(11), and then to reconstruct the effective pressure from Eq. (7) as

\[ p_e = -\alpha^{-1} \ln(e^{-\alpha q_0} - \alpha \mu_0 q_e). \]

To accomplish that, we need to write the above problem in curvilinear coordinates (x) and then to use standard perturbation technique. Before proceeding, we would like to establish the following:

The contravariant basis is given by the relation \( \mathbf{a'} \cdot \mathbf{a}_j = \delta_{ij} \) and it reads

\[ a^1 = -[a\varepsilon/(a^2 - ax_2 + x^2)]\mathbf{e}_x + \varepsilon^2/(a^2 - ax_2 + x^2)\mathbf{i}, \]

\[ a^2 = -[a\varepsilon x_2/\sqrt{a^2 + \varepsilon^2(a^2 - ax_2 + x^2)}]\mathbf{e}_x + \varepsilon^2 x_3/\sqrt{a^2 + \varepsilon^2(a^2 - ax_2 + x^2)}\mathbf{i}, \]

\[ a^3 = [\varepsilon \sqrt{a^2 + \varepsilon^2}/(a^2 - ax_2 + x^2)]\mathbf{e}_x + [(a - x_2) \sqrt{a^2 + \varepsilon^2}/(a^2 - ax_2 + x^2)]\mathbf{i}. \]

Due to the specific pipe’s geometry, the pair of basis \( \{\mathbf{a'}, \mathbf{a}\} \) can not be employed in further analysis (note the expression for \( \mathbf{a}_1 \) containing negative power of \( \varepsilon \)). In view of that and following Ref. 16, we return to Frenet basis \( \{t_e, n_e, b_e\} \) and note that \( \mathbf{b}_e \rightarrow \mathbf{i} \) while \( |t_e + \mathbf{e}_x| \rightarrow 0, \) uniformly in \( x_1. \) Thus, we take the basis \( \{\mathbf{e}_x, n_e, i\} \) and write the equations in such basis. For that reason, we introduce \( \mathbf{U}_e = u_e \otimes \mathbf{e}_x = U^e_x \mathbf{e}_x + U^e_\mathbf{n}_e + U^e_\mathbf{i}, \quad Q_e = q_e \otimes \mathbf{e}_x. \) Denoting \( \mathbf{B} = [\mathbf{e}_x \ n_e \ i] \) and taking into account that \( x_2, x_3 = \mathcal{O}(\varepsilon), \) we derive the expressions for each differential operator appearing in the transformed equations (see Ref. 16 for technical details) as

\[ (\nabla q_e)^T \otimes \mathbf{e}_x = \mathbf{B} \left[ -\varepsilon \frac{\partial Q_e}{\partial x_1} + \frac{\varepsilon}{a} \frac{\partial Q_e}{\partial x_2} + \frac{\varepsilon}{a} \frac{\partial Q_e}{\partial x_3} \right] + \mathcal{O}(\varepsilon^2), \]

\[ (\nabla u_e) \otimes \mathbf{e}_x = \mathbf{B} \mathbf{A} \mathbf{B}^T + \mathcal{O}(\varepsilon^2), \quad (\Delta u_e) \otimes \mathbf{e}_x = \mathbf{B} \mathbf{C} + \mathcal{O}(\varepsilon^2), \]
where

\[
A = \begin{bmatrix}
- \epsilon \frac{\partial U_1^3}{\partial x_1} + \epsilon \frac{\partial U_1^1}{\partial x_3} - \frac{U_2^1}{a} + \frac{U_1^1}{a} \\
- \epsilon \frac{\partial U_2^1}{\partial x_1} + \epsilon \frac{\partial U_2^1}{\partial x_3} + \frac{U_1^1}{a} \\
- \epsilon \frac{\partial U_3^1}{\partial x_1} + \epsilon \frac{\partial U_3^1}{\partial x_3} + \frac{U_1^1}{a}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
\frac{\partial^2 U_1^1}{\partial x_2^2} + \frac{\partial^2 U_1^1}{\partial x_3^2} - \frac{1}{a} \frac{\partial U_1^1}{\partial x_2} + \frac{2\epsilon}{a^2} \left( \frac{\partial U_2^1}{\partial x_1} - \frac{\partial U_2^1}{\partial x_3} \right)
\frac{\partial^2 U_2^1}{\partial x_2^2} + \frac{\partial^2 U_2^1}{\partial x_3^2} - \frac{1}{a} \frac{\partial U_2^1}{\partial x_2} + \frac{2\epsilon}{a^2} \frac{\partial U_2^1}{\partial x_1}
\frac{\partial^2 U_3^1}{\partial x_2^2} + \frac{\partial^2 U_3^1}{\partial x_3^2} - \frac{1}{a} \frac{\partial U_3^1}{\partial x_2} + \frac{2\epsilon}{a^2} \frac{\partial U_3^1}{\partial x_1}
\end{bmatrix}
\]

Taking the trace in the first item of Eq. (15), we get

\[
\text{div} \mathbf{u} \cdot \Phi = \frac{\partial U_2^1}{\partial x_2} + \frac{\partial U_3^1}{\partial x_3} - \frac{1}{a} \frac{\partial U_2^1}{\partial x_2} + \frac{2\epsilon}{a^2} \left( \frac{\partial U_2^1}{\partial x_1} - \frac{\partial U_2^1}{\partial x_3} \right)
\]

Thus, Eqs. (8) and (9) written in curvilinear coordinates have the following form

\[
- \left[ \frac{\partial^2 U_1^1}{\partial x_2^2} + \frac{\partial^2 U_1^1}{\partial x_3^2} - \frac{1}{a^2} \frac{\partial U_1^1}{\partial x_2} + \frac{2\epsilon}{a^3} \left( \frac{\partial U_2^1}{\partial x_1} - \frac{\partial U_2^1}{\partial x_3} \right) \right] \frac{\epsilon}{a} \frac{\partial Q}{\partial x_1} + \frac{\epsilon}{a} \frac{\partial Q}{\partial x_3} = 0,
\]

\[
- \left( \frac{\partial^2 U_2^1}{\partial x_2^2} + \frac{\partial^2 U_2^1}{\partial x_3^2} - \frac{1}{a^2} \frac{\partial U_2^1}{\partial x_2} + \frac{2\epsilon}{a^3} \frac{\partial U_2^1}{\partial x_1} \right) \frac{\partial Q}{\partial x_2} = 0,
\]

\[
- \left( \frac{\partial^2 U_3^1}{\partial x_2^2} + \frac{\partial^2 U_3^1}{\partial x_3^2} - \frac{1}{a^2} \frac{\partial U_3^1}{\partial x_2} + \frac{2\epsilon}{a^3} \frac{\partial U_3^1}{\partial x_1} \right) \frac{\partial Q}{\partial x_3} = 0
\]

To simplify the notation, we do not write the higher-order terms and omit the nonlinear term in the momentum equation since it does not contribute to the macroscopic model.

**Asymptotic expansion** Now we employ two-scale expansion technique and expand the unknowns in powers of small parameter \( \epsilon \). A priori estimates suggest\(^9\)

\[
U_{i e}^1(x) = \epsilon^3 U_0^1(x_1, x_2 / \epsilon, x_3 / \epsilon) + \epsilon^4 U_1^1(x_1, x_2 / \epsilon, x_3 / \epsilon) + ..., \\
U_{i e}^i(x) = \epsilon^i U_0^i(x_1, x_2 / \epsilon, x_3 / \epsilon) + \epsilon^{i+1} U_1^i(x_1, x_2 / \epsilon, x_3 / \epsilon) + ..., \quad i = 2, 3,
\]

\[
Q_{e}(x) = Q_0(x_1) + \epsilon^3 Q_1(x_1, x_2 / \epsilon, x_3 / \epsilon) + ... 
\]

In the sequel we introduce the fast variable \( \mathbf{y}' = (y_2, y_3) = (x_2 / \epsilon, x_3 / \epsilon) \) and use the following
notation
\[ \nabla_y = \frac{\partial}{\partial y_2} J + \frac{\partial}{\partial y_3} k, \quad \Delta_y \mathbf{V} = \frac{\partial^2 \mathbf{V}}{\partial y_2^2} + \frac{\partial^2 \mathbf{V}}{\partial y_3^2}, \quad \text{div}_y \mathbf{V} = \frac{\partial V^2}{\partial y_2} + \frac{\partial V^3}{\partial y_3}, \quad \mathbf{V} = V^1 J + V^2 J + V^3 k. \]

Plugging Eq. (20) in Eqs. (16)–(19) and collecting the terms with equal powers of \( \varepsilon \), we get
\[ \varepsilon : -\Delta_y U_0^1 - a^{-1} dQ_0 / dx_1 = 0, \]
\[ \varepsilon^2 : -\Delta_y U_0^1 + \partial Q_1 / \partial y_i = 0, \quad i = 2, 3, \]
\[ \varepsilon^3 : \text{div}_y U_0 + a^{-1} \partial U_0^1 / \partial y_3 = 0 \quad \text{in} \quad T = [0, \ell] \times B. \]

Considering boundary conditions (10) and (11) for the velocity and pressure, we can solve the above system by taking
\[ U_0^1 = (e^{-\alpha p_0} - e^{-\alpha p_i})(1 - |y'|^2)/(4a\mu_0\alpha \ell), \]
\[ Q_0 = (e^{-\alpha p_0} - e^{-\alpha p_i})/(\mu_0 \alpha) + x_1 (e^{-\alpha p_0} - e^{-\alpha p_i})/(\mu_0 \alpha \ell), \]
\[ U_0^3 = 0, \quad U_0^3 = -U_0^1 / a, \quad Q_1 = y_3 (e^{-\alpha p_0} - e^{-\alpha p_i})/(a^2 \mu_0 \alpha \ell). \]

The next term from the momentum equations (16)–(18) yields
\[ \varepsilon^2 : \Delta_y U_1^1 + a^{-1} \partial U_0^1 / \partial y_2 = 0, \quad \text{(21)} \]
\[ \varepsilon^3 : -\Delta_y U_1^1 - a^{-1} \partial U_0^1 / \partial y_2 + \partial Q_2 / \partial y_2 = 0, \quad \text{(22)} \]
\[ \varepsilon^3 : -\Delta_y U_1^1 - a^{-1} \partial U_0^1 / \partial y_2 - 2a^{-2} \partial U_0^1 / \partial y_2 + \partial Q_2 / \partial y_3 = 0 \quad \text{in} \quad T. \quad \text{(23)} \]

From Eq. (21) we immediately conclude \( U_1^1 = -y_2 U_0^1 / (4a) \). In view of that, the next term from the divergence equation reads
\[ \varepsilon^4 : \text{div}_y U_1 = a^{-1} (U_0^1 + \partial U_0^1 / \partial x_1 - \partial U_0^1 / \partial y_3) = (1/4)a^{-2} y_2 \partial U_0^1 / \partial y_3. \quad \text{(24)} \]

Equations (22)–(24) will be satisfied if we take \( U_1^2 = 0, U_1^3 = y_2 U_0^1 / (4a^2), Q_2 = 0 \). Now we write the solution of the transformed problem (8)–(11). For the velocity we have
\[ \mathbf{u}^{\text{approx}}(z) = \mathbf{U}(x), \quad z = \Phi(z), \]
\[ \mathbf{U} = (e^{3} U_0^1 + e^{4} U_1^1) e_x + (e^{4} U_0^2 + e^{5} U_1^2) n_x + (e^{4} U_0^3 + e^{5} U_1^3) i \]
\[ = e^2 \ell e (4\ell^2 a^2 \mu_0 \alpha)^{-1}(e^{-\alpha p_0} - e^{-\alpha p_i})(e^2 - x_2^2 - x_3^2)(1 - x_2^2 / (4a)) \mathbf{i} e(x_1), \]

where \( \ell = \ell / \sqrt{1 + a^2 / e^2} \) denotes the pipe’s length. For other unknowns \( q_0 \) we obtain
\[ q^{\text{approx}}(z) = q_0(x), \quad z = \Phi(z), \]
\[ q_0 = Q_0 + e^3 Q_1 = (e^{-\alpha p_0} - e^{-\alpha p_i})/(\mu_0 \alpha) + (e^{-\alpha p_0} - e^{-\alpha p_i})(x_1 + e^2 x_3 / a^2) / (\mu_0 \alpha \ell). \]

**Asymptotic approximation for the original problem** Now we go back to the original
problem and reconstruct the effective pressure as (see Eq. (12))

\[ p_{\text{approx}}^e = -\alpha^{-1} \ln(e^{-\alpha q_0} - \alpha \mu_0 q_{\text{approx}}^e). \]

Thus, the asymptotic approximation for the pressure distribution reads

\[ p_{\text{approx}}^e(z) = \mathcal{P}_e(x), \quad z = \Phi_e(x), \]

\[ \mathcal{P}_e(x) = -\alpha^{-1} \ln\{e^{-\alpha p_0} + [(e^{-\alpha p_\ell} - e^{-\alpha p_0})/\ell](x_1 + \epsilon^2 x_3/a^2)\}. \]

Since \( p_0 > p_\ell \), it follows \( e^{-\alpha p_\ell} > e^{-\alpha p_0} \) implying that the effective pressure, given by Eqs. (25) and (26), is well-defined. Moreover, observe that the effective pressure does not depend on the parameter \( q_0 \) at all. That justifies our transformation procedure, namely the choice of the parameter \( q_0 \) such that Eq. (13) holds. Comparing \( p_{\text{approx}}^e \) with the effective pressure for straight-pipe flow derived in Ref. 15

\[ \mathcal{P}(x_1) = -\alpha^{-1} \ln(e^{-\alpha p_0} + x_1(e^{-\alpha p_\ell} - e^{-\alpha p_0})/\ell), \]

we clearly detect the correction coming due to the pipe’s distortion. In addition, it is important to observe the following.

**Remark**  In Ref. 9, helical pipe flow has been also addressed but with constant viscosity (Newtonian case). The asymptotic solution for the pressure has been (rigorously) derived and it reads

\[ p_{\text{approx}}^e(z) = \Pi_e(x), \quad z = \Phi_e(x), \]

\[ \Pi_e(x) = q_0 + [(q_\ell - q_0)/\ell](x_1 + \epsilon^2 x_3/a^2). \]

Taking into account the effective pressure derived in the present paper (see Eqs. (25) and (26)), we can easily detect the difference coming due to the effects of the pressure-dependent viscosity.

Last but not least, the approximate fluid velocity reads

\[ u_{\text{approx}}^e(z) = U_e(x), \quad z = \Phi_e(x), \]

\[ U_e(x) = \epsilon^2 \ell_e (4 \ell^2 a^2 \mu_0 \alpha)^{-1} (e^{-\alpha p_\ell} - e^{-\alpha p_0})(e^{-2} - x_2^2 - x_3^2)[1 - x_2/(4a)]t_e(x_1). \]

The influence of the viscosity-pressure dependence can be easily spotted.

In the main part of the paper, we formally derived an asymptotic model describing non-Newtonian fluid flow through a thin pipe with helical shape. Assuming that the pressure-dependent viscosity satisfies Barus formula (4) and imposing physically relevant Dirichlet boundary conditions, we obtain the explicit expressions (25)–(28) for the pressure and velocity distribution through the pipe. The effects of the specific pipe’s geometry on the effective pressure and velocity are clearly detected by comparing the obtained result with the straight-pipe flow. The difference between Newtonian and non-Newtonian case is commented as well. From the strictly mathematical point of view, we should provide some kind of convergence result linking our formally obtained solution with the original solution. The best way to accomplish that is to evaluate
the difference between those two solutions by using the appropriate norm of the corresponding functional space. To accomplish that, we need to employ the functional analysis enabling us to understand the structure of the space in which we prove the error estimates. Though it is out of the scope of the present paper, let us mention that, in order to obtain satisfactory error estimates acknowledging the correction in the effective pressure, we need to compute few more terms in the velocity and pressure expansion (20). Being of a lower-order, those terms would not influence on the effective flow, but are essential for the convergence proof. Then, using similar tools as in Ref. 9 (adapted to our pressure-dependent viscosity setting) and the continuity of the inverse transformation (12) we would be in position to prove satisfactory error estimates.

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