Minimum Norm Solutions of Single Stiff Linear Analytic Differential Equations*

ILKKA KARASALO

Department of Information Processing, The Royal Institute of Technology,
S-100 44 Stockholm 70, Sweden

Submitted by G. Dahlquist

The $\ell_2$-norm of the infinite vector of the terms of the Taylor series of an analytic function is used to measure the “unsmoothness” of the function. The sets of solutions to the scalar differential equations $y'(t) = \lambda y(t) + f(t)$ and $y'(t) = q(t) y(t) + f(t)$ are analyzed with respect to this norm. A number of results on the particular solution with minimum norm are given.

1. INTRODUCTION

In a recent paper Dahlquist [1] discusses the possibility of quantifying “unsmoothness” of analytic functions. He introduces, as a measure of this quantity, the $\ell_2$-norm of the vector of Taylor coefficients of the function and studies from the point of view of this unsmoothness measure the set of solutions of “stiff” systems of linear differential equations with analytic coefficients.

The purpose of the present paper is to prove and extend some of the results announced in [1]. In particular, we carry out an analysis, in terms of one of the above mentioned norms, of the sets of solutions to the scalar linear analytic differential equations

$$y' = \lambda y + f(t)$$

and

$$y' = q(t) y + f(t).$$

In [1] such results are used in an iterative construction and analysis of smooth solutions of more general stiff linear analytic systems of differential equations.

* This work was supported by the Swedish Institute for Applied Mathematics.
2. Unsmoothness Norm

Let \( y(t) \) be analytic in a neighborhood of the point \( t_0 \) in the \( t \)-plane. With no loss of generality we assume \( t_0 = 0 \) in the sequel. Then the Taylor series

\[
y(t) = \sum_{k=0}^{\infty} y_k t^k
\]

converges in an open disc with center at \( t_0 = 0 \). Let \( R > 0 \) and introduce

\[
y \|_R = \left( \sum_{k=0}^{\infty} |y_k R^k|^2 \right)^{1/2}.
\]

We choose the norms (2.1) to be the unsmoothness norms to be studied in this paper. We call the norm (2.1) the \((R, 2)\)-norm and the associated linear space the \((R, 2)\)-space. For notational convenience we drop the index \( R \) in \( \|y\|_R \) whenever possible without risk of confusion. The \((R, 2)\)-space is a Hilbert space with the scalar product

\[
(z, y)_R = \sum_{k=0}^{\infty} \bar{z}_k y_k R^{2k}.
\]

Sometimes we shall also need to consider the more general norms

\[
\|y\|_{R, p} = \left( \sum_{k=0}^{\infty} |y_k R^k|^p \right)^{1/p},
\]

with \( p \in [1, \infty] \). These norms and the associated linear spaces are called the \((R, p)\)-norms and the \((R, p)\)-spaces, respectively. Some observations are rather immediate:

1. \( \|y\|_{R, p} \) is a nondecreasing function of \( R \) and, by Jensen's inequality for the \( \ell_p \)-norm, a nonincreasing function of \( p \), \( p \in [1, \infty] \).

2. \( \|y\|_{R, 1} \geq \max_{|t| \leq R} |y(t)| \) and there are functions \( y \) for which equality is attained.

3. If \( y \in (R, p)\)-space then \( y \) is analytic in the open disc \(|t| < R\). On the other hand, if \( y \) is analytic in the open disc \(|t| < R'\), then \( y \in (R, p)\)-space for all \( R < R' \).

4. Assume \( y \in (\rho, p)\)-space. Then \( \int_0^t y(\tau) \, d\tau \in (\rho, p)\)-space as well whereas in general \( y' \) need not belong to the \((\rho, p)\)-space. However \( y^{(k)} \in (R, p)\)-space for all \( k \) if \( R < \rho \). Generally, though, we may have
\[ \| y^{(k)} \|_{R, \rho} \to \infty \text{ as } k \to \infty . \] The rate of this divergence is bounded, for example, by
\[ \| y^{(k)} \|_{R, \rho} \leq \frac{\rho k!}{(\rho - R)^{k+1}} \| y \|_{\rho, \rho} . \] (2.4)

A transformation between the \((R_1, \rho)\) and the \((R_2, \rho)\) norms is equivalent to a rescaling of the unit of length in the \(t\)-plane. In our applications we shall want to measure unsmoothness of functions relative to a timescale determined by the rate of change of a particular solution to a certain initial value problem. Therefore we find it convenient to have a parameter representing the choice of time-scale explicitly present in the results. If \(h\) is the current stepsize in a step-by-step integration, a value of \(R\) in the range \((8h, 16h)\) may perhaps be a typical choice of this parameter. In this way the unsmoothness measure is chosen to depend on the behavior of the function on a "semilocal" time range.

We shall say, that the solution \(y_1\) of a certain initial value problem is smoother at the point \(t_0\) with respect to the time range \(R\) than \(y_2\) if \(\| y_1 \| < \| y_2 \| \). It should be noted, that a trivial rescaling of the dependent variable and the right-hand side of the differential equation with the same constant will change the unsmoothness norms with the same factor without adding to the complexity of the set of solutions. Thus, only the relative sizes of the norms of the different solutions and the right-hand side are of interest.

The possibility of choosing bases other than \(\{t^k\}\) for the space of functions analytic in a region around the origin has been given some attention in the investigations. One such choice could be the system of Chebyshev polynomials of \(t\), \(\{T_k(t)\}_{k=0}^\infty\). With this choice, the corresponding \(\ell_\rho\)-norms will depend on the behavior of the function in an elliptic region with center at the origin. However, the complexity of the analysis with bases other than \(\{t^k\}_{k=0}^\infty\) seems to be prohibitive.

### 3. Constant Coefficient Case

In this section we shall discuss the smoothness in terms of (2.1) of solutions to the equation
\[ y' = \lambda y + f(t) , \] (3.1)
where \(\lambda\) is a complex constant and unless otherwise stated \(f \in (\rho, 2)\)-space.

Denote by \(A\) the unbounded linear operator \(D - \lambda\) where \(D\) is the differentiation operator. Then Eq. (3.1) reads
\[ Ay = f . \] (3.2)
By inspection it is seen that the null-space of $A$ is spanned by the single vector $v$ corresponding to the function $v(t) = e^{At}$ and that the operator $V$ in

$$Vf(t) = \int_0^t e^{A(t-\tau)}f(\tau) \, d\tau$$

(3.3)

is a right-sided inverse of $A$. Hence all solutions $y$ of (3.2) are of the form

$$y = \alpha v + Vf,$$

(3.4)

where $\alpha$ is a complex constant. Now $v \in (R, 2)$-space for all $R > 0$. Further, using Lemma 4.1 below and property 4, Section 2, of the $(R, 2)$-norm, the operator $V$ can be shown to be bounded on the $(R, 2)$-space for all $R > 0$. Hence all solutions $y$ to (3.2) will be in the $(\rho, 2)$-space.

The following result is a straightforward consequence of standard results of functional analysis (see e.g. [7, pp. 22 and 24]).

**Lemma 3.1.** Let $V$ be the linear operator in (3.3) and $v$ the vector corresponding to $e^{At}$. Assume $f \in (\rho, 2)$-space. Then for every $R \in (0, \rho]$, Eq. (3.2) has a unique solution $y_R$ with minimal $(R, 2)$-norm. This solution is given by

$$y_R = \hat{\alpha} v + Vf,$$

where

$$\hat{\alpha} = -\frac{(v, Vf)_R}{\|v\|_R^2}.$$

This lemma has the immediate consequence as follows.

**Corollary.** The solution $y_R$ of (3.2) with minimal $(R, 2)$-norm is a linear function of $f$.

The linear operator in this corollary is a generalized inverse of the operator $A$, (cf. [10]). We denote this operator by

$$y_R = A^+(R)f.$$

(3.5)

Whenever possible without risk for confusion, we drop the index $R$ in the notation (3.5). Some properties of $A^+$ follow immediately from the definition. First

$$A^+ = P_{v^\perp} V,$$

(3.6)

where $P_{v^\perp}$ denotes the orthogonal projection onto the orthogonal complement
of $v$ in the $(R, 2)$-space. $I^*$ is some right-sided inverse to $A$, e.g., the operator (3.3). Further,

$$A^*A^* = I, \text{ identity operator,} \quad (3.7)$$

and, $A^+Af$ being the solution to $Ay := Af$ which is orthogonal to $v$,

$$A^+A = P_{v^*}. \quad (3.8)$$

The next result, which shows a relation between the operators $A^{+(R)}$ for different $R$, will be useful in the sequel. Introduce temporarily the constant $\lambda$ in (3.1) as a lower index in the notation for $A^{+(R)}$ by writing $A^{+(R)} = A^{+(R)}_\lambda$. Then the following lemma holds.

**Lemma 3.2.**

$$[A^{+(R)}_{\lambda}f(t)](t) = R \cdot [A^{+(R)}_{\lambda}f(Rt)] \left( \frac{t}{R} \right).$$

**Proof.** The transformation $T_R$ defined by

$$f(t) \in (R, 2)\text{-space} \Leftrightarrow f(Rt) \in (1, 2)\text{-space}$$

is bijective, linear, and norm-invariant. By the same transformation of variable in (3.1), $A^{+(R)}_{\lambda}$ is seen to correspond to $RA^{+(R)}_{\lambda}$ through the isomorphism $T_R$. This immediately proves the lemma.

Before stating our next result, we need to look at the relation (3.2) more in detail. By (3.1), (3.2) is seen to be equivalent to the recurrence relation

$$(k + 1)y_{k+1} - \lambda y_k = f_k, \quad k = 0, 1, 2, \ldots, \quad (3.9)$$

for the Taylor coefficients of $y$ and $f$, respectively. Hence, in the $(R, 2)$-space the operator $A$ is represented by the infinite bidiagonal matrix

$$A = \begin{pmatrix}
-\lambda & 1 & 0 & 0 & \cdots \\
0 & -\lambda & 2 & 0 & \cdots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}. \quad (3.10)$$

We are now able to formulate the following theorem.

**Theorem 3.1.** Let $A_N$ be the $N \times (N + 1)$ submatrix in the "northwest" corner of $A$ in (3.10). Let the least positive singular value of $A_N$ be $\sigma_N(\lambda)$. 

Then

$$\sigma(\lambda) = \lim_{N \to \infty} \sigma_N(\lambda) \quad \text{exists.}$$

$$\sigma(\lambda) > 0, \quad \forall \lambda, \quad \text{and} \quad \| A^{+(1)} \|_1 = \frac{1}{\sigma(\lambda)}.$$  

Proof. Throughout this proof we shall denote $A^{+(1)}$ by $A^+$ and both the $(1, 2)$-norm and the finite-dimensional euclidean norm by $\| \cdot \|$. By the notation $x_N$ we mean the finite-dimensional vector of the first $N$ Taylor coefficients of $x \in (1, 2)$-space. First, we shall show the inequality

$$\sigma_N(\lambda) \geq \frac{1}{\| A^+ \|} ; \quad N = 1, 2, 3, \ldots \quad (3.11)$$

Assume $N$ to be given. With the above notation, the first $N$ equations in (3.9) can be written

$$A_N y_{N+1} = f_N . \quad (3.12)$$

The solution $\tilde{y}_{N+1}$ of this equation with smallest euclidean norm can be expressed using the Moore–Penrose pseudoinverse of $A_N$

$$\tilde{y}_{N+1} = A^+_N f_N . \quad (3.13)$$

Choose $f_N$ to be the eigenvector to the smallest eigenvalue, $\sigma_N^2(\lambda)$, of $A_N A_N^H$. (Note, that the rows of $A_N$ are linearly independent for all $\lambda$.) Then all solution vectors $y_{N-1}$ to (3.12) fulfill

$$\| y_{N+1} \| \geq \| A^+_N f_N \| = \frac{1}{\sigma_N(\lambda)} \| f_N \| . \quad (3.14)$$

Denote by $f(N)$ the $N$th degree polynomial in the $(1, 2)$-space, whose coefficients are given by $f_N$ in (3.14). All solutions $y$ to (3.2) with $f = f(N)$ then fulfill

$$\| y \| \geq \| y_{N+1} \| \geq \frac{1}{\sigma_N(\lambda)} \| f_N \| = \frac{1}{\sigma_N(\lambda)} \| f(N) \| .$$

In particular, this is true for the solution $y = A^+ f(N)$. Hence

$$\frac{1}{\sigma_N(\lambda)} \leq \frac{\| A^+ f(N) \|}{\| f(N) \|} \leq \sup \frac{\| A^+ f \|}{\| f \|} = \| A^+ \| , \quad (3.15)$$

and (3.11) holds true. By 3.4 and Lemma 3.1 the operator $A^+$ is bounded. Thus the positive sequence $\{ \sigma_N(\lambda) \}_1^\infty$ is bounded below by a positive constant.
Furthermore, as a consequence of the extremal properties of the eigenvalues of $A_N A_N^+$ [3, p. 99], the sequence is nonincreasing. Thus

$$\lim_{N \to \infty} \sigma_N(\lambda) = \sigma(\lambda)$$

exists, and by (3.11),

$$\sigma(\lambda) \geq \frac{1}{\|A^+\|} > 0. \quad (3.16)$$

Next, we shall show that

$$\frac{\|A^+ f\|}{\|f\|} \leq \frac{1}{\sigma(\lambda)} \quad (3.17)$$

holds for an arbitrary $f$ in the $(1, 2)$-space. Assume $f$ to be given. Let $y(N)$ be the particular solution of (3.2) whose first $N + 1$ Taylor coefficients are given by (3.13) and denote by $y_k(N)$, $k = 0, 1, 2, \ldots$ the Taylor coefficients of $y(N)$. Then

$$\|y(N)\|^2 = \|A_N^+ f_N\|^2 + \sum_{k=N+2}^{\infty} |y_k(N)|^2 \quad (3.18)$$

From (3.9) we get, using the parallelogram law

$$|y_{k+1}(N)|^2 \leq \frac{2}{(k + 1)^2} |y_k^2(N)| + \frac{2}{(k + 1)^2} \|f_k\|^2, \quad k = 0, 1, 2, \ldots$$

By summing this last inequality from $k = N + 1$ to $k = \infty$ we obtain an upper bound, valid for example for $N + 2 > 2 |\lambda|$, for the last sum in (3.18). We end up with

$$\|y(N)\|^2 \leq \frac{\|f_N\|^2}{\sigma_N^2(\lambda)} + \frac{C}{(N + 2)^2} \|f\|^2; \quad N > 2 |\lambda| - 2, \quad (3.19)$$

where $C$ depends on $\lambda$ but not on $N$ or $f$. The right-hand side of (3.19) can be made arbitrarily close to $\|f\|^2/\sigma^2(\lambda)$ by choosing $N$ large enough. Now $\{y(N)\}$ is a subset of the set of all solutions to (3.2). Hence $\|f\|/\sigma(\lambda)$ must be an upper bound for the norm of the minimal-norm solution $\|A^+ f\|$. Thus (3.17) is valid, and hence

$$\|A^+\| = \sup \frac{\|A^+ f\|}{\|f\|} \leq \frac{1}{\sigma(\lambda)}. \quad (3.20)$$
By combining (3.16) and (3.20) we conclude

$$\| A^+ \| = \frac{1}{\sigma(\lambda)},$$

which completes the proof of Theorem 3.1.

The following generalization of Theorem 3.1 can be proved by straightforward application of Lemma 3.2.

COROLLARY. Let $\sigma(\lambda)$ be defined as in Theorem (3.1). Then

$$\| A^{+(R)} \|_R = \frac{R}{\sigma(\lambda R)}.$$

By Theorem 3.1 and the corollary, numerical values of $\| A^{+(R)} \|$ can be obtained by calculating the least singular value of $A_N$ for sufficiently large $N$. In the next theorem we shall give rather simple analytical bounds for $\| A^{+(R)} \|$ which are reasonably sharp for all $\lambda$. We were guided to the proof by observing from numerical calculations that the eigenvector to the least eigenvalue of $A_N A_N^H$ seems for large $N$ to be almost parallel to the vector of the first $N$ Taylor coefficients of the function $(e^{it} - 1)/t$.

THEOREM 3.2. For $\| A^{+(R)} \|$ the following bounds are valid

$$\left( \frac{R}{|\lambda|} \right)^{1/2} \left( \frac{I_0(2 |\lambda| R) - 1}{2I_1(2 |\lambda| R) - |\lambda| R} \right)^{1/2} \leq \| A^{+(R)} \| \leq \begin{cases} \frac{R}{\{1 + [(|\lambda| R)^2/2]\}^{1/2}}, & |\lambda| \leq \frac{2}{R} \\ \frac{R}{(2 |\lambda| R - 1)^{1/2}}, & |\lambda| \geq \frac{2}{R}. \end{cases}$$

Note. The $I_k$ are modified Bessel functions of the first kind. The lower bound for $\| A^{+(R)} \|$ in this theorem has the following asymptotical behavior (see e.g. [2, formulas 9.6.10 and 9.7.1]):

Lower bound $= \frac{R}{[1 + \{3(|\lambda| R^2/4) + O(|\lambda|^4)]^{1/2}}, \quad |\lambda| \to 0,$

and

Lower bound $= \frac{R}{[2 |\lambda| R - \frac{1}{2} + O(1/|\lambda|)]^{1/2}}, \quad |\lambda| \to \infty.$

Graphs of the bounds in Theorem 3.2 together with numerically determined values of $\| A^{+(R)} \|$ are shown in Fig. 1 below in the case $R = 1$. 409/51/3-3
Proof of Theorem 3.2. First we note that it means no lack of generality to assume \( \lambda \) real and nonnegative. The general case is brought to this form, with \(|\lambda|\) replacing \( \lambda \) in (3.1), by the transformation \( x(t) = e^{i\lambda t}(e^{-t\lambda}) \) where \( \alpha = \arg(\lambda) \). This is a unitary and diagonal transformation in the \((R, 2)\)-space which, consequently, does not change the \((R, 2)\)-norm. Second, we intend to carry out the proof for \( R = 1 \) and then use Lemma 3.2 to obtain the result for an arbitrary \( R \). Throughout the rest of this proof, we adopt the same conventions of notation for norms, vectors, and the operator \( A^{(1)} \) as in the proof of Theorem 3.1.

Assume \( \lambda \) real and nonnegative. Let \( \sigma(\lambda) \) be defined as in Theorem 3.1. We shall show that

\[
\lambda \frac{2I_0(2\lambda) - \lambda}{I_0(2\lambda) - 1} \geq \sigma^2(\lambda) \geq \begin{cases} 1 - (\lambda^2/2), & \lambda \leq 2 \\ (2\lambda - 1), & \lambda > 2. \end{cases}
\]  

(3.21)

Denote by \( u \) the function \((e^{t} - 1)/t\) in the \((1, 2)\)-space \( u \) has the Taylor coefficients \( u_k = \lambda^{k+1}/(k + 1)! \), \( k = 0, 1, 2, ... \). We intend to use this function to show both the inequalities in (3.21).

For the upper bound we use the extremal properties of the singular values [3, p. 99] to get

\[
\sigma_N^2(\lambda) = \min_{x \in \mathbb{R}^N} \frac{\| A_N^* x \|^2}{\| x \|^2} \leq \frac{\| A_N^* u_N \|^2}{\| u_N \|^2}
\]

Using the power series expansion of the \( I_k \), see e.g. [2, formula 9.6.10], it is seen that

\[
\lim_{N \to \infty} \frac{\| A_N^* u_N \|^2}{\| u_N \|^2} = \cdots = \frac{2I_1(2\lambda) - \lambda}{I_0(2\lambda) - 1}.
\]
Hence the upper bound in (3.21) holds for

\[ \sigma(\lambda) = \lim_{N \to \infty} \sigma_N(\lambda). \]

To obtain the lower bound we note that by definition \( \sigma_N^2(\lambda) \) is the least eigenvalue of the matrix \( \Lambda_N \Lambda_N^T \), which is seen to be a Stieltjes matrix. The theory for such matrices [9, p. 85] suggests a diagonal similarity transformation by the matrix

\[ \Delta_N = \text{diag}(u_0, u_1, \ldots, u_{N-1}) \]

and the use of the Gerschgorin inclusion theorem to get a lower bound for \( \sigma_N^2(\lambda) \). Hence we form

\[
\Lambda_N^{-1} \Lambda_N \Lambda_N^T \Lambda_N = \begin{pmatrix}
\lambda^2 + 1 & -\lambda^2/2 & 0 \\
-2 & \lambda^2 + 4 & -2\lambda^2/3 \\
0 & -3 \cdot 2 & \lambda^2 + 9 & -3\lambda^2/4 \\
& & & & \ddots
\end{pmatrix}
\]

The intersection of the \( k \)th Gerschgorin circle of this matrix and the real axis (note that all the eigenvalues are real) does not contain points to the left of

\[ \varphi(\lambda, k) = \lambda^2 + k^2 - \frac{k}{k + 1} \lambda^2 - k(k - 1) = k + \frac{\lambda^2}{k + 1}; \quad k = 1, \ldots, N. \]

Consequently,

\[ \sigma_N^2(\lambda) \geq \min_{k=1 \ldots N} \varphi(\lambda, k) \geq \begin{cases} 1 + (\lambda^2/2), & \lambda \leq 2 \\
(2\lambda - 1), & \lambda \geq 2 \end{cases} \]

holds for \( N = 1, 2, \ldots \). As the right membrum in (3.2) is independent of \( N \), it is a lower bound for \( \sigma(\lambda) \) as well.

We have shown the inequalities (3.21). The statement of Theorem 3.2 for \( R = 1 \) follows from (3.21) by use of Theorem 3.1. By Lemma 3.2 the relation

\[ \| \Lambda_\lambda^{(R)} \|_R = R \| \Lambda_\lambda^{(R)} \|_1 \] (3.22)

is seen to hold. From (3.22) and Theorem 3.2 for \( R = 1 \) the statement of Theorem 3.2 follows.
Next, we shall be concerned with the existence and some properties of a function \( f^*_\lambda, R \) such that
\[
\| A^{+}(R) \| = \| A^{+}(R) f^*_\lambda, R \| \| f^*_\lambda, R \|.
\]
First we establish the following lemma.

**Lemma 3.3.** The operator \( A^{+}(R) \) is completely continuous on the \((R, 2)\)-space.

**Proof.** We prove the result for \( R = 1 \). The generalization to an arbitrary \( R \) is immediate by Lemma 3.2. Here, too, we adopt the notation used in the proof of Theorem 3.1 for norms, vectors, and the operator \( A^{*}(I) \). Put \( f = t^N \) in (3.2) and consider the solution \( y \) with \( y(0) = 0 \). From (3.9) \( y \) is seen to have the Taylor coefficients
\[
y_k = \begin{cases} 0; & k = 0, \ldots, N \\
\frac{\lambda^{k-N-1}}{(N + 1)(N + 2) \cdots k}; & k = N + 1, N + 2, \ldots,
\end{cases}
\]
from which \( \| y \| \leq C/N, \ N > N_0(\lambda) \). \( C = C(\lambda) \) is seen to hold. Hence, in particular
\[
\| A^{+} t^N \| \leq C/N, \quad N > N_0(\lambda). \quad (3.23)
\]
Denote by \( P_N \) the orthogonal projection operator onto the subspace of polynomials of degree \( \leq N \) in the \((1, 2)\)-space. The operators \( A^{+} P_N \) are uniformly bounded on the \((1, 2)\)-space, \( N = 0, 1, 2, \ldots \). Furthermore, \( A^{+} P_N \) has a finite-dimensional range (the dimensionality does not exceed that of the range of \( P_N \)). By a well-known theorem of functional analysis, see e.g. [4, p. 114], \( A^{+} \) will be completely continuous if we can show that
\[
\lim_{N \to \infty} \| A^{+} - A^{+} P_N \| = 0. \quad (3.24)
\]
Let \( x \in (1, 2)\)-space and form
\[
(A^{+} - A^{+} P_N) x = \sum_{k=N+1}^{\infty} x_k A^{+} t^k.
\]
Using, in turn, the triangle inequality, the Cauchy–Schwartz inequality, and (3.23) we obtain
\[
\|(A^{+} - A^{+} P_N) x\|^2 \leq \left( \sum_{k=N+1}^{\infty} |x_k| \| A^{+} t^k \| \right)^2 \leq \sum_{k=N+1}^{\infty} |x_k|^2 \sum_{k=N+1}^{\infty} \| A^{+} t^k \|^2
\]
\[
\leq \sum_{k=N+1}^{\infty} |x_k|^2 \sum_{k=N+1}^{\infty} \frac{C^2}{k^2} \leq \frac{C^2 \| x \|^2}{N}.
\]
This proves (3.24) from which the lemma follows. We intend to use this lemma to establish the following.

**Corollary.** There exists a function \( f^* = f_{*,R}^* \) in the \((R, 2)\)-space, such that

\[
\| A^+(R) \| = \frac{\| A^+(R)f^* \|}{\| f^* \|} .
\]

**Proof.** It is a standard result, that the adjoint operator of a completely continuous operator is completely continuous and that the product of two completely continuous operators is completely continuous \([5, p. 275]\) and \([4, vol. 1, p. 115]\). Thus \( A^+HA^+ \) is completely continuous, self-adjoint, and positive semidefinite. Then \([5, p. 335]\) there exists a vector \( f^* \) such that

\[
A^+HA^+f^* = \| A^+HA^+ \| f^*. \tag{3.25}
\]

Moreover, by Cauchy's inequality we have for all \( f \)

\[
\| A^+f \|^2 = (f, A^+HA^+f) \leq \| A^+HA^+ \| \| f \|^2. \tag{3.26}
\]

(3.25) and (3.26) together give

\[
\| A^+ \|^2 = \| A^+HA^+ \| = \frac{\| A^+f^* \|^2}{\| f^* \|^2},
\]

which is the statement of the corollary.

In the next lemmas we establish some properties of the adjoint operator \( \Lambda^H \) of \( \Lambda \). (For simplicity, we use a notation which does not indicate the \( R \)-dependence of the adjoint.) Although these lemmas seem rather simple, we have not been able to derive them from similar general results of functional analysis. Note that the operator \( \Lambda \) is unbounded. The domain of \( \Lambda \) contains the complete orthonormal sequence \( \{ t_i \}_0^\infty \). Hence we define the linear operator \( \Lambda^H \) by the requirement

\[
(t^i, \Lambda^H, t^j)_R = (At^i, t^j)_R; \quad i, j = 0, 1, 2, \ldots
\]

\( \Lambda^H \) will be unbounded. In particular, for \( R = 1 \), \( \Lambda^H \) is represented by the conjugate transpose of the matrix in (3.10) and, hence, in the \((1, 2)\)-space,

\[
\Lambda^Hf = t(D(tf)) - \overline{\Lambda}f.
\]

**Lemma 3.4.** Assume \( f, g, \Lambda f, \) and \( \Lambda^Hg \) to be in the \((R, 2)\)-space. Then

\[
(\Lambda^Hg, f)_R = (g, \Lambda f)_R.
\]
Proof. For simplicity, we prove the result only for $R = 1$. From the matrix representations of $A^H$ and $A$ we get, respectively,

\[
\text{Left membrum} = \sum_{k=0}^{\infty} (k\bar{g}_{k-1} - \lambda \bar{g}_k) f_k = \sum_{k=1}^{\infty} k\bar{g}_{k-1} f_k - \lambda \sum_{k=0}^{\infty} \bar{g}_k f_k ,
\]

and

\[
\text{Right membrum} = \sum_{k=0}^{\infty} \bar{g}_k[-\lambda f_k + (k + 1) f_{k+1}]
= \sum_{k=0}^{\infty} \bar{g}_k(k + 1) f_{k+1} - \lambda \sum_{k=0}^{\infty} \bar{g}_k f_k .
\]

The convergence of these sums follows from the assumption. Hence the left and right membria are equal.

Lemma 3.5. Assume $f \in (R, 2)$-space and $(e^{it}, f) = 0$. Then the equation

\[A^H x = f\]

has a unique solution $x$ in the $(R, 2)$-space.

Proof. Again, we prove the result only in the case $R = 1$. Here, too, we denote the $(1, 2)$-norm with $\| \|$ . From the matrix representation of $A^H$ it is seen that the recurrence relation

\[
x_0 = -f_0/\lambda
x_{k+1} = [(k + 1) x_k - f_{k+1}]/\lambda, \quad k = 0, 1, 2, \ldots,
\]

is necessary and sufficient for $\{x_k\}_{0}^{\infty}$ to be the Taylor coefficients of a formal solution $x$ to the equation. These formulas define the sequence $\{x_k\}_{0}^{\infty}$ uniquely, in fact

\[
x_N = -\frac{N!}{\lambda^{N+1}} \left( f_0 + \frac{\lambda}{1!} f_1 + \frac{\lambda^2}{2!} f_2 + \cdots + \frac{\lambda^N}{N!} f_N \right).
\]

From the assumption $f \perp e^{it}$ it follows, that the sum within the brackets is equal to

\[
(\cdots) = -\sum_{k=N+1}^{\infty} \frac{\lambda^k}{k!} f_k,
\]

and hence

\[
|x_N| \leq \frac{C}{N + 1} \| f \|, \quad N > N_0(\lambda),
\]
where \( C \) depends on \( N_0 \) and \( \lambda \) but not on \( N \). Hence \( x \in (1, 2)\)-space, which completes the proof.

We shall use Lemmas 3.3–3.5 to prove the following regularity property of the "worst function" \( f_{\lambda,R}^* \) in the \((R, 2)\)-space (cf. the corollary of Lemma 3.3). Note that we have not shown \( f_{\lambda,R}^* \) to be unique.

**Theorem 3.3.** Any function \( f_{\lambda,R}^* \in (R, 2)\)-space, such that

\[
\| A^+(R) \| = \| A^+(R)f_{\lambda,R}^* \|/\| f_{\lambda,R}^* \|
\]

is analytic in the whole \( t \)-plane.

**Proof.** We carry out the proof assuming \( R = 1 \). The generalization to arbitrary \( R \) follows from Lemma 3.2. Again, we drop all indices that point out \( R = 1 \) from the notation.

Note, that \( f = \alpha A^{-H}A^+f \), where \( \alpha \) is the scalar \( 1/\| A^{-H}A^+ \| \), is necessary for equality in (3.26). Hence the requirement \( \| A^+ \| = \| A^+f^* \|/\| f^* \| \) is equivalent to

\[
A^{+H}A^+f^* = \| A^{+H}A^+f^* = f^*/\sigma^2(\lambda),
\]

where the last equality follows from Theorem 3.1 and the last formula in the proof of the corollary of Lemma 3.3. Put \( g^* = A^+f^* \). Then \( g^* \neq 0 \) and we get by multiplying with \( A^+ \) from the left

\[
A^+A^+g^* = \sigma^{-2}(\lambda) g^*.
\]

By (3.6), \( g^* \perp e^t \) and hence, by Lemma 3.5,

\[
g^* = A^Hh^*,
\]

where \( h^* \in (1, 2)\)-space is uniquely determined by \( g^* \). Insert (3.28) into (3.27) and multiply with \( A \) from the left. Note, that \( Ag^* \in (1, 2)\)-space by (3.7).

\[
A^{+H}A^Hh^* = \sigma^{-2}(\lambda) A^Hh^*.
\]

Now, for an arbitrary \( z \in (1, 2)\)-space,

\[
(A^{+H}A^Hh^*, z) = (A^Hh^*, A^+z)
\]

by definition of \( A^{+H} \) (which is a bounded operator). Further, by (3.7) and (3.28) the vectors \( A(A^+z) \), \( A^Hh^* \) belong to the \((1, 2)\)-space. From (3.7) and Lemma 3.4 it then follows

\[
(A^{+H}A^Hh^*, z) - (h^*, AA^+z) = (h^*, z).
\]
Hence $A^H A H h^* = h^*$ and, by (3.29), $A A H h^* = \sigma^2(\lambda) h^*$. Thus $h^* \in (1, 2)$-space is a nontrivial solution to the differential equation

$$t^2 h' + (3t - \lambda + \lambda t^2) h' + (1 - \lambda t + |\lambda|^2) h = \sigma^2(\lambda) h.$$  

(3.30)

$h^*$ is analytic in $|t| < 1$ by property 3, Section 2, of the $(1, 2)$-norm. However, (3.30) is a linear differential equation with coefficients which are analytic and single-valued everywhere except at the origin. Hence [8, p. 341] the origin is the only singular point possible for any analytic function which satisfies (3.30) in some open region in the t-plane. Thus $h^*$ is analytic in the whole t-plane. By (3.28), (3.7), and the definition of $g^*$,

$$f^* = A A H h^* = \sigma^2(\lambda) h^*.$$  

The last equality follows from the differential equation. Hence $f^*$ is analytic in the whole t-plane. This completes the proof of Theorem 3.3.

We note that the "worst case" in (3.1) is actually given by

$$\sup || A^{+R} f ||_R / || f ||_R$$

where the supremum is taken over the $(\rho, 2)$-subspace of the $(R, 2)$-space, $R \leq \rho$. The result of Theorem 3.3 shows that the worst case (possibly the worst cases) in the whole $(R, 2)$-space is (are) in fact contained in such a subspace regardless of the value of $\rho$. Hence the largest quotient $|| A^{+R} f ||_R / || f ||_R$ possible in (3.1) is independent of $\rho$, $\rho \geq R$.

Next, we shall study the solutions $A^{+R} f$ of (3.1) for functions $f$ that are not necessarily close to the worst function $f^*_A, R$. In particular, we want to investigate the asymptotic behavior of $A^{+R} f$ in the case when $|\lambda|$ tends to infinity and $f$ is a fixed function in the $(\rho, 2)$-space, $\rho > R$. First we give a useful upper bound for $|| (e^{At}, f^{(k)})_R ||$, $k = 1, 2, ...$.

**Lemma 3.6.** Assume $f \in (\rho, 2)$-space. Then, for any $R > 0$ and any integer $k \geq 0$

$$| (e^{At}, f^{(k)})_R | \leq \frac{1}{\rho^k} || f ||_\rho k! e^{|\lambda|(R^2/\rho)} L_k(- |\lambda| R^2/\rho),$$

where $L_i$, $i = 0, 1, 2, ...$ are the Laguerre polynomials.

**Proof.** By (2.2)

$$e^{At}, f^{(k)})_R = \sum_{i=0}^{\infty} \frac{\lambda^i (i + k) \cdots (i + 1)}{i!} f_{i+k} R^{2t}. $$

(3.31)
MINIMUM NORM SOLUTIONS

By (2.1), \( f \in (\rho, 2) \)-space implies

\[
|f_i| \leq \|f\|_\rho / \rho^i; \quad i = 0, 1, 2, \ldots
\]  

(3.32)

Use the triangle inequality and insert (3.32) in (3.31)

\[
|e^{i\lambda t}, f^{(k)}|_R \leq \frac{1}{\rho^k} \|f\|_\rho \sum_{i=0}^{\infty} \frac{|\lambda|^i (i + k) \cdots (i + 1)}{i!} \left( \frac{R^2}{\rho} \right)^i
\]

\[
= \frac{1}{\rho^k} \|f\|_\rho \left( -\frac{d^k}{dt^k} (t^k e^{\lambda t}) \right)_{t=R/|\lambda|}.
\]

By definition of the Laguerre polynomials, see e.g. [2, formulas 22.11.6 and 22.5.16], the lemma follows.

Next, we use the operator \( A^{+(R)} \) to construct a special sequence of solutions to (3.1). When \(|\lambda|\) is large, the first solutions in this sequence are smooth and have a number of smooth derivatives. We formulate this in the following lemma.

**Lemma 3.7.** Assume \( R < \rho \). Then for any nonnegative integer \( N \) there is a solution \( y_N = y_N(\lambda, t) \) to (3.1) such that

\[
y_N^{(k)} = -\frac{f^{(k)}}{\lambda} - \frac{f^{(k+1)}}{\lambda^2} - \cdots - \frac{f^{(N)}}{\lambda^{N-k+1}} = \lambda^k r_N; \quad k = 0, 1, \ldots, N,
\]

where

\[
\|r_N\|_R \leq \frac{\|A^{+(R)}\|_R \rho (N + 1)!}{|\lambda|^{N+1} (\rho - R)^{N+2}} \|f\|_\rho.
\]  

(3.33)

**Proof.** Put \( y = -f/\lambda - f'/\lambda^2 - \cdots - f^{(N)}/\lambda^{N+1} + z \) in (3.1). Then \( z \) is seen to fulfill

\[
z' = \lambda z + f^{(N+1)}/\lambda^{N+1}.
\]

Choose \( r_N \) to the particular solution \( A^{+(R)}f^{(N+1)}/\lambda^{N+1} \) of this equation. The inequality (3.33) then follows from (2.4).

Thus the lemma holds for \( k = 0 \). Assume it to hold for \( k = p < N \). By differentiating \( p \) times in Eq. (3.1) for \( y_N \) we get

\[
y_N^{(p+1)} = \lambda y_N^{(p)} + f^{(p)}.
\]

Inserting the expression for \( y_N^{(p)} \) from the lemma, we see that the lemma holds for \( k = p + 1 \). By induction then, it holds for \( k = 0, 1, \ldots, N \).

The following theorem shows that \( A^{+(R)}f \) is very close in the \((R, 2)\)-norm to the solutions \( y_N(\lambda, t) \) in Lemma 3.7 when \(|\lambda|\) is large.
Theorem 3.4. Assume $R < \rho$ and let $\gamma_N = \gamma_N(\lambda, t)$ be defined as in Lemma 3.7. Then for any integer $N \geq 0$

\[
\frac{d^k}{dt^k} \left( A^{(R)} f \right) = y_N^{(k)} + \lambda^k c_N e^{\lambda t}, \quad k = 0, 1, 2, \ldots
\]

where

\[
| c_N | \leq \| e^{\lambda t} \|_R^{-2} e^{(\lambda)^R R^{R/\rho}} \| f \|_\rho \sum_{k=0}^{N} \frac{k!}{\lambda^{k+1}} L_k(-|\lambda| R^{2/\rho}).
\]

Note. From formula (A.3) in the appendix it is seen that this bound for $| c_N |$ will impose upon $\| c_N e^{\lambda t} \|_R$ a bound

\[
\| c_N e^{\lambda t} \|_R \leq e^{-|\lambda| R[1-(R/\rho)]} \| f \|_\rho O_N(|\lambda|),
\]

where $O_N(|\lambda|) = O(|\lambda|^{-3/4})$, $|\lambda| \to \infty$. The right member in (3.34) decreases to zero faster than any negative power of $|\lambda|$ as $|\lambda| \to \infty$.

Proof of Theorem 3.4. By Lemma 3.1, the theorem holds for $k = 0$ and

\[
c_N = -\| e^{\lambda t} \|_R^{-2} (e^{\lambda t}, y_N)_R.
\]

By inserting the expression for $y_N$ from Lemma 3.7 and noting that $\gamma_N = A^{(R)} f^{(N+1)}/\lambda^{N+1}$ is orthogonal to $e^{\lambda t}$ we obtain

\[
c_N = \| e^{\lambda t} \|_R^{-2} \sum_{k=0}^{N} \frac{(e^{\lambda t}, f^{(k)})_R}{\lambda^{k+1}}.
\]

Using Lemma 3.6, the bound for $| c_N |$ in the theorem is obtained. Hence the theorem holds for $k = 0$. By differentiation it is seen to hold also for all positive integers $k$.

We have seen that for any $f$ with $\rho > R$ the function $A^{(R)} f$ has the asymptotic expansion in Lemma 3.7 as $|\lambda| \to \infty$. Lemma 3.7 and Theorem 3.4 also provide strict remainder terms for this asymptotic expansion. It is easily seen that, for example, the condition

\[
\left( \frac{\| f^{(k)} \|}{|\lambda|^{k+1}} \right)^{1/k} \leq \beta < 1 \quad \text{for } k > k_0
\]

is sufficient and that the condition

\[
\frac{\| f^{(k)} \|}{|\lambda|^{k+1}} \to 0, \quad k \to \infty
\]

is necessary for convergence in norm of the expansion in Lemma 3.7. In particular, for $f(t) = e^{\lambda t}$ the expansion is convergent in norm by these
conditions iff $|\mu| < |\lambda|$. In general, however, the expansion is not convergent. The bound (2.4) for the $(R, 2)$-norm of the derivatives of $f$ is easily seen to violate the above necessary condition for convergence in norm. The example $f(t) = \log(\rho + t)$ shows that the bound (2.4) is essentially of the best possible type. Note also, that in cases of convergence of the asymptotic expansion, the limiting function does not in general coincide with $A_{+}^{+}(R)f$.

As a last result on Eq. (3.1) we give a theorem concerning the pointwise closeness of two solutions in the circle $|t| \leq R < \rho$.

**Theorem 3.5.** For any two solutions $y_1$ and $y_2$ to (3.1)

$$|y_{1}^{(k)}(t) - y_{2}^{(k)}(t)| \leq \frac{|e^{it}| \rho^{k}!}{\|e^{it}\|_{R}} (\|y_{1}\|_{p} + \|y_{2}\|_{p}), \quad k = 0, 1, 2,...$$

The result follows directly from differentiating $k$ times in the relation

$$y_{1}(t) - y_{2}(t) = \gamma e^{it}$$

where $\gamma$ is a constant, taking $(R, 2)$-norms and using (2.4). For large $|\lambda|$ the formula (A.3) of the Appendix can be used to estimate the right member.

**4. Time-Dependent Coefficient Case**

In this section we discuss minimum norm solutions to

$$y' = q(t)y + f(t), \quad (4.1)$$

where $f \in (\rho, 2)$-space and $q \in (\rho, 1)$-space. Throughout this section we put $q(0) = \lambda$. First we show a useful result on the $(R, p)$-norm of the product of two functions.

**Lemma 4.1.** Assume $g \in (R, 1)$-space and $f \in (R, p)$-space for $p = 1$ or $2$. Then $fg \in (R, p)$-space and

$$\|fg\|_{R,p} \leq \|g\|_{R,1} \|f\|_{R,p} \cdot$$

**Proof.** For $p = 1$ this follows from

$$\|fg\|_{R,1} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |f_{k-j}g_{j}| R^{k} \leq \sum_{k=0}^{\infty} \sum_{j=0}^{k} |f_{k-j}| R^{k-j} |g_{j}| R^{j}$$

$$= \|f\|_{R,1} \|g\|_{R,1} \cdot$$  (4.2)
Assume \( p = 2 \), and note that the \((R, 2)\)-norm (2.1) can be written
\[
\| f \|_{R, 2} = \left( \frac{1}{2\pi} \int_{|t| = R} |f|^2 \, d\phi \right)^{1/2},
\]
where \( \phi = \text{arg}(t) \). Thus
\[
\| fg \|_{R, 2}^2 = \frac{1}{2\pi} \int_{|t| = R} |f|^2 |g|^2 \, d\phi \leq \left( \max_{|t| \leq R} |g(t)| \right)^2 \frac{1}{2\pi} \int_{|t| = R} |f|^2 \, d\phi
\]
\[
\leq \| g \|_{R, 1} \| f \|_{R, 2}^2.
\]
which completes the proof of the lemma. We also need the following.

**Corollary.** Assume \( g \in (R, 1)\)-space. Then \( e^{g(t)} \in (R, 1)\)-space and
\[
\| e^g \|_{R, 1} < \exp \| g \|_{R, 1}.
\]

**Proof.** Form
\[
\| e^g \|_{R, 1} = \left\| \sum_{k=0}^{\infty} \frac{g^k}{k!} \right\|_{R, 1} \leq \sum_{k=0}^{\infty} \frac{\| g \|_{R, 1}^k}{k!} = \exp (\| g \|_{R, 1}),
\]
where we have used the triangle inequality and Lemma 4.1 for \( p = 1 \).

Denote, in analogy with (3.5) the solution \( j \) of (4.1) with the smallest \((R, 2)\)-norm by
\[
\hat{j} = Q^+ f.
\]
For brevity, we write \( Q^+ \) instead of \( Q^{+,(R)} \). Then \( Q^+ \) is seen to be a bounded linear operator on the \((R, 2)\)-space by the same argument as applied to \( A^+ \).

We are interested in obtaining information analogous to Theorem 3.2 for \( Q^+ \). At present, our main result in this direction is rather weak.

**Theorem 4.1.** Let \( \eta_R = \min(1, \| A^+ \|, |q - \lambda|_{R, 1}) \). Then
\[
\| Q^+ \| \leq \| A^+ \| \min[1 - \eta_R^{-1}, \exp(R \| q - \lambda \|_{R, 1})].
\]

**Proof.** First, assume \( \eta_R < 1 \). Then it follows from Lemma 1.2 in [1] that for any \( f \) in the \((R, 2)\)-space, Eq. (4.1) has a solution \( y \) such that
\[
\| y \| \leq \frac{1}{1 - \eta_R} \| A^+ f \|.
\]
Hence
\[
\| Q^+ \| \leq \frac{\| A^+ \|}{1 - \eta_R}, \quad \eta_R < 1.
\]
Next, even if $\eta_R = 1$, put

$$y(t) = \exp \left( \int_0^t [q(\tau) - \lambda] \, d\tau \right) z(t)$$

in (4.1). Then $z$ is seen to fulfill

$$z' = \lambda z + \exp \left( - \int_0^t [q(\tau) - \lambda] \, d\tau \right) f.$$

Thus (4.1) has a solution

$$y = \exp \left( \int_0^t [q(\tau) - \lambda] \, d\tau \right) A^+ \left[ \exp \left( - \int_0^t [q(\tau) - \lambda] \, d\tau \right) f(t) \right]. \tag{4.6}$$

Note that $\int_0^t [q(\tau) - \lambda] \, d\tau \in (R, 1)$-space by property 4, Section 2, of the $(R, 1)$-norm. Further, as $q(\tau) - \lambda$ lacks a zero-degree term,

$$_{R,1} \left\| \int_0^t [q(\tau) - \lambda] \, d\tau \right\| \leq \frac{R}{2} \| q(t) - \lambda \|_{R,1}. \tag{4.7}$$

Use Lemma 4.1 and the corollary in (4.6). Insert (4.7). We obtain

$$\| y \| \leq \exp[R \| q(t) - \lambda \|_{R,1}] \| A^+ \| \| f \|.$$

Hence, independent of $\eta_R$

$$\| Q^+ \| \leq \| A^+ \| \exp[R \| q(t) - \lambda \|_{R,1}] \tag{4.8}.$$

From (4.5) and (4.8) the theorem follows.

Next, we look upon the expansion corresponding to Lemma 3.7 of solutions to 4.1. Assume $R < \rho$ and put

$$y_N = -q^{-1}[1 + Dq^{-1} + (Dq^{-1})^2 + \cdots + (Dq^{-1})^N] f + z, \tag{4.9}$$

where $D$ is the differentiation operator (note that $D$ does not commute with $q^{-1}$). Then

$$z' = q(t) z + (Dq^{-1})^{N+1} f. \tag{4.10}$$

Choose, in particular

$$z = Q^+(Dq^{-1})^{N+1} f.$$

For fixed $f$ and $q \lambda$, $\| z \|$ is seen from Theorem 4.1 to be of the order
\[ O(|\lambda|^{-N-3/2}) \text{ as } |\lambda| \to \infty. \text{ Also, } \| (Dq^{-1})^k f \| = O(|\lambda|^{-k}) \text{ as } |\lambda| \to \infty. \text{ Thus the solution } \gamma_N \text{ of } 4.1 \text{ fulfills}
\]
\[ y_N^{(k)} = D^k \left( -\frac{f}{q} \right) + O \left( \frac{1}{|\lambda|^2} \right), \quad |\lambda| \to \infty, \quad k = 0, 1, \ldots, N. \quad (4.11) \]

This formula corresponds to Lemma 3.7 for the constant coefficient case. The strict remainder term in (4.11) can in principle be calculated from (4.9) and (4.10). However, the expressions are rather messy and are omitted. Further, using the same technique as in Lemma 3.6 and Theorem 3.4, it can be shown that (4.9) is the correct asymptotic expansion of \( O^+ f \) as \( |\lambda| \to \infty \), other functions being fixed, and that the relation (4.11) holds for all finite derivatives of \( O^+ f \).

Finally, we give a result corresponding to Theorem 3.5 for Eq. (4.1).

**Theorem 4.2.** Let \( y_1 \) and \( y_2 \) be two solutions of (4.1). Then

\[ |y_1(t) - y_2(t)| \leq \frac{\exp \left( \int_0^t q(\tau) \, d\tau \right) \| q - \lambda \|_{R,1}}{\| e^{t \lambda} \|_R} (\| y_1 \|_R + \| y_2 \|_R). \]

**Proof.** As \( y_1 \) and \( y_2 \) are solutions of (4.1),

\[ y_1(t) - y_2(t) = \gamma \exp \left( \int_0^t q(\tau) \, d\tau \right) = \gamma e^{t \lambda} \exp \left( \int_0^t [q(\tau) - \lambda] \, d\tau \right), \quad (4.12) \]

for some constant \( \gamma \). Multiply with \( \exp \{ - \int_0^t [q(\tau) - \lambda] \, d\tau \} \) and take norms. Lemma 4.1 and the corollary can be used in the same way as in the proof of Theorem 4.1 to obtain an upper bound for \( |\gamma| \). Inserting this upper bound in (4.12) the theorem follows.

Again, formula (A.3) of the appendix will provide means of estimating the right member for large \( |\lambda| \).

5. **Summary**

We look upon Theorems 3.1–3.4 as the main results of the present investigation. The bounds obtained for \( \| A^+ \| \) are satisfactorily sharp for all \( \lambda \) (cf. Fig. 1 above).

Using Theorem 3.2, the behavior in terms of \( \lambda, \| q - \lambda \|_{R,1}, \) and \( R, \) of the upper bound for \( \| Q^+ \| \) given in Theorem 4.1 can be examined. For large \( |\lambda| \) this upper bound is seen to behave roughly like

\[ R^{1/2}((2 |\lambda|^{1/2} - R^{1/2} \| q - \lambda \|_{R,1})^{-1}. \]
At present, the question whether $\| Q^+ \|$ really shows this kind of behavior for large $|\lambda|$ has not been answered. One of the lines of work preceding this paper has been attempts to settle this question by a direct approach, similar to that used in Theorems 3.1 and 3.2, to the $\ell_2$-space formulation of Eq. (4.1). Although some numerical indications of a possible improvement of Theorem 4.1 are at hand, no results on this point are yet ready for reporting.

**APPENDIX**

In this appendix we list some formulas on the norms $\| e^{At} \|_{R,p}$ for $p = 1, 2,$ and $\infty$.

$p = 1$

\[ \| e^{At} \|_{R,1} = e^{\lambda |R|}. \]  

$p = 2$

\[ \| e^{At} \|_{R,2} = [I_0(2 |\lambda| R)]^{1/2}. \]  

$I_0$ is the modified Bessel function of the first kind of order zero. From, for example, [2, p. 377, formula 9.7.11], it is seen that

\[ \| e^{At} \|_{R,2} = \frac{e^{\lambda |R|}}{(4\pi |\lambda| R)^{1/4}} \left[ 1 + O \left( \frac{1}{|\lambda| R} \right) \right], \quad |\lambda R| \to \infty. \]  

$p = \infty$

\[ \| e^{At} \|_{R,\infty} = \frac{(|\lambda| R)^{|\lambda| R}}{|\lambda| |R|^!}. \]  

$[x]$ denotes the integer part of $x$. From Stirling’s formula, see for example [2, p. 257, formula 6.1.37], it follows

\[ \| e^{At} \|_{R,\infty} = \frac{e^{\lambda |R|}}{(2\pi |\lambda| R)^{1/2}} \left[ 1 + O \left( \frac{1}{|\lambda R|} \right) \right], \quad |\lambda R| \to \infty. \]  

For small $|\lambda|$:

\[ \lim_{|\lambda R| \to 0} \| e^{At} \|_{R,p} = 1, \quad p = 1, 2, \infty. \]  

**ACKNOWLEDGMENT**

The author wishes to thank Professor Germund Dahlquist for stimulating and helpful discussions during the preparation of this paper.
REFERENCES