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Orthogonal rational functions on the real half line with poles in $[-\infty, 0]$

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Abstract

The main objective is to generalize previous results obtained for orthogonal Laurent polynomials and their application in the context of Stieltjes moment problems to the multipoint case. The measure of orthogonality is supposed to have support on $[0, \infty)$ while the orthogonal rational functions will have poles that are assumed to be “in the neighborhood of 0 and ∞”. In this way orthogonal Laurent polynomials will be a special case obtained when all the poles are at 0 and ∞. We shall introduce the restrictions on the measure and the locations of the poles gradually and derive recurrence relations, Christoffel–Darboux relations, and the solution of the rational Stieltjes moment problem under appropriate conditions.

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1. Preliminaries

In [6, Chapter 11] we studied orthogonal rational functions with prescribed poles on the real line and a measure of orthogonality whose support is also on the real line. Important special cases include a situation where the measure is supported on a finite interval, $[-1, 1]$ say, in which case the poles can be outside that interval, and another so called Stieltjes situation is obtained when the measure is supported on the positive real line and the poles are all located on the negative real line. The most obvious choice in this context is the case of orthogonal Laurent polynomials where all the poles are at 0 or at ∞. These orthogonal Laurent polynomials were extensively studied in the context of the so called strong Stieltjes moment problem. See [2,14–17,21] and the references given there. When allowing the point at ∞ or the origin to proliferate into more points in $[-\infty, 0]$ which may be coinciding or not, we obtain multipoint rational generalizations of the Laurent polynomials. Several aspects were already discussed in papers like [7–9,11]. We note however that in these papers a pole at ∞ was excluded for technical reasons, thus excluding the Laurent polynomials as a special case. The results concerning quadrature formulas based on Laurent polynomials discussed in [3–5,12] were generalized to the multipoint case in [10]. In this paper we give several technical generalizations of the Laurent polynomials to orthogonal rational functions trying to include the Laurent polynomials as a special case when deriving recurrence relations, Christoffel–Darboux relations, and rational Stieltjes moment problems.

We start from the most general situations and consider a finite positive measure μ on the real line $\mathbb{R}$ and spaces of rational functions whose poles are in a prescribed set of points $\{\zeta_k\}_{k=1}^{\infty}$ all on the *extended real line* $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. This can be seen as a generalization of the Laurent polynomials who have all their poles in the set $\{0, \infty\}$. If all the poles are at ∞, the ordinary polynomials result as a special case. We shall be especially interested in the case where the support of μ is part of the real half line $[0, \infty)$ and the poles are in $[-\infty, 0]$. In [10], we started including the cases where $\zeta_k = 0$ and $-\infty$ as a special case. However, there we mainly concentrated on Gaussian-type quadrature formulas and multipoint Padé approximation. In this paper, we shall include the Laurent polynomial case from the beginning and we shall build up the Stieltjes situation more gradually. This is mainly a technical matter which will be dealt with systematically in the present paper.

We start out with a sequence $\{\zeta_n\}_{n=1}^{\infty}$ of points in the extended real line $\hat{\mathbb{R}}$.

It will be convenient to think of $\{\zeta_n\}$ as being the merging of two subsequences $\{\alpha_j\}$ and $\{\beta_k\}$. The α_k are multipoint equivalents of the origin and the β_k are the multipoint generalizations of the point at infinity. Our discussion will cover cases where the ζ_k are chosen alternatingly from the set of α's or the set of β's, but this needs not be the case; any sequence is possible, with or without repetitions. The matching between ζ's, α's and β's can be described as follows. Let $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ be two nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = n$ for all n. Thus we have exactly one of the following two possibilities

$$p(n) = p(n-1) + 1 \quad \text{and } q(n) = q(n-1) \text{ and then we set } \zeta_n = \alpha_{p(n)},$$

$$q(n) = q(n-1) + 1 \quad \text{and } p(n) = p(n-1) \text{ and then we set } \zeta_n = \beta_{q(n)}.$$

We define the factors $r_n(z)$ as follows:

$$r_n(z) = \begin{cases} 1 & \text{if } \zeta_n = \infty \\ \zeta_n - z & \text{if } \zeta_n \neq \infty \end{cases}, \quad n = 1, 2, \ldots . \tag{1.1}$$

The functions $D_n(z)$ are defined by

$$D_n(z) = r_1(z)r_2(z)\cdots r_n(z), \quad n = 1, 2, \ldots \quad \text{and} \quad D_0(z) = 1. \tag{1.2}$$

Here and in the rest of this paper, Π_n denotes the space of polynomials of degree at most n.

The spaces $\mathscr{L}_n$, $n = 0, 1, \ldots$ and $\mathscr{L}$ are defined by

$$\mathscr{L}_n = \left\{ \frac{\pi(z)}{D_n(z)} : \pi \in \Pi_n \right\}, \quad \mathscr{L} = \bigcup_{n=0}^{\infty} \mathscr{L}_n. \tag{1.3}$$

The spaces $\mathscr{L}_n$ have then dimension $n + 1$.

Although ζ_0 does not feature in these definitions, it will be convenient to assume $\zeta_0 = \infty$. For example in the previous definitions, we would not need a separate treatment of $n = 0$.

As a special case one may choose to place all poles at ∞. This means that $r_n(z) = 1$ for all $n = 1, 2, \ldots$. Clearly, in this case $\mathscr{L}_n = \Pi_n$, and we are in the polynomial case.

Another familiar case will appear if we set all $\alpha_n = 0$ and all $\beta_n = \infty$. This means that $\mathscr{L}_n = \Lambda_{-p(n),q(n)} = \text{span}\{z^k : -p(n) \leqslant k \leqslant q(n)\}$ is a subspace of Laurent polynomials. Since there are only two points used (0 and ∞), it is sometimes referred to as the two-point situation. The order of the introduction of the points $\alpha_k = 0$ and $\beta_k = \infty$ may be arbitrary. The special case where 0 and ∞ are alternating, i.e. $p(n)$ is the integer part of $n/2$ or of $(n+1)/2$, is called the balanced two-point situation.

2. Orthogonal functions

Let μ be a positive measure on $\mathbb{R}$ such that all functions in $\mathscr{L} \cdot \mathscr{L} = \{fg : f, g \in \mathscr{L}\}$ are absolutely integrable. Then μ is finite and we have an inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f \overline{g} \, \mathrm{d}\mu \tag{2.1}$$

in $\mathscr{L}$. So we can construct an *orthonormal* sequence $\{\varphi_n\}_{n=0}^{\infty}$ in $\mathscr{L}$ such that $\varphi_0 \in \mathscr{L}_0$ and $\varphi_n \in \mathscr{L}_n \backslash \mathscr{L}_{n-1}$ for $n = 1, 2, \ldots$. Notice that the φ_n are unique up to constant factors with modulus 1. Since all functions D_n are real valued on $\mathbb{R}$ and μ is real, we may normalize the φ_n such that all φ_n are real on $\mathbb{R}$.

For convenience we normalize μ such that $\int \mathrm{d}\mu = 1$.

We write

$$\varphi_n(z) = \frac{P_n(z)}{D_n(z)}, \quad P_n(z) = \mu_n^{(n)} z^n + \mu_{n-1}^{(n)} z^{n-1} + \cdots + \mu_0^{(n)}. \tag{2.2}$$

Note that by construction, $P_n(\zeta_n) \neq 0$ if $\zeta_n \neq \infty$, $\mu_n^{(n)} \neq 0$ if $\zeta_n = \infty$.

3. Recurrence relation

From now on we assume that the sequence $\{\varphi_n\}$ is *regular*. This means that

$$P_n(\zeta_{n-1}) \neq 0 \quad \text{if } \zeta_{n-1} \neq \infty \quad \text{and} \quad \mu_n^{(n)} \neq 0 \quad \text{if } \zeta_{n-1} = \infty. \tag{3.1}$$

We shall use the following *sign normalization*:

$$\begin{aligned}
&\operatorname{sgn} P_n(\zeta_{n-1}) = \operatorname{sgn} P_{n-1}(\zeta_{n-1}) \quad \text{if } \zeta_{n-1} \neq \infty\\
&\operatorname{sgn} \mu_n^{(n)} = \operatorname{sgn} \mu_{n-1}^{(n-1)} \quad \text{if } \zeta_{n-1} = \infty\\
&\mu_1^{(1)} > 0.
\end{aligned} \tag{3.2}$$

Let us now derive a recurrence relation for the functions $\{\varphi_n\}$.

Theorem 3.1. *Given a sequence* $\{p(n)\}$, *we associate the spaces* $\mathscr{L}_n$ *as in* (1.3). *For a positive measure* μ *on* $\mathbb{R}$ *we define the inner product as in* (2.1). *Suppose the sequence of orthogonal rational functions* $\{\varphi_n\}$ *with* $\varphi_n \in \mathscr{L}_n \backslash \mathscr{L}_{n-1}$ *and* $\varphi_n \perp \mathscr{L}_{n-1}$ *is regular. Then there exist numbers* $\{U_n\}$, $\{V_n\}$ *and* $\{W_n\}$ *such we have the recurrence* (*recall the definition of* r_n *from* (1.1))

$$\varphi_n(z) = \frac{U_n z^{\gamma_{n-1}} + V_n r_{n-1}(z)}{r_n(z)} \varphi_{n-1}(z) + W_n \frac{r_{n-2}(z)}{r_n(z)} \varphi_{n-2}(z), \quad n = 1, 2, \ldots \tag{3.3}$$

with initial conditions

$$r_0 = 1, \; r_{-1} = 1; \quad \varphi_0 = 1, \; \varphi_{-1} = 0; \quad \gamma_0 = 1, \; \gamma_{-1} = 1, \tag{3.4}$$

where

$$\gamma_n = \begin{cases} 1 & if \;\; \zeta_n = \infty, \\ 0 & if \;\; \zeta_n \neq \infty. \end{cases} \tag{3.5}$$

Proof. Consider the function

$$\begin{aligned}
h_n(z) &= \frac{1}{r_{n-2}(z)} [r_n(z)\varphi_n(z) - A_n z^{\gamma_{n-2}} \varphi_{n-1}(z)]\\
&= \frac{1}{D_{n-1}(z)} \frac{[P_n(z) - A_n z^{\gamma_{n-2}} P_{n-1}(z)]}{r_{n-2}(z)}.
\end{aligned}$$

This function will be in $\mathscr{L}_{n-1}$ if

$$A_n = \begin{cases} P_n(\zeta_{n-2})/P_{n-1}(\zeta_{n-2}) & \text{if } \zeta_{n-2} \neq \infty, \\ \mu_n^{(n)}/\mu_{n-1}^{(n-1)} & \text{if } \zeta_{n-2} = \infty. \end{cases}$$

On the other hand h_n is orthogonal to $\mathscr{L}_{n-3}$ because for $f \in \mathscr{L}_{n-3}$

$$\langle f, h_n \rangle = \int f(x) \frac{r_n(x)}{r_{n-2}(x)} \varphi_n(x) \, \mathrm{d}\mu(x) - A_n \int f(x) \frac{x^{\gamma_{n-2}}}{r_{n-2}(x)} \varphi_{n-1}(x) \, \mathrm{d}\mu(x) = 0,$$

since

$$f(x) \frac{r_n(x)}{r_{n-2}(x)} \in \mathscr{L}_{n-2} \quad \text{and} \quad f(x) \frac{x^{\gamma_{n-2}}}{r_{n-2}(x)} \in \mathscr{L}_{n-2}.$$

Thus we may write h_n as

$$h_n(z) = B_n\varphi_{n-1}(z) + C_n\varphi_{n-2}(z) + \sum_{k=0}^{n-3} D_k\varphi_k(z) = B_n\varphi_{n-1}(z) + C_n\varphi_{n-2}(z).$$

Therefore

$$\varphi_n(z) = \left[A_n\frac{z^{\gamma_{n-2}}}{r_n(z)} + B_n\frac{r_{n-2}(z)}{r_n(z)}\right]\varphi_{n-1}(z) + C_n\frac{r_{n-2}(z)}{r_n(z)}\varphi_{n-2}(z). \tag{3.6}$$

Since every element in Π_1 may be written in the form $U_n z^{\gamma_{n-1}} + V_n r_{n-1}(z)$, the recurrence may also be written as (3.3). The initial conditions (3.4) are easily verified. □

Note that so far W_1 appears with a coefficient zero and hence it may be chosen arbitrarily.

By rewriting the previous recurrence relation in terms of the numerators P_n of the orthogonal rational functions, we get immediately

Corollary 3.2. *Under the same conditions as in the previous theorem and letting P_n denote the numerator of the orthogonal functions, i.e., $\varphi_n = P_n/D_n$, we have*

$$P_n(z) = [U_n z^{\gamma_{n-1}} + V_n r_{n-1}(z)]P_{n-1}(z) + W_n r_{n-1}(z) r_{n-2}(z) P_{n-2}(z), \quad n = 1, 2, \dots . \tag{3.7}$$

with the initial conditions

$$r_0 = 1, \ r_{-1} = 1; \quad P_0 = 1, \ P_{-1} = 0; \quad \gamma_0 = 1, \ \gamma_{-1} = 1. \tag{3.8}$$

4. The coefficients U_n and W_n

The coefficients U_n and W_n of the recurrence relation have some special properties that we shall prove in this section.

Theorem 4.1. *Let $\{U_n\}$ and $\{W_n\}$ be the coefficients appearing in the recurrence relation. Then we have*

$$U_n > 0 \quad for\ n = 1, 2, \dots$$

and, provided that (*recall the notation of* (2.2))

$$P_{n-2}(\zeta_{n-1}) \neq 0 \quad if\ \zeta_{n-1} \neq \infty,$$

$$\mu_{n-2}^{(n-2)} \neq 0 \quad if\ \zeta_{n-1} = \infty,$$

we also have

$$W_n = -U_n/U_{n-1} < 0 \quad for\ n = 2, 3, \dots . \tag{4.1}$$

Proof. For U_n, it follows easily from (3.4)–(3.7) that

$$U_n = \frac{P_n(\zeta_{n-1})}{P_{n-1}(\zeta_{n-1})} \quad \text{if } \zeta_{n-1} \neq \infty,$$

$$U_n = \frac{\mu_n^{(n)}}{\mu_{n-1}^{(n-1)}} \quad \text{if } \zeta_{n-1} = \infty. \tag{4.2}$$

The sign normalization (3.2) then immediately implies that

$$U_n > 0 \quad \text{for } n = 1, 2, \ldots \; .$$

For the W_n we proceed as follows. First we establish a relation of the form $U_n I_n + W_n J_n = 0$. Then we find expressions for the numbers I_n and J_n. Once these expressions are found, the desired result follows.

First, multiplication of (3.3) by $\frac{r_n(z)}{r_{n-1}(z)}\varphi_{n-2}(z)$ gives

$$\frac{r_n(t)}{r_{n-1}(t)}\varphi_{n-2}(t)\varphi_n(t) = U_n \frac{t^{\gamma_{n-1}}}{r_{n-1}(t)}\varphi_{n-2}(t)\varphi_{n-1}(t) + V_n\varphi_{n-2}(t)\varphi_{n-1}(t) + W_n\frac{r_{n-2}(t)}{r_{n-1}(t)}\varphi_{n-2}(t)\varphi_{n-2}(t). \tag{4.3}$$

Clearly $\frac{r_n(t)}{r_{n-1}(t)}\varphi_{n-2}(t) \in \mathscr{L}_{n-1}$, hence by setting

$$I_n = \int \frac{t^{\gamma_{n-1}}}{r_{n-1}(t)}\varphi_{n-2}(t)\varphi_{n-1}(t)\,\mathrm{d}\mu(t)$$

and

$$J_n = \int \frac{r_{n-2}(t)}{r_{n-1}(t)}\varphi_{n-2}(t)\varphi_{n-2}(t)\,\mathrm{d}\mu(t),$$

we find after integration of (4.3)

$$U_n I_n + W_n J_n = 0. \tag{4.4}$$

We may write

$$\frac{t^{\gamma_{n-1}}\varphi_{n-2}(t)}{r_{n-1}(t)} = c_n\varphi_{n-1}(t) + d_n\varphi_{n-2}(t) + \cdots \tag{4.5}$$

and observe that $I_n = c_n$. We now find an expression for c_n.

Multiplication of (4.5) by $D_{n-1}(t)$ gives

$$t^{\gamma_{n-1}}P_{n-2}(t) = c_n P_{n-1}(t) + d_n r_{n-1}(t)P_{n-2}(t) + r_{n-1}(t)r_{n-2}(t)T_n(t), \tag{4.6}$$

with $T_n \in \Pi_{n-3}$.

For $\zeta_{n-1} \neq \infty$ we then get

$$I_n = c_n = \frac{P_{n-2}(\zeta_{n-1})}{P_{n-1}(\zeta_{n-1})} \tag{4.7}$$

while for $\zeta_{n-1} = \infty$ we get

$$I_n = c_n = \frac{\mu_{n-2}^{(n-2)}}{\mu_{n-1}^{(n-1)}} \tag{4.8}$$

by comparing coefficients of t^{n-1}.

We also have a representation

$$\frac{r_{n-2}(t)}{r_{n-1}(t)}\varphi_{n-2}(t) = e_n\varphi_{n-1}(t) + f_n\varphi_{n-2}(t) + \cdots, \tag{4.9}$$

and see that $J_n = f_n$.

Multiplication of (4.9) by $D_{n-1}(t)$ gives

$$r_{n-2}(t)P_{n-2}(t) = e_n P_{n-1}(t) + f_n r_{n-1}(t)P_{n-2}(t) + r_{n-1}(t)r_{n-2}(t)H_{n-3}(t), \tag{4.10}$$

with $H_{n-3} \in \Pi_{n-3}$. We need to determine the value of f_n.

We consider the various cases for $n = 2, 3, \ldots$.

(1) $\underline{\zeta_{n-1} \neq \infty}$.

(i) $\zeta_{n-2} \neq \zeta_{n-1}, \zeta_{n-2} \neq \infty$.

We get from (4.10)

$$r_{n-2}(\zeta_{n-1})P_{n-2}(\zeta_{n-1}) = e_n P_{n-1}(\zeta_{n-1}) \tag{4.11}$$

and

$$0 = r_{n-2}(\zeta_{n-2})P_{n-2}(\zeta_{n-2}) = e_n P_{n-1}(\zeta_{n-2}) + f_n r_{n-1}(\zeta_{n-2})P_{n-2}(\zeta_{n-2}). \tag{4.12}$$

Taking into account that $r_{n-2}(\zeta_{n-1}) = -r_{n-1}(\zeta_{n-2})$ we find that

$$J_n = f_n = \frac{P_{n-1}(\zeta_{n-2})P_{n-2}(\zeta_{n-1})}{P_{n-2}(\zeta_{n-2})P_{n-1}(\zeta_{n-1})}. \tag{4.13}$$

It follows from (4.4), (4.7), (4.2), and (4.13) that

$$W_n = -\frac{U_n I_n}{J_n} = -\frac{U_n}{U_{n-1}}, \tag{4.14}$$

provided that $P_{n-2}(\zeta_{n-1}) \neq 0$.

(ii) $\zeta_{n-2} = \infty$.

We get from (4.10)

$$P_{n-2}(\zeta_{n-1}) = e_n P_{n-1}(\zeta_{n-1}) \tag{4.15}$$

and

$$0 = e_n\mu_{n-1}^{(n-1)} - f_n\mu_{n-2}^{(n-2)} \tag{4.16}$$

(recall that $r_{n-1}(t) = \zeta_{n-1} - t$ and $r_{n-2}(t) = 1$).

Hence

$$J_n = f_n = \frac{P_{n-2}(\zeta_{n-1})\mu_{n-1}^{(n-1)}}{P_{n-1}(\zeta_{n-1})\mu_{n-2}^{(n-2)}} \tag{4.17}$$

and consequently

$$W_n = -\frac{U_n I_n}{J_n} = -\frac{U_n}{U_{n-1}}, \tag{4.18}$$

provided $P_{n-2}(\zeta_{n-1}) \neq 0$.

(iii) $\zeta_{n-2} = \zeta_{n-1}$.
From the definition of J_n and because the φ_n are normalized, we get $J_n = 1$. Since $J_n = f_n$ we therefore have

$$J_n = f_n = 1. \tag{4.19}$$

Consequently (since in this case $I_n = 1/U_{n-1}$)

$$W_n = -\frac{U_n I_n}{J_n} = -\frac{U_n}{U_{n-1}}. \tag{4.20}$$

(2) $\underline{\zeta_{n-1} = \infty}$.

(i) $\zeta_{n-2} \neq \infty$.
We get from (4.10)

$$-\mu_{n-2}^{(n-2)} = e_n \mu_{n-1}^{(n-1)} \tag{4.21}$$

and

$$0 = r_{n-2}(\zeta_{n-2}) P_{n-2}(\zeta_{n-2}) = e_n P_{n-1}(\zeta_{n-2}) + f_n P_{n-2}(\zeta_{n-2}) \tag{4.22}$$

(recall that $r_{n-1}(t) = 1$, $r_{n-2}(t) = \zeta_{n-2} - t$).
From this follows that

$$J_n = f_n = \frac{\mu_{n-2}^{(n-2)} P_{n-1}(\zeta_{n-2})}{\mu_{n-1}^{(n-1)} P_{n-2}(\zeta_{n-2})}. \tag{4.23}$$

It follows from (4.4), (4.8), (4.23) that

$$W_n = -\frac{U_n I_n}{J_n} = -U_n/U_{n-1} \tag{4.24}$$

provided $\mu_{n-2}^{(n-2)} \neq 0$.

(ii) $\zeta_{n-2} = \infty$.
We immediately see from (4.9) that $g_n = 0$. In (4.10) $H_{n-3} \in \Pi_{n-3}$, and hence $f_n = 1$. Thus

$$J_n = f_n = 1 \tag{4.25}$$

and consequently

$$W_n = \frac{U_n I_n}{J_n} = -\frac{U_n}{U_{n-1}}. \tag{4.26}$$

Altogether we have found that $W_n = -U_n/U_{n-1}$ for $n = 2, 3, \ldots,$ provided $P_{n-2}(\zeta_{n-1}) \neq 0$ when $\zeta_{n-1} \neq \infty$, or that $\mu_{n-2}^{(n-2)} \neq 0$ when $\zeta_{n-1} = \infty$. □

5. Christoffel–Darboux formula

In this section we establish the Christoffel–Darboux formula.

Theorem 5.1. *Assuming in addition to regularity of all $\varphi_n(z)$ that $P_n(z)$ for all n satisfy the following condition* (*recall* (2.2)):

$$\begin{aligned} &P_{n-2}(\zeta_{n-1}) \neq 0 \quad if\ \zeta_{n-1} \neq \infty, \\ &\mu_{n-2}^{(n-2)} \neq 0 \quad if\ \zeta_{n-1} = \infty, \end{aligned} \tag{5.1}$$

and using our previously introduced notation, we have the following Christoffel–Darboux formula (*recall* (1.1))

$$r_n(x)\varphi_n(x)r_{n-1}(y)\varphi_{n-1}(y) - r_n(y)\varphi_n(y)r_{n-1}(x)\varphi_{n-1}(x) = U_n(x-y)\sum_{k=0}^{n-1}\varphi_k(x)\varphi_k(y). \tag{5.2}$$

The confluent form when $x = y$ is

$$r_n(x)\varphi_n(x)[r_{n-1}(x)\varphi_{n-1}(x)]' - [r_n(x)\varphi_n(x)]' r_{n-1}(x)\varphi_{n-1}(x) = -U_n\sum_{k=0}^{n-1}\varphi_k(x)^2. \tag{5.3}$$

Proof. We recall from (4.1) that in this situation we have $W_n = -U_n/U_{n-1} < 0$ for $n = 2, 3, \ldots$.

The recurrence relation (3.3) may be written as

$$r_n(x)\varphi_n(x) = U_n x^{\gamma_{n-1}}\varphi_{n-1}(x) + V_n r_{n-1}(x)\varphi_{n-1}(x) + W_n r_{n-2}(x)\varphi_{n-2}(x). \tag{5.4}$$

Similarly we have

$$r_n(y)\varphi_n(y) = U_n y^{\gamma_{n-1}}\varphi_{n-1}(y) + V_n r_{n-1}(y)\varphi_{n-1}(y) + W_n r_{n-2}(y)\varphi_{n-2}(y). \tag{5.5}$$

Multiplying (5.4) by $r_{n-1}(y)\varphi_{n-1}(y)$ and (5.5) by $r_{n-1}(x)\varphi_{n-1}(x)$ and subtracting, we get

$$\Delta_n = -W_n\Delta_{n-1} + U_n[x^{\gamma_{n-1}}r_{n-1}(y) - y^{\gamma_{n-1}}r_{n-1}(x)]\varphi_{n-1}(x)\varphi_{n-1}(y), \tag{5.6}$$

where

$$\Delta_n = r_n(x)\varphi_n(x)r_{n-1}(y)\varphi_{n-1}(y) - r_n(y)\varphi_n(y)r_{n-1}(x)\varphi_{n-1}(x). \tag{5.7}$$

For $\zeta_{n-1} \neq \infty$ we have

$$x^{\gamma_{n-1}} r_{n-1}(y) - y^{\gamma_{n-1}} r_{n-1}(x) = (\zeta_{n-1} - y) - (\zeta_n - x) = x - y,$$

and for $\zeta_{n-1} = \infty$ we have

$$x^{\gamma_{n-1}} r_{n-1}(y) - y^{\gamma_{n-1}} r_{n-1}(x) = x - y.$$

Thus we have obtained

$$\Delta_n = U_n (x-y) \varphi_{n-1}(x) \varphi_{n-1}(y) - W_n \Delta_{n-1}. \tag{5.8}$$

Taking into account (4.1) we may write

$$\frac{\Delta_k}{U_k} - \frac{\Delta_{k-1}}{U_{k-1}} = (x-y) \varphi_{k-1}(x) \varphi_{k-1}(y).$$

Summation yields

$$\frac{\Delta_n}{U_n} - \frac{\Delta_1}{U_1} = (x-y) \sum_{k=2}^{n} \varphi_{k-1}(x) \varphi_{k-1}(y) = (x-y) \sum_{k=1}^{n-1} \varphi_k(x) \varphi_k(y). \tag{5.9}$$

Also note that $\Delta_1 = r_1(x)\varphi_1(x) - r_1(y)\varphi_1(y)$ (recall (3.4)) and $r_1(t)\varphi_1(t) = U_1 t + V_1$.

Thus $\Delta_1/U_1 = (x-y) = (x-y)\varphi_0(x)\varphi_0(y)$, so that we have obtained the Christoffel–Darboux formula (5.2).

Differentiating with respect to y and setting $y = x$ further yields the confluent formula (5.3). □

This implies an interlacing property of the zeros of the orthogonal functions.

Corollary 5.2. *Two consecutive orthogonal functions can have no common zeros. Moreover the zeros of φ_n are simple. In other words if $\varphi_n(x) = 0$, then $\varphi_{n+1}(x)\varphi_{n-1}(x)\varphi_n'(x) \neq 0$.*

Proof. This follows immediately from the confluent Christoffel–Darboux formula (5.3). If x is a zero of φ_n, hence of $f_n = r_n \varphi_n$, then it follows from (5.3) that $f_{n-1}(x) f_n'(x) \neq 0$. Thus $\varphi_{n-1}(x) \neq 0$ and $\varphi_n'(x) \neq 0$. □

6. Quadrature formulas

Here we assume that we have the same conditions as in Section 5, i.e., we have regularity of the $\{\varphi_n\}$ and condition (5.1) holds. Moreover, we assume that φ_n has n simple distinct zeros in the interior of the smallest interval containing supp μ.

To estimate the integral

$$I_\mu(f) = \int f(x)\, \mathrm{d}\mu(x),$$

we choose a quadrature formula of the form

$$I_n(f) = \sum_{k=1}^{n} \lambda_k f(x_k).$$

The $\{\lambda_k\}_{k=1}^n$ are called the coefficients or weights and $\{x_k\}_{k=1}^n$ are called the nodes of the quadrature formula. These weights and nodes could be chosen so as to have exactness in a space of the form $\mathscr{L}_n \cdot \mathscr{L}_r = \{f = g \cdot h : g \in \mathscr{L}_n, g \in \mathscr{L}_r\}$ with $r \geqslant 0$. This means that we want the nodes to be such that $I_\mu(f) = I_n(f)$ for all f in this space. Clearly, when we choose n nodes $\{x_k\}_{k=1}^n$ in $\operatorname{supp}(\mu)$, then the weights can be chosen such that the quadrature formula is exact for all $f \in \mathscr{L}_{n-1}$ (note that $\dim \mathscr{L}_{n-1} = n$). This is an *interpolatory quadrature formula* since it is obtained by integrating the unique function $R \in \mathscr{L}_{n-1}$ that interpolates in the points $\{(x_k, f(x_k))\}_{k=1}^n$.

We shall now choose the nodes to get exactness in a larger space $\mathscr{L}_n \cdot \mathscr{L}_r$ of dimension $n + r + 1 > n$ where we shall make r as large as possible. It is well known that in the polynomial case, we can obtain exactness by the Gauss formulas, i.e., formulas where the n nodes are chosen as the zeros of the nth orthogonal polynomial. Then it is possible to choose the weights such that the quadrature formula is exact in $\Pi_{2n-1} = \Pi_n \cdot \Pi_{n-1}$.

Note that in general $\mathscr{L}_n \cdot \mathscr{L}_{n-1} \neq \mathscr{L}_{2n-1}$, except under certain restrictive conditions. However, this is an indication that the maximal r that can be attained allowing exactness in $\mathscr{L}_n \cdot \mathscr{L}_r$ is $n - 1$, which is indeed the case as shown next.

We have the following result.

Theorem 6.1. *Let $I_n(f) = \sum_{k=1}^n \lambda_k f(x_k)$ be a quadrature formula with nodes $\{x_k\}_{k=1}^n \subset \operatorname{supp}(\mu)\backslash\{\zeta_j\}_{j=1}^\infty$. Then $I_n(f)$ is exact in $\mathscr{L}_n \cdot \mathscr{L}_r$ with $r \geqslant 0$ if and only if*

(1) *$I_n(f)$ is exact in $\mathscr{L}_{n-1}$.*
(2) *$\langle R_n, g\rangle = 0, \forall g \in \mathscr{L}_r$ where $R_n(x) = \prod_{j=1}^n (x - x_j)/D_n(x) \in \mathscr{L}_n$.*

Proof. If the quadrature formula is exact in $\mathscr{L}_n \cdot \mathscr{L}_r$, then it is obviously exact in $\mathscr{L}_{n-1}$. It also implies that $\langle R_n, \varphi_k\rangle = \sum_{j=1}^n \lambda_j R_n(x_j)\varphi_k(x_j) = 0$ for $k = 0, 1, \ldots, r$ which means that $R_n \perp \mathscr{L}_r$.

For the opposite implication, we have to prove that the quadrature formula is exact for all $f \in \mathscr{L}_n \cdot \mathscr{L}_r$. Suppose $\ell_j \in \mathscr{L}_{n-1}$ is defined by

$$\ell_j(x_k) = \delta_{jk}, \quad 1 \leqslant j, k \leqslant n.$$

Then the interpolating function from $\mathscr{L}_{n-1}$ for the nodes $\{x_j\}_{j=1}^n$ is given by $f_n(x) = \sum_{j=1}^n \ell_j(x) f(x_j)$. Thus, the interpolation error is $e_n = f - f_n$. Obviously $e_n \in \mathscr{L}_n \cdot \mathscr{L}_r$. Because it vanishes in all points $\{x_j\}_{j=1}^n$, it should have the form

$$e_n(x) = \frac{P(x)}{D_n(x) D_r(x)}$$

with $P(x) \in \Pi_{n+r}$ a polynomial vanishing in all $\{x_j\}_{j=1}^n$. Thus we may write

$$e_n(x) = R_n(x) f_r(x), \quad f_r \in \mathscr{L}_r.$$

Because $\int e_n(x)\,\mathrm{d}\mu(x) = \langle R_n, f_r\rangle = 0$ by (2), the integration error will be zero and hence the quadrature formula is exact in $\mathscr{L}_n \cdot \mathscr{L}_r$. $\square$

Note that exactness in $\mathscr{L}_n \cdot \mathscr{L}_n$ is impossible, because this would imply that $\langle \varphi_n, \varphi_n \rangle = \sum_{k=1}^{n} \lambda_k \varphi_n^2(x_k) = 0$. Thus, the maximal space of exactness one might hope for is $\mathscr{L}_n \cdot \mathscr{L}_{n-1}$. This can indeed be reached as is shown by the following result.

Corollary 6.2. *Let $x_1, \ldots, x_n$ be the n distinct zeros of the nth orthogonal rational function φ_n. Then there exist positive weights $\lambda_1, \ldots, \lambda_n$ such that*

$$I_n(f) = \sum_{k=1}^{n} A_k f(x_k) = I_\mu(f), \quad \forall f \in \mathscr{L}_n \cdot \mathscr{L}_{n-1}.$$

Proof. If the x_j are the zeros of φ_n, then condition (2) of Theorem 6.1 is satisfied for $r = n - 1$. Choosing the weights to make the quadrature formula of interpolatory type satisfies condition (1) of that theorem. Thus we have exactness in $\mathscr{L}_n \cdot \mathscr{L}_{n-1}$.

It only remains to show that all $\lambda_k > 0$. We therefore set $\ell_j \in \mathscr{L}_{n-1}$ such that

$$\ell_j(x_k) = \delta_{j,k}, \quad 1 \leqslant j, k \leqslant n.$$

Clearly $\ell_j^2 \in \mathscr{L}_{n-1} \cdot \mathscr{L}_{n-1} \subset \mathscr{L}_n \cdot \mathscr{L}_{n-1}$. Hence, since the quadrature formula is exact in the latter space, we get

$$0 < I_\mu(\ell_j^2) = \sum_{k=1}^{n} \lambda_k \ell_j^2(x_k) = \lambda_j, \quad j = 1, \ldots, n.$$

This proves the positivity of the weights. □

We shall refer to an exact formula in $\mathscr{L}_n \cdot \mathscr{L}_{n-1}$ as a *rational Gauss formula* for the measure μ. Note that the formula also depends on both the poles $\{\alpha_k\}$, $\{\beta_k\}$, and on the nesting of the spaces, i.e., on the sequence $\{p(n)\}_{k=1}^{\infty}$.

Theorem 6.3. *The weights of the rational Gauss formulas are given by* (*recall* (1.1))

$$\lambda_j = \frac{U_n}{r_{n-1}(x_j) r_n(x_j) \varphi_n'(x_j) \varphi_{n-1}(x_j)} = -\frac{U_{n+1}}{r_{n+1}(x_j) r_n(x_j) \varphi_{n+1}(x_j) \varphi_n'(x_j)} = \frac{1}{\sum_{k=0}^{n-1} \varphi_k^2(x_j)}.$$

Proof. We recall from [10, Theorem 3.4] that these weights are given by

$$\lambda_j = \frac{1}{\varphi_n'(x_j)} \int \frac{\varphi_n(x)}{x - x_j} \, d\mu(x). \tag{6.1}$$

Using the Christoffel–Darboux formula, we can derive the alternative expressions given above. Indeed, from (5.2) with $y = x_j$ we get

$$\varphi_{n-1}(x_j) r_n(x) \varphi_n(x) r_{n-1}(x_j) = U_n(x - x_j) \sum_{k=0}^{n-1} \varphi_k(x) \varphi_k(x_j).$$

Hence, because of orthonormality

$$\frac{r_{n-1}(x_j)\varphi_{n-1}(x_j)}{U_n}\int\frac{r_n(x)\varphi_n(x)}{x-x_j}\,\mathrm{d}\mu(x)=1.$$

From the definition of r_n and orthogonality, it follows that

$$\int\frac{r_n(x)\varphi_n(x)}{x-x_j}\,\mathrm{d}\mu(x)=r_n(x_j)\int\frac{\varphi_n(x)}{x-x_j}\,\mathrm{d}\mu(x).$$

Thus

$$\int\frac{\varphi_n(x)}{x-x_j}\,\mathrm{d}\mu(x)=\frac{U_n}{r_{n-1}(x_j)r_n(x_j)\varphi_{n-1}(x_j)},$$

so that, using (6.1)

$$\lambda_j=\frac{U_n}{r_{n-1}(x_j)r_n(x_j)\varphi_n'(x_j)\varphi_{n-1}(x_j)}. \tag{6.2}$$

Next, we use the Christoffel–Darboux relation (5.2) with n replaced by $n+1$ and get as before, setting $y=x_j$ that

$$-r_{n+1}(x_j)\varphi_{n+1}(x_j)r_n(x)\varphi_n(x)=U_{n+1}(x-x_j)\sum_{k=0}^{n}\varphi_k(x)\varphi_k(x_j),$$

yielding

$$-\frac{r_{n+1}(x_j)\varphi_{n+1}(x_j)}{U_{n+1}}\int\frac{r_n(x)\varphi_n(x)}{x-x_j}\,\mathrm{d}\mu(x)=1.$$

Thus we now obtain from (6.1)

$$\lambda_j=-\frac{U_{n+1}}{r_{n+1}(x_j)r_n(x_j)\varphi_{n+1}(x_j)\varphi_n'(x_j)}. \tag{6.3}$$

By our assumptions on the sequence $\{\varphi_n\}_{n=0}^{\infty}$ and Corollary 5.2 we get

$$\varphi_{n+1}(x_j)\varphi_{n-1}(x_j)\varphi_n'(x_j)\neq 0,\quad 1\leqslant j\leqslant n,\quad n\geqslant 1.$$

So both formulas (6.2) and (6.3) are meaningful.

Finally, we use the confluent form of the Christoffel–Darboux formula (5.3) for $x=x_j$ and we get

$$[r_n(x)\varphi_n(x)]'_{x=x_j}r_{n-1}(x_j)\varphi_{n-1}(x_j)=U_n\sum_{k=0}^{n-1}\varphi_k^2(x_j)$$

or equivalently

$$r_n(x_j)\varphi_n'(x_j)r_{n-1}(x_j)\varphi_{n-1}(x_j)=U_n\sum_{k=0}^{n-1}\varphi_k^2(x_j).$$

Thus, by (6.2) the third expression for the weights is obtained. □

7. Tridiagonal matrix and eigenvalue problem

In the polynomial case, the nodes and weights of the Gauss quadrature formula can be obtained via the solution of an eigenvalue problem for the tridiagonal Jacobi matrix. A generalization can be obtained in the case of rational Gauss formulas. We therefore write the recurrence relation, assuming regularity, as

$$z^{\gamma_n}\varphi_n(z) = d_{n-1}r_{n-1}(z)\varphi_{n-1}(z) + c_n r_n(z)\varphi_n(z) + d_n r_{n+1}(z)\varphi_{n+1}(z),$$

where $c_n = -V_{n+1}/U_{n+1}$ and $d_n = 1/U_{n+1}$. Writing this out for $n=0, 1, \ldots,$ using the following notation:

$$J = \operatorname{tridiag}\begin{pmatrix} d_0, d_1, \ldots \\ c_0, c_1, \ldots \\ d_0, d_1, \ldots \end{pmatrix}, \qquad \begin{array}{l} R = \operatorname{diag}(r_0(z), r_1(z), \ldots), \\ L = \operatorname{diag}(z^{\gamma_0}, z^{\gamma_1}, \ldots), \end{array} \qquad \Phi = \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \end{bmatrix},$$

we get $(JR - L)\Phi = 0$. Truncating this to the first $n+1$ rows and columns, we get the finite analog

$$(J_n R_n - L_n)\Phi_n = -d_n r_{n+1}(z)\varphi_{n+1}(z)\mathbf{e}_n,$$

with $\mathbf{e}_n = [0, \ldots, 0, 1]^{\mathrm{T}}$ the $(n+1)$th unit vector. If in this relation, we replace z by a zero λ of φ_{n+1}, we find that the right-hand side vanishes and hence the zeros are found as the solutions for λ of

$$[J_n R_n(\lambda) - L_n(\lambda)]\Phi_n(\lambda) = 0.$$

Since this is not in the familiar form of a (generalized) eigenvalue problem, we will make the equation more explicit. To this end define $I_n^{\gamma} = \operatorname{diag}(\gamma_0, \gamma_1, \ldots, \gamma_n)$. It is a matrix indicating where the infinite poles are. The "complementary" matrix $I_n^{1-\gamma} = I_n - I_n^{\gamma}$ indicates where the finite poles are. To represent R_n we introduce the matrix $Z_n = \operatorname{diag}(\tau_0, \tau_1, \ldots, \tau_n)$ with $\tau_k = \zeta_k$ if ζ_k is finite and $\tau_k = 1$ otherwise. Then the previous relation becomes

$$\{[J_n Z_n - I_n^{1-\gamma}] - \lambda[J_n I_n^{1-\gamma} + I_n^{\gamma}]\}\Phi_n = 0.$$

This is the generalized eigenvalue problem to solve. Each eigenvalue is a node of the quadrature formula and if $E_n = [e_{0,n}, \ldots, e_{n,n}]^{\mathrm{T}}$ is the corresponding normalized eigenvector ($E_n^{\mathrm{T}} E_n = 1$), then the weight for that node is given by $e_{0,n}^2/c_{0,0}$ just like in the polynomial case.

8. Stieltjes situation

We now assume the following:

$$\operatorname{supp}\mu \subset [0, \infty) \quad \text{and} \quad \zeta_n \in [-\infty, 0] \quad \text{for all } n. \tag{8.1}$$

We shall call this the *Stieltjes situation*.

Theorem 8.1. *In the Stieltjes situation, the numerator P_n of the nth orthogonal function φ_n has n simple zeros in $(0, \infty)$.*

Proof. Let $t_1, \ldots, t_\lambda$ denote the zeros of $P_n(t)$ of odd order in $(0, \infty)$, each counted only once. We prove that $\lambda < n$ will lead to a contradiction. If $\lambda < n$, then the function (recall (1.2))

$$F(t) = \frac{(t - t_1) \ldots (t - t_\lambda)}{D_{n-1}(t)}$$

belongs to $\mathscr{L}_{n-1}$, hence $\int F(t)\varphi_n(t)\,\mathrm{d}\mu(t) = 0$. On the other hand

$$F(t)\varphi_n(t) = \frac{(t - t_1) \ldots (t - t_\lambda) P_n(t)}{[D_{n-1}(t)]^2 r_n(t)}.$$

Here $(t - t_1) \ldots (t - t_\lambda) P_n(t)$ has constant sign in $(-\infty, \infty)$, while $r_n(t)$ has constant sign in $(0, \infty)$.

It follows that $\int F(t)\varphi_n(t)\,\mathrm{d}\mu(t) \neq 0$, which contradicts the previous conclusion that $\int F(t)\varphi_n(t)\,\mathrm{d}\mu(t) = 0$. □

Note that *we did not use regularity or condition* (5.1). However, it follows from the obtained properties of the zeros that all $P_n(t)$ have constant sign in $(-\infty, 0]$ and that the degree of $P_n(z)$ is exactly n. In particular $P_n(\zeta_k) \neq 0$ for all n and k. Consequently *all $\varphi_n(z)$ are automatically regular and satisfy condition* (5.1). Moreover we have an interlacing property of the zeros of two consecutive orthogonal functions.

Corollary 8.2. *Suppose that we are in the Stieltjes case. Then φ_n has only simple zeros in* $(0, \infty)$ *and the zeros of φ_n and φ_{n-1} interlace.*

Proof. That the zeros are simple and in $(0, \infty)$ (in fact they are in the interior of the convex hull of $\mathrm{supp}(\mu)$) is a classical result since the numerator polynomials P_n form an orthogonal polynomial sequence with respect to a varying measure. Since all the ζ_k are in $[-\infty, 0]$, none of the zeros of P_n can coincide with any of the ζ_k.

Set $f_n = r_n\varphi_n$. Since $r_n(t)$ has no zeros in $(0, \infty)$, the result will follow from the confluent Christoffel–Darboux formula (5.3) which states that $f_n(x)f'_{n-1}(x) - f_{n-1}(x)f'_n(x) < 0$ is true for all $x > 0$. Indeed, suppose x_j and x_{j+1} are two consecutive zeros of P_n, hence of f_n. Then $f'_n(x_j)f'_n(x_{j+1}) < 0$ since the zeros are simple. On the other hand, the confluent Christoffel–Darboux formula then implies that $f_{n-1}(x_j)f'_n(x_j) > 0$ and $f_{n-1}(x_{j+1})f'_n(x_{j+1}) > 0$, from which we conclude that $f_{n-1}(x_j)f_{n-1}(x_{j+1}) < 0$. In other words f_{n-1} or equivalently φ_{n-1} will have at least one zero between x_j and x_{j+1} by Rolle's theorem. Because this holds for every pair of consecutive zeros of φ_n, there can be at most (hence exactly one) zero of φ_{n-1} between two consecutive zeros of φ_n. □

9. Balanced Stieltjes situation

We shall now consider a special case of the Stieltjes situation, *the balanced Stieltjes situation*:

$$p(2m+1) > p(2m), \quad q(2m) > q(2m-1). \tag{9.1}$$

This means that

$$\zeta_{2m} = \beta_m, \quad \zeta_{2m+1} = \alpha_{m+1}. \tag{9.2}$$

We further assume that

$$-\infty \leqslant \beta_j < \alpha_k \leqslant 0 \quad \text{for all } j, k. \tag{9.3}$$

In this case, $\zeta_{n-1} \neq \zeta_{n-2}$ and at most one of them can be infinite. Therefore $\{r_{n-2}, r_{n-1}\}$ forms a basis for Π_1 the polynomials of degree at most 1. So we may then write the recurrence relation in the form

$$\varphi_n(z) = \frac{Q_n r_{n-2}(z) + R_n r_{n-1}(z)}{r_n(z)} \varphi_{n-1}(z) + W_n \frac{r_{n-2}(z)}{r_n(z)} \varphi_{n-2}(z). \tag{9.4}$$

For these Q_n and R_n we have the following properties.

Theorem 9.1. *In the balanced Stieltjes situation, assume that the recurrence relation takes the form* (9.4). *Then we have*

When $\zeta_{2m-2} = \beta_{m-1} \neq \infty$:

$$Q_{2m} < 0, \quad R_{2m} > 0, \tag{9.5}$$

$$Q_{2m-1} > 0, \quad R_{2m-1} < 0. \tag{9.6}$$

When $\zeta_{2m-2} = \beta_{m-1} = \infty$:

$$Q_{2m} > 0, \quad R_{2m} > 0, \tag{9.7}$$

$$Q_{2m-1} < 0, \quad R_{2m-1} < 0. \tag{9.8}$$

Proof. Multiplying by $D_n(z)$ and writing out for n even and odd we get

$$\begin{aligned} P_{2m}(z) = {} & [Q_{2m} r_{2m-2}(z) + R_{2m}(\alpha_m - z)] P_{2m-1}(z) \\ & + W_{2m} r_{2m-2}(z)(\alpha_m - z) P_{2m-2}(z), \end{aligned} \tag{9.9}$$

$$\begin{aligned} P_{2m+1}(z) = {} & [Q_{2m+1}(\alpha_m - z) + R_{2m+1} r_{2m}(z)] P_{2m}(z) \\ & + W_{2m+1} r_{2m}(z)(\alpha_m - z) P_{2m-1}(z), \end{aligned} \tag{9.10}$$

$$\begin{aligned} P_{2m-1}(z) = {} & [Q_{2m-1}(\alpha_{m-1} - z) + R_{2m-1} r_{2m-2}(z)] P_{2m-2}(z) \\ & + W_{2m-1} r_{2m-2}(z)(\alpha_{m-1} - z) P_{2m-3}(z). \end{aligned} \tag{9.11}$$

By substituting appropriate values we find when $\zeta_{2m-2} = \beta_{m-1} \neq \infty$:

$$Q_{2m} = \frac{P_{2m}(\alpha_m)}{(\beta_{m-1} - \alpha_m) P_{2m-1}(\alpha_m)}, \tag{9.12}$$

$$R_{2m} = \frac{P_{2m}(\beta_{m-1})}{(\alpha_m - \beta_{m-1}) P_{2m-1}(\beta_{m-1})}, \tag{9.13}$$

$$Q_{2m-1} = \frac{P_{2m-1}(\beta_{m-1})}{(\alpha_{m-1} - \beta_{m-1}) P_{2m-2}(\beta_{m-1})}, \tag{9.14}$$

$$R_{2m-1} = \frac{P_{2m-1}(\alpha_{m-1})}{(\beta_{m-1} - \alpha_{m-1}) P_{2m-2}(\alpha_{m-1})}. \tag{9.15}$$

Similarly when $\zeta_{2m-2} = \beta_{m-1} = \infty$ we get

$$Q_{2m} = \frac{P_{2m}(\alpha_m)}{P_{2m-1}(\alpha_m)}, \tag{9.16}$$

$$R_{2m} = -\frac{\mu_{2m}^{(2m)}}{\mu_{2m-1}^{(2m-1)}}, \tag{9.17}$$

$$Q_{2m-1} = -\frac{\mu_{2m-1}^{(2m-1)}}{\mu_{2m-2}^{(2m-2)}}, \tag{9.18}$$

$$R_{2m-1} = \frac{P_{2m-1}(\alpha_{m-1})}{P_{2m-2}(\alpha_{m-1})}. \tag{9.19}$$

We know that $P_n(t)$ has constant sign in $(-\infty, 0)$, and $P_n(t)$ and $P_{n-1}(t)$ has opposite sign in $(-\infty, 0)$ if and only if $\operatorname{sgn} \mu_n^{(n)} = \operatorname{sgn} \mu_{n-1}^{(n-1)}$. Taking into account the sign normalization (3.2) we find the sign of Q_n and R_n as indicated in the theorem. $\square$

10. Associated functions

We make the same assumptions as in Section 8, i.e., we assume the Stieltjes situation. We define the *associated functions* σ_n by

$$\sigma_n(z) = \int_0^\infty \frac{\varphi_n(t) - \varphi_n(z)}{t - z}\, \mathrm{d}\mu(t), \quad n = 0, 1, 2, \ldots \tag{10.1}$$

Note that (6.1) and (10.1) imply that the weights of the rational Gauss quadrature can also be written as $\lambda_j = \sigma_n(x_j)/\varphi_n'(x_j)$.

Recall that until now, the coefficient W_1 was arbitrary. We now fix it to be $W_1 = -U_1$ to get the following.

Theorem 10.1. *Suppose we are in the Stieltjes situation and define $W_1 = -U_1$. Then the associated functions satisfy* (*recall* (1.1))

$$\sigma_n(z) = \frac{U_n z^{\gamma_{n-1}} + V_n r_{n-1}(z)}{r_n(z)}\, \sigma_{n-1}(z) + W_n \frac{r_{n-2}(z)}{r_n(z)}\, \sigma_{n-2}(z), \quad n = 1, 2, 3, \ldots \tag{10.2}$$

with initial conditions

$$r_0 = 1,\ r_{-1} = 1; \quad \gamma_{-1} = 1,\ \gamma_0 = 1; \quad \sigma_{-1} = -1,\ \sigma_0 = 0. \tag{10.3}$$

Proof. Clearly $\sigma_0 = 0$.

If $\zeta_1 \neq \infty$, then $\varphi_1(z) = \frac{U_1 z + V_1}{\zeta_1 - z}$ by (3.3) and we easily find

$$\sigma_1(z) = \frac{\zeta_1 U_1 + V_1}{\zeta_1 - z} c_{0,1} \quad \text{where } c_{0,1} = \int_0^\infty \frac{\mathrm{d}\mu(t)}{\zeta_1 - t}. \tag{10.4}$$

If $\zeta_1 = \infty$, then $\varphi_1(z) = U_1 z + V_1$, so that $\sigma_1 = U_1$. Moreover $c_{0,1} = 1$.

A standard argument (consisting of plugging the recurrence relation for the φ_n into the definition of the σ_n) now shows that $\{\sigma_n\}$ satisfies the recurrence

$$\sigma_n(z) = \frac{U_n z^{\gamma_{n-1}} + V_n r_{n-1}(z)}{r_n(z)} \sigma_{n-1}(z) + W_n \frac{r_{n-2}(z)}{r_n(z)} \sigma_{n-2}(z), \quad n = 2, 3, \ldots \tag{10.5}$$

with initial conditions

$$r_0 = 1, \; r_{-1} = 1; \quad \gamma_0 = 1; \quad \sigma_0 = 0 \tag{10.6}$$

and σ_1 as described above.

Note that since $\int \varphi_1(t)\,\mathrm{d}\mu(t) = 0$, we get for $\zeta_1 \neq \infty$

$$0 = \int_0^\infty \frac{U_1 t + V_1}{\zeta_1 - t}\,\mathrm{d}\mu(t) = -U_1 + (U_1\zeta_1 + V_1)c_{0,1}.$$

If we want the recurrence for σ_n to hold for $n = 1$, we should define σ_{-1}. Still assuming $\zeta_1 \neq \infty$, it should follow from (10.4) and (10.5)–(10.6) that

$$\sigma_{-1} W_1 = c_{0,1}(\zeta_1 U_1 + V_1) = U_1. \tag{10.7}$$

If $\zeta_1 = \infty$, this relation becomes

$$\sigma_{-1} W_1 = \sigma_1 = U_1. \tag{10.8}$$

Since W_1 has been an arbitrary constant so far, we can now use it to fix

$$\sigma_{-1} = -1, \tag{10.9}$$

from which follows in both cases (ζ_1 infinite or not) that $W_1 = -U_1$. Thus by choosing $W_1 = -U_1$, we can take $\sigma_{-1} = -1$ as initial condition and the recurrence (10.2) is valid for $n = 1, 2, \ldots$.

Note that, because $U_1 > 0$,

$$W_1 < 0 \tag{10.10}$$

so that it is line with all the other W_k being negative. Also relation (4.1) holds for $n = 1, 2, \ldots$ if we set $U_0 = 1$.

If we assume that the conditions of Section 9 are satisfied, i.e., we *consider the balanced Stieltjes case*, it then follows immediately that we can rewrite the recurrence relation in terms of the coefficients $\{Q_n\}$ and $\{R_n\}$. We give the result without further proof. □

Theorem 10.2. *Suppose we are in the balanced Stieltjes situation. Then the associated functions* $\{\sigma_n\}$ *satisfy the recurrence* (*recall* (1.1))

$$\sigma_n(z) = \frac{Q_n r_{n-2}(z) + R_n r_{n-1}(z)}{r_n(z)} \sigma_{n-1}(z) + W_n \frac{r_{n-2}(z)}{r_n(z)} \sigma_{n-2}(z), \quad n = 1, 2, \ldots \tag{10.11}$$

$$\sigma_0 = 0, \quad \sigma_{-1} = -1. \tag{10.12}$$

If

$$a_n(z) = W_n \frac{r_{n-2}(z)}{r_n(z)}, \quad n = 1, 2, \ldots \tag{10.13}$$

$$b_n(z) = \frac{Q_n r_{n-2}(z) + R_n r_{n-1}(z)}{r_n(z)}, \quad n = 1, 2, \ldots \tag{10.14}$$

with

$$r_0 = 1, \quad r_{-1} = 1, \tag{10.15}$$

then $\{\sigma_n\}$ *and* $\{\varphi_n\}$ *satisfy*

$$\begin{bmatrix} \sigma_n(z) & \sigma_{n-1}(z) \\ \varphi_n(z) & \varphi_{n-1}(z) \end{bmatrix} = \begin{bmatrix} \sigma_{n-1}(z) & \sigma_{n-2}(z) \\ \varphi_{n-1}(z) & \varphi_{n-2}(z) \end{bmatrix} \begin{bmatrix} b_n(z) & 1 \\ a_n(z) & 0 \end{bmatrix}, \quad n = 1, 2, \ldots \tag{10.16}$$

$$\begin{bmatrix} \sigma_0(z) & \sigma_{-1}(z) \\ \varphi_0(z) & \varphi_{-1}(z) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{10.17}$$

The approximants of the continued fraction defined by (10.16)–(10.17) *are given by the quotients* $\sigma_n(z)/\varphi_n(z)$.

11. The separated balanced Stieltjes situation

We shall now assume all the conditions in Section 9 satisfied, and in addition the following: There exist numbers α and β such that

$$\beta_j \leqslant \beta < \alpha \leqslant \alpha_k \quad \text{for all } j, k. \tag{11.1}$$

We call this the *separated balanced Stieltjes situation*.

We shall consider the behavior of the approximants $\sigma_n(z)/\varphi_n(z)$ of the continued fraction (10.16)–(10.17). We shall in particular consider $z \in (\beta, \alpha)$.

For convenience we introduce the expression

$$\delta_n(t) = Q_n r_{n-2}(t) + R_n r_{n-1}(t). \tag{11.2}$$

We then have

$$b_n(z) = \frac{\delta_n(z)}{r_n(z)}. \tag{11.3}$$

We recall that $\zeta_{2m+1} \neq \infty$ for all m (see (9.2)) and consequently $r_{2m+1}(t) = \alpha_{m+1} - t$; hence $r_{2m+1}(t) > 0$ for $t \in (\beta, \alpha)$. For $\zeta_{2m} \neq \infty$, $r_{2m}(t) < 0$ for $t \in (\beta, \alpha)$, while for $\zeta_{2m} = \infty$, we have $r_{2m}(t) = 1$, thus $r_{2m}(t) > 0$.

Combining these results with (9.5)–(9.6) we obtain

Theorem 11.1. *Suppose we are in the separated balanced Stieltjes situation and suppose the* $b_n(z)$, *defined in* (10.14), *are denoted as in* (11.3). *Then we have*

- *For* $\zeta_{2m-2} = \beta_{m-1} \neq \infty$

$$\delta_{2m}(t) > 0 \quad for \ t \in (\beta, \alpha), \tag{11.4}$$

$$\delta_{2m-1}(t) > 0 \quad for \ t \in (\beta, \alpha). \tag{11.5}$$

- *For* $\zeta_{2m-2} = \beta_{m-1} = \infty$

$$\delta_{2m}(t) > 0 \quad for \ t \in (\beta, \alpha), \tag{11.6}$$

$$\delta_{2m-1}(t) < 0 \quad for \ t \in (\beta, \alpha). \tag{11.7}$$

12. Monotonicity of even and odd approximants

We make the same assumptions in this section as in Section 11, i.e., we consider the separated balanced Stieltjes situation.

Theorem 12.1. *Suppose we are in the separated balanced Stieltjes situation. Then the sequence* $\{\sigma_{2m}(t)/\varphi_{2m}(t)\}$ *is decreasing on* (β, α), *while the sequence* $\{\sigma_{2m+1}(t)/\varphi_{2m+1}(t)\}$ *is increasing on* (β, α).

Proof. By taking determinants in (10.16)–(10.17) we get some standard formulas for approximants of continued fractions [18], namely

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-1}(z)}{\varphi_{n-1}(z)} = (-1)^n \frac{a_1(z)\cdots a_n(z)}{\varphi_n(z)\varphi_{n-1}(z)} \tag{12.1}$$

and further standard methods yield

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)} = \frac{\varphi_{n-1}(z)}{\varphi_n(z)} b_n(z) \left[\frac{\sigma_{n-1}(z)}{\varphi_{n-1}(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)} \right]. \tag{12.2}$$

Here a_n and b_n are defined in (10.13) and (10.14), respectively. Combining these formulas and taking into account (10.13)–(10.17) we obtain

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)} = \frac{(-1)^{n+1} b_n(z) W_1 \cdots W_{n-1}}{\varphi_n(z)\varphi_{n-2}(z) r_{n-1}(z) r_{n-2}(z)} \tag{12.3}$$

and thus

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)} = \frac{(-1)^{n+1} \delta_n(z) W_1 \cdots W_{n-1} [D_{n-2}(z)]^2}{r_{n-2}(z) P_n(z) P_{n-2}(z)}. \tag{12.4}$$

Written out this gives

$$\frac{\sigma_{2m}(z)}{\varphi_{2m}(z)} - \frac{\sigma_{2m-2}(z)}{\varphi_{2m-2}(z)} = \frac{-\delta_{2m}(z)W_1 \cdots W_{2m-1}[D_{2m-2}(z)]^2}{r_{2m-2}(z)P_{2m}(z)P_{2m-2}(z)}, \tag{12.5}$$

$$\frac{\sigma_{2m+1}(z)}{\varphi_{2m+1}(z)} - \frac{\sigma_{2m-1}(z)}{\varphi_{2m-1}(z)} = \frac{\delta_{2m+1}(z)W_1 \cdots W_{2m}[D_{2m-1}(z)]^2}{r_{2m-1}(z)P_{2m+1}(z)P_{2m-1}(z)}. \tag{12.6}$$

In the following let $t \in (\beta, \alpha)$.

- Let $\zeta_{2m-2} = \beta_{m-1} \neq \infty$.
  Then $r_{2m-2}(t) < 0$, and $\delta_{2m}(t) > 0$ by (11.4). Furthermore it follows from the sign normalizations (3.2) that $P_{2m}(t)$ and $P_{2m-2}(t)$ have the same sign. Since all W_k are negative, we conclude that

$$\frac{\sigma_{2m}(t)}{\varphi_{2m}(t)} - \frac{\sigma_{2m-2}(t)}{\varphi_{2m-2}(t)} < 0. \tag{12.7}$$

- Let $\zeta_{2m-2} = \beta_{m-1} = \infty$.
  Then $r_{2m-2}(t) > 0$, and $\delta_{2m}(t) > 0$ by (11.6). It now follows from the sign normalizations (3.2) that $P_{2m}(t)$ and $P_{2m-2}(t)$ have opposite sign. Again we find that

$$\frac{\sigma_{2m}(t)}{\varphi_{2m}(t)} - \frac{\sigma_{2m-2}(t)}{\varphi_{2m-2}(t)} < 0. \tag{12.8}$$

- Let $\zeta_{2m} = \beta_m \neq \infty$.
  We have $r_{2m-1}(t) > 0$, and $\delta_{2m+1}(t) > 0$ by (11.5). The sign normalizations (3.2) implies that $P_{2m+1}(t)$ and $P_{2m-1}(t)$ have the same sign. Hence

$$\frac{\sigma_{2m+1}(t)}{\varphi_{2m+1}(t)} - \frac{\sigma_{2m-1}(t)}{\varphi_{2m-1}(t)} > 0. \tag{12.9}$$

- Let $\zeta_{2m} = \beta_m = \infty$.
  Again $r_{2m-1}(t) > 0$, while $\delta_{2m+1}(t) < 0$ by (11.7). The sign normalizations (3.2) now implies that $P_{2m+1}(t)$ and $P_{2m-1}(t)$ have opposite sign. Thus we again find that

$$\frac{\sigma_{2m+1}(t)}{\varphi_{2m+1}(t)} - \frac{\sigma_{2m-1}(t)}{\varphi_{2m-1}(t)} > 0. \tag{12.10}$$

Altogether we conclude that the theorem is correct. □

A further argument shows that these even and odd sequences will converge.

Theorem 12.2. *Suppose we are in the separated balanced Stieltjes situation. Then there exist functions* F_0 *and* F_∞, *analytic outside* $[0, \infty)$, *such that for all* $z \in \mathbb{C}\backslash[0, \infty)$

$$\lim_{m\to\infty} \frac{\sigma_{2m}(z)}{\varphi_{2m}(z)} = F_\infty(z) \quad \textit{and} \quad \lim_{m\to\infty} \frac{\sigma_{2m+1}(z)}{\varphi_{2m+1}(z)} = F_0(z).$$

Proof. We return to formula (12.1) which may be written as

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-1}(z)}{\varphi_{n-1}(z)} = (-1)^n \frac{W_1 \cdots W_n [D_{n-1}(z)]^2}{r_{n-1}(z) P_n(z) P_{n-1}(z)}. \tag{12.11}$$

Thus

$$\frac{\sigma_{2m}(z)}{\varphi_{2m}(z)} - \frac{\sigma_{2m-1}(z)}{\varphi_{2m-1}(z)} = \frac{W_1 \cdots W_{2m} [D_{2m-1}(z)]^2}{(\alpha_m - z) P_{2m}(z) P_{2m-1}(z)} \tag{12.12}$$

and

$$\frac{\sigma_{2m+1}(z)}{\varphi_{2m+1}(z)} - \frac{\sigma_{2m}(z)}{\varphi_{2m}(z)} = -\frac{W_1 \cdots W_{2m+1} [D_{2m}(z)]^2}{r_{2m}(z) P_{2m+1}(z) P_{2m}(z)}. \tag{12.13}$$

Let again $t \in (\beta, \alpha)$. Since all W_k are negative and $\operatorname{sgn} P_{2m}(t) = \operatorname{sgn} P_{2m-1}(t)$ by (3.2), we conclude that

$$\frac{\sigma_{2m}(t)}{\varphi_{2m}(t)} - \frac{\sigma_{2m-1}(t)}{\varphi_{2m-1}(t)} > 0. \tag{12.14}$$

Furthermore, if $\zeta_{2m} = \beta_m \neq \infty$, then $r_{2m}(t) < 0$ and $\operatorname{sgn} P_{2m+1}(t) = \operatorname{sgn} P_{2m}(t)$, and if $\zeta_{2m} = \beta_m = \infty$, then $r_{2m}(t) > 0$ and $\operatorname{sgn} P_{2m+1}(t) = -\operatorname{sgn} P_{2m}(t)$. Hence in both cases

$$\frac{\sigma_{2m+1}(t)}{\varphi_{2m+1}(t)} - \frac{\sigma_{2m}(t)}{\varphi_{2m}(t)} < 0. \tag{12.15}$$

Let j, k be arbitrary indices and let $m > \max(j, k)$. Then by (12.7)–(12.10) and (12.14)–(12.15) we have

$$\frac{\sigma_{2j}(t)}{\varphi_{2j}(t)} > \frac{\sigma_{2m}(t)}{\varphi_{2m}(t)} > \frac{\sigma_{2m-1}(t)}{\varphi_{2m-1}(t)} > \frac{\sigma_{2k-1}(t)}{\varphi_{2k-1}(t)} \tag{12.16}$$

and

$$\frac{\sigma_{2j+1}(t)}{\varphi_{2j+1}(t)} < \frac{\sigma_{2m+1}(t)}{\varphi_{2m+1}(t)} < \frac{\sigma_{2m}(t)}{\varphi_{2m}(t)} < \frac{\sigma_{2k}(t)}{\varphi_{2k}(t)}. \tag{12.17}$$

In particular the sequence $\{\sigma_{2m}(t)/\varphi_{2m}(t)\}$ is bounded below and the sequence $\{\sigma_{2m+1}(t)/\varphi_{2m+1}(t)\}$ is bounded above. Consequently there exist functions $F_0(t)$ and $F_\infty(t)$ on (β, α) such that

$$\lim_{m\to\infty} \frac{\sigma_{2m}(t)}{\varphi_{2m}(t)} = F_\infty(t) \quad \text{and} \quad \lim_{m\to\infty} \frac{\sigma_{2m+1}(t)}{\varphi_{2m+1}(t)} = F_0(t), \quad \forall t \in (\beta, \alpha). \tag{12.18}$$

Furthermore, by arguments of normal families, it follows that $\{\sigma_{2m}(z)/\varphi_{2m}(z)\}$ and $\{\sigma_{2m+1}(z)/\varphi_{2m+1}(z)\}$ converge to analytic functions $F_\infty(z)$ and $F_0(z)$ outside $[0, \infty)$, extending the functions (12.18). □

13. Stieltjes transforms

Suppose we have a linear functional M defined on $\mathscr{L} \cdot \mathscr{L}$. If $\{\Omega_n\}_{n=0}^{\infty}$ is a basis for $\mathscr{L}$, then the moments

$$c_{k,l} = M[\Omega_k \Omega_l], \quad k, l = 0, 1, \ldots \tag{13.1}$$

are well defined. Note the symmetry $c_{k,l} = c_{l,k}$. A positive measure μ with infinite support in $[0, \infty)$ is said to solve the (*rational*) *Stieltjes moment problem on* $\mathscr{L} \cdot \mathscr{L}$ if it satisfies

$$c_{k,l} = \int_0^\infty \Omega_k(x)\Omega_l(x)\,\mathrm{d}\mu(x), \quad \forall k, l = 0, 1, \ldots$$

and it solves the (rational) Stieltjes moment problem on $\mathscr{L}$ if

$$c_{k,0} = \int_0^\infty \Omega_k(x)\,\mathrm{d}\mu(x), \quad \forall k = 0, 1, \ldots \;.$$

We now place our previous results in the above context, i.e., we assume that the Stieltjes moment problem on $\mathscr{L} \cdot \mathscr{L}$ has at least one solution and that the measure μ considered in previous sections is such a solution. Define its Stieltjes transform by

$$S(z, \mu) = \int_0^\infty \frac{\mathrm{d}\mu(t)}{z - t}. \tag{13.2}$$

Since we are in the Stieltjes situation, the orthogonal rational function φ_n shall have n simple zeros in $(0, \infty)$ and we can define a rational Gauss quadrature formula for μ.

Theorem 13.1. *Suppose that μ is a solution of the rational Stieltjes moment problem in $\mathscr{L} \cdot \mathscr{L}$. Suppose also that μ_n is the discrete measure that represents the n-point rational Gauss quadrature formula that is exact on $\mathscr{L}_n \cdot \mathscr{L}_{n-1}$. Then we have*

$$S(z, \mu_n) = \frac{\sigma_n(z)}{\varphi_n(z)}. \tag{13.3}$$

Furthermore (*recall the definition of r_n from* (1.1))

$$\frac{\sigma_n(z)}{\varphi_n(z)} - S(z, \mu) = \frac{1}{r_n(z)\varphi_n(z)^2} \int_0^\infty \frac{\varphi_n(t)^2 r_n(t)\,\mathrm{d}\mu(t)}{t - z}. \tag{13.4}$$

Proof. Formula (13.3) follows from the definition of σ_n. The integrand of (10.1) belongs to $\mathscr{L}_n$ and therefore the rational Gauss quadrature is exact. Since the nodes are zeros of φ_n, formula (13.3) is immediate.

Again using the definition of $\sigma_n(z)$ we then get

$$\frac{\sigma_n(z)}{\varphi_n(z)} - S(z, \mu) = \int_0^\infty \frac{\varphi_n(t)\,\mathrm{d}\mu(t)}{\varphi_n(z)(t - z)}. \tag{13.5}$$

For $\zeta_n \neq \infty$ we have

$$1 = \frac{r_n(t)}{r_n(z)} + \frac{t - z}{r_n(z)}$$

and hence

$$\int_0^\infty \frac{\varphi_n(t)\,\mathrm{d}\mu(t)}{\varphi_n(z)(t - z)} = \int_0^\infty \frac{r_n(t)\varphi_n(t)\,\mathrm{d}\mu(t)}{r_n(z)\varphi_n(z)(t - z)} + \frac{1}{r_n(z)\varphi_n(z)} \int_0^\infty \varphi_n(t)\,\mathrm{d}\mu(t), \tag{13.6}$$

from which follows

$$\int_0^\infty \frac{\varphi_n(t)\,\mathrm{d}\mu(t)}{\varphi_n(z)(t-z)} = \frac{1}{r_n(z)\varphi_n(z)} \int_0^\infty \frac{r_n(t)\varphi_n(t)}{t-z}\,\mathrm{d}\mu(t), \tag{13.7}$$

by orthogonality. For $\zeta_n = \infty$, the formula obviously holds.

Furthermore we may write

$$\int_0^\infty \left[\frac{\varphi_n(t)^2}{\varphi_n(z)^2} - \frac{\varphi_n(t)}{\varphi_n(z)}\right] \frac{r_n(t)}{r_n(z)} \frac{\mathrm{d}\mu(t)}{t-z} = \frac{1}{r_n(z)\varphi_n(z)^2} \int_0^\infty \frac{[\varphi_n(t)-\varphi_n(z)]r_n(t)\varphi_n(t)}{t-z}\,\mathrm{d}\mu(t). \tag{13.8}$$

The function

$$f_z(t) = \frac{[\varphi_n(t)-\varphi_n(z)]r_n(t)}{t-z}$$

belongs to $\mathscr{L}_{n-1}$, hence the integral in (13.8) vanishes. Thus

$$\int_0^\infty \frac{\varphi_n(t)r_n(t)\,\mathrm{d}\mu(t)}{\varphi_n(z)r_n(z)(t-z)} = \int_0^\infty \frac{\varphi_n(t)^2 r_n(t)\,\mathrm{d}\mu(t)}{\varphi_n(z)^2 r_n(z)(t-z)}. \tag{13.9}$$

Combining (13.5), (13.7), and (13.9) we obtain (13.4). □

We now make the same assumptions as in Sections 11 and 12, i.e., we consider the separated balanced Stieltjes situation.

Theorem 13.2. *Suppose we are in the separated balanced Stieltjes situation. Then for any measure that solves the rational Stieltjes moment problem on* $\mathscr{L} \cdot \mathscr{L}$, *we have*

$$\frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)} < S(x,\mu) < \frac{\sigma_{2m}(x)}{\varphi_{2m}(x)} \quad for\ x \in (\beta, \alpha). \tag{13.10}$$

Proof. Let $x \in (\beta, \alpha)$. The integral

$$\int_0^\infty \frac{\varphi_n(t)^2 r_n(t)\,\mathrm{d}\mu(t)}{t-x}$$

is negative if $\zeta_n \neq \infty$ and positive if $\zeta_n = \infty$. The factor $r_{2m}(x)$ is negative if $\zeta_{2m} \neq \infty$, positive if $\zeta_{2m} = \infty$. The factor $r_{2m+1}(x)$ is positive. Thus we find that

$$\frac{\sigma_{2m}(x)}{\varphi_{2m}(x)} - S(x,\mu) > 0, \tag{13.11}$$

$$\frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)} - S(x,\mu) < 0. \tag{13.12}$$

This completes the proof. □

In particular, if the sequence $\{\sigma_n(x)/\varphi_n(x)\}$ converges for $x \in (\beta, \alpha)$, then $S(x, \mu)$ and hence $S(z, \mu)$ for $z \notin [0, \infty)$ is unique, which implies that μ is unique.

Thus: if $\{\sigma_n(x)/\varphi_n(x)\}$ converges for $x \in (\beta, \alpha)$, then the inner product in $\mathscr{L}$ has a unique representing measure μ.

By a standard compactness argument, the functions $F_0(z)$ and $F_\infty(z)$ are Stieltjes transforms of measures $\mu^{(0)}$ and $\mu^{(\infty)}$ representing the functional M on $\mathscr{L}$.

14. A canonical basis

We continue to study the separated balanced Stieltjes situation, and consider the basis $\{\Omega_0, \Omega_1, \ldots, \Omega_n\}$ for $\mathscr{L}_n$ defined as follows (with r_n as in (1.1)):

$$\Omega_0 = 1, \tag{14.1}$$

$$\Omega_{2m}(z) = \frac{r_1(z)r_3(z)\cdots r_{2m-1}(z)}{(z/\zeta_2 - 1)(z/\zeta_4 - 1)\cdots(z/\zeta_{2m} - 1)}, \tag{14.2}$$

$$\Omega_{2m+1}(z) = \frac{(z/\zeta_2 - 1)(z/\zeta_4 - 1)\cdots(z/\zeta_{2m} - 1)}{r_1(z)r_3(z)\cdots r_{2m+1}(z)}. \tag{14.3}$$

We may also write (with D_n as in (1.2))

$$\Omega_n = \frac{T_n}{D_n}, \tag{14.4}$$

where

$$T_{2m+1}(z) = \frac{1}{\kappa_{2m}}\,[r_2(z)r_4(z)\cdots r_{2m}(z)]^2, \tag{14.5}$$

$$T_{2m}(z) = \kappa_{2m}[r_1(z)r_3(z)\cdots r_{2m-1}(z)]^2 \tag{14.6}$$

with

$$\kappa_{2m} = \prod_{\zeta_{2k}\neq\infty;\, k\leqslant m} (-\zeta_{2k}). \tag{14.7}$$

Note that κ_{2m} is positive.

The orthonormal function φ_n has an expansion according to $\{\Omega_k\}$ with leading coefficient v_n:

$$\varphi_n = v_n\Omega_n + \cdots. \tag{14.8}$$

This implies (recall $\varphi_n = P_n/D_n$)

$$P_n(z) = v_n T_n(z) + r_n(z)[\cdots]. \tag{14.9}$$

Thus if $\zeta_n \neq \infty$, we have

$$P_n(\zeta_n) = v_n T_n(\zeta_n), \tag{14.10}$$

while if $\zeta_{2m} = \infty$, we have

$$\mu_{2m}^{(2m)} = v_{2m}\kappa_{2m}, \tag{14.11}$$

(the leading coefficient of T_{2m} being κ_{2m}).

In order to derive a condition under which the rational Stieltjes moment problem is determinate, i.e., has a unique solution, we shall analyse the behavior of the difference

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)}.$$

It follows from (12.3) that we may write

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)} = \frac{\Delta_n(x)}{\varphi_n(z)\varphi_{n-2}(z)}, \tag{14.12}$$

where

$$\Delta_n(z) = \frac{(-1)^{n+1} b_n(z) W_1 \cdots W_{n-1}}{r_{n-1}(z) r_{n-2}(z)}. \tag{14.13}$$

We want to obtain lower bounds for $|\Delta_n(t)|$ for $t \in (\beta, \alpha)$ in terms of the coefficients v_n. This will be our aim in this section and the next one.

Taking into account the equality $W_k = -U_k/U_{k-1}$ obtained in Section 4, we find

$$\Delta_n(z) = \frac{U_{n-1}[Q_n r_{n-2}(z) + R_n r_{n-1}(z)]}{r_n(z) r_{n-1}(z) r_{n-2}(z)}. \tag{14.14}$$

Let $t \in (\beta, \alpha)$. It follows from (9.5)–(9.8) that $\operatorname{sgn} Q_n r_{n-2}(t) = \operatorname{sgn} R_n r_{n-1}(t)$. Consequently

$$|\Delta_n(t)| \geqslant \frac{|R_n U_{n-1}|}{|r_n(t) r_{n-2}(t)|}. \tag{14.15}$$

The rest of this section is devoted to finding expressions for $R_n U_{n-1}$. From (4.2), (9.13), (9.15), (14.10), (14.11), it follows that we may write $R_n U_{n-1}$ in terms of v_n and v_{n-2} in the following way:

- When $\zeta_{2m-2} \neq \infty$, $\zeta_{2m} \neq \infty$:

$$\begin{aligned} R_{2m+1} U_{2m} &= \frac{P_{2m+1}(\zeta_{2m-1}) P_{2m}(\zeta_{2m-1})}{r_{2m}(\zeta_{2m-1}) P_{2m}(\zeta_{2m-1}) P_{2m-1}(\zeta_{2m-1})} \\ &= \frac{T_{2m+1}(\zeta_{2m+1})}{T_{2m-1}(\zeta_{2m-1}) r_{2m}(\zeta_{2m-1})} \cdot \frac{P_{2m+1}(\zeta_{2m-1})}{P_{2m+1}(\zeta_{2m+1})} \cdot \frac{v_{2m+1}}{v_{2m-1}}, \end{aligned} \tag{14.16}$$

$$\begin{aligned} R_{2m} U_{2m-1} &= \frac{P_{2m}(\zeta_{2m-2}) P_{2m-1}(\zeta_{2m-2})}{r_{2m-1}(\zeta_{2m-2}) P_{2m-1}(\zeta_{2m-2}) P_{2m-2}(\zeta_{2m-2})} \\ &= \frac{T_{2m}(\zeta_{2m})}{T_{2m-2}(\zeta_{2m-2}) r_{2m-1}(\zeta_{2m-2})} \cdot \frac{P_{2m}(\zeta_{2m-2})}{P_{2m}(\zeta_{2m})} \cdot \frac{v_{2m}}{v_{2m-2}}, \end{aligned} \tag{14.17}$$

- When $\zeta_{2m-2}=\infty$, $\zeta_{2m}=\infty$:

$$R_{2m+1}U_{2m}=\frac{P_{2m+1}(\zeta_{2m-1})P_{2m}(\zeta_{2m-1})}{P_{2m}(\zeta_{2m-1})P_{2m-1}(\zeta_{2m-1})} = \frac{T_{2m+1}(\zeta_{2m+1})}{T_{2m-1}(\zeta_{2m-1})}\cdot\frac{P_{2m}(\zeta_{2m-1})}{P_{2m+1}(\zeta_{2m+1})}\cdot\frac{\upsilon_{2m+1}}{\upsilon_{2m-1}}, \tag{14.18}$$

$$R_{2m}U_{2m-1}=-\frac{\mu_{2m}^{(2m)}\mu_{2m-1}^{(2m-1)}}{\mu_{2m-1}^{(2m-1)}\mu_{2m-2}^{(2m-2)}} = -\frac{\kappa_{2m}}{\kappa_{2m-2}}\cdot\frac{\upsilon_{2m}}{\upsilon_{2m-2}}=-\frac{\upsilon_{2m}}{\upsilon_{2m-2}}, \tag{14.19}$$

since in this case $\kappa_{2m}=\kappa_{2m-2}$.

- When $\zeta_{2m-2}\neq\infty$, $\zeta_{2m}=\infty$:

$$R_{2m+1}U_{2m}=\frac{P_{2m+1}(\zeta_{2m-1})P_{2m}(\zeta_{2m-1})}{P_{2m}(\zeta_{2m-1})P_{2m-1}(\zeta_{2m-1})} = \frac{T_{2m+1}(\zeta_{2m+1})}{T_{2m-1}(\zeta_{2m-1})}\cdot\frac{P_{2m+1}(\zeta_{2m-1})}{P_{2m+1}(\zeta_{2m+1})}\cdot\frac{\upsilon_{2m+1}}{\upsilon_{2m-1}}. \tag{14.20}$$

For $R_{2m}U_{2m-1}$ we obtain in this case a more complicated expression.
We write (14.9) in more detail as (recall that $r_{2m}(z)=1$):

$$P_{2m}(z)=\upsilon_{2m}T_{2m}(z)+w_{2m}T_{2m-1}(z) + u_{2m}r_{2m-1}(z)T_{2m-2}(z)+r_{2m-1}(z)r_{2m-2}(z)[\cdots]. \tag{14.21}$$

We find that

$$P_{2m}(\zeta_{2m-2})=\upsilon_{2m}T_{2m}(\zeta_{2m-2})+u_{2m}r_{2m-1}(\zeta_{2m-2})T_{2m-2}(\zeta_{2m-2}). \tag{14.22}$$

Writing

$$T_n(z)=\sum_{k=0}^{n}t_k^{(n)}z^k, \tag{14.23}$$

we find by comparing coefficients of the terms with z^{2m-1} in (14.21):

$$\mu_{2m-1}^{(2m)}=\upsilon_{2m}t_{2m-1}^{(2m)}-u_{2m}\kappa_{2m-2}. \tag{14.24}$$

Here we made use of the facts that deg $T_{2m-1}\leqslant 2m-2$, $t_{2m-2}^{(2m-2)}=\kappa_{2m-2}$ and the degree of the term between square brackets in (14.21) is at most $2m-4$.

Substituting from (14.24) into (14.22) and using (4.2), (9.13) we find

$$R_{2m}U_{2m-1} = \frac{P_{2m}(\zeta_{2m-2})P_{2m-1}(\zeta_{2m-2})}{r_{2m-1}(\zeta_{2m-2})P_{2m-1}(\zeta_{2m-2})P_{2m-2}(\zeta_{2m-2})} = \frac{\upsilon_{2m}T_{2m}(\zeta_{2m-2}) + u_{2m}r_{2m-1}(\zeta_{2m-2})T_{2m-2}(\zeta_{2m-2})}{r_{2m-1}(\zeta_{2m-2})T_{2m-2}(\zeta_{2m-2})\upsilon_{2m-2}} \quad (14.25)$$

and further

$$R_{2m}U_{2m-1} = \frac{T_{2m}(\zeta_{2m-2})}{T_{2m-2}(\zeta_{2m-2})r_{2m-1}(\zeta_{2m-2})} \cdot \frac{\upsilon_{2m}}{\upsilon_{2m-2}} + \frac{\upsilon_{2m}t_{2m-1}^{(2m)} - \mu_{2m-1}^{(2m)}}{\kappa_{2m-2}\upsilon_{2m-2}}. \quad (14.26)$$

Taking into account (14.6), we write this as

$$R_{2m}U_{2m-1} = r_{2m-1}(\zeta_{2m-2})\frac{\upsilon_{2m}}{\upsilon_{2m-2}} + \frac{\upsilon_{2m}t_{2m-1}^{(2m)} - \mu_{2m-1}^{(2m)}}{\kappa_{2m-2}\upsilon_{2m-2}}, \quad (14.27)$$

since $\kappa_{2m} = \kappa_{2m-2}$ in this case.

- Note that we have not considered the situation $\zeta_{2m-2} = \infty$, $\zeta_{2m} \neq \infty$.

15. Monotonicity of the interpolation points

We still consider the separated balanced Stieltjes situation, with the canonical basis and derived concepts introduced in Section 14.

In addition, we assume the following monotonicity property:

$$\alpha_{k+1} \geqslant \alpha_k, \quad \beta_{k+1} \leqslant \beta_k \quad \text{for all } k. \quad (15.1)$$

We may call this the *monotone separated balanced Stieltjes situation*.

We shall obtain lower bounds for the expression $\Delta_n(t)$ when $t \in (\beta, \alpha)$ in terms of quotients $\upsilon_m/\upsilon_{m-2}$. More precisely, we prove:

Theorem 15.1. *Suppose we are in the monotone separated balanced Stieltjes situation. Then there is a positive constant K such that, with υ_n as defined in* (14.9),

$$|\Delta_n(t)| \geqslant \frac{1}{K}\frac{\upsilon_n}{\upsilon_{n-2}}, \quad n = 2, 3, \ldots \quad (15.2)$$

for all $t \in (\beta, \alpha)$.

Proof. We start by noting that $|P_n(t)|$ tends to ∞ as t tends to $-\infty$, and that $P_n'(t)$ has no zeros in $(-\infty, 0)$ because of the properties of the zeros of $P_n(t)$. Consequently $|P_n(t)|$ decreases as t increases in $(-\infty, 0)$.

In the following we assume that $t \in (\beta, \alpha)$.

- Let $\zeta_{2m} \neq \infty$, $\zeta_{2m-2} \neq \infty$.
From (14.5), (14.15), (14.16), we conclude that

$$|\Delta_{2m+1}(t)| \geqslant \frac{\kappa_{2m-2}[B_m(\alpha_{m+1})]^2 |P_{2m+1}(\alpha_m)||v_{2m+1}|}{\kappa_{2m}(\alpha_{m+1}-t)(\alpha_m - t)[B_{m-1}(\alpha_m)]^2(\beta_m - \alpha_m)|P_{2m+1}(\alpha_{m+1})||v_{2m-1}|} \tag{15.3}$$

with $B_k(t) = (\beta_1 - t)\cdots(\beta_k - t)$. Since $|P_{2m+1}(t)|$ is decreasing in $(-\infty, 0)$ and $\alpha_m \leqslant \alpha_{m+1}$, it follows that $|P_{2m+1}(\alpha_m)| \geqslant |P_{2m+1}(\alpha_{m+1})|$. Since $\alpha_{m+1} - \beta_k \geqslant \alpha_m - \beta_k$ and $\alpha_k - t < \alpha - \beta$, we find

$$|\Delta_{2m+1}(t)| \geqslant \frac{\kappa_{2m-2}(\alpha_m - \beta_m)|v_{2m+1}|}{\kappa_{2m}(\alpha - \beta)^2 |v_{2m-1}|}. \tag{15.4}$$

Furthermore $\kappa_{2m-2}/\kappa_{2m} = -1/\beta_m$, and

$$\left|\frac{\beta_m - \alpha_{m+1}}{-\beta_m}\right| = \left|\frac{\beta_m - \alpha_{m+1}}{\beta_m}\right| = 1 - \frac{|\alpha_{m+1}|}{|\beta_m|} \geqslant 1 - \frac{|\alpha|}{|\beta|} = \frac{|\beta| - |\alpha|}{|\beta|}. \tag{15.5}$$

Consequently

$$|\Delta_{2m+1}(t)| \geqslant \frac{1}{(\alpha - \beta)|\beta|} \cdot \left|\frac{v_{2m+1}}{v_{2m-1}}\right|. \tag{15.6}$$

From (14.6), (14.15), (14.17), we get

$$|\Delta_{2m}(t)| \geqslant \frac{\kappa_{2m}[A_m(\beta_m)]^2 |P_{2m}(\beta_{m-1})||v_{2m}|}{\kappa_{2m-2}(t - \beta_m)(t - \beta_{m-1})[A_{m-1}(\beta_{m-1})]^2(\alpha_m - \beta_{m-1})|P_{2m}(\beta_m)||v_{2m-2}|} \tag{15.7}$$

with

$$A_k(t) = (\alpha_1 - t)\cdots(\alpha_k - t). \tag{15.8}$$

Here $t - \beta_{m-1} \leqslant \alpha_m - \beta_{m-1}$ and $t - \beta_m < 2|\beta_m|$, while $\kappa_{2m}/\kappa_{2m-2} = -\beta_m = |\beta_m|$. Thus

$$|\Delta_{2m}(t)| \geqslant \frac{[A_m(\beta_m)]^2 |P_{2m}(\beta_{m-1})||v_{2m}|}{2[A_m(\beta_{m-1})]^2 |P_{2m}(\beta_m)||v_{2m-2}|}. \tag{15.9}$$

To handle the expression in (15.9), we introduce the function

$$\rho_{2m}(t) = \frac{P_{2m}(t)}{[A_m(t)]^2}. \tag{15.10}$$

We find

$$\rho'_{2m}(t) = \frac{\psi_{2m}(t)}{[A_m(t)]^3} \tag{15.11}$$

with

$$\psi_{2m}(t) = P'_{2m}(t)A_m(t) - 2P_{2m}(t)A'_m(t). \tag{15.12}$$

Since the leading coefficients of the two terms in (15.12) both equal $(-1)^m 2m\mu_{2m}^{(2m)}$, it follows that $\psi_{2m}(t)$ is a polynomial of degree at most $3m-2$.

Let t_k and t_{k+1} be consecutive zeros of $P_{2m}(t)$. Then $P'_{2m}(t_k)$ and $P'_{2m}(t_{k+1})$ have opposite signs, while $A_m(t)$ has constant sign in $(0,\infty)$. Consequently, $\psi_{2m}(t)$ has a zero in (t_k, t_{k+1}). This accounts for at least $2m-1$ zeros.
Next, consider the intervals (α_k, α_{k+1}), $k=1,\ldots,m-1$. $A_m(t)$ has a simple zero at each of these points, and $A'_m(\alpha_k)$ and $A'_m(\alpha_{k+1})$ have opposite sign. $P_{2m}(t)$ has constant sign in $(-\infty, 0)$. Consequently, $\psi_{2m}(t)$ has at least one zero in (α_k, α_{k+1}). This accounts for at least $m-1$ zeros.
Thus all the zeros of $\psi_{2m}(t)$ are accounted for. It follows that $\rho_{2m}(t)$ is monotone for $t<\alpha_1$. Since $|\rho_{2m}(t)|$ tends to ∞ as t tends to α_1, we conclude that $|\rho_{2m}(t)|$ is increasing in $(-\infty, \alpha_1)$.
We may write (15.9) as

$$|\Delta_{2m}(t)| \geqslant \frac{1}{2} \cdot \frac{|\rho_{2m}(\beta_{m-1})|}{|\rho_{2m}(\beta_m)|} \cdot \frac{|\upsilon_{2m}|}{|\upsilon_{2m-2}|}. \tag{15.13}$$

Because of (15.1) and the fact that $|\rho_{2m}(t)|$ is increasing, we conclude that

$$|\Delta_{2m}(t)| \geqslant \frac{1}{2} \left| \frac{\upsilon_{2m}}{\upsilon_{2m-2}} \right|. \tag{15.14}$$

- Next, let $\zeta_{2m-2}=\infty$, and $\zeta_{2m}=\infty$.
We find that

$$|\Delta_{2m+1}(t)| \geqslant \frac{[B_m(\alpha_{m+1})]^2 \kappa_{2m-2} |P_{2m+1}(\alpha_m)||\upsilon_{2m+1}|}{(\alpha_{m+1}-t)(\alpha_m-t)\kappa_{2m}[B_{m-1}(\alpha_m)]^2 |P_{2m+1}(\alpha_{m+1})||\upsilon_{2m-1}|}, \tag{15.15}$$

with

$$B_l(t) = r_2(t) r_4(t) \cdots r_{2l}(t), \tag{15.16}$$

where each term $r_{2k}(t)$ is either of the form $r_{2k}(t)=\beta_k - t$ or $r_{2k}(t)=1$. In both cases $|r_{2k}(\alpha_{m+1})| \geqslant |r_{2k}(\alpha_m)|$ because of (15.1). Furthermore $\alpha_k - t < \alpha - \beta$ and $\kappa_{2m-2}=\kappa_{2m-1}$ since $\zeta_{2m}=\infty$. Finally, $P_{2m+1}(\alpha_m) \geqslant P_{2m+1}(\alpha_{m+1})$. Consequently

$$|\Delta_{2m+1}(t)| \geqslant \frac{1}{(\alpha-\beta)^2} \left| \frac{\upsilon_{2m+1}}{\upsilon_{2m-1}} \right|. \tag{15.17}$$

Similarly

$$|\Delta_{2m}(t)| \geqslant \frac{|\upsilon_{2m}|}{|r_{2m}(t) r_{2m-2}(t)||\upsilon_{2m-2}|}. \tag{15.18}$$

Since $r_{2m}(t) = r_{2m-2}(t) = 1$, this gives

$$|\Delta_{2m}(t)| \geqslant \left| \frac{\upsilon_{2m}}{\upsilon_{2m-2}} \right|. \tag{15.19}$$

- Finally let $\zeta_{2m}=\infty$ and $\zeta_{2m-2}\neq\infty$.
We have

$$|\Delta_{2m+1}(t)| \geqslant \frac{[B_m(\alpha_{m+1})]^2 \kappa_{2m-2} |P_{2m+1}(\alpha_m)||\upsilon_{2m+1}|}{\kappa_{2m}(\alpha_{m+1}-t)(\alpha_m-t)[B_{m-1}(\alpha_m)]^2 |P_{2m+1}(\alpha_{m+1})||\upsilon_{2m-1}|} \tag{15.20}$$

with $B_l(t)$ as in (15.16). Again $|r_{2k}(\alpha_{m+1})| \geqslant |r_{2k}(\alpha_m)|$, $\kappa_{2m} = \kappa_{2m-2}$, $|P_{2m+1}(\alpha_m)| > |P_{2m+1}(\alpha_{m+1})|$, $r_{2m}(t) = 1$, $\alpha_k - t < \alpha - \beta$. Thus

$$|\Delta_{2m+1}(t)| \geqslant \frac{1}{(-\beta)^2} \left| \frac{v_{2m+1}}{v_{2m-1}} \right|. \tag{15.21}$$

We also have

$$|\Delta_{2m}(t)| \geqslant \frac{|R_{2m} U_{2m-1}|}{|r_{2m}(t) r_{2m-2}(t)|} = \frac{|R_{2m} U_{2m-1}|}{(t - \beta_{m-1})}. \tag{15.22}$$

Recall formula (14.27) for $R_{2m}U_{2m-1}$. Note that $-t^{(2m)}_{2m-1}$ equals the sum of the zeros of $T_{2m}(t)/\kappa_{2m}$, i.e., $t^{(2m)}_{2m-1} = -2(\alpha_1 + \cdots \alpha_n)/\kappa_{2m}$, so that $t^{(2m)}_{2m-1} > 0$. Further recall that $v_{2m} = \mu^{(2m)}_{2m}$, so that $-\mu^{(2m)}_{2m-1}/v_{2m}$ equals the sum of the zeros of $P_{2m}(t)$. Thus v_{2m} and $\mu^{(2m)}_{2m-1}$ have opposite signs. It follows that $v_{2m} t^{(2m)}_{2m-1} - \mu^{(2m)}_{2m-1}$ is positive if v_{2m} is positive, and negative if v_{2m} is negative. Consequently (since $r_{2m-1}(\zeta_{2m-2}) = \alpha_m - \beta_{m-1}$), $r_{2m-1}(\zeta_{2m-2}) v_{2m}$ and $v_{2m} t^{(2m)}_{2m-1} - \mu^{(2m)}_{2m-1}$ have the same sign. Hence

$$|R_{2m} U_{2m-1}| \geqslant (\alpha_m - \beta_{m-1}) \left| \frac{v_{2m}}{v_{2m-2}} \right|. \tag{15.23}$$

Combining this with (15.22) we get

$$|\Delta_{2m}(t)| \geqslant \frac{\alpha_m - \beta_{m-1}}{(t - \beta_{m-1})} \left| \frac{v_{2m}}{v_{2m-2}} \right|. \tag{15.24}$$

Since $\alpha_m - \beta_{m-1} > t - \beta_{m-1}$, this finally gives

$$|\Delta_{2m}(t)| \geqslant \left| \frac{v_{2m}}{v_{2m-2}} \right|. \tag{15.25}$$

- Note that because of (15.1) the case $\zeta_{2m} \neq \infty$, $\zeta_{2m-2} = \infty$ cannot occur.

From (15.6), (15.14), (15.17), (15.19), (15.21), (15.22), we conclude that there exists a constant K such that (15.2) holds for all $t \in (\beta, \alpha)$ and all $n \geqslant 2$. □

16. Unique representation

We are now able to prove uniqueness.

Theorem 16.1. *Suppose we are in the monotone separated balanced Stieltjes situation. Assume*

$$\lim_{n\to\infty} \left| \frac{v_n}{v_{n-2}} \right|^{1/2} = \infty. \tag{16.1}$$

Then the rational Stieltjes moment problem on $\mathscr{L} \cdot \mathscr{L}$ is determinate.

Proof. Recall from (14.12) that

$$\Delta_n(z)=\left[\frac{\sigma_n(z)}{\varphi_n(z)}-\frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)}\right]\varphi_n(z)\varphi_{n-2}(z), \tag{16.2}$$

and from (15.2) that

$$\left|\frac{v_n}{v_{n-2}}\right|\leqslant K|\Delta_n(t)| \quad \text{for } t\in(\beta,\alpha). \tag{16.3}$$

Assume that

$$\sum_{m=1}^{\infty}\left|\frac{v_{2m}}{v_{2m-2}}\right|^{1/2}=\infty.$$

Then

$$\sum_{m=1}^{\infty}|\Delta_{2m}(t)|^{1/2}=\infty \quad \text{for } t\in(\beta,\alpha).$$

The boundedness and monotonicity results in Section 12 imply convergence of

$$\sum_{m=1}^{\infty}\left[\frac{\sigma_{2m-2}(t)}{\varphi_{2m-2}(t)}-\frac{\sigma_{2m}(t)}{\varphi_{2m}(t)}\right].$$

Thus by the Schwarz inequality we conclude

$$\left\{\sum_{m=1}^{\infty}\left[\frac{\sigma_{2m-2}(t)}{\varphi_{2m-2}(t)}-\frac{\sigma_{2m}(t)}{\varphi_{2m}(t)}\right]\right\}\left\{\sum_{m=1}^{\infty}|\varphi_{2m}(t)\varphi_{2m-2}(t)|\right\}=\infty. \tag{16.4}$$

Similarly, if $\sum_{m=1}^{\infty}|v_{2m+1}/v_{2m-1}|=\infty$, then

$$\left\{\sum_{m=1}^{\infty}\left[\frac{\sigma_{2m+1}(t)}{\varphi_{2m+1}(t)}-\frac{\sigma_{2m-1}(t)}{\varphi_{2m-1}(t)}\right]\right\}\left\{\sum_{m=1}^{\infty}|\varphi_{2m+1}(t)\varphi_{2m-1}(t)|\right\}=\infty \tag{16.5}$$

for $t\in(\beta,\alpha)$. From this we conclude (again applying the Schwarz inequality to $\sum|\varphi_{2m}(t)\varphi_{2m-2}(t)|$ or $\sum|\varphi_{2m+1}(t)\varphi_{2m-1}(t)|$) that if $\sum_{n=2}^{\infty}|v_n/v_{n-2}|^{1/2}=\infty$, then at least one of

$$\sum_{m=0}^{\infty}|\varphi_{2m}(t)|^2=\infty \quad \text{or} \quad \sum_{m=0}^{\infty}|\varphi_{2m+1}(t)|^2=\infty$$

holds for all $t\in(\beta,\alpha)$. Thus

$$\sum_{n=0}^{\infty}|\varphi_n(t)|^2=\infty \quad \text{for all } t\in(\beta,\alpha).$$

Let μ be an arbitrary measure representing the functional M on $\mathscr{L} \cdot \mathscr{L}$. By (13.5) we have

$$\frac{\sigma_n(z)}{\varphi_n(z)} - S(z, \mu) = \frac{1}{\varphi_n(z)} \int_0^\infty \frac{\varphi_n(t)}{t-z} \, d\mu(t). \tag{16.6}$$

Let $x \in (\beta, \alpha)$. Then the function $t \mapsto (t-x)^{-1}$ is square integrable with respect to μ. We observe from (16.6) that its Fourier coefficient with respect to the system $\{\varphi_n\}$ is $[\sigma_n(x)/\varphi_n(x) - S(x, \mu)]\varphi_n(x)$. Hence by Bessel's inequality

$$\sum_{n=0}^{\infty} \left[\frac{\sigma_n(x)}{\varphi_n(x)} - S(x, \mu) \right]^2 \varphi_n(x)^2 < \infty. \tag{16.7}$$

Now assume that $\sum_{n=2}^{\infty} |v_n/v_{n-2}|^{1/2} = \infty$. It follows from the considerations above that $\sum_{m=0}^{\infty} |\varphi_{2m}(t)|^2 = \infty$ or $\sum_{m=0}^{\infty} |\varphi_{2m+1}(t)|^2 = \infty$ for all $t \in (\beta, \alpha)$. Assume first that $\sum_{m=0}^{\infty} |\varphi_{2m}(t)|^2 = \infty$. Then a subsequence of $[\sigma_{2m}(x)/\varphi_{2m}(x) - S(x, \mu)]^2$ tends to zero by (16.7), and hence by the monotonicity of $\{\sigma_{2m}(x)/\varphi_{2m}(x)\}$ we have

$$\lim_{m \to \infty} \frac{\sigma_{2m}(x)}{\varphi_{2m}(x)} = S(x, \mu) \quad \text{for } x \in (\beta, \alpha). \tag{16.8}$$

Similarly, if $\sum_{m=0}^{\infty} |\varphi_{2m+1}(t)|^2 = \infty$, then

$$\lim_{m \to \infty} \frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)} = S(x, \mu) \quad \text{for } x \in (\beta, \alpha). \tag{16.9}$$

In both cases, *all* representing measures on $\mathscr{L} \cdot \mathscr{L}$ have the same Stieltjes transform on (β, α), hence in $\mathbb{C} \setminus [0, \infty)$. Consequently, *the functional has a unique representing measure on* $\mathscr{L} \cdot \mathscr{L}$ *if* (16.1) *holds.* □

From this, Carleman-type conditions on the "moments" $\int_0^\infty \Omega_n(t)^2 \, d\mu(t)$ can be deduced. We have

Theorem 16.2. *Suppose we are in the monotone separated balanced Stieltjes situation. Let the moments* $c_{n,n} = \int_0^\infty \Omega_n^2(t) \, d\mu(t)$ *be defined as in* (13.1). *Then the rational Stieltjes moment problem in* $\mathscr{L} \cdot \mathscr{L}$ *is determinate if*

$$\sum_{n=0}^{\infty} \frac{1}{(c_{n,n})^{1/2n}} = \infty. \tag{16.10}$$

Proof. This proof of [7, Theorem 6.2] can be used without any change.

Note that when $\alpha_n = 0$ and $\beta_n = -\infty$ for all n, then $\Omega_{2m}(z) = z^m$ and $\Omega_{2m+1}(z) = z^{-(m+1)}$. If we set

$$c_n = \int_0^\infty t^n \, d\mu(t), \quad n = 0, \pm 1, \pm 2, \ldots, \tag{16.11}$$

then $c_{2m,2m} = c_{2m}$ and $c_{2m-1,2m-1} = c_{-2m}$. Condition (16.10) may thus be written as

$$\sum_{m=1}^{\infty} \frac{1}{(c_{-2m})^{\frac{1}{4m-2}}} + \sum_{m=0}^{\infty} \frac{1}{(c_{2m})^{\frac{1}{4m}}} = \infty.$$

The orthogonal rational functions become the orthogonal Laurent polynomials. The rational Stieltjes moment problem on $\mathscr{L} \cdot \mathscr{L}$ becomes the strong Stieltjes moment problem. Hence we recover a result that is essentially in [1] and which can also be found in [13,19,20]. □

Corollary 16.3. *Define the moments* c_n, $n = 0, \pm 1, \pm 2, \ldots$ *as in* (16.11). *Then*, *if at least one of*

$$\sum_{m=1}^{\infty} \frac{1}{(c_{-2m})^{\frac{1}{4m-2}}} = \infty \quad or \quad \sum_{m=0}^{\infty} \frac{1}{(c_{2m})^{\frac{1}{4m}}} = \infty$$

holds, *then the strong Stieltjes moment problem is determinate*.

References

[1] E. Aldén, A survey of weak and strong moment problems with generalizations, Preprint Series 2, Dept. Math., Umeå, 1985.

[2] C.M. Bonan-Hamada, W.B. Jones, Para-orthogonal Laurent polynomials and the strong Stieltjes moment problem, J. Comput. Appl. Math. 105 (1999) 175–185.

[3] A. Bultheel, C. Díaz-Mendoza, P. González-Vera, R. Orive, Quadrature on the half line and two-point Padé approximants to Stieltjes functions, Part II: convergence, J. Comput. Appl. Math. 77 (1997) 53–76.

[4] A. Bultheel, C. Díaz-Mendoza, P. González-Vera, R. Orive, Quadrature on the half line and two-point Padé approximants to Stieltjes functions, Part III: the unbounded case, J. Comput. Appl. Math. 87 (1997) 95–117.

[5] A. Bultheel, C. Díaz-Mendoza, P. González-Vera, R. Orive, Estimates of the rate of convergence for certain quadrature formulas on the half line, in: B. Berndt, F. Gesztesy (Eds.), Continued Fractions: From Number Theory to Constructive Approximation, vol. 236, Contemp. Math., American Mathematical Society, Providence, RI, 1999, pp. 85–100.

[6] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal Rational Functions, vol. 5, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1999.

[7] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Determinacy of a rational moment problem, J. Comput. Appl. Math. 133 (1–2) (2000) 241–252.

[8] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions and continued fractions, in: J. Bustoz, M. Ismail, S. Suslov (Eds.), Special Functions 2000: Perspective and Future Directions, vol. 30, NATO-ASI Series II: Mathematics, Physics and Chemistry, Kluwer, Dordrecht, 2001, pp. 87–109.

[9] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, A rational Stieltjes moment problem, Appl. Math. Comput. 128 (2–3) (2002) 217–235.

[10] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions and quadrature on the real half line, J. Complexity 19 (3) (2003) 212–230.

[11] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Positivity of continued fractions associated with rational Stieltjes moment problems, Rocky Mountain J. Math. 33 (2) (2003) 609–627.

[12] A. Bultheel, P. González-Vera, R. Orive, Quadrature on the half line and two-point Padé approximants to Stieltjes functions, Part I: algebraic aspects, J. Comput. Appl. Math. 65 (1995) 57–72.

[13] P. González-Vera, O. Njåstad, Convergence of two-point Padé approximants to series of Stieltjes, J. Comput. Appl. Math. 32 (1990) 97–105.

[14] W.B. Jones, O. Njåstad, Orthogonal Laurent polynomials and strong moment theory, J. Comput. Appl. Math. 105 (1999) 51–91.

[15] W.B. Jones, O. Njåstad, W.J. Thron, Two-point Padé expansions for a family of analytic functions, J. Comput. Appl. Math. 9 (1983) 105–124.

[16] W.B. Jones, W.J. Thron, Orthogonal Laurent polynomials and Gaussian quadrature, in: K. Gustafson, W.P. Reinhardt (Eds.), Quantum Mechanics in Mathematics, Chemistry and Physics, Plenum, New York, 1984, pp. 449–455.

[17] W.B. Jones, W.J. Thron, H. Waadeland, A strong Stieltjes moment problem, Trans. Amer. Math. Soc. 206 (1980) 503–528.

[18] L. Lorentzen, H. Waadeland, Continued Fractions with Applications, vol. 3, Studies in Computational Mathematics, North-Holland, Amsterdam, 1992.
[19] O. Njåstad, Convergence properties related to p-point Padé approximants of Stieltjes functions, J. Approx. Theory 73 (1993) 149–161.
[20] O. Njåstad, Extremal solutions of the strong Stieltjes moment problem, J. Comput. Appl. Math. 65 (1995) 309–318.
[21] S. Tokarzewski, Two-point Padé approximants to Laurent expansions of Stieltjes functions, Preprint, Inst. Fund. Technological Research, Warsaw, 1994.