SUFFICIENT CONDITIONS FOR ERGODICITY AND RECURRENCE OF MARKOV CHAINS ON A GENERAL STATE SPACE

Richard L. TWEEDIE
Division of Mathematics and Statistics, CSIRO, Canberra, Australia

Received 25 May 1974
Revised 9 November 1974

Let \( \{X_n\} \) be a \( \phi \)-irreducible Markov chain on an arbitrary space. Sufficient conditions are given under which the chain is ergodic or recurrent. These extend known results for chains on a countable state space. In particular, it is shown that if the space is a normed topological space, then under some continuity conditions on the transition probabilities of \( \{X_n\} \) the conditions for ergodicity will be met if there is a compact set \( K \) and an \( \epsilon > 0 \) such that
\[
\mathbb{E} \{ \|X_{n+1}\| - \|X_n\| \mid X_n = x \} < -\epsilon \quad \text{whenever } x \text{ lies outside } K \text{ and }\]
\[
\mathbb{E} \{ \|X_{n+1}\| \mid X_n = x \} \text{ is bounded, } x \in K; \text{ whilst the conditions for recurrence will be met if there exists a compact } K \text{ with } \mathbb{E} \{ \|X_{n+1}\| - \|X_n\| \mid X_n = x \} \leq 0 \text{ for all } x \text{ outside } K. \text{ An application to queueing theory is given.}
\]

AMS Subj. Class.: 60J05

reversibility invariant measures
positive recurrence stationary measures
ergodicity waiting time
Markov chains dependent queues

1. Introduction

Suppose \( \{X_n\} \) is a Markov chain taking values in some arbitrary space \((X, \mathcal{F})\), with temporally homogeneous transition probabilities
\[
P^n(x, A) = \mathbb{P}[X_n \in A \mid X_0 = x], \quad x \in X, A \in \mathcal{F},
\]
where for fixed \( A \in \mathcal{F}, P^n(\cdot, A) \) is a measurable function on \( X \), and for fixed \( x \in X, P^n(x, \cdot) \) is a probability measure on the \( \sigma \)-field \( \mathcal{F} \). We will assume throughout that \( \{X_n\} \) is \( \phi \)-irreducible for some non-trivial \( \sigma \)-finite measure \( \phi \) on \( \mathcal{F} \); that is, whenever \( \phi(A) > 0 \),
\[
\sum_n 2^{-n} P^n(x, A) > 0 \quad \text{for every } x \in X.
\]
Under the assumption of $\phi$-irreducibility, it can be shown that, analogously with the familiar countable state space case, the following dichotomy holds: either

(i) for every $x \in \mathcal{X}$ and every $A \in \mathcal{F}$ with $\phi(A) > 0$,

$$\sum_{n=1}^{\infty} P^n(x, A) = \infty ; \quad (1.1)$$

or

(ii) it is possible to find a countable collection of sets $A(j)$ with

$$A(j) \cap A(k) = \emptyset, \quad j \neq k,$$

$$\bigcup_{j} A(j) = \mathcal{X}$$

(that is, a partition of $\mathcal{X}$) such that, for every $x$ and every $A \in \mathcal{F}$ with $A \subseteq A(j)$ for some $j$,

$$\sum_{n=1}^{\infty} P^n(x, A) < \infty .$$

(For a proof of this, see either [23, Theorem 11], or, under a slightly stronger irreducibility condition, [22].) We shall call any chain satisfying (1.1) recurrent; if $\{X_n\}$ is non-recurrent we call $\{X_n\}$ transient.

We shall call a $\sigma$-finite non-trivial measure $\mu$ on $\mathcal{F}$ subinvariant for $\{X_n\}$ if it satisfies the subinvariant equations

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dy) P(y, A) , \quad A \in \mathcal{F} . \quad (1.2)$$

For $\phi$-irreducible chains, some subinvariant measure always exists (cf. [9, Section 2] or [23, Section 3]); if $\{X_n\}$ is recurrent, this solution is unique (up to constant multiples), and satisfies (1.2) with equality for all $A \in \mathcal{F}$ (cf. [23, Section 3]). If this unique solution is a finite measure, we shall call $\{X_n\}$ ergodic (such an $\{X_n\}$ is often called positive recurrent [23]), and denote the unique subinvariant probability measure by $\pi$.

When using a $\phi$-irreducible Markov chain as a model, it is often of great importance to know whether the model is ergodic. A necessary and sufficient condition for ergodicity is the existence of a probability measure $\pi$ satisfying (1.2), but this is often a difficult criterion to verify. In the general state space context, a sufficient condition for ergodicity that has been used is the Doeblin condition, a form of which (given by Doob [3]) states that $\{X_n\}$ is ergodic if there exists a probability measure $\theta$ on $\mathcal{F}$, a fixed integer $\kappa > 0$, and a $\delta > 0$, such that, whenever $\theta(A) \leq \delta$,

$$P^{\kappa}(x, A) \leq 1 - \delta \quad \text{for all } x \in \mathcal{X}.$$
This condition is, however, rather strong. When $\mathcal{X}$ is countable, it is closely related to the condition that the transition matrix is what is called in [7] a **Markov matrix**; that is, a matrix with the elements of one column bounded from zero. As noted by Miller [15] this suffices not merely for ergodicity, but also for geometric ergodicity in the aperiodic case; that is, $P^n(t, j) \to \pi(j)$ geometrically fast as $n \to \infty$. The Doeblin condition implies ergodicity basically because it insists that from anywhere in the state space the chain moves, with high probability, in a fixed finite number of steps to a region near the 'centre' of the state space. It is unsatisfactory because it does not cover chains which, rather than mimicking this 'renewal' type behaviour, are closer to the random walk type, and are ergodic because they 'drift' consistently back to the 'centre' of the state space, even though the number of steps they take to reach the 'centre' depends essentially on how far from the 'centre' the starting point of the chain is.

In this paper we find sufficient conditions on general state space chains which ensure ergodicity, and which cover chains of random walk type. These extend known results in the countable state space case, due originally to Foster [5] and extended somewhat by Mauldon [14]. We also show how the results specialize when $\mathcal{X}$ is a topological space and the transition law of $\{X_n\}$ satisfies certain continuity conditions; this makes more explicit the notion of the 'centre' of the space mentioned above. Specializing further, we show that if $\mathcal{X}$ is a normed space, simple conditions couched in terms of the norm are sufficient for the conditions of the previous sections to hold, and hence give readily verifiable criteria for ergodicity. In the special case of finite-dimensional Banach spaces, these reduce to the condition that, under some continuity conditions, $\{X_n\}$ is ergodic if there exists $\epsilon > 0, M > 0$ such that, for all $x$ with $\|x\| > M$,

$$E(\|X_{n+1}\| - \|X_n\| \mid X_n = x) \leq -\epsilon,$$

and that $E(\|X_{n+1}\| \mid X_n = x) < \infty$ for all $x \in \mathcal{X}$. This condition corresponds closely to that known for random walk.

The condition of recurrence, whilst not perhaps of such importance as ergodicity in modelling, is nonetheless of great importance in the theory of Markov chains. In conjunction with the results on ergodicity mentioned above, we give sufficient conditions for recurrence in the general, topological and Banach space contexts. These again are analogous to those known in the case of countable $\mathcal{X}$, first given by Foster [5] and in an extended form by Pakes [18].

The paper is concluded by an application of the results to a queueing
theory problem, and by a brief note on conditions for ergodicity and recurrence previously known in the case of countable $\mathcal{X}$.

2. Notation and preliminaries

Whether $\{X_n\}$ is transient or recurrent, there is at least one solution to the subinvariant equations (1.2). For the remainder of this paper, we let $\mu$ denote a fixed subinvariant measure for $\{X_n\}$ (with the understanding that if $\{X_n\}$ is ergodic, $\mu$ is taken to be $\pi$); and we let

$$\mathcal{F}_\mu = \{ A \in \mathcal{F} : 0 < \mu(A) < \infty \} .$$

The set $\mathcal{F}_\mu$ is useful in that it allows us to discriminate between recurrent and transient chains; from [24, Proposition 10.3], we have:

**Lemma 2.1.** Either $\{X_n\}$ is recurrent, or $\sum P^n(x, A) < \infty$ for every $x \in \mathcal{X}$ and every $A \in \mathcal{F}_\mu$. $\Box$

We write, for $A, B \in \mathcal{F}$,

$$A^P(x, B) = P[X_n \in B, X_s \notin A, s = 1, \ldots, n-1 \mid X_0 = x] \quad (2.1)$$

for the $n$-step transition probabilities of the chain $\{X_n\}$, and put

$$F(x, A) = \sum_{n=1}^{\infty} A^P(x, A) \quad (2.2)$$

for the probability that $\{X_n\}$ ever reaches $A$ from $x$. A recurrence condition stronger than (1.1) is

$$F(x, A) = 1, \quad x \in \mathcal{X}, \quad A \in \mathcal{F}, \quad \phi(A) > 0 ; \quad (2.3)$$

if $\{X_n\}$ satisfies (2.3) we call $\{X_n\}$ $\phi$-recurrent (cf. [17, p.4]).

We need the following connections between the two definitions of recurrence:

**Lemma 2.2.** (i) If $\{X_n\}$ satisfies $F(x, A) = 1$ for some $A \in \mathcal{F}_\mu$ and all $x \notin A$, then $\{X_n\}$ is recurrent; in particular, therefore, $\phi$-recurrence implies recurrence.

(ii) If $\{X_n\}$ is recurrent, then for any $A \in \mathcal{F}$ with $\phi(A) > 0$ there is a $\phi$-null set $N(A)$ such that $F(x, A) = 1$ for all $x \notin N(A)$; if $\mathcal{F}$ is countably generated, there is a $\phi$-null set $N$ such that $N(A) \subseteq N$ for all $A$. 

Proof. (i) Since, for $x \in A$,

$$F(x, A) = P(x, A) + \int_{A^c} P(x, dy) F(y, A),$$

the condition that $F(x, A) \equiv 1, x \in A^c$, implies $F(x, A) \equiv 1, x \in \mathcal{X}$. A last exit decomposition then gives, for all $x$,

$$\sum_{n=1}^{\infty} P^n(x, A) = F(x, A) + \int \sum_{n=1}^{\infty} P^n(x, dy) F(y, A)$$

$$= 1 + \sum_{n=1}^{\infty} P^n(x, A), \quad (2.4)$$

and so $\sum_{n=1}^{\infty} P^n(x, A) = \infty$. If $A \in \mathcal{F}$, Lemma 2.1 shows that $\{X_n\}$ is recurrent. If $\{X_n\}$ is $\phi$-recurrent for some $\phi$, $\sigma$-finiteness and non-triviality of $\mu$ and $\phi$, together with the fact that $\mu \geq \phi$ from subinvariance of $\mu$, imply that $F(x, A) \equiv 1$ for some $A \in \mathcal{F}$, and so $\{X_n\}$ is recurrent.

(ii) If $\{X_n\}$ is recurrent, and $\mathcal{F}$ is countably generated, then from [9, Theorem 2] there is a $\phi$-null set $N$ such that $F(x, A) \equiv 1$ for $x \notin N$ and all $A \in \mathcal{F}$ with $\phi(A) > 0$. If $\mathcal{F}$ is not countably generated, and $A \in \mathcal{F}$, $\phi(A) > 0$, then there exists a countably generated $\mathcal{F}_0 \subseteq \mathcal{F}$ with $A \in \mathcal{F}_0$, and $\{X_n\}$ can be taken as a chain on $(\mathcal{X}, \mathcal{F}_0)$ (cf. [17, p. 7]); applying the previous result to the chain on this admissible $\sigma$-field shows that a $\phi$-null set $N(A)$ exists such that $F(x, A) \equiv 1, x \notin N(A)$. \( \square \)

If $\{X_n\}$ is ergodic, then the invariant measure $\pi$ satisfies similar limit properties to those familiar in the countable state space case; in particular, for every $A$, if $N(A)$ is the null set of Lemma 2.2(ii), then (see [23, Theorem 6])

$$\frac{1}{n} \sum_{i=1}^{n} P^i(y, A) \to \pi(A) \quad \text{as } n \to \infty \quad \text{for all } y \notin N(A). \quad (2.5)$$

[If $\{X_n\}$ is actually $\phi$-recurrent and ergodic, then (see [19, Theorem 7.1]) (2.5) can be replaced by the stronger statement that for any initial distribution $\nu$ on $\mathcal{F}$,

$$\left\| \frac{1}{n} \int_{\mathcal{X}} \nu(dy) \sum_{i=1}^{n} P^i(y, \cdot) - \pi(\cdot) \right\| \to 0 \quad \text{as } n \to \infty, \quad (2.6)$$

where $\|\cdot\|$ denotes total variation.]

On the other hand, if $\{X_n\}$ is recurrent but not ergodic, then (see [23, Theorem 6]) there is a partition $\{K_i\}$ of $\mathcal{X}$ such that for every
$A \in F, A \subseteq K_l$ for some $j$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} P^*(x, A) = 0 \quad \text{for all } x \in X. \quad (2.7)$$

The set $F_\mu$ is again useful for distinguishing between ergodic and non-ergodic recurrent chains; from the proof of [23, Theorem 6], using the fact that $\mu$ is the unique invariant measure for recurrent chains, we have:

**Lemma 2.3.** If $\{X_n\}$ is recurrent, then either $\{X_n\}$ is ergodic, or (2.7) holds for every $A \in F_\mu$. □

**3. The conditions for a general state space**

**Theorem 3.1.** A sufficient condition for $\{X_n\}$ to be ergodic is the existence of a $K \in F_\mu$ and a non-negative measurable function $g$ on $X$ such that

$$\int_X P(x, dy) g(y) \leq g(x) - 1, \quad x \notin K, \quad (3.1)$$

and, for some fixed $B > 0$,

$$\int_X P(x, dy) g(y) = \lambda(x) \leq B < \infty, \quad x \in K. \quad (3.2)$$

**Theorem 3.2.** A sufficient condition for $\{X_n\}$ to be recurrent is the existence of a $K \in F_\mu$ and a non-negative measurable function $g$ on $X$ such that

1. $\int_X P(x, dy) g(y) \leq g(x), \quad x \notin K$;
2. $g$ is strictly unbounded, in the sense that there exists, for every sufficiently large $M$, a set $K_M \in F_\mu$ with

$$g(x) > M, \quad x \in K_M^c. \quad (3.3)$$

**Proof of Theorem 3.1.** If $g$ is as in the theorem, write, for each $n \geq 1$,

$$g^{(n+1)}(x) = \int_X P^n(x, dy) g(y).$$

If $K$ satisfies (3.1) and (3.2), then
\[
g^{(n+2)}(x) = \int_{\mathcal{X}} P^n(x, dw) \left[ \int_{\mathcal{X}} P(w, dy) g(y) \right] \\
\leq \int_{K} P^n(x, dw) \lambda(w) + \int_{K^c} P^n(x, dw) [g(w) - 1] \\
\leq \int_{K} P^n(x, dw) [\lambda(w) + 1] + \int_{K^c} P^n(x, dw) g(w) - P^n(x, \mathcal{X}) \\
\leq (B + 1)P^n(x, K) + g^{(n+1)}(x) - 1. \quad (3.4)
\]

Iterating (3.4) and dividing by \( n \) gives

\[
n^{-1}g^{(n+2)}(x) \leq (B + 1) \left[ n^{-1} \sum_{m=1}^{n} P^m(x, K) \right] + n^{-1}g^{(2)}(x) - 1. \quad (3.5)
\]

Now Lemma 2.1, Lemma 2.3 and (2.5) guarantee that, regardless of the recurrence or transience of \( \{X_n\} \), for almost all \( x \),

\[
\lim_{n \to \infty} n^{-1} \sum_{m=1}^{n} P^m(x, K) = \pi(x, K) \quad (3.6)
\]

exists and is finite when \( K \in \mathcal{F}_\mu \). Moreover, \( \pi(x, K) \equiv 0 \) unless \( \{X_n\} \) is ergodic, when \( \pi(x, K) = \pi(K) \) for almost all \( x \).

Let \( n \to \infty \) in (3.5); we get, by the non-negativity of \( g^{(n)} \) and the finiteness of \( g^{(2)}(x) \) (guaranteed by (3.1) and (3.2)) that

\[
\pi(x, K) \geq [B+1]^{-1} > 0,
\]

and, from the above remarks, \( \{X_n\} \) is ergodic. \( \square \)

**Proof of Theorem 3.2.** Let \( g, K \) and \( K_M \) be as in the theorem for all sufficiently large \( M \), and put, for \( A \in \mathcal{F} \),

\[
\hat{P}(x, A) = \begin{cases} 
    P(x, A), & x \notin K, \\
    \delta(x, A), & x \in K.
\end{cases}
\]

From (i), \( g \) satisfies

\[
\int_{\mathcal{X}} \hat{P}(x, dy) g(y) \leq g(x) \quad \text{for all } x \in \mathcal{X}. \quad (3.7)
\]

Write \( \hat{P}^n(x, A) \) for the \( n \)-step transition probabilities of a chain \( \{\hat{X}_n\} \) with transition law \( \hat{P}(\cdot, \cdot) \). Since each point in \( K \) is absorbing for the chain \( \{\hat{X}_n\} \), we have

\[
\hat{P}^n(x, K) \leq \hat{P}^{n+1}(x, K),
\]
and since $\hat{p}^n(x, K) = \sum_{i=1}^{n} K P^i(x, K), x \notin K$, we have

$$F(x, K) = \lim_{n \to \infty} \hat{p}^n(x, K), \quad x \notin K. \quad (3.8)$$

Now iterating (3.7) gives, for any $n$,

$$g(x) \geq \int \hat{p}^n(x, dy) g(y)$$

$$\geq \int_{K \setminus M} \hat{p}^n(x, dy) g(y)$$

$$\geq M \left[ 1 - \hat{p}^n(x, K_M) \right]$$

$$\geq M \left[ 1 - \hat{p}^n(x, K) \right] - M \hat{p}^n(x, K_M \setminus K). \quad (3.9)$$

Assume that $\{X_n\}$ is transient. Since

$$\hat{p}^n(x, K_M \setminus K) \leq P^n(x, K_M)$$

and $K_M \in \mathcal{T}_\mu$, Lemma 2.1 implies that letting $n \to \infty$ in (3.9) gives, for $x \in K^c$,

$$g(x) \geq M \left[ 1 - \lim_{n \to \infty} \hat{p}^n(x, K) \right]$$

$$= M \left[ 1 - F(x, K) \right]. \quad (3.10)$$

But $g(x)$ is finite for each $x$, and $M$ can be chosen arbitrarily large, from (3.3); and so (3.10) implies that $F(x, K) = 1$ for all $x \in K^c$. So, from Lemma 2.2, $\{X_n\}$ is recurrent. This contradicts our assumption that $\{X_n\}$ is transient (assumed in order to most easily eliminate the last term of (3.9) as $n \to \infty$), and so the theorem is proved. $\Box$

4. The conditions for a topological space

In most applications of Markov chains, the state space will not be completely general, but will be equipped with a topology $\mathcal{T}$. In order for the topology to be associated with the Markov chain in a reasonable way, one must impose some $\mathcal{T}$-continuity condition on the transition probabilities of the chain.

In the countable state space case, Theorems 3.1 and 3.2 hold when
\( K = \{0, 1, ..., N\} \) for any fixed \( N \), because finite sets are in \( \mathcal{F}_\mu \) in this case. The natural analogue of finiteness is, under most conditions, compactness; one thus wants a continuity condition on the transition function which ensures that compact sets are of finite \( \mu \)-measure. Such conditions have been studied in [19]. We say that the transition law \( \{P(x, \cdot)\} \) is strongly continuous if, for every \( A \in \mathcal{F} \), \( P(x, \cdot) \) is a continuous function in \( x \). (This form of continuity is sometimes expressed by calling \( \{X_n\} \) strongly Feller.) It is well known that \( \{P(x, \cdot)\} \) is strongly continuous if and only if \( \int P(x, dy) g(y) \) is a continuous bounded function of \( x \) whenever \( g \) is a bounded measurable function on \( \mathcal{X} \). In [19] the following result is proved:

**Lemma 4.1.** If \( \{P(x, \cdot)\} \) is strongly continuous, and \( \mu \) is subinvariant for \( \{X_n\} \), then every compact set has finite \( \mu \)-measure. \( \square \)

We write \( \mathcal{K} \) for the set of compact sets of positive \( \phi \)-measure, where \( \{X_n\} \) is \( \phi \)-irreducible. Note that, by iterating (1.2), \( \mu(K) \) is thus positive for \( K \in \mathcal{K} \) since \( \mu \) is subinvariant; thus Lemma 4.1 says that \( \mathcal{K} \subseteq \mathcal{F}_\mu \). Applying this to the results of the previous section yields immediately:

**Theorem 4.2.** Let \( \{X_n\} \) be a \( \phi \)-irreducible Markov chain on a topological space \( (\mathcal{X}, \mathcal{F}) \). If \( \{P(x, \cdot)\} \) is strongly continuous, a sufficient condition for \( \{X_n\} \) to be ergodic is the existence of a \( K \in \mathcal{K} \) and a non-negative measurable function \( g \) on \( \mathcal{X} \) such that

1. \( \int g(y) \leq g(x) - 1 \), \( x \notin K \);
2. \( \int g(y) = \lambda(x) \leq B < \infty \), \( x \in K \), for some fixed \( B > 0 \). \( \square \)

**Theorem 4.3.** Under the same strong continuity condition as in Theorem 4.1, a sufficient condition for recurrence of \( \{X_n\} \) is the existence of \( K \in \mathcal{K} \) and a non-negative measurable function \( g \) satisfying

1. \( \int g(y) \leq g(x) \), \( x \notin K \);
2. for every sufficiently large \( M \), there is a set \( K_M \in \mathcal{K} \) such that \( g(x) > M \) on \( K_M \). \( \square \)

Instead of strong continuity of \( \{P(x, \cdot)\} \), one might require weak continuity, defined by requiring that \( \int P(x, dy) g(y) \) be a continuous bounded function of \( x \) whenever \( g \) is a continuous bounded function. Example 2 in [19] shows that weak continuity is not enough to ensure that the conditions of Theorems 4.2 and 4.3 suffice for either ergodicity or recurrence. However, reasonable additional conditions suffice for the
theorems to hold under weak continuity of \( \{P(x, \cdot)\} \); for example, as shown in [19, Theorem 2], if there exists a pair of sets \( A_1 \subseteq A_2, A_1, A_2 \in \mathcal{F} \), with \( 0 < \mu(A_1) \leq \mu(A_2) < \infty \), and a continuous function \( g \) on \( \mathcal{X} \) whose support contains \( A_1 \) and is contained in \( A_2 \), then the theorems hold with weak in place of strong continuity. (See note added in proof.)

5. The conditions on a normed space

Suppose now that the space \( \mathcal{X} \) is equipped with a norm, which we denote by \( \| \cdot \| \), that \( \mathcal{F} \) is the topology generated by the norm, and that \( \mathcal{F} \) is the Borel \( \sigma \)-field on \( (\mathcal{X}, \mathcal{F}) \). In such a space one has a much better idea of what 'drift back to the centre' of the space may mean, and one can often check that the particular function \( g(x) = \|x\| \) (or \( g(x) = \|x\|/c \) for some constant \( c \)) satisfies the conditions of the theorems in the preceding section.

Let us write

\[
\gamma_x = E\{\|X_{n+1}\| - \|X_n\| \mid X_n = x\}. \tag{5.1}
\]

In terms of \( \gamma_x \), we find:

**Theorem 5.1.** If \( \{P(x, \cdot)\} \) is strongly continuous, then a sufficient condition for ergodicity is that there exists a compact set \( K \) in \( \mathcal{F} \) with \( \phi(K) > 0 \) and a constant \( c > 0 \) such that

\[
\gamma_x \leq -c, \quad \text{for all } x \notin K;
\]

and, for some \( B > 0 \),

\[
\gamma_x \leq B < \infty, \quad \text{for all } x \in K. \tag{5.3}
\]

**Proof.** Since we can write (5.1) as

\[
\gamma_x = \int P(x, dy) \|y\| - \|x\|,
\]

(5.2) shows that \( g(x) = \|x\|/c \) satisfies Theorem 4.2(i); whilst (5.3), together with the observation that \( g \) is continuous, and hence bounded, on compact \( K \), shows that \( g \) satisfies Theorem 4.2(ii). \( \square \)

If \( \mathcal{X} \) is a finite-dimensional Banach space, then we can identify the compact sets further, and we have:
Corollary 5.2. If \( \{P(x, \cdot)\} \) is strongly continuous and \( X \) is a finite-dimensional Banach space, then a sufficient condition for ergodicity is the existence of constants \( \alpha, c \) such that for all \( x \) with \( \|x\| > \alpha \),

\[
\gamma_x \leq -c,
\]

and for all \( x \) with \( \|x\| \leq \alpha \), \( \gamma_x \) is bounded above.

Proof. This utilises the facts that \( \{x: \|x\| \leq \alpha\} \) is compact when \( X \) is finite-dimensional, and that for some \( \alpha' > \alpha \), \( \phi(x: \|x\| \leq \alpha') > 0 \), since the sequence of sets \( A_n = \{x: \|x\| \leq n\} \) increases to \( X \). \( \square \)

Similar ideas prove:

Theorem 5.3. If \( \{P(x, \cdot)\} \) is strongly continuous, then a sufficient condition for recurrence is the existence of a compact \( K \) with \( \phi(K) > 0 \) such that \( \gamma_x < 0 \) for all \( x \not\in K \). \( \square \)

Corollary 5.4. If \( X \) is also finite-dimensional, a sufficient condition for recurrence is the existence of \( \alpha > 0 \) such that \( \gamma_x < 0 \) for all \( x \) with \( \|x\| > \alpha \). \( \square \)

6. Necessary conditions for ergodicity

In this section we present a converse to Theorem 3.1, and then show by an example the rather surprising result that it is not possible in general to weaken the boundedness in (3.2) to finiteness and still retain ergodicity.

Suppose that \( \{X_n\} \) is recurrent, and define

\[
R(x, A) = \sum_{n=1}^{\infty} n \cdot P^n(x, A), \quad x \in X, \ A \in \mathcal{F},
\]

where \( A \cdot P^n(x, A) \) is defined by (2.1); from [23, Theorem 7], if \( \{X_n\} \) is ergodic and \( \pi \) is the invariant probability measure for \( \{X_n\} \), then \( R(x, A) \) will be finite for \( \pi \)-almost all \( x \in A \).

Theorem 6.1. If \( \{X_n\} \) is ergodic and some set \( K \in \mathcal{F}_\pi \) exists such that \( R(x, K) \) is bounded for all \( x \in K \), then there is a non-negative solution to

\[
\int_X P(x, dy) g(y) \leq g(x) - 1 \quad \text{for } \pi \text{-almost all } x \in K^c; \tag{6.1}
\]

\[
\int_X P(x, dy) g(y) = \lambda(x) \leq B < \infty \quad \text{for all } x \in K. \tag{6.2}
\]
Proof. From the definition, we have formally that, for all \(x,\)

\[
R(x, K) = P(x, K) + \sum_{n=2}^{\infty} n \int_{K^c} P(x, dy) K P^{n-1}(y, K)
\]

\[
= P(x, K) + \int_{K^c} P(x, dy) [R(y, K) + F(y, K)] .
\] (6.3)

Now ergodicity implies recurrence, which implies \(F(y, K) = 1\) for at least \(\pi\)-almost all \(y \in \mathcal{X}\) (since \(\{X_n\}\) is \(\pi\)-irreducible), from Lemma 2.2(ii); since, from the stationarity equations, \(\pi(A) = 0\) implies \(\pi\{x: P(x, A) > 0\} = 0\), we have therefore from (6.3) that, for \(\pi\)-almost all \(x,\)

\[
R(x, K) = 1 + \int_{K^c} P(x, dy) R(y, K) .
\] (6.4)

Further, iterating (6.4) gives, for \(\pi\)-almost all \(x\) and any \(n \geq 1,\)

\[
R(x, K) \geq \int_{K^c} K^n P(x, dy) R(y, K) ;
\] (6.5)

since, by assumption, \(R(x, K)\) is bounded for \(x \in K,\) (6.5) ensures that \(R(y, K)\) is at least finite for \(\pi\)-almost all \(y.\) Now set \(g(\cdot) = R(\cdot, K)\) for \(y \in K^c\) with \(R(y, K) < \infty,\) and \(g(y) = 0\) elsewhere. It then follows from (6.4) that (6.1) and also, because we have assumed \(R(x, K)\) bounded on \(K,\) (6.2) hold for this \(g.\) \(\square\)

This theorem shows that, under fairly general conditions, (3.1) and (3.2) are in fact equivalent to ergodicity. The existence of the solution \(g(x) = R(x, K)\) for countable chains is noted by Foster [5]; this solution also shows that the type of solution to (3.1) and (3.2) considered in Theorem 5.1 (that is, a strictly unbounded solution) may not always exist if the chain under consideration is actually of a renewal type, and so it may be profitable in such cases to try and construct bounded solutions to (3.1) and (3.2). This situation contrasts with that of Theorem 3.2.

We now construct an example of a chain \(\{X_n\}\) satisfying

\[
\int_{x} P(x, dy) g(y) \leq g(x) - 1, \quad x \in K^c ,
\] (6.6)

\[
\int_{x} P(x, dy) g(x) = \lambda(x) < \infty, \quad x \in K ,
\] (6.7)

but for which \(\lambda(x)\) is not bounded on \(K;\) and which is recurrent, but not ergodic. The basis for this construction is the observation that if \(K\) is a set such that \(R(x, K) < \infty\) for all \(x\) not in \(K,\) and such that
for all \( x \in K \), then as in the proof of Theorem 6.1, provided \( F(y, K) \equiv 1 \) for \( y \in K^c \), the function \( g(x) \) defined as \( g(x) \equiv 0 \) on \( K \) and \( g(x) = R(x, K) \) on \( K^c \) will satisfy (6.6) and (6.7). If, on the other hand, there is a set \( A \in \mathcal{F}_\mu \) (where \( \mu \) is now the unique invariant measure, since \( \{X_n\} \) is recurrent because of the assumption that \( F(y, K) \equiv 1 \), from Lemma 2.2(i)) such that \( R(x, A) = \infty \) for every \( x \in A \), it follows from [23, Theorem 7] that \( \{X_n\} \) is not ergodic.

Our example is a renewal-type chain on \([0, \infty)\), and has the transition function given by the following conditions (where \(|A|\) denotes Lebesgue measure):

(a) for \( x \geq 4 \),
\[
P(x, A) = |A|, \quad A \subseteq [x - 2, x - 1];
\]
(b) for \( x \in [3,4] \),
\[
P(x, A) = (x - 3) |A|, \quad A \subseteq [x - 2, x - 1],
\]
\[
P(x, \{0\}) = (4 - x);
\]
(c) for \( x \in [1,3] \),
\[
P(x, \{0\}) = 1;
\]
(d) for \( x \in [\frac{1}{2}, 1] \),
\[
P(x, \{0\}) = 2x - 1,
\]
\[
P(x, A) = (2 - 2x) |A|, \quad A \subseteq [(2 - 2x)^{-2}, (2 - 2x)^{-2} + 1];
\]
(e) for \( x \in [0, \frac{1}{2}] \),
\[
P(x, A) = 2x|A|, \quad A \subseteq [1, 2],
\]
\[
P(x, A) = (1 - 2x) |A|, \quad A \subseteq [0, 1].
\]

It is easily verified that:
\( \{P(x, \cdot)\} \) is strongly continuous, so that \([0,1] \in \mathcal{F}_\mu;\)
\( \{X_n\} \) is \( \phi \)-irreducible with \( \phi(A) = |A| + \delta(0, A); \)
\( F(x, [0,1]) \equiv 1 \) for \( x \notin [0,1] \), so that \( \{X_n\} \) is recurrent;
\( R(x, [0,1]) \leq x \) for \( x \notin [0,1] \), and \((1 - x)^{-1} - 1 \leq R(x, [0,1]) \leq (2 - 2x)^{-1} + 2 \) for \( x \in [0,1] \), so that (6.6) and (6.7) hold but (3.2) fails;
but \( \{0\} \in \mathcal{F}_\mu \) and \( R(0, \{0\}) = \infty \), so that \( \{X_n\} \) is not ergodic.
7. An application to queueing theory

The known results on a countable state space, which our results extend, have been used frequently to prove the ergodicity of various Markov chains occurring in queueing theory (for recent examples see [1, 2, 8]). In [2], Callahan denotes by $W_n$ the waiting time of the $n^{th}$ customer in a queue where the service time depends on the waiting time, and, assuming that both the (conditional) service times and the interarrival times have discrete non-negative distributions (so that the sequence $\{W_n\}$ is a Markov chain on a countable space), uses the results of [18] to show that $\{W_n\}$ is ergodic under mild conditions. As Callahan remarks, the restriction to discrete service time and interarrival time distributions (necessitated by the desire to have $\{W_n\}$ a chain on the integers) is a serious one. He conjectures that his theorem holds in general, and we now show that under some continuity conditions this is true, using the conditions of Section 5.

Let $T$ be a non-negative random variable with finite mean, and let $S(w)$, for each $w \geq 0$, be a class of non-negative random variables. Assume that the interarrival times of customers in a single server queue are independently and identically distributed as $T$, and that the service time of an arriving customer is $S(w)$, conditional on the customer's waiting time being $w$. Let $W_n$ denote the waiting time of the $n^{th}$ arriving customer, so that $W_n$ is given recursively by

$$W_n = \max(0, W_{n-1} + S(W_{n-1}) - T). \tag{7.1}$$

From (7.1), $\{W_n\}$ is a Markov chain on $[0, \infty)$. We assume that the distributions of $S(x)$ and $T$ have densities with respect to Lebesgue measure $l$, given respectively by $s_x(\cdot)$ and $t(\cdot)$, and that these are such that $\{W_n\}$ is $l$-irreducible. A sufficient condition for this, paralleling that used by Callahan, is that the density $(s_x * (-t))$ of $S(x) - T$ be positive for all $y$ in a region $(-\delta, \delta)$ containing the origin, where $\delta$ is fixed and independent of $x$.

**Theorem 7.1.** Sufficient conditions for the ergodicity of the Markov chain $\{W_n\}$ are that

(i) the densities $s_x(y)$ of the service times are, for each fixed $y$, continuous functions of $x$;

(ii) there exists an $N > 0$ such that

$$E\{S(w)\} \leq B < \infty, \quad w \leq N; \tag{7.2}$$

$$E\{S(w)\} \leq E\{T\} - \epsilon, \quad w > N. \tag{7.3}$$
Proof. The first condition ensures that the transition law \( \{P(x, \cdot)\} \) of \( \{W_n\} \) is strongly continuous. This can be seen by using (7.1) to get (when \( 0 \notin A \))

\[
P(x, A) = \int_0^\infty P[S(x) \in A - x + y] t(y) \, dy
= \int_0^\infty \left[ \int_{A-x+y} s_x(u) \, du \right] t(y) \, dy,
\]

(7.4)

and noting that, if \( x_n \to x \), the continuity of \( s_x(u) \) together with the continuity of the integral operator gives

\[
\int_{A-x_n-y} s_{x_n}(u) \, du \to \int_{A-x-y} s_x(u) \, du,
\]

and that the limit can be taken inside the integral in (7.4) by dominated convergence. A similar result holds when \( A = \{0\} \). That (7.2) and (7.3) imply that both

\[
E\{W_n - W_{n-1} \mid W_{n-1} = x\} \leq \beta, \quad x \leq N,
\]

and

\[
E\{W_n - W_{n-1} \mid W_{n-1} = x\} \leq -\varepsilon, \quad x > N
\]

(and hence ergodicity by Theorem 5.1) follows as in Callahan's proof [2]. 

Corollary 7.2. If \( \{W_n\} \) satisfies the continuity condition (i) of Theorem 7.1 and also

(i)' the mean service time \( \mu(x) = E\{S(x)\} \) is continuous in \( x \), and finite for all \( x \),

then a sufficient condition for ergodicity is that

\[
\limsup_{x \to \infty} E\{S(x)\} < E\{T\}.
\]

(7.5)

Proof. The inequality (7.5) implies that for some \( N > 0 \) there exists \( \varepsilon > 0 \) such that \( E\{S(x)\} < E\{T\} - \varepsilon \) for all \( x > N \). The continuity of \( \mu(x) \), and its finiteness, then imply that \( \mu(x) \) is bounded on the (compact) set \([0, N]\), and the conditions of the theorem hold. 

8. Historical note on ergodicity and recurrence conditions

In the special case of a countable state space, the results of Sections 3–5 have been known in one form or another for some years. However, a certain amount of confusion has on occasion been caused by the similarity of these results to other, earlier, results which give sufficient conditions for an analogue of ergodicity for reducible chains, and which are actually vacuous in the irreducible case. The object of this final section is to try to clarify this situation somewhat, whilst also pointing out the historical development of the results.

Suppose then that \( \{X_n\} \) is a chain on \( N = \{0, 1, \ldots\} \) (the denumerable case usually studied) with transition matrix \( P = (p_{ij}) \) and that \( \{X_n\} \) is irreducible in the classical sense. The analogues of our results are:

**Theorem 8.1.** (i) A sufficient condition for \( \{X_n\} \) to be ergodic is the existence of a sequence \( \{y_j\}, \infty > y_j \geq 0 \), and an integer \( N \) such that

\[
\sum_{k=0}^{\infty} p_{ik} y_k < y_i - 1, \quad i \geq N; \tag{8.1}
\]

\[
\sum_{k=0}^{\infty} p_{ik} y_k < \infty, \quad i < N. \tag{8.2}
\]

(ii) For (8.1) and (8.2) to hold, it suffices that

\[
\lim \sup_{i \to \infty} E\{X_n - X_{n-1} \mid X_{n-1} = i\} < 0, \tag{8.3}
\]

\[
E\{X_n \mid X_{n-1} = i\} < \infty \quad \text{for all } i. \quad \Box \tag{8.4}
\]

**Theorem 8.2.** (i) A sufficient condition for \( \{X_n\} \) to be recurrent is the existence of a sequence \( \{y_j\}, \infty > y_j \geq 0 \), and an integer \( N \) such that

\[
\sum_{k=0}^{\infty} p_{ik} y_k < y_i, \quad i \geq N; \tag{8.5}
\]

\[
y_j \to \infty \quad \text{as } j \to \infty. \tag{8.6}
\]

(ii) For (8.5) and (8.6) to hold it suffices that

\[
\lim \sup_{i \to \infty} E\{X_n - X_{n-1} \mid X_{n-1} = i\} < 0. \quad \Box
\]

Both Theorems 8.1(i) and 8.2(i) are given by Foster [5], for the special case \( N = 1 \), although in the discussion of [11] he gives Theorem 8.1(i)
for the case of general $N$; and the general case is also quoted soon after by, for example, Moustafa [16]. Mauldon [14] proves Theorem 8.1(i) in the somewhat more general context of reducible Markov chains. He shows that (8.1) and (8.2) then suffice for $\{X_n\}$ to be non-dissipative; that is, from any starting point, the chain is eventually absorbed by an ergodic class.

Kingman [12] also mentions that Foster's proof extends easily to general $N$. He constructs some two-dimensional lattice random walks with boundaries where the solution indicated is not the usual Euclidean norm that our Section 5 would suggest. Kingman's methods are continuous rather than lattice in type, and would fit much more happily in the context of our Section 4 than in the discrete framework needed for the Foster–Mauldon methods. Pakes [18] gives both the theorems in the form above, and mentions explicitly the solution $y_i = i/\varepsilon$ (given by Theorems 8.1(ii) and 8.2(ii)), which is usually the most easily checked candidate to solve (8.1)–(8.2) or (8.5)–(8.6) for 'random-walk-like' chains. A form of Theorem 8.2 for the reducible case is in [25], where it is shown that (8.5) and (8.6) imply that the chain is eventually absorbed by a recurrent class.

There seems to be some confusion between these theorems and the following:

**Theorem 8.3.** A sufficient condition for $\{X_n\}$ to be non-dissipative is that there exists $\{y_i\}, \infty > y_i \geq 0$, such that

$$\sum_{k=0}^{\infty} p_{ik} y_k < y_i \quad \text{for all } i \in \mathbb{N}. \quad (8.7)$$

$$y_i \to \infty \quad \text{as } i \to \infty. \quad (8.8)$$

This is first proved by Foster [4], for the special case $y_i = i$, and by Kendall [10] in the above form. These seem to be the earliest of the sequence of papers we have discussed, and the point which must be made is that Theorem 8.3 is vacuous if $\{X_n\}$ is irreducible. Consequently, it is useless as a condition for ergodicity of irreducible chains; yet it has been quoted in this context by, for example, Sarymsakov [21], and also as recently as 1971 by Pruitt [20] (who point out correctly, that Mauldon’s proof of Theorem 8.1 in [14] predates that of Pakes [18]; but who also claims, incorrectly, that Pakes’ proof of Theorem 8.2 is predated by Theorem 8.3). To see that the conditions (8.7) and (8.8) can never be met in the irreducible case, it suffices to note Kendall’s
remark in [10] that, if $S$ denotes the set on which $\{y_j\}$ satisfying (8.7) and (8.8) attains its minimum, then $S$ is finite and all transitions out of $S$ have probability zero.

Finally, we remark that our ergodic conditions in Theorem 5.1 can be seen as a direct descendant of the famous result of Lindley [13] for random walk on a half-line; and that a generalization of Lindley's result for countable state spaces, giving a 'stochastic comparison' criterion for ergodicity, given by Gnedenko and Kovalenko [6, p. 118], is in fact a special case of Theorem 8.1(ii) (although the criterion as stated in [6] is not complete; it implies only (8.3), whilst some extra condition such as (8.4) is also necessary, as can be seen by considering examples such as that in [15, Section 6]).

Note added in proof.

In a sequel to [19] we show that in Lemma 4.1 strong continuity can be replaced by weak continuity provided that the irreducibility measure $\phi$ is regular and has support of second category. These latter conditions on $\phi$ are automatically satisfied for the normed spaces $\mathcal{X}$: Section 5 provided that $\mathcal{X}$ is complete in the norm, that is, when $\mathcal{X}$ is a Banach space.

Acknowledgements

I am very grateful to Professor D. Vere-Jones, who brought this problem to my attention and with whom I had some stimulating conversations on these topics; and to Dr. E. Seneta and Dr. D. Daley, both of whom brought to my attention invaluable references. This work was done whilst I held an Australian National University Postdoctoral Fellowship in the Department of Statistics of the Institute of Advanced Studies, Canberra.

References


