On Modules with Weyl Filtration for Schur Algebras of Finite Type

Corina Sáenz

Facultad de Matemáticas, Univ. de Guanajuato, Apdo. Postal 402, Guanajuato, Guanajuato, Mexico
E-mail: corina@fractal.cimat.mx

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1. INTRODUCTION

In the study of highest weight categories arising in the representation theory of Lie algebras and algebraic groups, the category \( \mathcal{R}(\Delta) \) of modules which have a filtration by Weyl modules is of major interest.

In this paper we deal with the highest weight categories corresponding to blocks of Schur algebras which are of finite representation type. First we shall give a complete description of all indecomposable modules with Weyl filtration. This is done by determining the relative Auslander–Reiten quiver. (See Figs. 1 and 2 in Section 4.) We also show that such blocks are characterized as follows: They are precisely the quasi-hereditary algebras which have a duality fixing the simple modules, such that their Weyl modules are multiplicity-free and that \( \tau(\Delta(i)) = \Delta(i + 2) \), for all \( 1 \leq m - 2 \) where \( \tau \) is the Auslander–Reiten translation. Here the simple modules are labelled by \( \{1, 2, \ldots, m\} \) and \( \leq \) is the natural order.

The algebra structure of these blocks was determined in [3]. Any such block is Morita equivalent to a quasi-hereditary algebra \( A_m \), which has a presentation

\[
\begin{array}{cccccccc}
1 & \beta_1 & \cdots & \alpha_{m-2} & \alpha_{m-1} & \beta_{m-1} & \cdots & \beta_1 \\
\end{array}
\]

modulo the ideal \( I \) of the path algebra generated by all paths \( \alpha_{t+1} \beta_t \beta_{t+1}, \beta_t \alpha_{t+1} \alpha_t, \beta_t - \beta_{t+2}, \alpha_{m-1} \alpha_{m-2} \) for \( 1 \leq t \leq m - 2 \) and \( \alpha_{m-1} \beta_{m-1} \).
Moreover, there is a one to one correspondence between the indecomposable (non-projective) modules with a Weyl filtration and the set containing the highest weights of their $\Delta$-quotients; see Corollary 9.

2. PRELIMINARIES

Let $A$ be a finite-dimensional algebra over an algebraically closed field $K$ and let $X$ be a finite poset in bijective correspondence with the isomorphism classes of simple $A$-modules. For each $\lambda \in X$, let $E(\lambda)$ be a simple module in the isomorphism class corresponding to $\lambda$ and $P(\lambda)$ be a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$, called a standard module. Dually, let $Q(\lambda)$ be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$; we call $\nabla(\lambda)$ a costandard module.

Let $\Delta$ denote the subcategory of all standard $A$-modules $\Delta(\lambda)$, $\lambda \in X$, and let $\mathcal{R}(\Delta)$ denote the full subcategory of all the good $A$-modules, that is, the subcategory of all $A$-modules having a filtration with factors in $\Delta$.

The algebra $A$ is said to be quasi-hereditary with respect to $(X, \leq)$ provided that for each $\lambda \in X$ we have:

1. $\text{End}(\Delta(\lambda)) = K$
2. $P(\lambda)$ belongs to the category $\mathcal{R}(\Delta)$, and moreover, $P(\lambda)$ has a $\Delta$-filtration with quotient $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

Let $p$ denote the characteristic of the field $K$ and let $B$ denote a block component of finite representation type of a Schur algebra. By [3] the algebra $B$ is Morita equivalent to $A_m$ for some $m \geq 1$, as defined above.

The ordering of the simple $A_m$-modules appearing in the block $B$ is given by the dominance order of the partitions of the block (see [3]). Then the standard modules are the usual Weyl modules, and $\nabla(i) = \Delta(i)^\omega$ where $(\cdot)^\omega$ denotes a contravariant duality. This dominance order for partitions in such a block is linear; and this is the only possible ordering which makes the algebra $A_m$ quasi-hereditary. This follows easily from the definition of a quasi-hereditary algebra.

We shall now summarize the results of this paper. We determine the relative Auslander–Reiten quiver $\Gamma_{\mathcal{R}(\Delta)}$ of the algebra $A_m$. In Section 3 we describe the $\tau$-orbit of the standard and costandard $A_m^\tau$-modules, where $\tau$ denotes the (usual) Auslander–Reiten translate, and we show that the
middle term $M$ of the (usual) Auslander–Reiten sequence

$$0 \to \tau(N) \to M \to N \to 0$$

is indecomposable, where $N$ is either a standard or a costandard $A_n$-module ($N \neq \nabla(2)$ and $m > 2$). As a consequence of this, we shall prove that, in the Auslander–Reiten quiver $\Gamma(A_m)$, there are two wings of good and cogood $A_m$-modules, respectively. In Section 4, we describe the relative Auslander–Reiten quiver $\Gamma_{\mathcal{R}(\Delta)}$ for the subcategory $\mathcal{R}(\Delta)$ of $A_m$-mod and prove that the good $A_m$-modules in the two wings of the Auslander–Reiten quiver $\Gamma(A_m)$ together with the indecomposable projective $A_m$-modules $P(1), \ldots, P(m - 1)$ are all the indecomposable good $A_m$-modules. The last section contains a characterisation of these blocks.

From now on, $A$ is a quasi-hereditary algebra. We will consider (finitely generated left) $A$-modules; maps between $A$-modules will be written on the right hand of the argument. Thus the composition of two maps $f: M_1 \to M_2$, $g: M_2 \to M_3$ will be denoted by $fg$. The category of all $A$-modules will be denoted by $A$-mod. All subcategories considered will be full and closed under isomorphisms, so usually we will describe subcategories by just specifying their objects (up to isomorphisms).

We review some basic facts of the subcategory $\mathcal{R}(\Delta)$ of $A$-mod. For the proofs, we refer to [7].

(P.1) $\text{Ext}^1_A(\Delta(i), \Delta(j)) = 0$, for $i \geq j$.

(P.2) The projective $A$-modules belong to the category $\mathcal{R}(\Delta)$ and the injective $A$-modules belong to the category $\mathcal{R}(\nabla)$.

(P.3) We have $\text{Ext}^t_A(X, Y) = 0$ for all $X \in \mathcal{R}(\Delta)$, $Y \in \mathcal{R}(\nabla)$, and all $t \geq 1$.

(P.4) The Ext-projective modules in $\mathcal{R}(\Delta)$ are the projective $A$-modules; the Ext-injective modules in $\mathcal{R}(\Delta)$ are the $A$-modules in the subcategory $\omega = \mathcal{R}(\Delta) \cap \mathcal{R}(\nabla)$.

(P.5) The category $\mathcal{R}(\Delta)$ has relative Auslander–Reiten sequences.

(P.6) $[P(j): \Delta(i)] = [\Delta(i): L(j)]$. Here $[M: \Delta(i)]$ denotes the filtration multiplicity, for $M$ in $\mathcal{R}(\Delta)$, and for any $N$, $[N: L(j)]$ denotes the composition multiplicity.

We shall need the following definition. Let us consider the quiver $Q$ of the path algebra $A_n$. If $w = w_1w_2 \cdots w_n$ (or $w = 1_w$) is a string of length $n \geq 0$, let $Q_w$ be the quiver with underlying graph $A_{n+1}$,
where the edge labelled with $w_i$ is an arrow pointing to the left if $w_i$ is an arrow, and it is labelled with an arrow pointing to the right if $w_i$ is a formal inverse of an arrow. Then we refer to the representation of $Q_w$ as the string $A_m$-module where at each vertex the space is $K$, and for each arrow, the map is the identity map (see [2]).

3. AUSLANDER–REITEN QUIVER

Consider the algebra $A_m$ ($m \geq 2$). The following can be deduced directly from the presentation of the algebra.

For $2 \leq i \leq m - 1$ we have $P(i) = \hat{Q}(i)$ and $\text{rad } P(i)/\text{soc } P(i) = E(i + 1) \oplus E(i - 1)$ for $2 \leq i \leq m - 1$. The standard and costandard $A_m$-modules are given by

$$
\Delta(1) = \nabla(1) = E(1), \quad \Delta(i) = \begin{array}{c} i \end{array}, \quad \text{and} \quad \nabla(i) = \begin{array}{c} i - 1 \\ \beta_{i-1} \\ \alpha_{i-1} \\ i - 1 \\ i \end{array}
$$

for $2 \leq i \leq m$.

By (P.4), the indecomposable Ext-injective $A_m$-modules are $\Delta(1)$ and $P(i - 1), 2 \leq i \leq m$.

In order to describe the $\tau$-orbit of the standard and costandard $A_m$-modules we need the following proposition. (Here $\tau$ denotes the usual Auslander–Reiten translate; see [1].)

From now on we will assume $m > 2$ (unless otherwise specified).

**Proposition 1.** For the standard and costandard $A_m$-modules we have

1. The $A_m$-module $\tau(\Delta(i))$ is isomorphic to the standard $A_m$-module $\Delta(i + 2)$ for $1 \leq i \leq m - 2$.

2. The $A_m$-module $\tau(\Delta(m - 1))$ is isomorphic to the simple $A_m$-module $E(m)$.

3. The $A_m$-module $\tau^{-1}(\nabla(i))$ is isomorphic to the costandard $A_m$-module $\nabla(i + 2)$ for $1 \leq i \leq m - 2$.

4. The $A_m$-module $\tau^{-1}(\nabla(m - 1))$ is isomorphic to the simple $A_m$-module $E(m)$.

5. The $A_m$-module $\tau^{-1}(\Delta(2))$ is isomorphic to the costandard $A_m$-module $\nabla(2)$.

**Proof.** This follows from [5, p. 74].
So, for $m$ even we see that the $\tau$-orbit containing the standard and costandard $A_m$-module $E(1)$, is given by the diagram

$$E(m) \xleftarrow{\tau} \Delta(m - 1) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \Delta(3) \xleftarrow{\tau} \Delta(1) \xleftarrow{\tau} \nabla(3) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \nabla(m - 1) \xleftarrow{\tau} E(m).$$

Similarly, the $\tau$-orbit containing the standard $A_m$-module $\Delta(2)$, is given by the diagram

$$\Delta(m) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \Delta(2) \xleftarrow{\tau} \nabla(2) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \nabla(m).$$

In the case when $m$ is odd the $\tau$-orbit containing the standard and costandard $A_m$-module $E(1)$ is given by

$$E(m) \xleftarrow{\tau} \Delta(m - 1) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \Delta(3) \xleftarrow{\tau} \Delta(1) \xleftarrow{\tau} \nabla(3) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \nabla(m - 1) \xleftarrow{\tau} E(m).$$

and the $\tau$-orbit containing the standard $A_m$-module $\Delta(2)$ is given by

$$E(m) \xleftarrow{\tau} \Delta(m - 1) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \Delta(2) \xleftarrow{\tau} \nabla(2) \xleftarrow{\tau} \cdots \xleftarrow{\tau} \nabla(m - 1) \xleftarrow{\tau} E(m).$$

Now we establish the position of the standard and costandard modules in the Auslander–Reiten quiver $\Gamma(A_m)$. We use Proposition 1, and we consider only the case of the standard $A_m$-modules; the other case is analogous.

From Proposition 1 and cases (1), (2), and (5) we have the Auslander–Reiten sequences

$$0 \to \Delta(i + 2) \xrightarrow{f_i} M_i \xrightarrow{g_i} \Delta(i) \to 0 \quad \text{for } 1 \leq i \leq m - 2, \quad (2)$$

$$0 \to E(m) \xrightarrow{f_{m-1}} M_{m-1} \xrightarrow{g_{m-1}} \Delta(m - 1) \to 0 \quad (3)$$

and

$$0 \to \Delta(2) = \text{rad } P(1) \xrightarrow{} E(2) \oplus P(1) \xrightarrow{} \nabla(2) \to 0. \quad (4)$$

The following proposition shows that the $A_m$-module $M_i$ for $1 \leq i \leq m - 1$ is indecomposable. Therefore the standard $A_m$-modules are either at the top or at the bottom of the Auslander–Reiten quiver $\Gamma(A_m)$ (see Fig. 1 in Section 4), and the same is true for the costandard $A_m$-modules.

**Proposition 2.** The $A_m$-module $M_i$ for $1 \leq i \leq m - 1$ defined in (2) and (3), respectively, is indecomposable.

**Proof.** This follows for example from the fact that $\mathcal{S}(\Delta)$ is closed under direct summands. $\blacksquare$
COROLLARY 3. We have two wings of good $A_m$-modules and two wings of cogood $A_m$-modules in the Auslander–Reiten quiver $\Gamma(A_m)$. (For the definition of a wing, see [5].)

Proof. The proof follows from Propositions 1 and 2.

So for $m$ even the two wings of the good $A_m$-modules are given by the diagrams

\[
\begin{aligned}
\Delta(m - 1) &\quad \xleftarrow{\tau} \quad \Delta(m - 3) \quad \cdots \quad \Delta(3) &\quad \xrightarrow{\tau} \quad \Delta(1) \\
\Delta(m - 3) &\quad \xrightarrow{\tau} \quad \Delta(1) \\
\Delta(m - 1) &\quad \xrightarrow{\tau} \quad \Delta(3) \\
\cdots &
\end{aligned}
\]

and

\[
\begin{aligned}
\Delta(m - 2) &\quad \xrightarrow{\tau} \quad \Delta(m - 2) \quad \cdots \quad \Delta(4) &\quad \xleftarrow{\tau} \quad \Delta(2) \\
\Delta(m) &\quad \xrightarrow{\tau} \quad \Delta(m) \\
\Delta(m) &\quad \xrightarrow{\tau} \quad \Delta(m) \\
\cdots &
\end{aligned}
\]

where we write

\[
\begin{aligned}
\Delta(1) \\
\vdots \\
\Delta(m - 1)
\end{aligned}
\]

meaning the string $A_m$-module

\[
\begin{array}{cccc}
m - 1 & m - 3 & 3 & 1 \\
\beta_a & \beta & \beta_a & \\
m - 2 & m - 4 & 2 & \\
\end{array}
\]

and similarly the other modules with such notation.
Using long exact homology sequences one proves:

**Proposition 4.** Let \( m \geq 2 \). For the quasi-hereditary algebra \( A_m \) we have

\[
\dim_k \text{Ext}^1_{A_m}(\Delta(i), \Delta(j)) = \begin{cases} 
1, & j = i + 1 \\
1, & j = i + 2 \\
0, & \text{otherwise}
\end{cases}
\]

There is a dual result, for costandard \( A_m \)-modules.

### 4. Relative Auslander–Reiten Quiver

In this section we describe the relative Auslander–Reiten quiver \( \Gamma_{\mathcal{F}(\Delta)} \) for the subcategory \( \mathcal{F}(\Delta) \) of \( A_m \)-mod. Our result will be that the \( A_m \)-modules lying in the two wings of the Auslander–Reiten quiver \( \Gamma(A_m) \) of \( A_m \) plus the indecomposable projective-injective \( A_m \)-modules are all the good \( A_m \)-modules of \( \mathcal{F}(\Delta) \). For the definitions of a relative irreducible map in \( \mathcal{F}(\Delta) \) and of a relative Auslander–Reiten quiver we refer to [6]. In the proof of Theorem 1 and in Fig. 1 we assume \( m > 2 \) to be even (for \( m = 2 \) see Proposition 7).

We start by giving a brief summary of the definitions and results that we shall use; we refer to [7] for the proofs.

To find relative Auslander–Reiten sequences, Ringel has mentioned the following:

**Lemma 5.** Consider the (usual) Auslander–Reiten sequence of the \( A \)-module \( Z, 0 \rightarrow \tau(Z) \rightarrow Y \rightarrow Z \rightarrow 0 \), and suppose \( Z \in \mathcal{F}(\Delta) \). Let \( h : B \rightarrow Y \) be a right \( \mathcal{F}(\Delta) \)-approximation of the \( A \)-module \( Y \) and \( \gamma = hg : B \rightarrow Z \). Then the right minimal part of \( \gamma \) is a relative left almost split homomorphism for \( Z \).

(Recall from [7] that a right \( \mathcal{F}(\Delta) \)-approximation of an \( A \)-module \( M \) is an \( A \)-homomorphism \( \gamma : X \rightarrow M \) with \( X \in \mathcal{F}(\Delta) \) such that for any \( A \)-homomorphism \( \gamma' : X' \rightarrow M \) with \( X' \in \mathcal{F}(\Delta) \) there exists an \( A \)-homomorphism \( e : X' \rightarrow X \) satisfying \( \gamma' = e\gamma \).)

To find a right \( \mathcal{F}(\Delta) \)-approximation of an \( A \)-module \( M \), one may use:

**Lemma 6.** Let \( M \in A \)-mod and suppose

\[
0 \rightarrow Y \rightarrow X \xrightarrow{\gamma} M \rightarrow 0
\]

is exact with \( X \in \mathcal{F}(\Delta) \) and \( Y \in \mathcal{F}(\nabla) \). Then \( \gamma \) is a right \( \mathcal{F}(\Delta) \)-approximation of \( M \).
The following results describe the relative Auslander–Reiten quiver for the category \( \mathcal{A}(\Delta) \):

**Proposition 7.** For \( m = 2 \), there is only one relative Auslander–Reiten sequence for the category \( \mathcal{A}(\Delta) \) of good \( A_2 \)-modules, and it is given by

\[
0 \to \Delta(2) \to P(1) \to \Delta(1) \to 0.
\]

**Proof.** This follows by (P.4) and Proposition 4. \( \blacksquare \)

**Proposition 8.** Let \( m > 2 \) and let \( m \) be even. Then

1. For \( i \) odd and \( i \neq 1, m - 1 \), the relative Auslander–Reiten sequence is of the form

\[
\begin{align*}
\Delta(2) & \quad \Delta(2) & \quad \Delta(m - i + 2) \\
0 \to & : \quad \to : \quad \oplus P(m - i) \oplus : \\
\Delta(m - i + 1) & \quad \Delta(m - i - 1) & \quad \Delta(m - 1) \\
\end{align*}
\]

\[
\Delta(m - i) \\
\Delta(m - i + 2) \\
\vdots \\
\Delta(m - 1) \to 0.
\]

2. For \( i = 1 \), the relative Auslander–Reiten sequence is of the form

\[
\begin{align*}
\Delta(2) & \quad \Delta(4) & \quad \Delta(2) \\
0 \to & : \quad \to P(m - 1) \oplus : \quad \to \Delta(m - 1) \to 0. \\
\Delta(m) & \quad \Delta(m - 2) \\
\end{align*}
\]

3. For \( i = m - 1 \), the relative Auslander–Reiten sequence is of the form

\[
\begin{align*}
\Delta(3) & \quad \Delta(1) \\
0 \to & \Delta(2) \to P(1) \oplus : \quad \to \Delta(3) \to 0. \\
\Delta(m - 3) & \quad \Delta(m - 1) \\
\Delta(m - 1) & \quad \Delta(m - 1)
\end{align*}
\]
(4) For \( i \) even \( i \neq m - 2,0 \), the relative Auslander–Reiten sequence is of the form

\[
\begin{array}{ccc}
\Delta(1) & \Delta(1) & \Delta(m - i + 2) \\
0 & \rightarrow & \rightarrow \\
\Delta(m - i + 1) & \Delta(m - i - 1) & \Delta(m) \\
\rightarrow & \rightarrow & 0 \\
\Delta(m) & \\
\end{array}
\]

\( \oplus P(m - i) \oplus \)

(5) For \( i = m - 2 \), the relative Auslander–Reiten sequence is of the form

\[
\begin{array}{ccc}
\Delta(1) & \Delta(1) & \Delta(2) \\
0 & \rightarrow & \rightarrow \\
\Delta(3) & \Delta(1) & \oplus P(2) & \rightarrow \\
\rightarrow & \rightarrow & 0 \\
\Delta(m) & \Delta(m) & \\
\end{array}
\]

(6) If \( i = 0 \), then \( \Delta(m) \) is projective.

Proof. This follows by Lemma 5 and Lemma 6, respectively.

The following theorem follows directly from the previous proposition.

**Theorem 1.** For \( m > 2, m \) even, the relative Auslander–Reiten quiver for the category \( \mathcal{F}(\Delta) \) is given in Fig. 1. In particular the indecomposable \( A_m \)-modules of the category \( \mathcal{F}(\Delta) \) are the modules in the two wings (5) and (6) of the Auslander–Reiten quiver \( \Gamma(A_m) \) of \( A_m \) plus the projective–injective indecomposable \( A_m \)-modules \( P(1), \ldots, P(m - 1) \).

In Fig. 1 we have repeated two columns in the right and left sides, and we have omitted the valuation \( (d', d'') \) since it is always \( (1, 1) \) (see [6, Theorem in Chap. 6]).

Now consider the case when \( m \) is odd. If \( m = 1 \) then the relative Auslander–Reiten quiver consists of one vertex only; the only indecomposable module in the category \( \mathcal{F}(\Delta) \) is \( \Delta(1) \).

**Theorem 2.** For \( m > 1, m \) odd, the relative Auslander–Reiten quiver for the category \( \mathcal{F}(\Delta) \) is given by Fig. 2. In particular the indecomposable \( A_m \)-modules of the category \( \mathcal{F}(\Delta) \) are the modules in the two wings (5) and (6) of the Auslander–Reiten quiver \( \Gamma(A_m) \) of \( A_m \) plus the projective–injective indecomposable \( A_m \)-modules \( P(1), \ldots, P(m - 1) \).
In Fig. 2 we have repeated two columns on the right and left sides, and we have omitted the valuation \((d', d'')\) since it is always \((1, 1)\) (see [6, Theorem in Chap. 6]).

So, we have that for the case \(m\) even, the number of \(\tau\)-orbits is \(1 + (m - 1) = m\). For \(m\) odd, the number of \(\tau\)-orbits is \((m + 1)/2 + (m - 1)/2\).

Let \(M \in \mathcal{R}(\Delta)\) be an indecomposable non-projective \(A_m\)-module. Then the highest weights of its \(\Delta\)-quotients are given by the set \(H_{i,j} = \{E(j), E(j + 2), \ldots, E(j + 2l)\}\) where \(j\) is the smaller \(i\) such that \([M, \Delta(j)] = 1\) and \(j + 2l\) is the larger \(i\) such that \([M, \Delta(j + 2l)] = 1\). So, in this case, the highest weights have the same parity and we have:

**Corollary 9.** There is a one-to-one correspondence between the sets \(H_{i,j}\) and the indecomposable (non-projective) modules in \(\mathcal{R}(\Delta)\).

## 5. CONDITION FOR FINITE REPRESENTATION TYPE

In this section we give a condition for a block of a Schur algebra to be of finite representation type. For doing so, let us fix \(A\) to be a quasi-hereditary algebra with a duality fixing the simple \(A\)-modules. Let \(E(1) \ldots E(m)\) be the simple \(A\)-modules and let \(\tau\) denote the usual Auslander–Reiten translate. We recall Ringel’s formula (see [5, p. 74]). Let \(M\) be an indecomposable \(A\)-module and let

\[
0 \to M'' \to P_1 \to P_0 \to M \to 0
\]

be a minimal projective presentation for \(M\) (extended by \(M''\)); then

\[
\dim \tau(M) = (\dim M)\phi - (\dim M'')\phi + \dim \nu M,
\]

where \(\phi = -C^{-T}C\) and \(C\) is the Cartan matrix of \(A\) and \(\nu = D \text{Hom}_A(-, A)\). Since we assume a duality fixing the simple \(A\)-modules, \(C\) is symmetric; therefore \(\phi = -\text{id}\). With the above notation we have the following result:

**Theorem 3.** Let \(A\) be a quasi-hereditary algebra with a duality fixing the simple modules. then the following are equivalent:

1. \(\tau(i) = \Delta(i + 2)\) for all \(1 \leq i \leq m - 2\) and \([P(i) : \Delta(j)] \leq 1\) for all \(1 \leq i, j \leq m\).

2. The algebra \(A\) is Morita equivalent to the algebra \(A_m\).

**Proof.** (2) \(\Rightarrow\) (1) holds by Theorems 1 and 2. We assume \(m \geq 4\) and \(m\) even (the case \(m = 2\) is trivial and the case for \(m\) odd is completely
analogous). We prove the implication (1) → (2) in three steps. First we show that the socle of each \( \Delta(i) \) is equal to the simple \( E(i - 1) \), for \( 2 \leq i \leq m \). Then we show that the \( \Delta \)-filtration of each projective indecomposable \( A \)-module \( P(i) \) is given by \( \Delta(i)^{\Delta(i)} \). Finally we prove that each projective indecomposable \( A \)-module \( P(i) \) has simple socle. Note that the \( \Delta(j) \) are multiplicity-free if (1) is assumed by (P.6).

**Step 1.** We use induction on \( i \), for \( 2 \leq i \leq m \). Assume (for a contradiction) that \( \Delta(2) = E(2) \). Then \([\Delta(2):E(1)] = 0 = [P(1):\Delta(2)] \) and we have the minimal projective resolution for \( \Delta(1) \),

\[
P'' \to P(1) \to \Delta(1) \to 0,
\]

with \( P'' \mid \bigoplus_{i=3}^{m} P(i)^{\mu_i} \). Since \( \tau(\Delta(1)) = \Delta(3) \) we have an exact sequence \( 0 \to \Delta(3) \to Q'' \to Q(1) \) with \( Q'' \mid \bigoplus_{i=3}^{m} Q(i)^{\mu_i} \); therefore \( \text{soc} \Delta(3) \subseteq \bigoplus_{i=3}^{m} E(i)^{\mu_i} \). Then \( \text{soc} \Delta(3) = E(3) \) and therefore \( \Delta(3) = \nabla(3) = E(3) \). It follows that \( Q'' = Q(3) \), so \( P'' = P(3) \) and \([\Delta(3):E(1)] = 0 = [P(1):\Delta(3)] \). The minimal projective resolution for \( \Delta(1) \) shows that \( P'' \mid \bigoplus_{i=4}^{m} P(i) \) which is a contradiction to \( P'' = P(3) \). So \( \text{soc} \Delta(2) = E(1) \) (in particular \( \Delta(2) \) is not simple).

Assume \( \text{soc} \Delta(r - 1) = E(r - 2) \) for \( 3 \leq r \leq m \). By induction we know that \([P(r - 2):\Delta(r - 1)] = [\Delta(r - 1):E(r - 2)] = 1 \), so we have the minimal projective resolution

\[
P'' \oplus P(r - 1) \to P(r - 2) \to \Delta(r - 2) \to 0,
\]

with \( P'' \mid \bigoplus_{i=r}^{m} P(i)^{\mu_i} \), and since \( \tau(\Delta(r - 2)) = \Delta(r) \), then we have the minimal injective coresolution \( 0 \to \Delta(r) \to Q'' \oplus Q(r - 1) \to Q(r - 2) \ldots \) with \( Q'' \mid \bigoplus_{i=r}^{m} Q(i)^{\mu_i} \), so \( \text{soc} \Delta(r) = E(r - 1) \), \( Q'' = 0 \), and \( P'' = 0 \). In particular we have the minimal projective resolution

\[
\to P(r - 1) \to P(r - 2) \to \Delta(r - 2) \to 0
\]

and the minimal injective coresolution

\[
0 \to \Delta(r) \to Q(r - 1) \to Q(r - 2) \to \ldots
\]

**Step 2.** We use induction on \( i \), for \( 1 \leq i \leq m - 1 \). For \( i = m - 1 \), we know that \([P(m - 1):\Delta(m)] = [\Delta(m):E(m - 1)] = 1 \), so \( P(m - 1) \) has \( \Delta \)-filtration \( \Delta(m)^{\Delta(m - 1)} \).

Let us assume the result valid for \( P(r + 1) \). We have that for \( 3 \leq l \leq m - r(P(r):\Delta(r + l)] = 0 \), because we have the minimal projective resolution

\[
\cdots \to P(r + 1) \to P(r) \to \Delta(r) \to 0
\]
and by the inductive hypothesis $P(r+1)$ has $\Delta$-filtration
\[
\Delta(r+1) \\
\Delta(r+2).
\]

If we assume that $[P(r):\Delta(r+2)] = 1$ then $[\Delta(r+2):E(r)] = 1$, and using the minimal projective resolution for $\Delta(r)$ and the inductive hypothesis we have the exact sequence
\[
0 \to P(r+1) \to P(r) \to \Delta(r) \to 0.
\]

Now, we know that
\[
\dim r[\Delta(r)] = [-,-,\ldots,1,1,0\ldots,0] = \dim \Delta(r+2).
\]

So using Ringel’s formula for $M = \Delta(r)$ we have (see [5, p. 74])
\[
\dim \tau(\Delta(r)) = -\dim \Delta(r) + 0 + \dim \nu[\Delta(r)] \\
= [-,-,\ldots,1,1,0\ldots,0] + [b_1,b_2,\ldots,b_{r+2},\ldots,b_m].
\]

Then $b_{r+2} = 1 = \dim \text{Hom}(\Delta(r),P(r+2))$, but $\text{soc}P(r+2) \subseteq E(r+2) \oplus E(r+1)$ by the inductive hypothesis, which is a contradiction. Therefore $P(r)$ has $\Delta$-filtration
\[
\Delta(r) \\
\Delta(r+1).
\]

In particular, we have that $1 = [\Delta(1):E(j)] = [P(j):\Delta(i)]$ for $2 \leq i \leq m$ and $j = i, i - 1$ and $[\Delta(i):E(j)] = 0$ otherwise.

**Step 3.** We use induction on $i$. For $i = 1$, we know that the socle of $P(1)$ is equal to the socle of the $A$-module $\Delta(2)$, that is, $E(1)$. So $P(1)$ is contained in the indecomposable injective $A$-module $Q(1)$ and since they have same composition factors, they are equal.

For $2 \leq r \leq m-1$ assume that the projective indecomposable $A$-module $P(r)$ does not have simple socle; then $0 \neq \text{Hom}(\Delta(r-1),P(r))$ and since we have the exact sequence
\[
0 \to \Delta(r+1) \to P(r) \to P(r-1) \to \Delta(r-1) \to 0
\]
by Ringel’s formula
\[
\dim \tau[\Delta(r-1)] = -\dim \Delta(r-1) + \dim \nu\Delta(r-1) + \dim \Delta(r+1).
\]
This implies \( \dim \Delta(r - 1) = \dim \Delta(r - 1) \) (since by hypothesis \( \tau[\Delta(r - 1)] = \Delta(r + 1) \)). We know that \( [\Delta(r - 1): E(r)] = 0 \); therefore \( \Hom(\Delta(r - 1), P(r)) = 0 \), a contradiction. So \( P(r) \) is contained in the indecomposable injective \( A \)-module \( Q(r) \), and since they both have the same composition factors, they are equal. 

Combining this with the results in [3] we have:

**Theorem 4.** Assume \( A \) is Morita equivalent to a block of a Schur algebra. Then the following are equivalent:

1. \( \tau(\Delta(i)) = \Delta(i + 2) \), all \( 1 \leq i \leq m - 2 \), and \( \Delta(j) \) is multiplicity-free for all \( j \).
2. \( A \) is Morita equivalent to the algebra \( A_m \).
3. \( A \) is of finite representation type.

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**References**