Formal Invariants for Nonsolvable Subgroups of $\text{Diff}^\omega(\mathbb{C}, 0)$

Frank Loray

Laboratoire Du C.N.R.S., U.M.R. 8524, U.F.R. De Mathématiques,
Université Lille I, 59655 Villeneuve d’Ascq Cedex, France
E-mail: loray@gat.univ-lille1.fr

Communicated by Michel Broué

Received February 7, 2001

Let $G$ be a nonsolvable subgroup of $\text{Diff}^\omega(\mathbb{C}, 0)$. We derive an infinite sequence of additive subgroups $G_k \subset \text{Diff}^\omega(\mathbb{C}, 0)$, depending only on the conjugacy class of $G$ within $\text{Diff}^\omega(\mathbb{C}, 0)$. As an application, we prove that $G$ is conjugate within $\text{Diff}^\omega(\mathbb{C}, 0)$ to a subgroup of $\text{Diff}^\omega(\mathbb{R}, 0)$ if and only if all $G_k$ are real.

1. DEFINITIONS AND RESULT

We denote by $\text{Diff}^\omega(\mathbb{C}, 0)$ the group (for composition law) of invertible germs of holomorphic mappings fixing $0 \in \mathbb{C}$

$$\text{Diff}^\omega(\mathbb{C}, 0) = \left\{ f(z) = \sum_{n \geq 0} a_n z^{n+1} \in \mathbb{C}\{z\}; \ a_0 \neq 0 \right\}.$$  

Let $J^k f = \sum_{n=0}^{k} a_n z^{n+1}$ be the jet of $f = \sum_{n \geq 0} a_n z^{n+1}$ of order $k \in \mathbb{N}$, and set

$$V^k(f) := \{ g \in \text{Diff}^\omega(\mathbb{C}, 0); J^k(g) = J^k(f) \} \subset \text{Diff}^\omega(\mathbb{C}, 0).$$

The subsets $V^k(f)$ form a basis of open neighborhoods of $f$ for the Krull topology. More precisely, a sequence $(f_i = \sum_{n \geq 0} a_{i,n} z^{n+1})_{i \in \mathbb{N}}$ will converge to some element $f = \sum_{n \geq 0} a_n z^{n+1}$ in $\text{Diff}^\omega(\mathbb{C}, 0)$ if and only if, for any $n \in \mathbb{N}$, we have $a_{i,n} = a_n$ for $i \gg 0$ sufficiently large. The neighborhoods $V^N(f)$ are open and closed.
Denote by $\widehat{\text{Diff}}(\mathbb{C}, 0)$ the closure of $\text{Diff}^\omega(\mathbb{C}, 0)$ in the Krull topology

$$\widehat{\text{Diff}}(\mathbb{C}, 0) = \left\{ \hat{f}(z) = \sum_{n=0}^\infty a_n z^{n+1} \in \mathbb{C}[[z]]; \; a_0 \neq 0 \right\}.$$ 

Let $[f, g]$ be the commutator of two elements $f, g \in \widehat{\text{Diff}}(\mathbb{C}, 0)$

$$[f, g] = f \circ g \circ f^{(n-1)} \circ g^{(n-1)}.$$

For any $a, b \in \mathbb{C}^*$, $c, d \in \mathbb{C}$, and $p, q \in \mathbb{N}^*$, an obvious computation shows that

1. $[az + \cdots, bz + \cdots] = z + \cdots$
2. $[z + cz^{p+1} + \cdots, z + dz^{q+1} + \cdots] = z + (q - p)cdz^{p+q+1} + \cdots$.

Let $G$ be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}, 0)$. The central series of $G$ is defined by

$$G^{(0)} := G, \quad G^{(1)} := [G, G], \quad \ldots \quad G^{(n+1)} := [G^{(n)}, G^{(n)}], \quad \ldots$$

where $[G^{(n)}, G^{(n)}]$ stands for the normal subgroup generated by commutators of elements of $G^{(n)}$. Recall that $G$ is said to be solvable if $G^{(n)} = \{\text{identity}\}$ for some $n \geq 0$, metabelian if, moreover, $n = 2$, and finally abelian if $n = 1$.

The following definitions and remarks originated from a private discussion with Felipe Cano and Dominique Cerveau. For any $k \in \mathbb{N}^*$, we define

$$G_k := \{ c \in \mathbb{C}; \exists g = z + cz^{k+1} + \cdots \in G \}.$$

The subsets $G_k \subset \mathbb{C}$ are additive subgroups of $\mathbb{C}$ and satisfy

$$\varphi = az + \cdots \in \widehat{\text{Diff}}(\mathbb{C}, 0) \Rightarrow (\varphi^*G)_k = a^k G_k.$$

In this sense, the $G_k$ are almost formal invariants. We also define the set of contacts

$$K(G) := \{ k \in \mathbb{N}^*; G_k \neq \{0\} \}.$$

The sequence of integers $K(G)$ is a formal invariant and satisfies

$$\begin{cases} k_1, k_2 \in K(G) \\ \text{with } k_1 < k_2, \end{cases} \Rightarrow \begin{cases} k_1 + k_2 \in K(G) \text{ and} \\ (k_2 - k_1)G_{k_1}G_{k_2} \subset G_{k_1+k_2}. \end{cases}$$

In this sense, $K(G)$ is almost a semi-group.

**Preliminary Proposition.** If $K(G)$ is finite, then $G$ is discrete with regard to the Krull topology. In this case, $\#K(G) \leq 1$, and $G$ is metabelian.

If $K(G)$ is infinite, then its closure $\overline{G}$ in $\widehat{\text{Diff}}(\mathbb{C}, 0)$ with regard to the Krull topology is not countable. In this case, $G$ is not solvable.
Proof. Suppose that $K(G)$ is finite, and set $k := \sup(K(G))$ (set $k = 0$ in the case $K(G) = \emptyset$). Then, any element of $G$ is given by its Taylor jet of order $k$

$$\forall f \in G, \quad J^k(f) = z \quad \Rightarrow \quad f = z.$$ 

This means that $G$ is discrete with regard to the Krull topology. From (1) and (2), we deduce that $G$ is metabelian (abelian if $K(G) = \emptyset$).

If $\#K(G) \geq 2$, then $K(G)$ is infinite following (4). In this case, (2) implies that $\#K(G^{(1)}) = \infty$. By induction, we have $\#K(G^{(n)}) = \infty$ for all $n \in \mathbb{N}$. Thus $G$ cannot be solvable. Choose some (strictly) increasing sequence $k_1 < k_2 < \cdots < k_n < \cdots$ in $K(G)$. For each $n \in \mathbb{N}$, choose some element $f_n \in G$ realizing the corresponding contact $k_n \in K(G)$ to the identity $f_n(z) = z + c_n z^{k_n+1} + \cdots$ with $c_n \neq 0$. Then for any sequence $(\varepsilon_n) \in \{0, 1\}^{\mathbb{N}}$, the power series $f_1^{\varepsilon_1} \circ f_2^{\varepsilon_2} \circ \cdots \circ f_n^{\varepsilon_n} \circ \cdots$ belongs to $\overline{G}$. Hence, $\overline{G}$ contains $\{0, 1\}^{\mathbb{N}}$.

The data of $K(G)$ and all the $G_k$ do not provide a complete list of formal invariants. Indeed, many other groups have the same invariants. For instance, $K(\overline{G}) = K(G)$ and $\overline{G_k} = G_k$ although $G$ is not formally conjugate to its closure $\overline{G}$ in general. Nevertheless, these invariants are sufficient to characterize some natural geometrical properties of $G$. For instance, suppose that all elements of $G$ are power series with real coefficients; we will briefly say that $G$ is real and denote $G \subset \text{Diff}^{\omega}(\mathbb{R}, 0)$. Then each $G_k$ is actually a subgroup of $\mathbb{R}$. After conjugacy by a generic element $\varphi \in \text{Diff}^{\omega}(\mathbb{C}, 0)$, $G$ will be no longer real. Nevertheless, each $G_k$ will still be contained in some real line $L_k \subset \mathbb{C}$ following (3). By the same way, suppose now that $G$ is lifting a subgroup $\tilde{G} \subset \text{Diff}^{\omega}(\mathbb{C}, 0)$ via the branched covering $z \mapsto z^d$ where $d \in \mathbb{N}^*$; we then say that $G$ is ramification of order $d \in \mathbb{N}^*$ of $\tilde{G}$. In particular, any element of $G$ takes the form $f = \sum_{n \geq 0} a_n z^{nd+1}$ and $K(G) \subset d\mathbb{Z}$. Only the latter property is preserved by local changes of coordinates.

Felipe Cano, Dominique Cerveau, and the author conjectured a converse of these remarks. We answer by the following:

**Theorem.** Let $G$ be a nonsolvable subgroup of $\text{Diff}^{\omega}(\mathbb{C}, 0)$ and let $d$ be the greatest common divisor of $K(G)$. Suppose that each $G_k$ is contained in some real line $L_k \subset \mathbb{C}$. Then, up to conjugacy by some element $\varphi \in \text{Diff}^{\omega}(\mathbb{C}, 0)$, $G$ is the ramification of order $d$ of some real subgroup of $\text{Diff}^{\omega}(\mathbb{R}, 0)$, i.e., any element takes the form $f = \sum_{n \geq 0} a_n z^{nd+1}$ with $a_n \in \mathbb{R}$.

If every $G_k$ is contained in some given subring $A \subset \mathbb{C}$, our approach suggests that elements of $G$ have Taylor coefficients in $A$ after a convenient change of coordinates. Similarly, if each $G_k$ belongs to a one dimensional lattice, can we derive suitable coordinate which conjugates simultaneously all elements of $G$ to power series with integral coefficients? Our proof does
not adapt immediately to this context: for instance, the fact that \( A \) is not invariant by homotheties leads to additional difficulties. We did not pursue in this direction.

2. DYNAMICAL MOTIVATIONS

Subgroups of \( \text{Diff}^\omega(\mathbb{C}, 0) \) naturally appear in the study of singular foliations associated to complex differential equations (see [C,M], for instance). The holonomy pseudo-group \( \mathcal{G} \) of a leaf belonging to a codimension one holomorphic foliation consists of local diffeomorphisms fixing \( 0 \in \mathbb{C} \). Their dynamics reflect the complexity of the foliation in a neighborhood of this leaf. After germification, we obtain a subgroup \( G \in \text{Diff}^\omega(\mathbb{C}, 0) \), namely the group of Taylor series at \( 0 \) of elements of \( \mathcal{G} \). This group \( G \) is well defined by the foliation up to conjugacy in \( \text{Diff}^\omega(\mathbb{C}, 0) \). Following our proposition, such a group is either metabelian or nonsolvable. In the latter case, the main theorem of [N] provides the following dichotomy

**Theorem [N].** Let \( \mathcal{G} \) be a holomorphic pseudo-group acting on \( \mathbb{C} \), and fixing \( 0 \). Suppose that the associated group of germs \( G \in \text{Diff}^\omega(\mathbb{C}, 0) \) is not solvable. Then

- either \( \mathcal{G} \) acts minimally on any sufficiently small pointed disc,
- or \( \mathcal{G} \) preserves a finite collection \( \Sigma \) of real analytic smooth curves (called separatrices) at \( 0 \). In this case, in any sufficiently small disc, \( \mathcal{G} \) acts minimally on each connected component of the complement of \( \Sigma \) and on each connected component of \( \Sigma \setminus \{0\} \).

Moreover, the latter case is characterized by the existence of an analytic coordinate which conjugates \( G \) to the ramification at the order \( d \) of some real subgroup of \( \text{Diff}^\omega(\mathbb{R}, 0) \). The total separatrix is then given by \( \Sigma = \{z; z^d \in \mathbb{R}\} \).

The goal of the present note is to provide a formal algorithm to decide whether a nonsolvable subgroup \( G \in \text{Diff}^\omega(\mathbb{C}, 0) \) has a separatrix \( \Sigma \) or not. Given an element \( g \in \text{Diff}^\omega(\mathbb{C}, 0) \) (resp. a subgroup \( G \subset \text{Diff}^\omega(\mathbb{C}, 0) \)), the existence of a germ of real smooth analytic curve \( \Sigma \) at \( 0 \) invariant by \( g \) (resp. \( G \)) is equivalent to the existence of a coordinate \( \varphi \in \text{Diff}^\omega(\mathbb{C}, 0) \) in which \( g \) (resp. any element of \( G \)) is expressed as a real Taylor series. Indeed, it is well known that such a curve \( \Sigma \) is sent into the real line after holomorphic change of coordinate. Of course, any complex diffeomorphism preserving the real line must have real Taylor coefficients. We recall that a germ \( g(z) = z + \cdots \in \text{Diff}^\omega(\mathbb{C}, 0) \) tangent to the identity is conjugate to a real germ if and only if the Écalle–Voronin invariants (see [C,M]) of \( g \) are all real. However, none of these invariants can be computed algebraically from a finite jet of \( g \): they are transcendental functions of all the Taylor coefficients. Surprisingly, our theorem above shows that, for a nonsolvable subgroup
\[ \text{G} \in \text{Diff}^\omega(\mathbb{C}, 0) \] (even if it consists of germs tangent to the identity!), all obstructions to the existence of a smooth invariant curve are expressed by algebraic functions of finite numbers of coefficients: indeed, for each \( k \), \( G_k \) may be computed with the jet \( f^k(G) \) of order \( k \) of \( G \). In particular, the genericity of the absence of such separatrix becomes clear from our result.

3. THE PROOF

The following technical lemma has been communicated to the author by Jean Écalle. The original application was the construction of pairs of noncommuting germs tangent to the identity with prescribed relation. We borrow it with thanks.

**Lemma 1 [É].** Let \( f(z) = z + a_p z^{p+1} + \cdots \) and \( g(z) = z + b_q z^{q+1} + \cdots \) be elements of \( \text{Diff}^\omega(\mathbb{C}, 0) \) tangent to the identity with distinct contacts \( p \neq q \). Consider

\[ h_1(z) = [g, [f, [f, f, g]]] \quad \text{and} \quad h_2(z) = [f, [g, [f, f, g]]]. \]

Then

\[ h_1(z) = z + m_1 a_p b_q^2 z^{3p+2q+1} + \cdots \quad \text{with} \quad \frac{m_1}{m_2} = \frac{3p + 3q}{2p + 4q}. \]

\[ h_2(z) = z + m_1 a_p b_q^2 z^{3p+2q+1} + \cdots \quad \text{with} \quad \frac{m_1}{m_2} = \frac{3p + 3q}{2p + 4q}. \]

Set \( h = h_1^{(2p+4q)} \circ h_2^{(-3p-3q)} \). Then \( h(z) = z + \sum_{k \geq 0} c_{3p+2q+k} z^{3p+2q+k+1} \) with

\[ c_{3p+2q+k} = \delta_{(p, q, k)} c_{3p+2q+k} \]

where \( c_{3p+2q+k} \in \mathbb{Z}[a_p, \ldots, a_{p+k}, b_q, \ldots, b_{q+k}] \) and \( \delta_{(p, q, k)} \neq 0 \) for \( k > 0 \).

This statement is fundamental for our proof. It will insure that the \( G_k \) depend on all the coefficients of any element of \( G \) tangent to the identity.

**Proof.** It is a straightforward computation which may be better understood in the associated Lie algebra. Set

\[ X := \log f = (a_p z^{p+1} + \cdots) \frac{\partial}{\partial z} \]

and

\[ Y := \log g = (b_q z^{q+1} + \cdots) \frac{\partial}{\partial z}. \]

Then, we have

\[ \log[g, [f, [f, f, g]]] = \{Y, \{X, \{X, \{X, Y\}\}\}\} + \cdots \]

\[ \log[f, [g, [f, f, g]]] = \{X, \{Y, \{X, \{X, Y\}\}\}\} + \cdots. \]
The iterated brackets \( \{ Y, \{ X, \{ X, \{ X, Y \} \} \} \} \) and \( \{ X, \{ Y, \{ X, \{ X, Y \} \} \} \} \) are the shortest ones in the free Lie algebra generated by \( X \) and \( Y \) that satisfy the following property. They are linearly independent and they have the same weight in \( X \) and \( Y \). This is the key of the proof.

Denoting \( \hat{X}_p = z^{p+1} \frac{\partial}{\partial z} \) for \( p \in \mathbb{N} \), we have

\[
\{ X_{p_1}, \{ X_{p_2}, \ldots \} \} = C_{(p_2, \ldots, p_1)} z^{p_1 + p_2 + \cdots + p_1 + 1} \frac{\partial}{\partial z}
\]

for any finite sequence \( p_k, \ldots, p_2, p_1 \in \mathbb{N} \) where

\[
C_{(p_k, \ldots, p_1)} := (p_1 - p_2)(p_1 + p_2 - p_3) \cdots (p_1 + p_2 + \cdots + p_{k-1} - p_k).
\]

The coefficient of \( z^{3p+2q+k+1} \) in the iterated bracket \( \{ Y, \{ X, \{ X, Y \} \} \} \), \( k > 0 \), is a polynomial in variables \( a_p, \ldots, a_{p+k}, b_q, \ldots, b_{q+k} \) with coefficients in \( \mathbb{Z} \). The monomials in which variables \( a_{p+k} \) and \( b_{q+k} \) occur are

\[
[C(q, p, p, p, q+k) + C(q+k, p, p, q)] a_p b_q b_{q+k} + [C(q, p, p, p+k, q) + C(q+k, p, p+k, q) + C(q+k, p, q+k)] a_p^2 a_{p+k} b_q^2.
\]

We obtain a similar identity for \( \{ X, \{ Y, \{ X, Y \} \} \} \), and \( \delta(p, q, k) \) is one of the discriminants

\[
\left| \begin{array}{cc}
C_{(q, p, p, p, q)} & C_{(q+k, p, p, p, q)} + C_{(q, k, p, p, p, q)} \\
C_{(p, p, q, p, p)} & C_{(p+k, q, p, p, p)} + C_{(p, k, q, p, p, p)}
\end{array} \right|.
\]

We obtain \( \delta(p, q, k) = (q - p)k(k^2 + kp + 6pq) \) which is nonzero by assumption.

**Lemma 2 [B].** Let \( f(z) = z + a_p z^{p+1} + \cdots \in \text{Diff}^\omega(\mathbb{C}, 0) \). Then there exists a formal change of coordinate \( \varphi \in \text{Diff}(\mathbb{C}, 0) \) after which \( f \) takes the special form \( f(z) = z + z^{p+1} + \lambda z^{2p+1} \), where \( \lambda \in \mathbb{Z}[a_p, \ldots, a_{2p}] \). This coordinate \( \varphi \) is unique modulo composition by an element of the centralizer \( \text{Cent}(f) \) within \( \text{Diff}(\mathbb{C}, 0) \). If \( f(z) = z + z^{p+1} + \lambda z^{2p+1} \) modulo \( z^{p+k+1} \), \( k > 0 \), then \( \varphi \) may be chosen tangent to the identity at the order \( k + 1 \). In this new coordinate, each element \( g \in \text{Cent}(f) \) takes the form

\[
g(z) = b_0 z + b_p z^{p+1} + \cdots \quad \text{with} \quad b_0^p = 1
\]

and the arrow

\[
\text{Cent}(f) \rightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{C}; \quad g \mapsto (b_0, b_p/b_0)
\]

is an isomorphism of groups.
Proof. Consider \( \varphi = \cdots \circ \varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1 \circ \varphi_0 \) with 
\[
\varphi_0(z) = b_0 z \quad \text{and} \quad \varphi_n(z) = z + b_n z^{n+1} \quad \text{for} \quad n > 0.
\]
One has
\[
(\varphi_0)_* f := \varphi_0 \circ f \circ \varphi_0^{(-1)} = [\varphi_0, f] \circ f = z + a_p (b_0^p - 1) z^{p+1} \quad \text{modulo} \quad z^{p+2}
\]
which determines \( b_0^p \). Now, for \( n > 0 \) one has 
\[
(\varphi_n)_* f := \varphi_n \circ f \circ \varphi_n^{(-1)} = [\varphi_n, f] \circ f = f + (p - n) a_p b_n z^{p+n+1} \quad \text{modulo} \quad z^{p+n+2}
\]
which determines \( \varphi_n \) except when \( n = p \). In particular, \( \varphi_p \) is arbitrary. \( \square \)

Lemma 3. Let \( f, g \in \text{Diff}^\omega(\mathbb{C}, 0) \) with \( f(z) = z + a_p z^{p+1} + \cdots \) real and \( g(z) = b_0 z + \cdots \) arbitrary. Then the following conditions are equivalent:

(i) \( g^* f = g^{\omega(-1)} \circ f \circ g \) is real;

(ii) \( [g^{\omega(-1)}, f] \) is real;

(iii) \( g = \tilde{f} \circ \tilde{g} \) where \( \tilde{f} \) commutes with \( f \) and \( \tilde{g} \) is real.

Proof. (i) \( \iff \) (ii) are straightforward. Let us show (ii) \( \Rightarrow \) (iii). Remark first that \( [g^{\omega(-1)}, f] = z + (b_0^p - 1) z^{p+1} + \cdots \). Hence, after composition of \( g \) by a convenient element of \( \text{Cent}(f) \), \( b_0 \) is real. Suppose \( g \) is not real. Then, write down \( g = \tilde{g} \circ \tilde{f} \) where \( \tilde{g} \) is real and \( \tilde{f}(z) = z + b_q z^{q+1} + \cdots \) with \( b_q \) is not real. Then \( g^* f = \tilde{f}^*(\tilde{g}^* f) \) is real if and only if \( [\tilde{f}^{\omega(-1)}, \tilde{g}^* f] = z + (q - p) a_p b_q z^{p+q+1} + \cdots \) is real. This is possible only if \( p = q \). But, then, one may again modify \( g \) by an element of \( \text{Cent}(f) \). \( \square \)

Lemma 4. Let \( f(z) = z + a_p z^{p+1} + \cdots \) and \( g(z) = z + b_q z^{q+1} + \cdots \) be two elements of \( G \) tangent to the identity. Suppose all \( G_k \) are real. If the \( p \) first (main) coefficients \( a_p, \ldots, a_{2p-1} \) of \( f \) are real, then the \( p \) first coefficients \( b_q, \ldots, b_{q+p-1} \) of \( g \) are so.

Proof. First, suppose \( p \neq q \). By assumption, \( b_q \in G_q \) is real. Lemma 1 applied to \( f \) and \( g \) asserts that \( b_{q+1} \) is real as well. Again, Lemma 1 ensures that \( b_{q+2}, \ldots, b_{q+p-1} \) are real if and only if the \( p - 1 \) first coefficients \( c_{3p+2q+1}, \ldots, c_{3p+2q+p-1} \) of \( h \) are so. Replacing \( g \) by \( h \) and applying again Lemma 1, we start an induction leading to the conclusion.

Suppose now \( p = q \). Lemma 3 asserts that the \( p \) first coefficients of \( g \) are real if and only if the first \( p \) coefficients of \( [f, g] \) are so. This leads back to the previous case. \( \square \)
Proof of the theorem. After preliminary conjugacy by a complex homothety, one may assume that all $G_k$ are real. Indeed, as soon as the first nontrivial one has been made real by such homothety, formula (4) implies that any other $G_k$ is so.

We begin dealing with the ramification problem. Let us construct a formal change of coordinate after which any element of $G$ takes the form $g(z) = z \tilde{g}(z^d)$. The construction is “step by step.” Suppose that the elements of $G$ already take the form
\[
f(z) = a_0 z + a_d z^{d+1} + \cdots + a_{Nd} z^{Nd+1} + a_{Nd+k} z^{Nd+k+1} + \cdots \quad N, k \in \mathbb{N}^*, \quad k < d.
\]
Then the arrow $f \mapsto (a_0 z + a_{Nd+k})/a_0^{Nd+k}$ defines a morphism $\rho$ from $G$ into the affine group. The assumption $Nd + k \not\in K(G)$ means that $\rho(G)$ does not contain a nontrivial translation. Thus, $\rho(G)$ becomes linear after conjugacy by some convenient translation $z \mapsto z + t$, $t \in \mathbb{C}$. This implies that all the coefficients leading $z^{Nd+k+1}$ occurring in $G$ are simultaneously killed by the corresponding change of coordinate of $G_{Nd+k}(z) = z + tz^{Nd+k+1}$. The expected change of coordinate follows by iteration of this process. Convergence will be justified later.

If all elements of $G$ take the form $f(z) = z \tilde{f}(z^d)$, then $G$ is clearly the pull-back, by the ramification $z \mapsto z^d$, of a subgroup of $Diff^\infty(\mathbb{C}, 0)$. This new group (that we will still denote by $G$) still satisfies the assumptions of the theorem with $gcd(K(G)) = 1$.

Now, choose some element $f(z) = z + a_p z^{p+1} + \cdots$ of $G^{(1)} = [G, G]$ as well as a coordinate through which the $p$ first coefficients of $f$ are real; this is possible by Lemma 2. In this coordinate, $G^{(1)}$ is real modulo $z^{p+2}$. Indeed, if $g(z) = z + \cdots$ is any element of $G^{(1)}$, then its $p + 1$ first coefficients are real. This is a straightforward application of Lemma 4. Choose now some element $g(z) = z + b_q z^{q+1} + \cdots$ of $G^{(1)}$ with $q > p$. Again following Lemma 4, the $p$ first (main) coefficients $b_q, \ldots, b_{q+p-1}$ of $g$ are real. There exists, due to Lemma 2, some change of coordinate of the type $\varphi(z) = z + cz^{p+1}$ through which the $q$ first coefficients of $g$ become real. In this new coordinate, $G^{(1)}$ is real modulo $z^{q+2}$. Step by step, one constructs some formal coordinate through which $G^{(1)}$ is real. Let us verify that $G$ is actually real in this coordinate.

Let $h(z) = c_0 z + \cdots$ be any element of $G$. Give two elements $f(z) = z + a_p z^{p+1} + \cdots$ and $g(z) = z + b_q z^{q+1} + \cdots$ of $G^{(1)}$ with $p$ and $q$ relatively prime. This is possible since $gcd(K(G)) = 1$. Then $f$, $g$, $[h^{(-1)}, f]$ and $[h^{(-1)}, g]$ are real. Lemma 3 asserts that
\[
h = \tilde{f} \circ \tilde{h}_1 = \tilde{g} \circ \tilde{h}_2,
\]
where $\tilde{f}$ commute with $f$, $\tilde{g}$ commute with $g$, and $\tilde{h}_1$ and $\tilde{h}_2$ are real. Following Lemmas 2 and 4, $c_0^p$ and $c_0^q$ are real and thus $c$ is so. Therefore, we may
assume, up to modifying the decomposition \( h = \tilde{f} \circ \tilde{h}_1 \), that \( \tilde{f} \) is tangent to the identity at the order \( p \) (Lemma 2) and, as well, \( \tilde{g} \) is tangent to the identity at the order \( q \). Now, if (for instance) \( p < q \), one deduces that the coefficient leading \( z^{p+1} \) in \( \tilde{f} \) is real. Then \( \tilde{f} \) (Lemma 2) as well as \( h \) are real.

In order to finish, let us justify that the formal coordinates constructed above are convergent. The idea is already in [N]. The first step consists, a posteriori, in linearizing some formal periodic diffeomorphism \( f \in \text{Diff}(\mathbb{C}, 0) \) which commutes with \( G \). After the first step, this diffeomorphism is the rotation \( f: z \mapsto e^{2i\pi/p} \cdot z \). Following a theorem of [C,M], any formal conjugacy between nonsolvable subgroups of \( \text{Diff}^\omega(\mathbb{C}, 0) \) turn to be convergent. In particular, \( \tilde{f} \) is convergent. Now it is well known that any periodic element of \( \text{Diff}^\omega(\mathbb{C}, 0) \) is linearizable by analytic change of coordinates. After this change of coordinates, \( G \) commutes with the rotation and thus takes the special expected ramified form. The first step can be done in an analytic way. The second step consists in straightening a certain formal anti conformal involution \( \sigma \) to \( z \mapsto \bar{z} \). The formal map \( z \mapsto \sigma(\bar{z}) \) conjugates \( G \) to some other nonsolvable subgroup of \( \text{Diff}^\omega(\mathbb{C}, 0) \). Applying again the theorem of [C,M] to it shows that \( \sigma \) is convergent as well. As recalled in Section 2, any convergent anti conformal involution (or equivalently the smooth real analytic curve from which it is the reflection) is conjugate to \( z \mapsto \bar{z} \) by holomorphic change of coordinates. The second step is convergent as well.

ACKNOWLEDGMENT

We thank the referee for helpful remarks on our English.

REFERENCES