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A singular approach to discontinuous vector fields on the plane ☆

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Abstract

In this paper we deal with discontinuous vector fields on $\mathbb{R}^2$ and we prove that the analysis of their local behavior around a typical singularity can be treated via singular perturbation. The regularization process developed by Sotomayor and Teixeira is crucial for the development of this work.

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1. Introduction and statements of the main results

Let $K \subseteq \mathbb{R}^2$ be a compact set and $\Sigma \subseteq K$ given by $\Sigma = F^{-1}(0)$, where F is a smooth function $F : K \to \mathbb{R}$ having $0 \in \mathbb{R}$ as a regular value.

Designate by χ^r the space of C^r vector fields on K endowed with the C^r-topology with $r \geqslant 1$ or $r = \infty$, big enough for our purposes. Call $\Omega^r = \Omega^r(K, F)$ the space of vector fields $X : K \setminus \Sigma \to \mathbb{R}^2$ such that

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$$X(x,y)=\begin{cases} X_1(x,y), & \text{for } F(x,y)>0,\\ X_2(x,y), & \text{for } F(x,y)<0,\end{cases} \tag{1}$$

where $X_i=(f_i,g_i)\in\chi^r$, $i=1,2$. We write $X=(X_1,X_2)$, which we will accept to be multivalued in the points of Σ.

The trajectories of X are solutions of the autonomous differential equation system $\dot{q}=X(q)$, which has, in general, discontinuous right-hand side.

The study of differential equations with discontinuous right-hand side is motivated by its many applications mainly in mechanics, electrical engineering and general automatic control. Correlated problems can be obtained in [7,9,10,12] and basic results for differential equations in this context can be found in [6]. The regularization method introduced by Sotomayor and Teixeira in [10] gives the mathematical tool to study the stability of these systems, according to the program introduced by Peixoto. The method consists in the analysis of the regularized vector field which is a smooth approximation of the discontinuous vector field.

Definition 1. A C^∞ function $\varphi:\mathbb{R}\to\mathbb{R}$ is a transition function if $\varphi(x)=-1$ for $x\leqslant -1$, $\varphi(x)=1$ for $x\geqslant 1$ and $\varphi'(x)>0$ if $x\in(-1,1)$. The φ-regularization of $X=(X_1,X_2)$ is the 1-parameter family $X_\varepsilon\in C^r$ given by

$$X_\varepsilon(q)=\left(\frac{1}{2}+\frac{\varphi_\varepsilon(F(q))}{2}\right)X_1(q)+\left(\frac{1}{2}-\frac{\varphi_\varepsilon(F(q))}{2}\right)X_2(q), \tag{2}$$

with $\varphi_\varepsilon(x)=\varphi(x/\varepsilon)$, for $\varepsilon>0$.

Definition 2. Let $U\subseteq\mathbb{R}^2$ be an open subset and $\varepsilon\geqslant 0$. A singular perturbation problem in U (SP-problem) is a differential system which can be written like

$$x'=dx/d\tau=f(x,y,\varepsilon),\qquad y'=dy/d\tau=\varepsilon g(x,y,\varepsilon) \tag{3}$$

or equivalently, after the time rescaling $t=\varepsilon\tau$,

$$\varepsilon\dot{x}=\varepsilon\,dx/dt=f(x,y,\varepsilon),\qquad \dot{y}=dy/dt=g(x,y,\varepsilon), \tag{4}$$

with $(x,y)\in U$ and f,g smooth in all variables.

In what follows we will use the notation

$$X_iF(p)=\langle\nabla F(p),X_i(p)\rangle=\big((\partial F/\partial x)(p),(\partial F/\partial y)(p)\big)X_i(p)$$

for $i=1,2$.

Our first result is:

Theorem A. *Consider $X\in\Omega^r$, X_ε its φ-regularization, and $p\in\Sigma$. Suppose that φ is a polynomial of degree k in a small interval $I\subseteq(-1,1)$ with $0\in I$. Then the trajectories of X_ε in $V_\varepsilon=\{q\in K\colon\ F(q)/\varepsilon\in I\}$ are in correspondence with the solutions of an ordinary differential equation $z'=h(z,\varepsilon)$, satisfying that h is smooth in both variables and $h(z,0)=0$ for any $z\in\Sigma$. Moreover, if $((X_1-X_2)F^k)(p)\neq 0$ then we can take a C^{r-1}-local coordinate system $\{(\partial/\partial x)(p),(\partial/\partial y)(p)\}$ such that this smooth ordinary differential equation is a SP-problem.*

The understanding of the phase portrait of the vector field associated to a SP-problem is the main goal of the *geometric singular perturbation-theory* (GSP-theory). The techniques of GSP-theory can be used to obtain information on the dynamics of (3) for small values of $\varepsilon > 0$, mainly in searching limit cycles.

System (3) is called the *fast system*, and (4) the *slow system* of SP-problem. Observe that for $\varepsilon > 0$ the phase portraits of the fast and the slow systems coincide.

For $\varepsilon = 0$, let $\mathcal{S}$ be the set

$$\mathcal{S} = \{(x, y)\colon\ f(x, y, 0) = 0\} \tag{5}$$

of all singular points of (3). We call $\mathcal{S}$ the slow manifold of the singular perturbation problem and it is important to notice that Eq. (4) defines a dynamical system, on $\mathcal{S}$, called the *reduced problem*:

$$f(x, y, 0) = 0, \qquad \dot{y} = g(x, y, 0). \tag{6}$$

Combining results on the dynamics of these two limiting problems, with $\varepsilon = 0$, one obtains information on the dynamics of X_ε for small values of ε.

In this paper we consider those systems related to a discontinuous vector fields, like in Theorem A, and we refer to [5] for an introduction to the general theory of singular perturbations. Related problems can be seen in [2,4] and [11].

Our first result (Theorem A) says that we can transform a discontinuous vector field in a SP-problem. In general this transition cannot be done explicitly. Theorem B provides an explicit formula of the SP-problem for a suitable class of vector fields.

Consider $C = C(K) = \{\xi : K \subset \mathbb{R}^2 \to \mathbb{R}\colon\ \xi \in C^r,\ L(\xi) = 0\}$ where $L(\xi)$ denotes the linear part of ξ at $(0, 0)$.

Let $\Omega_d \subset \Omega^r$ be the set of vector fields $X = (X_1, X_2)$ in Ω^r such that there exists $\xi \in C$ that is a solution of

$$\nabla\xi(X_1 - X_2) = \Pi_i(X_1 - X_2), \tag{7}$$

where Π_i denote the canonical projections, for $i = 1$ or $i = 2$.

Our second result is:

Theorem B. *Consider $X \in \Omega_d$ and X_ε its φ-regularization. Suppose that φ is a polynomial of degree k in a small interval $I \subseteq \mathbb{R}$ with $0 \in I$. Then the trajectories of X_ε on $V_\varepsilon = \{q \in K\colon F(q)/\varepsilon \in I\}$ are solutions of a SP-problem.*

We remark that the singular problems discussed in the previous theorems, when $\varepsilon \searrow 0$, defines a dynamical system on the discontinuous set of the original problem. This fact can be very useful for problems in control theory.

Our third theorem deals with SP-problem such that the fast and the slow systems approach the discontinuous vector field. More specifically, the fast system approaches the discontinuous vector field whereas the slow system approaches the corresponding sliding vector field (see Section 2 for the definition).

Consider $X \in \Omega^r$ and $\rho : K \to \mathbb{R}$ with $\rho(x, y)$ being the distance between (x, y) and Σ. We denote by $\widehat{X}$ the vector field given by $\widehat{X}(x, y) = \rho(x, y)X(x, y)$.

In what follows we identify $\widehat{X}_\varepsilon$ and the vector field on $[(K \setminus \Sigma) \times \mathbb{R}] \subset \mathbb{R}^3$ given by $\widehat{X}(x, y, \varepsilon) = (\widehat{X}_\varepsilon(x, y), 0)$.

Theorem C. *If $p \in \Sigma$ then there exist an open set $U \subset \mathbb{R}^2$, $p \in U$, a 3-dimensional manifold M, a smooth function $\Phi : M \to \mathbb{R}^3$ and a SP-problem Y on M such that Φ sends orbits of $Y|_{\Phi^{-1}(U\times(0,+\infty))}$ in orbits of $\widehat{X}|_{(U\times(0,+\infty))}$.*

The paper is organized as follows. In Section 2 we prove Theorems A and B and present some usual definitions of discontinuous vector fields. Moreover, we make explicit the equations of the SP-problem obtained in Theorem B. In Section 3 we apply Theorem B to study the class of discontinuous vector fields $X = (X_1, X_2)$ with $X_i = (f_i, g_i)$ $(i = 1, 2)$ and $g_1 = g_2$. We prove that the trajectories of the sliding vector field X^Σ are solutions of the reduced problem of a SP-problem.

Let S be the set given by (5). We say that $q \in S$ is *normally hyperbolic* if $(\partial f/\partial x)(q, 0) \neq 0$.

The classical results of GSP-theory give a complete description of the flow of system (3) near a compact normally hyperbolic subset of S, when $\varepsilon \searrow 0$. See, for instance, [5]. In Sections 2 and 3 we prove that small perturbations of a discontinuous vector fields near a point $q \in \Sigma_2 \cup \Sigma_3$ (the sets of regular points of the kind escaping and sliding, respectively) can be analyzed like a perturbation of a normally hyperbolic point. For precise definitions see Section 2.

In Section 4 we prove Theorem C and analyze the regularizations of generic vector fields using the blow-up method.

The blowing up method transforms the regularized vector field $\widehat{X}(x, y, \varepsilon)$ into a new 3-dimensional vector field $Y(r, \theta, y)$ which has trajectories being the solutions of a SP-problem.

Finally we address the interested reader to the results of Denkowska and Roussarie in [3] where the method of blowing up for families of vector fields is extensively discussed.

We emphasize that if φ_1 and φ_2 are two distinct regularizations then the phase portraits on the blowing up loci are the same.

We remark that in this paper, as usual in GSP-theory, the notation $\dot{p}$ means that the time is $t \in \mathbb{R}$ and p' means that the time is $\tau = t/\varepsilon$. Moreover, in the phase portrait, double arrow means that the trajectories are of the fast dynamical system and simple arrow means that the trajectories are of the slow dynamical system.

2. The sliding vector field

We start this section with basic facts of discontinuous vector fields.

We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma_1 \subseteq \Sigma$ is the *sewing region* if $(X_1 F)(X_2 F) > 0$ on Σ_1.
(ii) $\Sigma_2 \subseteq \Sigma$ is the *escaping region* if $(X_1 F) > 0$ and $(X_2 F) < 0$ on Σ_2.
(iii) $\Sigma_3 \subseteq \Sigma$ is the *sliding region* if $(X_1 F) < 0$ and $(X_2 F) > 0$ on Σ_3.

Definition 3. Consider $X \in \Omega^r$. The sliding vector field associated to X is the vector field X^s tangent to Σ_3 and defined at $q \in \Sigma_3$ by $X^s(q) = m - q$ with m being the point where the segment joining $q + X_1(q)$ and $q + X_2(q)$ is tangent to Σ_3.

It is clear that if $q \in \Sigma_3$ then $q \in \Sigma_2$ for $-X$ and then we can define the *escaping vector field* on Σ_2 associated to X by $X^e = -(-X)^s$. In what follows we use the notation X^Σ for both cases.

A singular point $q \in \Sigma$ of X^{Σ} is a *saddle* provided one of the following conditions is satisfied: (i) $q \in \Sigma_2$ and q is an attractor for X^{Σ}, (ii) $q \in \Sigma_3$ and q is a repeller for X^{Σ}.

A point $q \in \Sigma$ is a *fold point* of X_i (for $i = 1$ or $i = 2$) if $X_i F(q) = 0$ but $X_i^2 F(q) \neq 0$.

Definition 4. Consider $X \in \Omega^r$. We say that $q \in \Sigma$ is a regular point if

(i) $X_1 F(q) X_2 F(q) > 0$, or
(ii) $q \in \Sigma_3$ (respectively on $q \in \Sigma_2$) and q is not a singular point of the sliding (respectively escaping) vector field.

Definition 5. Consider $X \in \Omega^r$. A point $q \in \Sigma$ is an elementary singular point of X if q is either a fold of X_i (for $i = 1$ or $i = 2$) or a hyperbolic singular point of X^{Σ}.

Proof of Theorem A. Consider $X \in \Omega^r$ with $X_i = (f_i, g_i) \in \chi^r$, $i = 1, 2$. Suppose that $a_1 t + \cdots + a_k t^k$ is the polynomial expression of φ on $I \subset \mathbb{R}$ with $0 \in I$. The trajectories of X_ε on V_ε are the solutions of the differential system

$$\dot{x} = (f_1 + f_2)/2 + \varphi(F/\varepsilon)(f_1 - f_2)/2,$$
$$\dot{y} = (g_1 + g_2)/2 + \varphi(F/\varepsilon)(g_1 - g_2)/2.$$

The time rescaling $\tau = t/\varepsilon^k$ gives

$$x' = h_1 = \varepsilon^k (f_1 + f_2)/2 + \big(a_1 F \varepsilon^{k-1} + \cdots + a_k F^k\big)(f_1 - f_2)/2,$$
$$y' = h_2 = \varepsilon^k (g_1 + g_2)/2 + \big(a_1 F \varepsilon^{k-1} + \cdots + a_k F^k\big)(g_1 - g_2)/2.$$

Thus we take $h = (h_1, h_2)$ and have that $h(x, y, 0) = 0$ for all $(x, y) \in \Sigma$. The eigenvalues of the linear part of h for $\varepsilon = 0$ are the solutions of the equation

$$\lambda^2 - (a_k/2)(X_1 - X_2) F^k(p) \lambda = 0.$$

It follows that zero is an eigenvalue of multiplicity at least one. From the other side our hypothesis ensures that there exists a non-zero eigenvalue. So we get a normally hyperbolic scenario and we may apply the Fenichel theory (see [5, Lemma 5.3, p. 67]) to get the desired coordinates. □

Lemma 6 *(Fundamental lemma). Consider $X \in \Omega_d$ and $\xi \in C$ satisfying $\nabla \xi (X_1 - X_2) = \Pi_2 (X_1 - X_2)$. If $\bar{x} = x$ and $\bar{y} = y - \xi(x, y)$ then the differential system*

$$\dot{x} = (f_1 + f_2)/2 + \varphi(F/\varepsilon)(f_1 - f_2)/2,$$
$$\dot{y} = (g_1 + g_2)/2 + \varphi(F/\varepsilon)(g_1 - g_2)/2 \tag{8}$$

is written as

$$\varepsilon \dot{\bar{x}} = \varepsilon\big[(\bar{f}_1 + \bar{f}_2)/2 + \overline{\varphi(F/\varepsilon)}(\bar{f}_1 - \bar{f}_2)/2\big],$$
$$\dot{\bar{y}} = (\bar{g}_1 + \bar{g}_2)/2 - \overline{(\partial \xi/\partial x)}(\bar{f}_1 + \bar{f}_2)/2 - \overline{(\partial \xi/\partial y)}(\bar{g}_1 + \bar{g}_2)/2,$$

where the line over the function means that the respective function is given in such new coordinates.

Proof. If $\bar{x}=x$ and $\bar{y}=y-\xi(x,y)$ we get

$$\dot{\bar{x}}=(\bar{f}_1+\bar{f}_2)/2+\overline{\varphi(F/\varepsilon)}(\bar{f}_1-\bar{f}_2)/2$$

and

$$\begin{aligned}\dot{\bar{y}}&=(\bar{g}_1+\bar{g}_2)/2-\overline{(\partial\xi/\partial x)}(\bar{f}_1+\bar{f}_2)/2-\overline{(\partial\xi/\partial y)}(\bar{g}_1+\bar{g}_2)/2\\&\quad-\overline{\varphi(F/\varepsilon)/2}\big(\nabla\xi(X_1-X_2)-\Pi_2(X_1-X_2)\big).\end{aligned}$$

Using the partial differential equation we get the desired formula. □

Analogously if $\nabla\xi(X_1-X_2)=\Pi_1(X_1-X_2)$ and $\bar{x}=x-\xi(x,y)$ and $\bar{y}=y$ then the differential system (8) is written as

$$\begin{aligned}\dot{\bar{x}}&=(\bar{f}_1+\bar{f}_2)/2-\overline{(\partial\xi/\partial x)}(\bar{f}_1+\bar{f}_2)/2-\overline{(\partial\xi/\partial y)}(\bar{g}_1+\bar{g}_2)/2,\\\varepsilon\dot{\bar{y}}&=\varepsilon\big[(\bar{g}_1+\bar{g}_2)/2+\overline{\varphi(F/\varepsilon)}(\bar{g}_1-\bar{g}_2)/2\big].\end{aligned}$$

Proof of Theorem B. Consider $X\in\Omega_d$ and suppose that φ is a polynomial of degree k in a small interval $I\subseteq\mathbb{R}$ with $0\in I$.

There exists $\xi:\mathbb{R}^2\to\mathbb{R}$ satisfying $\xi\in C^r$, $L(\xi)=0$ and the partial differential equation $\nabla\xi(X_1-X_2)=\Pi_i(X_1-X_2)$, for $i=1$ or $i=2$. We suppose without lost of generality that $i=2$.

The trajectories of the regularized vector field X_ε, on V_ε, are the solutions of the differential system (8).

We consider the coordinates $(\bar{x},\bar{y})$ given by $\bar{x}=x$, $\bar{y}=y-\xi(x,y)$, and then we apply Lemma 6. □

In order to the simplify the computation we suppose $\varphi(x)=x$ for $-1/2<x<1/2$.

Example 1. Consider the class of discontinuous vector fields $X\in\Omega^r$, $X_i=(f_i,g_i)$, $i=1,2$, with $F(x,y)=x$ and $g_1=g_2=g$. One can see that $\xi\equiv0$ is a solution of the partial differential equation (7). It implies that in canonical coordinates we get a SP-problem.

Example 2. Consider the class of discontinuous vector fields $X\in\Omega^r$, $X_i=(f_i,g_i)$, $i=1,2$, with $F(x,y)=y$, $f_1=f_2=f$, and $g_1=-g_2=g$. As before, $\xi\equiv0$ is a solution of the partial differential equation (7) and in canonical coordinates we get the SP-problem

$$\dot{x}=f(x,y),\qquad\varepsilon\dot{y}=yg(x,y).$$

Example 3. Take $X_1(x,y)=(1,x)$, $X_2(x,y)=(-1,-3x)$, and $F(x,y)=y$. The discontinuity set is $\{(x,0)\mid x\in\mathbb{R}\}$. We have $X_1F=x$, $X_2F=-3x$, and then the unique non-regular point is $(0,0)$. We apply Theorem B.

The vector field (2) is $X_\varepsilon(x, y) = (y/\varepsilon, 2xy/\varepsilon - x)$. The partial differential equation (7) with $i = 2$ becomes $2(\partial\xi/\partial x) + 4x(\partial\xi/\partial y) = 4x$. A solution $\xi \in C$ is given by $\xi(x, y) = x^2$. Thus we take the coordinate change $\bar{x} = x$, $\bar{y} = y - x^2$. The trajectories of X_ε in these coordinates are the solutions of the singular system

$$\varepsilon\dot{\bar{x}} = \bar{y} + \bar{x}^2, \qquad \dot{\bar{y}} = -\bar{x}.$$

3. A special subclass of discontinuous vector fields

In this section we consider the class of discontinuous vector fields $X_i = (f_i, g_i)$, $i = 1, 2$, satisfying the additional condition

$$F(x, y) = x, \qquad g_1 = g_2 = g. \tag{9}$$

For simplicity we assume that $\varphi(x) = x$ for $-1/2 < x < 1/2$.

Proposition 7. *Let $X \in \Omega^r$ satisfying the hypotheses of Theorem* A *and X_ε its φ-regularization written like* (3)*. We have*:

(i) *if $p \in \Sigma_3$ then p is an attractor of X_0 for the fast flow*;
(ii) *if $p \in \Sigma_2$ then p is a repeller of X_0 for the fast flow*;
(iii) *if $p \in \Sigma$ is a regular point then either p is an attractor, or p is a repeller or p is a sewing point.*

Proof. The linear part of the corresponding singular problem (3) at $p \in \Sigma$ and with $\varepsilon = 0$ has two eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = (1/2)(X_1 - X_2)F(p)$. The eigenvalue λ_1 determines the slow manifold (or discontinuous set, according the terminology used) and the eigenvalue λ_2 determines if p is attractor ($\lambda_2 < 0$) or repeller ($\lambda_2 > 0$). If $p \in \Sigma_3$ then $(X_1F)(p) < 0$ and $(X_2F)(p) > 0$ and thus $\lambda_2 < 0$. If $p \in \Sigma_2$ then $(X_1F)(p) > 0$ and $(X_2F)(p) < 0$ and thus $\lambda_2 > 0$. Suppose that $p \in \Sigma$ is not an attractor or a repeller. It means that $p \in (\Sigma_2 \cup \Sigma_3)^C$. Thus $(X_1F)(X_2F) > 0$ and then p is a sewing point. □

In what follows $X \in (\Omega_d)^*$ means that $X \in \Omega_d$ with the additional condition (9).

A direct computation gives the following result.

Corollary 8. *Consider $X \in (\Omega_d)^*$. The trajectories of X_ε given by* (2)*, on $|F(q)| < \varepsilon/2$ are the solutions of the singular system*

$$\varepsilon\dot{x} = f(x, y, \varepsilon), \qquad \dot{y} = g(x, y) \tag{10}$$

with $f(x, y, \varepsilon) = \varepsilon(f_1 + f_2)/2 + x(f_1 - f_2)/2$ and $g \in C^r$.

Proposition 9. *Consider $X \in (\Omega_d)^*$. We have*:

(a) *The slow manifold of* (10) *is the set*

$$\Sigma \cup \{(x, y):\ f_1(x, y) = f_2(x, y)\}.$$

(b) *$p \in \Sigma_1$ if and only if $f_1(p)f_2(p) > 0$.*

(c) $p \in \Sigma_2$ *if and only if* $f_1(p) > 0$ *and* $f_2(p) < 0$.
(d) $p \in \Sigma_3$ *if and only if* $f_1(p) < 0$ *and* $f_2(p) > 0$.
(e) *If* $p \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ *and* $f_1(p) \neq f_2(p)$ *then* p *is a normally hyperbolic point of* (10).

Proof. The slow manifold of (10) is the set $S = \{(x, y)\colon f(x, y, 0) = 0\}$ and thus $S = \{(x, y)\colon x(f_1(x, y) - f_2(x, y)) = 0\}$. To prove (b)–(d) we compute X_1F and X_2F:

$$X_iF = (f_i, g_i)F = (1, 0)(f_i, g_i) = f_i.$$

So we use the definition of Σ_1, Σ_2 and Σ_3. To finish the proof we have just to compute the linear part of

$$x' = f(x, y, \varepsilon), \qquad y' = \varepsilon g(x, y, \varepsilon)$$

at $(x, y, \varepsilon) = (0, y, 0)$ which has the eigenvalue $\lambda = (1/2)(f_1(0, y) - f_2(0, y))$ associated to eigenvector $(1, 0)$. □

Proposition 10. *Consider* $X \in (\Omega_d)^*$. *The trajectories of the sliding vector field* X^Σ *are the solutions of the reduced problem of the SP-problem* (10).

Proof. It is enough to observe that the y-component of all points on the line joining (f_1, g) and (f_2, g) is g and to apply the definitions. In fact, the trajectories of the sliding vector field X^Σ are the trajectories of the vector field $(0, g(x, y, 0))$ and the reduced problem of the SP-problem is

$$x = 0, \qquad \dot{y} = g(x, y, 0). \qquad \square$$

Proposition 11. *Consider* $X \in (\Omega_d)^*$. *We have that* $p \in \Sigma$ *is a fold point if and only if* $f_1(p) = 0$ *and the vector field* X_1 *is not tangent to the level* 0 *of the function* f_1.

Proof. We have that $X_1F = f_1$ and $(X_1)^2F = \nabla f_1.X_1 = 0$ if and only if $\nabla f_1 \perp X_1$. □

Theorem 12. *Consider* $X \in (\Omega_d)^*$ *and the* φ*-regularized system* X_ε *given by* (2).

(a) *If* $q \in \Sigma$ *is a regular point of* X *then there exist a neighborhood* $V \subseteq R^2$ *with* $q \in V$ *and* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, X_ε *does not have singular points in* V.
(b) *If* $q \in (\Sigma_2 \cup \Sigma_3)$ *is a hyperbolic singular point of* X^Σ *then there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, X_ε *has a saddle point or a node point near* q.

Lemma 13. *Consider* $X \in (\Omega_d)^*$ *and the* φ*-regularized system* X_ε *given by* (2). *We have that* $q \in \Sigma$ *is a regular point of* X *if and only if* $f_1(q)f_2(q) > 0$ *or* $q = (0, y_0)$ *with* $g(q) \neq 0$.

Proof. A point q is a regular point of X if and only if $(X_1FX_2F)(q) > 0$ or $q \in \Sigma_2 \cup \Sigma_3$ and it is not a singular point of X^Σ. The first condition means that $f_1(q)f_2(q) > 0$ and the second one follows from Proposition 10. In fact, a singular point of X^Σ is also a singular point of the reduced problem

$$x = 0, \qquad \dot{y} = g(0, y)$$

for every normally hyperbolic point. □

Consider X the vector field given by (3) and S its slow manifold given by (5). We assume that, for every normally hyperbolic $q \in S$, $(\partial f/\partial x)(q,0)$ has k^s eigenvalues with negative real part and k^u eigenvalues with positive real part.

Lemma 14. *Let $q \in S$ be a hyperbolic singular point of the slow flow with j^s-dimensional local stable manifold W^s and a j^u-dimensional local unstable manifold W^u. If X is normally hyperbolic at q then there exists an ε-continuous family q_ε such that $q_0 = q$ and q_ε has a $(j^s + k^s)$-dimensional local stable manifold W^s_ε and a $(j^u + k^u)$-dimensional local unstable manifold W^u_ε.*

For a proof see [5].

Proof of Theorem 12. (a) Suppose that $q \in \Sigma$ is a regular point. Lemma 13 implies that $f_1(q)f_2(q) > 0$ or $q = (0, y_0)$ with $g(q) \neq 0$. Suppose that $f_1(q)f_2(q) > 0$. So f_1 and f_2 have the same sign and it implies that $X_\varepsilon(q)$ is transversal to the line $x = 0$ at any $\bar{q}$ near q, for sufficiently small $\varepsilon > 0$. And we use the flow box theorem. If $g(q) \neq 0$ then $g(\bar{q}) \neq 0$ for any $\bar{q}$ near q and so $X_\varepsilon(q) \neq (0,0)$.

(b) Consider $q \in (\Sigma_2 \cup \Sigma_3)$. Without lost of generality assume that $q \in \Sigma_3$. Using Proposition 9 we have that q is normally hyperbolic. Lemma 14 implies that X_ε has a singular point q_ε which approaches q when $\varepsilon \searrow 0$. If $f_1(q) > f_2(q)$ then $k^s = 0$ and $k^u = 1$ and finally q_ε is a repelling node if $j^u = 1$, $j^s = 0$ or a saddle if $j^u = 0$, $j^s = 1$. If $f_1(q) < f_2(q)$ then $k^s = 1$ and $k^u = 0$ and then q_ε is an attracting node if $j^u = 0$, $j^s = 1$ or a saddle if $j^u = 1$, $j^s = 0$. The proof for the case $q \in \Sigma_2$ is similar. □

4. Blowing up

In this section the desingularization of generic vector fields is discussed by means of the blowing up method. We also present the proof of Theorem C.

Consider $X = (X_1, X_2) \in \Omega^r$ with $X_i = (f_i, g_i)$, $i = 1, 2$, and local coordinates around $p \in \Sigma$ such that $F(x,y) = x$ and $p = 0$. Thus $\widehat{X}(x,y) = |x| X(x,y)$ and the trajectories of its φ-regularization $\widehat{X}(x,y,\varepsilon)$ satisfy the differential system

$$\begin{aligned}
\dot{x} &= (f_1 + f_2)/2 + \varphi(x/\varepsilon)(f_1 - f_2)/2, \\
\dot{y} &= (g_1 + g_2)/2 + \varphi(x/\varepsilon)(g_1 - g_2)/2, \\
\dot{\varepsilon} &= 0
\end{aligned} \tag{11}$$

for $\varepsilon > 0$ and $(x,y) \in U \subseteq \mathbb{R}^2$, where U is a neighborhood of $(0,0)$. We observe that system (11) is not defined for $\varepsilon = 0$ and it is not an explicit SP-problem according to Definition 2. However if we consider the directional blow-up $\beta : \mathbb{R}^3 \to \mathbb{R}^3$ given by $(x, y, \varepsilon) = \beta(\bar{x}, y, \bar{\varepsilon}) = (\bar{x}\bar{\varepsilon}, y, \bar{\varepsilon})$, system (11) becomes

$$\begin{aligned}
\bar{\varepsilon}\dot{\bar{x}} &= (f_1 + f_2)/2 + \varphi(\bar{x})(f_1 - f_2)/2, \\
\dot{y} &= (g_1 + g_2)/2 + \varphi(\bar{x})(g_1 - g_2)/2, \\
\dot{\bar{\varepsilon}} &= 0.
\end{aligned} \tag{12}$$

Taking $\bar{\varepsilon}$ as a parameter system (12) is clearly an explicit SP-problem. Moreover, the parameter value $\bar{\varepsilon} = 0$ can be considered.

Proof of Theorem C. Consider $X \in \Omega^r$ as in the previous remark and $U \subseteq \mathbb{R}^2$ a small neighborhood of $(0,0)$. Let $V = U \times (0, +\infty)$ and $M = \beta^{-1}(V)$. Next, it is easy to deduce that system (12) defines a SP-problem Y on M and the map $\Phi = \beta|_M$ sends orbits of $Y|_{\beta^{-1}(V)}$ into orbits of $\widehat{X}|_V$. $\square$

4.1. The polar blow-up

Geometrically speaking, it is more convenient to consider the polar blow-up coordinates $\alpha : [0, +\infty) \times [0, \pi] \times \mathbb{R} \to \mathbb{R}^3$ given by $x = r\cos\theta$ and $\varepsilon = r\sin\theta$. The map α induces the vector field on $[0, +\infty) \times [0, \pi] \times \mathbb{R}$ given by

$$\begin{aligned}
\theta' &= -\sin\theta\big[(f_1 + f_2)/2 + \varphi(\cot\theta)(f_1 - f_2)/2\big],\\
y' &= r\big[(g_1 + g_2)/2 + \varphi(\cot\theta)(g_1 - g_2)/2\big],\\
r' &= r\cos\theta\big[(f_1 + f_2)/2 + \varphi(\cot\theta)(f_1 - f_2)/2\big]. \qquad (13)
\end{aligned}$$

The parameter value $\varepsilon = 0$ is now represented by $r = 0$ and the induced vector field is described below.

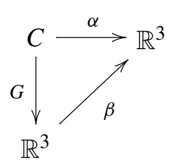
- On the region $(r, \theta, y) \in (\{0\} \times (0, \pi) \times \mathbb{R})$:

$$\theta' = -\sin\theta\big[(f_1 + f_2)/2 + \varphi(\cot\theta)(f_1 - f_2)/2\big], \qquad y' = 0. \qquad (14)$$

- On the region $(r, \theta, y) \in (\{0\} \times \{0\} \times \mathbb{R}) \cup (\{0\} \times \{\pi\} \times \mathbb{R})$:

$$\theta' = 0, \qquad y' = 0. \qquad (15)$$

We observe that the directional blow-up and the polar blow-up are essentially the same. In fact, if we consider the map $G : C = [0, +\infty) \times (0, \pi) \times \mathbb{R} \to \mathbb{R}^3$ given by $G(r, \theta, \pi) = (\cot\theta, y, r\sin\theta)$ then $\beta \circ G = \alpha$.



In what follows we present a rough geometrical description of the phase portrait of the system $\widehat{X}_\varepsilon$. Its trajectories on the region $\{(x, y, 0) \mid x < 0\}$ (respectively on the region $\{(x, y, 0) \mid x > 0\}$) are determined by the orbits of $-x.X_2(x, y)$ (respectively $x.X_1(x, y)$). We represent the set $\{(0, y, 0) \mid y \in \mathbb{R}\}$ by $S^1_+ \times \mathbb{R} = \{(\cos\theta, \sin\theta, y) \mid \theta \in (0, \pi),\ y \in \mathbb{R}\}$. The curves $\theta = 0$ and $\theta = \pi$ are composed by singular points. The fast flow on $S^1_+ \times \mathbb{R}$ is given by the solutions of system (14) and the slow flow is given by the solutions of

$$0 = -\sin\theta\big[(f_1 + f_2)/2 + \varphi(\cot\theta)(f_1 - f_2)/2\big],$$
$$\dot{y} = (g_1 + g_2)/2 + \varphi(\cot\theta)(g_1 - g_2)/2. \tag{16}$$

If necessary we may consider additional blowing ups to get normally hyperbolic points on the slow manifold. The following charts can be very useful to the understanding of the phase portrait of the system according to the techniques introduced in [4].

- The rescaling family:

$$\theta = s\bar{\theta}, \qquad y = s\bar{y}, \qquad r = s^l$$

  with $\bar{\theta}, \bar{y} \in \mathbb{R}^2$, $s \geqslant 0$ and $l \in \mathbb{N}$. In this case the circle $S^1 = \{(\bar{\theta}, \bar{y}, s);\ s = 0,\ \bar{\theta}^2 + \bar{y}^2 = 1\}$ is the boundary of the Poincaré disk.
- The phase-directional:

$$\theta = s\cos\psi, \qquad y = s\sin\psi \tag{17}$$

  with $\theta \in [0, 2\pi]$ and $s \geqslant 0$.

In this case we can apply the tools of GSP-theory (Flow-box theorem, center manifolds, Fenichel theory) in order to get the phase portrait of the regularized vector field.

4.2. *Generic discontinuous vector fields*

The dynamics of a discontinuous vector field X in a neighborhood of a point $p \in \Sigma$ can be analyzed by means of the regularization method. Our aim, in this section, is to apply the GSP-theory to this analysis.

First of all we analyze the regular points of the discontinuous vector field.

We discuss some normal forms presented in [10] representing the codimension zero singularities. For each one of these forms we blow-up the discontinuous set and describe the SP-problem in the blowing up locus. We have that $X_N = (X_1, X_2) \in (\Omega_d)^*$ and so equations of (13) become

$$r\dot{\theta} = -\sin\theta\big[(f_1 + f_2)/2 + \varphi(\cot\theta)(f_1 - f_2)/2\big], \qquad \dot{y} = g.$$

4.2.1. *Sewing*

Assume that $p = (0, 0) \in \Sigma_1$, $X_1(x, y) = (1, 1)$ and $X_2(x, y) = (2, 1)$. The SP-problem in the blowing up locus (13) is

$$r\dot{\theta} = -\sin\theta\big(3/2 - \varphi(\cot\theta)/2\big), \qquad \dot{y} = 1.$$

The composition $\varphi(\cot\theta)$ is a decreasing function with $\lim_{\theta\to 0^+}\varphi(\cot\theta) = 1$, and $\lim_{\theta\to\pi^-}\varphi(\cot\theta) = -1$. Thus $-\sin\theta(3/2 - \varphi(\cot\theta)/2) < 0$ for any $0 < \theta < \pi$. The fast flow, on the blowing up locus, is invariant on each section $y = \text{constant}$. The phase portrait of the fast and slow dynamics of the singular problem, for $\varepsilon = 0$, and the phase portrait of the regularized vector field for small $\varepsilon > 0$ are illustrated in Fig. 1.

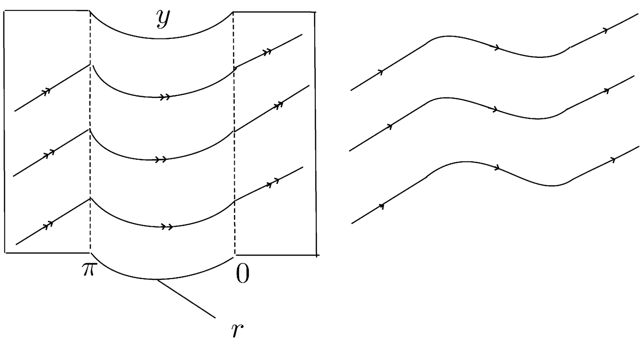


Fig. 1. Fast and slow dynamics of the SP-problem corresponding to the sewing case and its regularization.

4.2.2. Escaping

Assume that $p = (0,0) \in \Sigma_2$, $X_1(x,y) = (1,1)$ and $X_2(x,y) = (-1,1)$. The SP-problem in the blowing up locus (13) is

$$r\dot{\theta} = -\sin\theta\varphi(\cot\theta), \qquad \dot{y} = 1.$$

The slow manifold is $\{(\theta, y) \in (0,\pi) \times \mathbb{R};\ \theta = \theta_0,\ \varphi(\cot\theta_0) = 0\}$ and the reduced flow goes in the positive direction of the y-axis. For the fast flow we observe that any point on the slow manifold is a repelling singular point because the angular eigenvalue is $d/d\theta(-\sin\theta\varphi(\cot\theta))|_{\theta=\theta_0} = \varphi'(0) > 0$. The phase portrait of the fast and slow dynamics of the singular problem, for $\varepsilon = 0$, and the phase portrait of the regularized vector field for small $\varepsilon > 0$ are illustrated in Fig. 2.

4.2.3. Sliding

Assume that $p = (0,0) \in \Sigma_3$, $X_1(x,y) = (-1,-1)$ and $X_2(x,y) = (1,-1)$. The phase portrait of the fast and slow dynamics of the singular problem, for $\varepsilon = 0$, and the phase portrait of the regularized vector field for small $\varepsilon > 0$ are illustrated in Fig. 3.

The first non-regular case appears when p is a singular point of X^{Σ}. In this case $p \in \Sigma_2 \cup \Sigma_3$ and it is an attractor or a repeller, or p is a fold point. For each one of this subcases we have a normal form in $(\Omega_d)^*$.

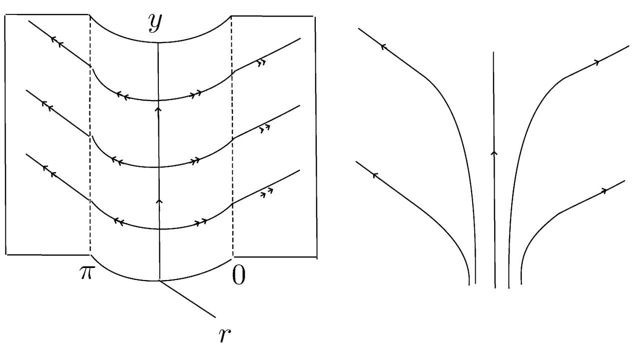


Fig. 2. Fast and slow dynamics of the SP-problem corresponding to the escaping case and its regularization.

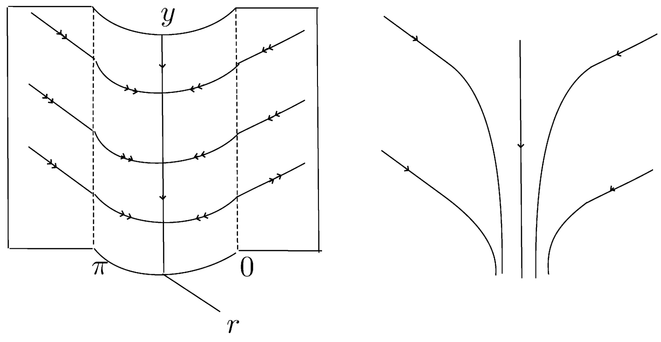


Fig. 3. Fast and slow dynamics of the SP-problem corresponding to the sliding case and its regularization.

4.2.4. Saddle

Assume that $p=(0,0)\in\Sigma_2$ (respectively $p=(0,0)\in\Sigma_3$), $X_1(x,y)=(x+1,-y)$ (respectively $X_1(x,y)=(-x-1,y)$) and $X_2(x,y)=(x-1,-y)$ (respectively $X_2(x,y)=(-x+1,y)$). The SP-problem in the blowing up locus (13) is

$$r\dot\theta=-r\sin\theta\cos\theta-\sin\theta\varphi(\cot\theta),\qquad \dot y=-y$$

(respectively

$$r\dot\theta=r\sin\theta\cos\theta+\sin\theta\varphi(\cot\theta),\qquad \dot y=y).$$

The slow manifold is $\{(\theta,y)\in(0,\pi)\times\mathbb{R};\ \theta=\theta_0,\ \varphi(\cot\theta_0)=0\}$. The reduced flow follows the positive (respectively negative) direction of the y-axis if $y<0$ and follows the negative (respectively positive) direction of the y-axis if $y>0$. For the fast flow we observe that any point on the slow manifold is a repelling (respectively an attracting) singular point because the angular eigenvalue is $\varphi'(0)>0$ (respectively $-\varphi'(0)<0$). The phase portrait of the fast and slow dynamics of the singular problem, for $\varepsilon=0$, and the phase portrait of the regularized vector field for small $\varepsilon>0$ are illustrated in Fig. 4.

The last elementary case that we consider is:

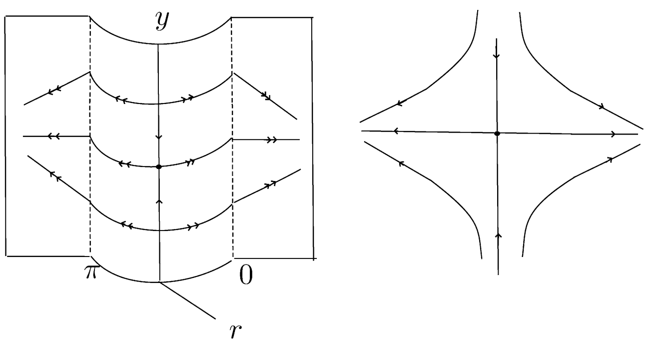


Fig. 4. Fast and slow dynamics of the SP-problem corresponding to the saddle case $p\in\Sigma_2$ and its regularization.

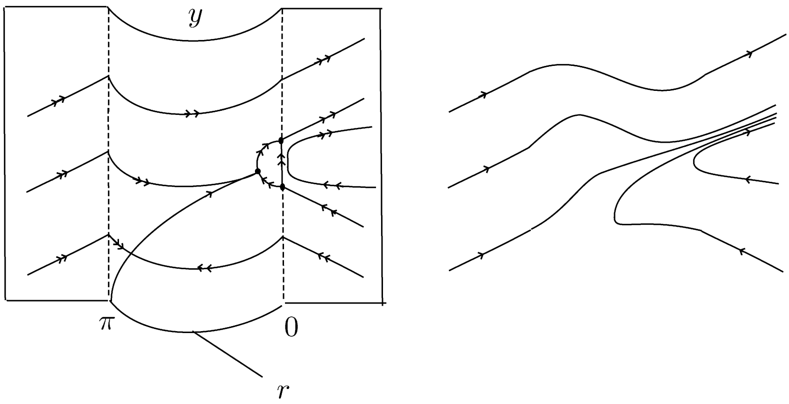


Fig. 5. Fast and slow dynamics of the SP-problem corresponding to the fold case and its regularization.

4.2.5. Fold

Assume that $p=(0,0)$, $X_1(x,y)=(y,1)$ and $X_2(x,y)=(1,1)$. The SP-problem in the blowing up locus (13) is

$$r\dot{\theta}=-\sin\theta\big((y+1)/2+\varphi(\cot\theta)(y-1)/2\big),\qquad \dot{y}=1.$$

The slow manifold is the curve $y(\varphi(\cot\theta)+1)=(\varphi(\cot\theta)-1)$. It is the graphic of a decreasing function which is 0 for $\theta=0$ and tends to $-\infty$ when $\theta\to\pi$. The reduced flow goes in the positive direction of the y-axis. For the fast flow we observe that $\dot{\theta}>0$, for $0<\theta<\theta(y)$ and $\dot{\theta}<0$, for $\theta(y)<\theta<\pi$, with $\theta(y)$ given implicitly by $y(\varphi(\cot\theta)+1)-(\varphi(\cot\theta)-1)=0$. The phase portrait of the fast and slow dynamics of the singular problem, for $\varepsilon=0$, and the phase portrait of the regularized vector field for small $\varepsilon>0$ are illustrated in Fig. 5.

Observe that the point $(\theta,y)=(0,0)$ is not a normally hyperbolic singular point since the following equality occurs

$$\frac{\partial}{\partial\theta}\big[-\sin\theta\big((y+1)/2+\varphi(\cot\theta)(y-1)/2\big)\big]\Big|_{(0,0)}=0.$$

We perform an additional blow-up at $(\theta,y,r)=(0,0,0)$. Such blow-up is defined by formulas (17) with $s\geqslant 0$ and $\frac{-\pi}{2}\leqslant\psi\leqslant\frac{\pi}{2}$. In these coordinates we have

$$s'=-\cos\psi\sin(s\cos\psi)G(s,\psi),\qquad \psi'=\frac{\sin\psi}{s}\sin(s\cos\psi)G(s,\psi)$$

with $G(s,\psi)=\frac{s\sin\psi+1}{2}+\varphi(\cot(s\cos\psi))\frac{s\sin\psi-1}{2}$.

One verifies that $G(0,\psi)=0$ and that $\frac{\partial G}{\partial s}|_{s=0}=\sin\psi$. In order to determine the flow on the blowing up locus we observe that, after a division by s, $\lim_{s\to 0}\psi'=\sin^2\psi\cos\psi$.

It means that the angle component is increasing for $\psi\in(-\frac{\pi}{2},\frac{\pi}{2})$ with a singular point at $\psi=0$.

4.3. SP-problems for the codimension 1 normal forms of singular points

In Section 2 we defined the set of regular points and the set of elementary singular points for a discontinuous vector field (X_1, X_2) having discontinuous set Σ. In this section we consider the break of the conditions which give the elementary singular set. More specifically we analyze the codimension 1 singular points.

Definition 15. We say that $q \in \Sigma$ is a codimension 1 singular point if one of the following conditions is satisfied.

(a) Saddle-node: $X_1F(q)X_2F(q) < 0$ and q is a non-hyperbolic singular point of X^Σ.
(b) Elliptical fold: $X_1F(q) = X_2F(q) = 0$, $X_1^2F(q) < 0$ and $X_2^2F(q) > 0$.
(c) Hyperbolic fold: $X_1F(q) = X_2F(q) = 0$, $X_1^2F(q) > 0$ and $X_2^2F(q) < 0$.
(d) Parabolic fold: $X_1F(q) = X_2F(q) = 0$, $X_1^2F(q) > 0$ and $X_2^2F(q) > 0$.

Next we discuss the codimension one normal forms of singularities presented in [10]. For each one of such form we blow-up the discontinuous set and describe the SP-problem in the blowing up locus.

4.3.1. Saddle-node

We take $q = (0, 0) \in \Sigma_3$, $X_1(x, y) = (-1, -y^2)$ and $X_2(x, y) = (1, 0)$. The SP-problem in the blowing up locus is

$$r\dot{\theta} = \sin\theta\varphi(\cot\theta), \qquad \dot{y} = -\big(y^2/2\big)\big(1 + \varphi(\cot\theta)\big).$$

The slow manifold is the set $\{(\theta, y) \in (0, \pi) \times \mathbb{R};\ \theta = \theta_0,\ \varphi(\cot\theta_0) = 0\}$. The reduced flow on the slow manifold $\theta = \frac{\pi}{2}$ is $\dot{y} = -\frac{y^2}{2}$. Moreover, there exists a reduced singular point at $(\theta, y) = (\theta_0, 0)$. The fast vector field is $(\theta', 0)$ with $\theta' > 0$ for $0 < \theta < \theta_0$ and $\theta' < 0$ for $\theta_0 < \theta < \pi$. The fast and the slow dynamics are represented in Fig. 6.

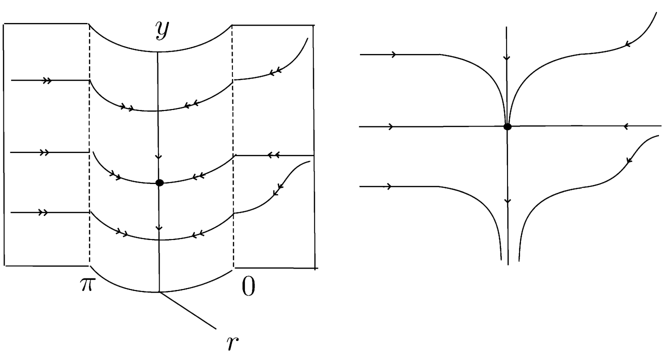


Fig. 6. Fast and slow dynamics of the SP-problem corresponding to the saddle-node case and its regularization.

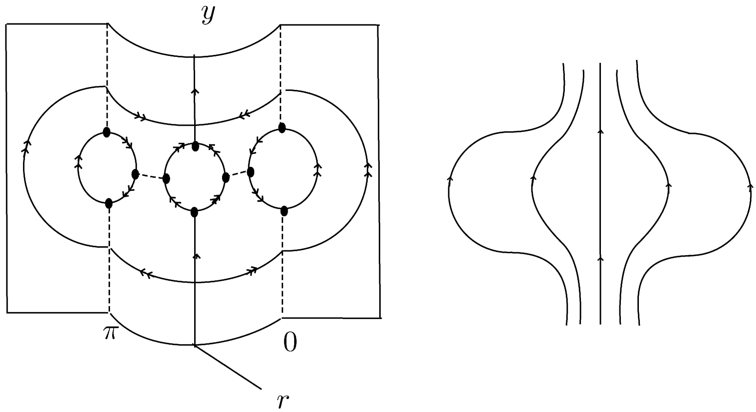


Fig. 7. Fast and slow dynamics of the SP-problem corresponding to the elliptical fold (+) case and its regularization.

4.3.2. Elliptical fold

We take $q = (0,0) \in \Sigma$, $X_1(x,y) = (-y,1)$ and $X_2^{\pm}(x,y) = (\pm y, \pm 1)$. For X_2^+ the SP-problem in the blowing up locus is

$$r\dot{\theta} = y \sin\theta \varphi(\cot\theta), \qquad \dot{y} = 1.$$

The slow manifold is given by $(\theta, y) = (\theta, 0)$ or $(\theta, y) = (\frac{\pi}{2}, y)$. The reduced flow, on $\frac{\pi}{2}$, follows the positive direction of the y-axis because $\dot{y} = 1 > 0$ and on $y = 0$ it is composed by singular points. The fast vector field is expressed by $(\theta', 0)$ with $\theta' > 0$ if and only if $y > 0$ and $0 < \theta < \frac{\pi}{2}$ or $y < 0$ and $\frac{\pi}{2} < \theta < \pi$. The fast and the slow dynamics are represented in Fig. 7. The regularized vector field does not have singular points.

For any $\theta \in [0, \pi]$, the point $(\theta, y) = (\theta, 0)$ is not a normally hyperbolic singular point because $\frac{\partial}{\partial\theta}[y \sin\theta\varphi(\cot\theta)]|_{(\theta,0)} = 0$. Since $(\theta, y, r) = (\frac{\pi}{2}, 0, 0)$ is a self intersection point of the slow manifold, we perform an additional blow-up on it. First of all we translate this point to the origin with $(\theta_1, y_1) = (\theta - \frac{\pi}{2}, y)$. Next we consider the blow-up defined by formulas (17) with $s \geqslant 0$ and $\psi \in [0, 2\pi]$. In these coordinates we have $\psi' = \frac{-\sin^2\psi}{2} G(s, \psi)$ with $G(s, \psi) = 2\sin(s\cos\psi + \frac{\pi}{2})\varphi(\cot(s\cos\psi + \frac{\pi}{2}))$. As before $G(0, \psi) = 0$ and now $\frac{\partial G}{\partial s}|_{s=0} = -3\varphi'(0)\cos\psi$. Thus after a division by s, we have $\lim_{s\to 0}\psi' = \frac{3}{2}\sin^2\psi\cos\psi\varphi'(0)$. It means that the angle component is increasing for $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and decreasing for $\psi \in (\frac{\pi}{2}, \frac{3\pi}{2})$.

We also consider an additional blow-up at $(0,0)$. We consider the blow-up defined by formulas (17) with $s \geqslant 0$ and $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. In these coordinates we have $\psi' = \frac{-\sin^2\psi}{2} G(s, \psi)$ with $G(s, \psi) = 2\sin(s\cos\psi)\varphi(\cot(s\cos\psi))$. We get $G(0, \psi) = 0$ and $\frac{\partial G}{\partial s}|_{s=0} = 2\cos\psi$. Thus after a division by s, we have $\lim_{s\to 0}\psi' = -\sin^2\psi\cos\psi$. It means that the angle component is decreasing for $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. A similar analysis can be considered at the point $(\pi, 0, 0)$.

For X_2^- the SP-problem in the blowing up locus is

$$r\dot{\theta} = y \sin\theta, \qquad \dot{y} = \varphi(\cot\theta).$$

The slow manifold is the set $y = 0$. The reduced flow on $y = 0$ is composed by singular points. The fast vector field is $(\theta', 0)$ with $\theta' > 0$ if and only if $y > 0$ and $\theta' < 0$ if and only if $y < 0$. As in the previous case, we can perform additional blow-ups at the points $(0,0,0)$ and $(\pi, 0, 0)$.

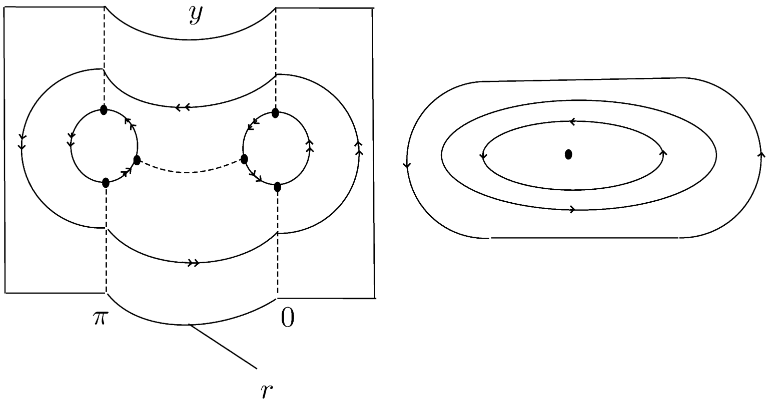


Fig. 8. Fast and slow dynamics of the SP-problem corresponding to the elliptical fold (−) case and its regularization.

Observing that $X_\varepsilon = (-y, \varphi(\frac{x}{\varepsilon}))$ is reversible with respect to involution $R(x, y) = (-x, y)$ we conclude that the origin is a center of the regularized system. The fast and the slow dynamics are represented in Fig. 8.

4.3.3. Hyperbolic fold

We take $q = (0,0) \in \Sigma$, $X_1(x, y) = (y, 1)$, and $X_2^+(x, y) = (2y, -1)$, and $X_2^-(x, y) = (-2y, 1)$. For $X = (X_1, X_2^+)$ we have that the SP-problem in the blowing up locus is

$$r\dot{\theta} = -(y/2)\sin\theta\big(3 - \varphi(\cot\theta)\big), \qquad \dot{y} = \varphi(\cot\theta).$$

The slow manifold is $y = 0$. The reduced flow is composed by singular points. The fast vector field is $(\theta', 0)$ with $\theta' > 0$ if $y > 0$ and with $\theta' < 0$ if $y < 0$. As before, we perform additional blow-ups at the points $(0, 0, 0)$ and $(\pi, 0, 0)$. The fast and the slow dynamics are represented in Fig. 9. The regularized vector field has only one singular point which is of the saddle type. For $X = (X_1, X_2^-)$ we have that the SP-problem in the blowing up locus is

$$r\dot{\theta} = -(y/2)\sin\theta\big(-1 + 3\varphi(\cot\theta)\big), \qquad \dot{y} = 1.$$

The slow manifold is $y = 0$ or $\theta = \theta_0$ with $\varphi(\cot\theta_0) = 1/3$. The reduced flow goes in the positive direction of the y-axis on $\theta = \theta_0$ and is composed by singular points on $y = 0$. The fast vector field is $(\theta', 0)$ with $\theta' < 0$ for $0 < \theta < \theta_0$ and $y > 0$ or $y < 0$ and $\theta_0 < \theta < \pi$; and $\theta' > 0$ for $0 < \theta < \theta_0$ and $y < 0$ or $\theta_0 < \theta < \pi$ and $y > 0$. Here we perform additional blow-ups at the points $(0, 0, 0)$, $(\theta_0, 0, 0)$ and $(\pi, 0, 0)$. The dynamics of the system at the slow manifold at $(\theta_0, 0, 0)$ can be easily visualized if a translation and the rescaling

$$\theta = s\theta, \qquad y = sy, \qquad r = s^3$$

are considered. The fast and the slow dynamics are represented in Fig. 10. The regularized vector field does not have singular points.

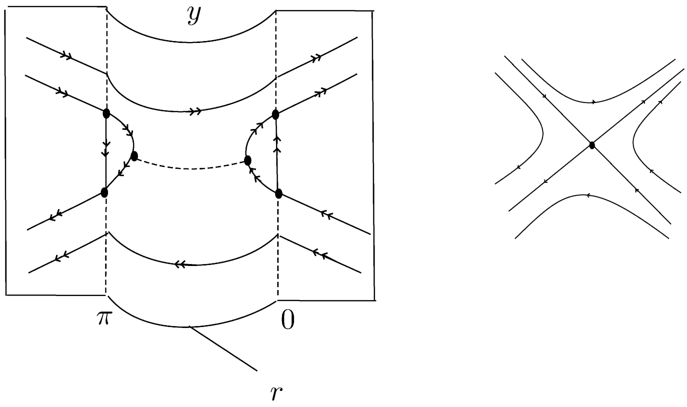


Fig. 9. Fast and slow dynamics of the SP-problem corresponding to the hyperbolic fold case with sign (+) and its regularization.

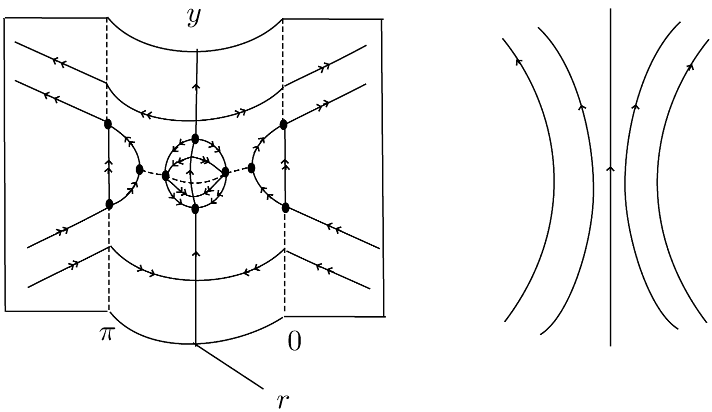


Fig. 10. Fast and slow dynamics of the SP-problem corresponding to the hyperbolic fold case with sign (−) and its regularization.

4.3.4. *Parabolic fold*

Initially we take $q = (0,0) \in \Sigma$, $X_1(x,y) = (-y,-1)$ and $X_2(x,y) = (2y,1)$. The SP-problem in the blowing up locus is

$$r\dot{\theta} = (y/2)\sin\theta\big(-1 + 3\varphi(\cot\theta)\big), \qquad \dot{y} = -\varphi(\cot\theta).$$

The slow manifold is $y = 0$ or $\theta = \theta_0$, $\varphi(\cot\theta_0) = 1/3$. The reduced flow is composed by singular points if $y = 0$ and it goes in the negative direction of the y-axis if $\theta = \theta_0$. The fast vector field can be obtained from the hyperbolic fold (−) case if a change of the orientation is considered. The fast and the slow dynamics are represented in Fig. 11. We have that $(\frac{\pi}{2}, 0)$ is the unique singular point of the regularized vector field. Moreover, it is a focus or a center. In fact, it is easy to see that the linear part of the regularization has eigenvalues $\lambda = \pm\sqrt{\frac{(\varphi)'(0)}{2\varepsilon}}\,i$.

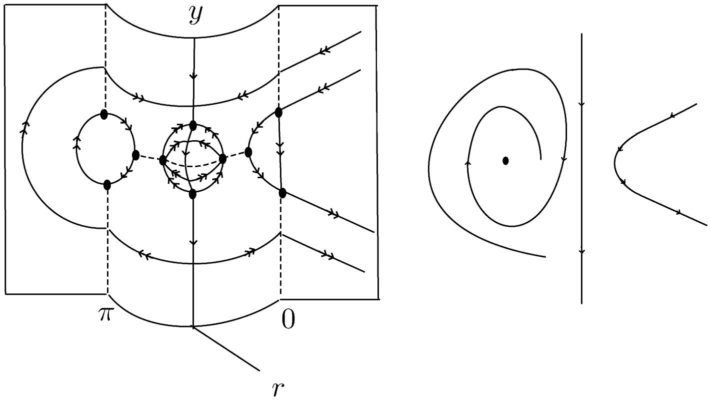


Fig. 11. Fast and slow dynamics of the SP-problem corresponding to the parabolic fold case and its regularization.

Suppose now that $a, b > 0$ and $X_1(x, y) = (-ay, -1)$ and $X_2(x, y) = (-by, -1)$. In this case the SP-problem in the blowing up locus is

$$r\dot{\theta} = -y\sin\theta\left(\frac{-a-b}{2} + \frac{b-a}{2}\varphi(\cot\theta)\right), \qquad \dot{y} = -1.$$

The slow manifold is $y = 0$ because $|\frac{-b-a}{b-a}| > 1$ and then there is no solution of $\varphi(\cot\theta) = \frac{-b-a}{b-a}$. The reduced flow is composed by singular points if $y = 0$ and the fast vector field is $(\theta', 0)$ with $\theta' > 0$ if $y > 0$ and $0 < \theta < \theta_0$ or $y < 0$. Observe that if we take $a = b = 1$ then the fast vector field is the same as the elliptical fold $(-)$ case. The fast and the slow dynamics are represented in Fig. 12.

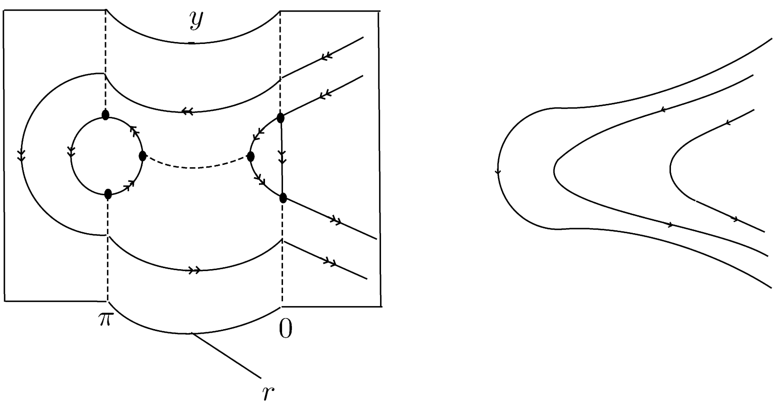


Fig. 12. Fast and slow dynamics of the SP-problem corresponding to the parabolic fold case with $a = b = 1$ and its regularization.

Example. In this example we try to give a rough idea how to get minimal sets by means of the regularization process. Consider $X_1(x,y) = (3y^2 - y - 2, 1)$ and $X_2(x,y) = (-3y^2 - y + 2, -1)$. As before we assume that $F(x,y) = x$. The regularized vector field is

$$X_\varepsilon(x,y) = \left(\frac{1}{2} + \frac{1}{2}\varphi\left(\frac{x}{\varepsilon}\right)\right)\left(3y^2 - y - 2, 1\right) + \left(\frac{1}{2} - \frac{1}{2}\varphi\left(\frac{x}{\varepsilon}\right)\right)\left(-3y^2 - y + 2, -1\right).$$

Applying Theorem C we get that the SP-problem in the blowing up locus is

$$r\dot{\theta} = -\sin\theta\big(-y + \varphi(\cot\theta)\big(3y^2 - 2\big)\big), \qquad \dot{y} = \varphi(\cot\theta).$$

The slow manifold is given implicitly by $\varphi(\cot\theta) = \frac{y}{3y^2-2}$ which defines two functions

$$y_1(\theta) = \frac{1 + \sqrt{1 + 24\varphi^2(\cot\theta)}}{6\varphi(\cot\theta)} \quad \text{and} \quad y_2(\theta) = \frac{1 - \sqrt{1 + 24\varphi^2(\cot\theta)}}{6\varphi(\cot\theta)}.$$

The function $y_1(\theta)$ is increasing,

$$y_1(0) = 1, \qquad \lim_{\theta \to \frac{\pi}{2}^-} y_1(\theta) = +\infty, \qquad \lim_{\theta \to \frac{\pi}{2}^+} y_1(\theta) = -\infty \quad \text{and} \quad y_1(\pi) = -1.$$

The function $y_2(\theta)$ is increasing,

$$y_2(0) = -\frac{2}{3}, \qquad \lim_{\theta \to \frac{\pi}{2}} y_2(\theta) = 0 \quad \text{and} \quad y_2(\pi) = \frac{2}{3}.$$

We can extend y_2 to $(0,\pi)$ as a differential function with $y_2(\frac{\pi}{2}) = 0$.

The fast vector field is $(\theta', 0)$ with $\theta' > 0$ if (θ, y) belongs to

$$\left[\left(0, \frac{\pi}{2}\right) \times \big(y_2(\theta), y_1(\theta)\big) \cup \left(\frac{\pi}{2}, \pi\right) \times \big(y_2(\theta), +\infty\big) \cup \left(\frac{\pi}{2}, \pi\right) \times \big(-\infty, y_1(\theta)\big)\right]$$

and with $\theta' < 0$ if (θ, y) belongs to

$$\left[\left(0, \frac{\pi}{2}\right) \times \big(y_1(\theta), +\infty\big) \cup \left(0, \frac{\pi}{2}\right) \times \big(-\infty, y_2(\theta)\big) \cup \left(\frac{\pi}{2}, \pi\right) \times \big(y_1(\theta), y_2(\theta)\big)\right].$$

The reduced flow has one singular point at $(0,0)$ and it goes in the positive direction of the y-axis if $y \in (-\frac{2}{3}, 0) \cup (1, \infty)$ and goes in the negative direction of the y-axis if $y \in (-\infty, -1) \cup (0, \frac{2}{3})$.

One can see that $(\theta, y, r) = (0, 1, 0)$ and $(\theta, y, r) = (0, -1, 0)$ are not normally hyperbolic points. So we perform additional blow-ups, as before. In Fig. 13 we have the fast and the slow dynamics of the SP-problem.

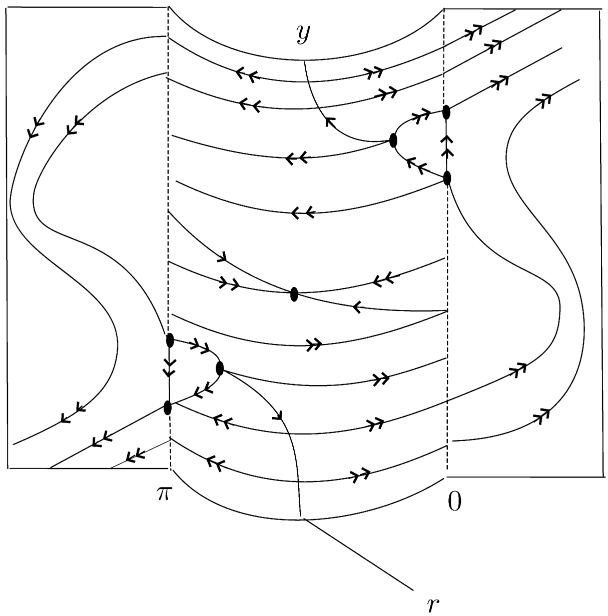


Fig. 13. Fast and slow dynamics of the SP-problem corresponding to the non-elementary graphics case.

5. Final remarks

In this section we discuss the existence of periodic orbits for the regularized system. The existence of periodic orbits can be expected in some cases:

- *X has a codimension* 1 *singular point* $q \in \Sigma$ *of parabolic fold kind.*

For simplicity this case will be discussed by means of the vector field written in the following normal form:

$$X_1(x, y) = \left(-y + o\left|(x, y, a)\right|^2, -1 + o\left|(x, y, a)\right|^2\right),$$
$$X_2(x, y) = \left(3y + o\left|(x, y, a)\right|^2, 1 + o\left|(x, y, a)\right|^2\right),$$

where $a \in \mathbb{R}$ is an additional parameter.

Suppose that $\varphi(t) = \sum_{n=1}^{\infty} a_n t^n$. Thus the trajectories of X_ε are the solutions of the differential system

$$\dot{x} = y + o\left|(x, y, a)\right|^2 + \varphi(x/\varepsilon)\left(-2y + o\left|(x, y, a)\right|^2\right),$$
$$\dot{y} = 0 + o\left|(x, y, a)\right|^2 + \varphi(x/\varepsilon)\left(-2 + o\left|(x, y, a)\right|^2\right).$$

We take the linear coordinate change $y = \sqrt{\frac{a_1}{\varepsilon}}\, y_1$, $x = x_1$. The differential system in these coordinates is

$$\dot{x}_1 = \sqrt{\frac{a_1}{\varepsilon}} y_1 + o\left|(x, y, a)\right|^2, \qquad \dot{y}_1 = -\sqrt{\frac{a_1}{\varepsilon}} x_1 + o\left|(x, y, a)\right|^2. \tag{18}$$

If we consider the polar coordinate change then the phase portrait of (18) is composed by the graphic of $r_a = f_\rho(\theta, a)$. The return map $\pi : \mathbb{R}^+ \to \mathbb{R}$ is given by $\pi(\rho, a) = f_\rho(2\pi, a) - \rho$, and the correspondent Lyapunov values are given by $V_k(a) = \frac{\pi^k(0,a)}{k!}$. It is known that if there exist $n \in \mathbb{N}$ and $a \in \mathbb{R}$ such that $\pi'(0, a) = \cdots = \pi^{n-1}(0, a) = 0$ and $\pi^n(0, a) \neq 0$ then n is an odd number. In this case we say that $\frac{n-1}{2}$ is the multiplicity of the focus and there exist $\lambda_0, \delta_0 > 0$ such that for any $\lambda < \lambda_0$, $\delta < \delta_0$ and $1 \leqslant s \leqslant n$ there exists a system which is δ-close to (18) and has n limit cycles in a λ-neighborhood of $(0, 0)$.

- *The discontinuous set in a neighborhood of $q \in \Sigma$ is the slow manifold of the regularized one and q is a canard point.*

The second way to get regularizations with limit cycles is to consider discontinuous vector fields for which the related singular problem presents a canard point q. It means that the singular problem must be written like (3) and satisfies the conditions:

$$\frac{\partial f}{\partial y}(q, 0) \neq 0, \quad \frac{\partial^2 f}{\partial x^2}(q, 0) \neq 0, \qquad g(q, 0) = 0, \quad \frac{\partial g}{\partial x}(q, 0) \neq 0.$$

"Canard Explosion" is a term used in chemical and biological literature to denote a very fast transition, upon variation of a parameter, from a small amplitude limit cycle to a big amplitude limit cycle. This phenomenon is related to the presence of a family of canard cycles and occurs for example in the Van der Pol equation

$$\dot{x} = y - \frac{x^2}{2} - \frac{x^3}{3}, \qquad \dot{y} = \varepsilon(a - x).$$

In this case $(x, y, \varepsilon, a) = (0, 0, 0, 0)$ is a canard point. There is a distinguished history of investigations on this context [1,8].

- *There exists a singular graph of X of elementary kind.*

Consider $X = (X_1, X_2) \in \Omega^r$ and $Q \subseteq K$ a compact and connected region. Moreover, suppose that

$$\Sigma \cap \partial Q = \{\alpha, \beta\}, \quad [\alpha, \beta] \subseteq \Sigma, \quad 0 \in (\alpha, \beta).$$

A singular graph $\gamma \subseteq Q$ is a set composed by an arc S_1 of Σ and a piece τ_0 of an orbit of X_1 in $\{(x, y) \in K\colon F(x, y) > 0\}$ such that X_1 does not have singular points in S_1. An elementary graph is a singular graph $\gamma_0 \cup \overline{OP}$, where γ_0 be the orbit of X_1 by 0 and P is such that γ_0 intersects Σ at $P \in (\alpha, \beta)$. Sotomayor and Teixeira proved that if γ is an elementary graph of X, then there exists a neighborhood B of γ in K and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the regularized vector field X_ε has a unique periodic orbit in B. For a precise statement see [10, Lemma 4.4, p. 220].

We observe that, according to the terminology used in GSP-theory, this result produces a class of singular problems for which closed singular orbits can be approached by regular orbits.

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