



# Club-guessing, stationary reflection, and coloring theorems

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## ABSTRACT

We obtain very strong coloring theorems at successors of singular cardinals from failures of certain instances of simultaneous reflection of stationary sets. In particular, the simplest of our results establishes that if  $\mu$  is singular and  $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ , then there is a regular cardinal  $\theta < \mu$  such that any fewer than  $\text{cf}(\mu)$  stationary subsets of  $S_{\geq \theta}^{\mu^+}$  must reflect simultaneously.

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## 1. Introduction

Our main results establish a strong connection between square-brackets partition relations at successors of singular cardinals and simultaneous reflection of stationary sets. The general question of whether the failure of simultaneous reflection of stationary subsets of a cardinal  $\kappa$  can be used to negate square-brackets partition relations at  $\kappa$  has been around since the 1980s — Todorčević [29] (see also Shelah [26]) established that  $\kappa \not\rightarrow [\kappa]_{\kappa}^2$  holds in the presence of a non-reflecting stationary set, and the question of whether the hypothesis can be weakened to a failure of simultaneous reflection is natural. Shelah's study of coloring theorems at small inaccessible cardinals (mainly in Chapters III and IV of [27]) contains many partial results in this vein; he obtains coloring theorems from failures of simultaneous reflection using some additional assumptions on the cardinal  $\kappa$ . Our techniques build on his work by modifying his techniques for inaccessible cardinals so that they can be applied to successors of singular cardinals. Our main theorem results from the union of these techniques with a generalization of our previous work in the area (much of which has been joint with Shelah). The payoff is a substantial improvement in our understanding: we get much stronger conclusions from much weaker hypotheses. Along the way, we obtain a version of the main result of [8] which holds even for successors of singular cardinals of countable cofinality. This is accomplished by changing the sort of club-guessing sequence used in those papers, and by refining our arguments so that they work in this new context. In forthcoming work, we use the techniques developed here to settle several other questions left open by [8, 10].

In this introductory section, our goal is to provide enough background that the preceding paragraph makes sense. We assume that the reader has an acquaintance with basic set-theoretic notation; any of the standard references (say [14, 20],

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or [16]) are more than sufficient. Some of the objects of importance to us have been denoted in a variety of ways throughout the literature, so we take a moment to set our conventions.

**Notation.** Let  $\lambda$  and  $\kappa$  be cardinals.

- (1)  $[\lambda]^\kappa := \{A \subseteq \lambda : |A| = \kappa\}$ .
- (2)  $S_\kappa^\lambda := \{\alpha < \lambda : \text{cf}(\alpha) = \kappa\}$ .

We also utilize minor variants of the above notation, but these should all be self-explanatory.

One more convention is quite important for us: by “ $I$  is an ideal on  $\kappa$ ”, we mean “ $I$  is a proper ideal on  $\kappa$  containing all bounded subsets of  $\kappa$ ”. We have chosen this notation for convenience, since it eliminates many trivialities.

We now move on to the background material; we begin with the square-brackets partition relation  $\kappa \rightarrow [\lambda]_\theta^\mu$  of Erdős, Hajnal, and Rado [11], which means that for any function  $F : [\kappa]^\mu \rightarrow \theta$ , (which we refer to as a coloring) we can find a set  $H \subseteq \kappa$  of cardinality  $\lambda$  for which

$$\text{ran}(F \upharpoonright [H]^\mu) \subsetneq \theta,$$

that is, when we restrict the function  $F$  to  $[H]^\mu$ , at least one color is omitted. These square-brackets partition relations arise naturally when one investigates the extent to which Ramsey’s Theorem generalizes to uncountable sets, as the statement “ $\kappa \rightarrow [\lambda]_\theta^\mu$ ” asserts that a very weak form of Ramsey’s Theorem holds at the cardinal  $\kappa$ .

The negations of these partition relations are quite strong combinatorial hypotheses in their own right, and it is for this reason that they have been studied extensively by set theorists:

The moral is that if one knows an ordinary negative partition relation then it is often worth asking whether a stronger assertion, a negative square-brackets relation, is also true [12].

Consider for example, the statement  $\kappa \not\rightarrow [\kappa]_\kappa^2$ . This asserts the existence of a function  $F : [\kappa]^2 \rightarrow \kappa$  (a coloring of the pairs from  $\kappa$  using  $\kappa$  colors) with the property that  $F$  assumes every possible value on (the pairs from) every unbounded subset of  $\kappa$ . This says that Ramsey’s Theorem fails in a very spectacular way at  $\kappa$ : it is possible to color the pairs of ordinals less than  $\kappa$  utilizing  $\kappa$  colors such that any subset of  $\kappa$  of size  $\kappa$  is completely inhomogeneous with respect to the coloring. Theorems that assert the existence of “complicated” colorings of pairs (or other finite sets) of ordinals (where the exact meaning of “complicated” depends on context) can be conveniently grouped under the sobriquet “coloring theorems”, and our main theorem is one such.

In Appendix 1 of [27], Shelah systematically studies several combinatorial principles stronger than the negated square-brackets relations discussed above, and our results are most naturally stated using his notation.

**Definition 1.1.** If  $\lambda$  is an infinite cardinal, and  $\kappa + \theta \leq \mu \leq \lambda$ , then  $\text{Pr}_1(\lambda, \mu, \kappa, \theta)$  asserts the existence of function  $c : [\lambda]^2 \rightarrow \kappa$  such that whenever we are given a collection  $\langle t_\alpha : \alpha < \mu \rangle$  of pairwise disjoint elements of  $[\lambda]^{<\theta}$  as well as an ordinal  $\zeta < \kappa$ , then there are  $\alpha < \beta$  for which  $c \upharpoonright t_\alpha \times t_\beta$  is constant with value  $\zeta$ .

Our concern in this paper is with the case where  $\lambda = \mu$  and  $\theta > 2$  in the above definition. In this situation, we see that  $\text{Pr}_1(\lambda, \lambda, \kappa, \theta)$  is a strengthened version of  $\lambda \not\rightarrow [\lambda]_\kappa^2$ , for one may take the sets  $t_\alpha$  to be singletons. One should think of  $\text{Pr}_1(\lambda, \lambda, \kappa, \theta)$  as establishing a version of  $\lambda \not\rightarrow [\lambda]_\kappa^2$  that works for “blocks” of size less than  $\theta$ .

We will break off our discussion of these relations for a moment to pick up another thread that is important for our work, namely, reflection of stationary sets. Jech’s [13] gives a nice introduction to this topic, and the paper [5] of Cummings, Foreman, and Magidor is also an excellent resource.

**Definition 1.2.** Let  $S$  be a stationary subset of an uncountable regular cardinal  $\kappa$ .

- (1) We say  $S$  reflects at  $\alpha$  if  $\alpha < \kappa$  has uncountable cofinality and  $S \cap \alpha$  is stationary in  $\alpha$ .
- (2)  $\text{Refl}(S)$  holds if every stationary subset of  $S$  reflects at some  $\alpha$ .
- (3)  $S$  is non-reflecting if  $S$  does not reflect at any  $\alpha$ .

We observe here that any cardinal of the form  $\kappa^+$  for regular  $\kappa$  has a non-reflecting stationary subset: consider the set  $S_{\kappa^+}^{\kappa^+}$  of all ordinals below  $\kappa^+$  of cofinality  $\kappa$ . This set does not reflect, as any  $\alpha < \kappa^+$  of uncountable cofinality contains a closed unbounded subset of ordinals each of cofinality less than  $\kappa$ . The situation at successors of singular cardinals is much more delicate, and we will have much to say about this later.

It has been known for a long time that there are connections between stationary reflection and coloring theorems. For example, Tryba [31] (and independently Hugh Woodin) established that  $\kappa \not\rightarrow [\kappa]_\kappa^{<\omega}$  whenever  $\kappa$  has a non-reflecting stationary subset. Stevo Todorćević [29] was able to use his technique of minimal walks to improve this to colorings of pairs:  $\kappa \not\rightarrow [\kappa]_\kappa^2$  whenever  $\kappa$  has a non-reflecting stationary set, and hence, in particular,  $\kappa^+ \not\rightarrow [\kappa^+]_{\kappa^+}^2$  whenever  $\kappa$  is a regular cardinal. Shelah [26] simplified Todorćević’s argument a bit and obtained some other generalizations.

Research on coloring theorems for successors of regular cardinals has continued. For example, Shelah [28] has established that  $\text{Pr}_1(\kappa^{+2}, \kappa^{+2}, \kappa^{+2}, \kappa)$  holds for every regular cardinal  $\kappa$ , while Justin Moore [23] obtained a significant strengthening of Todorćević’s result for  $\aleph_1$  and solved a long-standing open problem in general topology.

Given our understanding of successors of regular cardinals, the following question is natural:

**Question 1.** To what extent are analogous results true for successors of singular cardinals?

This question is still very mysterious, in large part due to the fact that the combinatorics of successors of singular cardinals is very sensitive to underlying assumptions in set theory. For example, if  $\mu$  is singular and  $\mu^+$  has a non-reflecting stationary subset, then the work of Todorćević can be brought to bear and we can conclude that  $\mu^+ \not\rightarrow [\mu^+]^2_{\mu^+}$ . However, assuming the existence of large cardinals, Magidor [22] established the consistency of  $\text{Refl}(\mu^+)$  for  $\mu$  singular, and so the situation for successors of singular cardinals differs greatly from that at successors of regular cardinals.

Back in 1978, Shelah [24] was able to establish  $\aleph_{\omega+1} \not\rightarrow [\aleph_{\omega+1}]^2_{\aleph_{\omega+1}}$  from the assumption that  $2^{\aleph_0} < \aleph_\omega$ . His later development of pcf theory let him eliminate the additional assumption (see Chapter II of [27]), while also extending the class of cardinals for which such results hold. Moreover, in ZFC, Shelah (Conclusion 4.1 in Chapter II of [27]) was able to prove

$$\text{Pr}_1(\mu^+, \mu^+, \text{cf}(\mu), \text{cf}(\mu)) \text{ holds for any singular cardinal } \mu. \tag{1.1}$$

Notice that this is a strong negative partition relation using  $\text{cf}(\mu)$  colors; the question of whether or not this can be improved to a coloring using  $\mu^+$  colors (that is, whether  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$  holds for  $\mu$  singular) is still very much open, as are the related questions concerning  $\mu^+ \not\rightarrow [\mu^+]^2_{\mu^+}$  and  $\mu^+ \not\rightarrow [\mu^+]^{\omega}_{\mu^+}$ . The main result of this paper obtains strong colorings under very weak assumptions, but to describe the precise situation, we need some notation from [5].

**Definition 1.3.** Let  $\kappa$  be a regular uncountable cardinal, and let  $S$  be a stationary subset of  $\kappa$ . We say  $\text{Refl}(\theta, S)$  holds if for every sequence  $\langle T_i : i < \theta \rangle$  of stationary subsets of  $S$ , there exists an  $\alpha < \kappa$  such that each  $T_i$  reflects at  $\alpha$ . Similarly, we say  $\text{Refl}(< \theta, S)$  holds if for every  $\sigma < \theta$  and every sequence  $\langle T_i : i < \sigma \rangle$  of stationary subsets of  $S$ , there exists an  $\alpha < \kappa$  such that each  $T_i$  reflects at  $\alpha$ .

We can now state our main theorem, a theorem illustrating the connection between failures of simultaneous reflection of stationary sets and the existence of complicated colorings.

**Main Theorem.** Assume  $\mu$  is a singular cardinal.

- (1) If  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for some  $\theta < \mu$ , then  $\text{Pr}_1(\mu^+, \mu^+, \theta, \text{cf}(\mu))$  holds.
- (2) If  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for arbitrarily large  $\theta < \mu$ , then we obtain both  $\text{Pr}_1(\mu^+, \mu^+, \mu, \text{cf}(\mu))$  and  $\mu^+ \not\rightarrow [\mu^+]^2_{\mu^+}$ .
- (3) If  $\text{cf}(\mu) > \aleph_0$  and  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for arbitrarily large  $\theta < \mu$ , then (2) can be improved to  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ .
- (4) If  $\text{cf}(\mu) = \aleph_0$  and  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for arbitrarily large  $\theta < \mu$ , then there is a function  $d : [\mu^+]^3 \rightarrow \mu^+$  such that whenever  $\langle t_\alpha : \alpha < \mu^+ \rangle$  is a pairwise disjoint family of finite subsets of  $\mu^+$  and  $\zeta < \mu^+$ , there are  $\alpha < \beta < \gamma$  such that

$$(\forall \epsilon \in t_\alpha)(\forall \zeta \in t_\beta)(\forall \xi \in t_\gamma)[d(\epsilon, \zeta, \xi) = \zeta].$$

The proof builds on work of the author and Shelah [9,10,8]. In particular, the main theorem of [8], when taken together with results in Sections 2 and 3 of the current paper, is strong enough to imply our main theorem in the case where  $\mu$  has uncountable cofinality. The paper [10] partially extended the results of [8] to the case where  $\mu$  has countable cofinality (the combination of the main theorem in [10] with the results of Sections 2 and 3 in the current paper results in something weaker than  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ ); the current paper gets around the obstacle in [10] caused by the countable cofinality of  $\mu$ .

The proof of most parts of the main theorem will proceed according to the following sketch:

Sketch of the argument

Let  $\mu$  be a singular cardinal.

- (1) In ZFC we prove that there are a function  $c : [\mu^+]^2 \rightarrow \mu^+$  and an ideal  $I$  (related to club-guessing) such that whenever  $\langle t_\alpha : \alpha < \mu^+ \rangle$  is a family of pairwise disjoint elements of  $[\mu^+]^{< \text{cf}(\mu)}$ , for almost every (in the sense of  $I$ )  $\beta^* < \mu^+$  there are  $\alpha < \beta$  such that  $c \upharpoonright t_\alpha \times t_\beta$  is constant with value  $\beta^*$ .
- (2) If the desired colorings fail to exist, then we can conclude that the ideal  $I$  from (1) possesses some strong combinatorial properties.
- (3) These properties are strong enough to imply the needed instances of simultaneous reflection of stationary subsets of  $\mu^+$ .

The main theorem of [8] establishes (1) for the case where the cofinality of  $\mu$  is uncountable, while the main theorem of [10] gives a conclusion weaker than (1). Statement (2) will be obtained from (1) by modifying one of the arguments from [8]. Statement (3) relies on extending some results from Section 3 of Chapter IV of Shelah’s [27]; the results seem to be new, though they have a “folklore-ish” flavor.

Moving on, we now give an account of the organization of this paper. In Sections 2 and 3, we prove a series of results culminating in a proof of (3) above. Section 4 proves a club-guessing result needed to define the ideal  $I$ . Section 5 pins down the assumptions that we need for our main theorems, and lays some groundwork for later arguments as well. Section 6 introduces much of the minimal walks machinery that we need, as well as providing a proof of a crucial preliminary result.

Section 7 furnishes some background material concerning scales and elementary submodels, and Section 8 provides a proof of (1), as well as a few other odds and ends that we use in the proof of our main theorem. Section 9 closes the paper by finishing the proof of our primary result.

Finally, some historical remarks are in order. Clearly, the research in this paper rests on that of Saharon Shelah, and although the main theorem is an advance in our knowledge, the proof is obtained via a synthesis of techniques drawn from several places in his vast body of work. The line of investigation which resulted in this paper originated in our work with Shelah in [9], where I noticed that the argument in Section 4 of Chapter III in [27] which purported to extend the main coloring theorem in that section to cardinals of the form  $\mu^+$  for  $\aleph_0 = \text{cf}(\mu) < \mu$  did not work. A close study of the problem led to the isolation of the coloring theorems in [8], and in the joint paper [10] we exploited a combinatorial trick to provide a partial rescue of the result from [27]. Some years earlier, Shelah had suggested the idea of “off-center” club-guessing as a possible way of repairing the error in [27], but nothing came of the idea at the time because the combinatorics did not work as we hoped. While writing up [10], the author realized that the “off-center” approach might be viable when combined with the combinatorics of that paper. The work in Sections 3–9 shows that this was indeed the case – in Theorem 6 we get a version of the main result from [8] which holds for successors of singular cardinals of countable cofinality. The jump from this to getting results on simultaneous reflection of stationary sets is based on other work of Shelah – Sections 2 and 3 of this paper can be viewed as teasing out additional consequences from arguments appearing in Section 3 of Chapter III from [27].

## 2. Weak saturation and indecomposability

Our goal in this section is a modest one – we will take two fairly well-known properties of ideals and show, using elementary arguments, that their conjunction is equivalent to several useful properties. The results appearing in this section have the flavor of folklore results; Shelah maps out some of the implications appearing here in Chapter III of his [27], but he stops short of proving that many of the properties that he was considering are in fact equivalent. Theorem 1 at the end of this section gathers the results in a single place.

Before proving anything, we take a moment to discuss some notational conventions and expressions that will appear over and over in our results. This terminology is fairly standard, so readers familiar with such things can just skip ahead.

**Definition 2.1.** Let  $I$  be an ideal on  $\kappa$ .

- (1)  $I^*$  denotes the filter dual to  $I$ .
- (2) Expressions of the form “ $\varphi$  holds for  $I$ -almost all  $\alpha < \kappa$ ” mean that the set of  $\alpha < \kappa$  for which  $\varphi$  holds is in  $I^*$ . If  $I$  is clear from context, we may omit explicit reference to it and say only “ $\varphi$  holds for almost all  $\alpha < \kappa$ ”.
- (3) If  $A$  and  $B$  are subsets of  $\kappa$ , then  $A \subseteq_I B$  ( $A$  is a subset of  $B$  modulo  $I$ ) means that  $A \setminus B \in I$ , and we say that  $A =_I B$  ( $A$  and  $B$  are equal modulo  $I$ ) if both  $A \subseteq_I B$  and  $B \subseteq_I A$ .
- (4) Similarly, if  $f$  and  $g$  are ordinal-valued functions with domain  $\kappa$ , we say that  $f \leq_I g$  if  $f(\alpha) \leq g(\alpha)$  for almost all  $\alpha < \kappa$ . The expression  $f =_I g$  is also given the obvious meaning.

Variants of the above notation should also be interpreted in the canonical fashion.

We look now at the first of two properties which form our main interest in this section.

**Definition 2.2.** Let  $I$  be an ideal on the cardinal  $\kappa$ , and let  $\theta$  be a cardinal. The ideal  $I$  is *weakly  $\theta$ -saturated* if there is no partition of  $\kappa$  into  $\theta$  disjoint  $I$ -positive sets.

Weak saturation provides a measure of how closely the filter  $I^*$  dual to  $I$  comes to being an ultrafilter. For example,  $I^*$  is an ultrafilter if and only if  $I$  is weakly 2-saturated, while the ideal of bounded subsets of  $\kappa$  fails to be weakly  $\kappa$ -saturated. This concept has not been studied as systematically as its better-known relative “saturation”, but we shall see that it is quite important in its own right. The following observation starts us on our way.

**Proposition 2.3.** Let  $I$  be an ideal on the cardinal  $\kappa$ . The following statements are equivalent for a regular cardinal  $\theta$ :

- (1)  $I$  is weakly  $\theta$ -saturated.
- (2) Any  $\subseteq$ -increasing  $\theta$ -sequence  $\langle A_i : i < \theta \rangle$  of subsets of  $\kappa$  is eventually constant modulo  $I$ .
- (3) If  $\theta \leq \text{cf}(\tau)$ , then any  $\subseteq$ -increasing  $\tau$ -sequence  $\langle A_i : i < \tau \rangle$  of subsets of  $\kappa$  is eventually constant modulo  $I$ .

**Proof.** The proof is trivial. For example, one proves that (1) implies (2) via contradiction: If (2) fails, then we can refine  $\langle A_i : i < \theta \rangle$  to a subsequence  $\langle B_i : i < \theta \rangle$  with the property that  $B_i \subseteq_I B_{i+1}$  and  $B_{i+1} \setminus B_i \notin I$  for each  $i < \theta$ . The collection  $\langle B_{i+1} \setminus B_i : i < \theta \rangle$  quickly leads to a contradiction of (1).  $\square$

The next concept that we need for our discussion is that of *indecomposability*, defined as follows:

**Definition 2.4.** Let  $I$  be an ideal on  $\kappa$ , and let  $\theta$  be a cardinal. The ideal  $I$  is said to be  *$\theta$ -indecomposable* if whenever  $\langle A_i : i < \theta \rangle$  is a  $\theta$ -sequence of subsets of  $\kappa$  with  $\bigcup_{i < \theta} A_i \notin I$ , there is a set  $w \subseteq \theta$  of cardinality less than  $\theta$  with  $\bigcup_{i \in w} A_i \notin I$ . In the case where  $\theta$  is a regular cardinal, this is equivalent to the statement that the ideal  $I$  is closed under *increasing unions* of length  $\theta$ .

Indecomposability has been considered many times in the literature (see [3,21] for example), although one usually finds the definition phrased in terms of filters rather than ideals, and very often the authors restrict themselves to considering ultrafilters instead of the more general case.

**Proposition 2.5.** *Let  $I$  be an ideal on the cardinal  $\kappa$ . The following two statements are equivalent for a regular cardinal  $\theta$ :*

- (1)  $I$  is  $\theta$ -decomposable.
- (2) There is a  $\theta$ -sequence  $\langle A_i : i < \theta \rangle$  of  $I$ -equivalent  $I$ -positive sets with

$$\bigcap_{i < \theta} \bigcup_{i \leq j < \theta} A_j = \emptyset. \tag{2.1}$$

**Proof.** Suppose  $I$  is  $\theta$ -decomposable. Since  $\theta$  is regular, this means there is an increasing sequence  $\langle B_i : i < \theta \rangle$  of elements of  $I$  whose union  $B$  is  $I$ -positive. If we define

$$A_i := B \setminus B_i,$$

then  $\langle A_i : i < \theta \rangle$  has the required properties.

For the other direction, suppose we are given  $\langle A_i : i < \theta \rangle$  as in (2). Let  $B = A_0$ , and define

$$B_i = B \setminus \bigcup_{i \leq j < \theta} A_j.$$

Note that  $B_i \in I$  as  $B$  and  $A_i$  are equivalent modulo  $I$ . The sequence  $\langle B_i : i < \theta \rangle$  is also increasing, and by (2.1) we have

$$B = \bigcup_{i < \theta} B_i.$$

Since  $B \notin I$ , we conclude that  $I$  is  $\theta$ -decomposable.  $\square$

Our main concern in this section is a consideration of the conjunction of “ $\theta$ -indecomposable” and “weakly  $\theta$ -saturated” for a regular cardinal  $\theta$ . The next proposition shows us that this combination is has some strength – it is equivalent to improvements of both Propositions 2.3 and 2.5.

**Proposition 2.6.** *Let  $I$  be an ideal on the cardinal  $\kappa$ . The following statements are equivalent for a regular cardinal  $\theta$ .*

- (1)  $I$  is weakly  $\theta$ -saturated and  $\theta$ -indecomposable.
- (2) Whenever  $\langle B_i : i < \theta \rangle$  is an increasing  $\theta$ -sequence of subsets of  $\kappa$ , there is an  $i^*$  such that

$$i^* \leq i < \theta \implies B_i =_I \bigcup_{j < \theta} B_j. \tag{2.2}$$

- (3) Whenever  $\langle A_i : i < \theta \rangle$  is a  $\theta$ -sequence of  $I$ -positive subsets of  $\kappa$ , we have

$$\bigcap_{i < \theta} \bigcup_{i \leq j < \theta} A_j \neq \emptyset. \tag{2.3}$$

**Proof.** Assume (1) holds, and let  $\langle B_i : i < \theta \rangle$  be an increasing  $\theta$ -sequence of subsets of  $\kappa$ . Since  $I$  is weakly  $\theta$ -saturated, we know that this sequence is eventually constant modulo  $I$ , so fix  $i^* < \theta$  with the property that

$$i^* \leq i < \theta \implies B_i =_I B_{i^*} \text{ mod } I.$$

Let  $B = \bigcup_{i < \theta} B_i$ , and note that (2) is established if we can prove that  $B \setminus B_{i^*}$  is in  $I$ . This is done through  $\theta$ -indecomposability, for  $B \setminus B_{i^*}$  can be expressed as the union of an increasing  $\theta$ -sequence of elements of  $I$ :

$$B \setminus B_{i^*} = \left( \bigcup_{i^* \leq i < \theta} B_i \right) \setminus B_{i^*} = \bigcup_{i^* \leq i < \theta} (B_i \setminus B_{i^*}).$$

Next, assume that condition (2) holds for our ideal, and assume by way of a contradiction that  $\langle A_i : i < \theta \rangle$  is a  $\theta$ -sequence of  $I$ -positive subsets of  $\kappa$  for which (2.3) fails. For each  $i < \theta$ , define

$$B_i := \kappa \setminus \bigcup_{i \leq j < \theta} A_j.$$

The sequence  $\langle B_i : i < \theta \rangle$  is increasing, and furthermore  $\bigcup_{i < \theta} B_i = \kappa$  because (2.3) fails. By (2), there is an  $i^*$  such that  $B_{i^*} =_I \kappa$ . This is a contradiction, as  $A_{i^*}$  is  $I$ -positive and disjoint to  $B_{i^*}$ .

To finish the proof, we show that the failure of (1) implies the failure of (3). This is easily done – if  $I$  is not weakly  $\theta$ -saturated then any partition of  $\kappa$  into  $\theta$  disjoint  $I$ -positive sets will contradict (3), and if  $I$  is  $\theta$ -decomposable then we contradict (3) by way of Proposition 2.5.  $\square$

Notice that condition (2) says something stronger than the conclusion of Proposition 2.3 – the sequence of sets is not only eventually constant modulo  $I$ , it is the case that eventually the individual sets  $B_i$  are equal to the union of the entire sequence modulo  $I$ .

Condition (3) says that any “point- $< \theta$ ” collection of  $I$ -positive sets has size less than  $\theta$ , for given a collection of  $\theta$  (or more)  $I$ -positive sets, there is a subcollection of size  $\theta$  with non-empty intersection. This is a much stronger conclusion than that obtained in Proposition 2.5 as we no longer need to assume that the sets involved are all  $I$ -equivalent.

Since prime ideals are weakly 2-saturated, we obtain the following (known) characterization of  $\theta$ -indecomposability in the context of ultrafilters.

**Corollary 2.7.** *Let  $\mathcal{U}$  be an ultrafilter on some cardinal  $\kappa$ , and let  $\theta$  be a regular cardinal. Then  $\mathcal{U}$  is  $\theta$ -decomposable if and only if it is  $(\theta, \theta)$ -regular.*

**Proof.** By definition  $\mathcal{U}$  is  $(\theta, \theta)$ -regular if and only if there is a family  $\{A_i : i < \theta\}$  of elements of  $\mathcal{U}$  with the property that the intersection of any subfamily of size  $\theta$  is empty. This is precisely the negation of condition (2.3), and so the result follows immediately as the ideal dual to  $\mathcal{U}$  is trivially weakly  $\theta$ -saturated.  $\square$

Our next move takes us to the realm of functions modulo ideals, on the cusp of pcf theory.

**Proposition 2.8.** *Let  $I$  be an ideal on the cardinal  $\kappa$ . The following two statements are equivalent for any regular cardinal  $\theta < \kappa$ :*

- (1)  $I$  is weakly  $\theta$ -saturated and  $\theta$ -indecomposable.
- (2) Suppose  $\langle S_\alpha : \alpha < \kappa \rangle$  is a sequence of sets of ordinals with  $|S_\alpha| < \theta$  for all  $\alpha$ . Then any  $\leq$ -increasing  $\theta$ -sequence of functions in  $\prod_{\alpha < \kappa} S_\alpha$  is eventually constant modulo  $I$ .

**Proof.** Both directions make use of condition (3) in Proposition 2.6. Let us suppose  $\langle S_\alpha : \alpha < \kappa \rangle$  is as in (2), and assume by way of a contradiction that  $\bar{f} = \langle f_i : i < \theta \rangle$  is a  $\leq$ -increasing sequence in  $\prod_{\alpha < \kappa} S_\alpha$  that is not eventually constant modulo  $I$ . By passing to a subsequence we may assume

$$A_i := \{\alpha < \kappa : f_i(\alpha) < f_{i+1}(\alpha)\} \notin I \text{ for all } i < \theta.$$

Fix  $\alpha < \kappa$ , and suppose  $\alpha \in A_i \cap A_j$  for some  $i < j < \theta$ . Since  $\bar{h}$  is  $\leq$ -increasing, we have

$$h_i(\alpha) < h_{i+1}(\alpha) \leq h_j(\alpha). \tag{2.4}$$

Since  $|S_\alpha| < \theta$ , we conclude that

$$|\{i < \theta : \alpha \in A_i\}| \leq |S_\alpha| < \theta,$$

and therefore

$$\bigcap_{i < \theta} \bigcup_{i \leq j < \theta} A_j = \emptyset. \tag{2.5}$$

Each  $A_i$  was assumed to be  $I$ -positive, so we have contradicted (1).

For the other direction, let us assume that (1) fails. By Proposition 2.6, we can find a sequence  $\langle A_i : i < \theta \rangle$  of  $I$ -positive subsets of  $\kappa$  such that

$$\bigcap_{i < \theta} \bigcup_{i \leq j < \theta} A_j = \emptyset,$$

that is,

$$|\{i < \theta : \alpha \in A_i\}| < \theta \text{ for all } \alpha < \kappa. \tag{2.6}$$

We define a sequence  $\bar{h} = \langle h_i : i < \theta \rangle$  of functions in  ${}^\kappa \text{Ord}$  by the following recursion:

Case 1: Initial stage

We define  $h_0$  to be identically 0.

Case 2: Successor stages

Given  $h_i$ , we define  $h_{i+1}$  via the formula

$$h_{i+1}(\alpha) = \begin{cases} h_i(\alpha) + 1 & \text{if } \alpha \in A_i \\ h_i(\alpha) & \text{otherwise.} \end{cases}$$

Case 3: Limit stages

If  $i$  is a limit ordinal, then we define  $h_i$  by setting  $h_i(\alpha) = \sup\{h_j(\alpha) : j < i\}$ .

Now let us define  $S_\alpha := \{h_i(\alpha) : \alpha < \theta\}$ . Since (2.6) holds, our construction guarantees that  $|S_\alpha| < \theta$  for all  $\alpha$ . Clearly  $\bar{h}$  is  $\leq$ -increasing, and since no  $A_i$  is in  $I$ , the sequence is also not eventually constant modulo  $I$ .  $\square$

The proof of the above is easily modified to yield the following slightly strengthened result:



**Corollary 2.9.** Suppose  $I$  is a weakly  $\theta$ -saturated and  $\theta$ -indecomposable ideal on  $\kappa$  for some regular cardinal  $\theta$ , and let  $\langle S_\alpha : \alpha < \kappa \rangle$  be a sequence of sets of ordinals with  $|S_\alpha| < \theta$ . If  $\bar{h} = \langle h_\beta : \beta < \tau \rangle$  is a  $\leq$ -increasing sequence of functions in  $\prod_{\alpha < \kappa} S_\alpha$  with  $\text{cf}(\tau) \geq \theta$ , then  $\bar{h}$  is eventually constant modulo  $I$ .

Another easy characterization obtained by slightly different methods is the following:

**Proposition 2.10.** The following statements are equivalent for an ideal  $I$  on  $\kappa$  and regular  $\theta < \kappa$ :

- (1)  $I$  is weakly  $\theta$ -saturated and  $\theta$ -indecomposable.
- (2) Any function  $f : \kappa \rightarrow \theta$  is bounded below  $\theta$  almost everywhere, that is, there is an ordinal  $\beta < \theta$  such that  $f(\alpha) < \beta$  for almost all  $\alpha < \kappa$ .

**Proof.** Assume  $f : \kappa \rightarrow \theta$ , and for each  $\beta < \theta$ , let us define

$$A_\beta := \{\alpha < \kappa : f(\alpha) < \beta\}.$$

The sequence  $\langle A_\beta : \beta < \theta \rangle$  is increasing with union  $\kappa$ , so by (2) of Proposition 2.6, there is a  $\beta < \theta$  such that  $\kappa \setminus A_\beta \in I$ , and therefore  $f(\alpha) < \beta$  for almost all  $\alpha < \kappa$ .

We prove the other direction by contrapositive – if (1) fails, then by (3) of Proposition 2.6 we can find a family  $\langle A_i : i < \theta \rangle$  of  $I$ -positive subsets of  $\kappa$  such that

$$\bigcap_{i < \theta} \bigcup_{i \leq j < \theta} A_j = \emptyset. \tag{2.7}$$

Now let us define a function  $f$  with domain  $\kappa$  by

$$f(\alpha) = \sup\{\beta < \theta : \alpha \in A_\beta\}.$$

Since (2.7) holds, it follows that  $f$  maps  $\kappa$  into  $\theta$ . However,  $f$  is not bounded below  $\theta$  almost everywhere, as for each  $\beta < \theta$ , we know

$$A_\beta \subseteq \{\alpha < \kappa : \beta \leq f(\alpha)\}$$

and  $A_\beta$  is  $I$ -positive.  $\square$

The proof of the above easily yields something slightly stronger:

**Corollary 2.11.** Suppose  $I$  is a weakly  $\theta$ -saturated  $\theta$ -indecomposable ideal on  $\kappa$  for some regular  $\theta < \kappa$ . If  $\delta$  is an ordinal of cofinality  $\theta$ , then any function  $f : \kappa \rightarrow \delta$  is bounded below  $\delta$  almost everywhere.

We close this section by formulating a theorem summarizing the above results.

**Theorem 1.** The following statements are equivalent for an ideal  $I$  on  $\kappa$  and a regular cardinal  $\theta < \kappa$ .

- (1)  $I$  is weakly  $\theta$ -saturated and  $\theta$ -indecomposable.
- (2) Whenever  $\langle B_i : i < \theta \rangle$  is an increasing  $\theta$ -sequence of subsets of  $\kappa$ , there is an  $i^*$  such that

$$i^* \leq i < \theta \implies B_i = \bigcup_{j < \theta} B_j. \tag{2.8}$$

- (3) Whenever  $\langle A_i : i < \theta \rangle$  is a  $\theta$ -sequence of  $I$ -positive subsets of  $\kappa$ , we have

$$\bigcap_{i < \theta} \bigcup_{i \leq j < \theta} A_j \neq \emptyset. \tag{2.9}$$

- (4) If  $\langle S_\alpha : \alpha < \kappa \rangle$  is a sequence of sets of ordinals with  $|S_\alpha| < \theta$  for all  $\alpha$ , then any  $\leq$ -increasing  $\theta$ -sequence of functions in  $\prod_{\alpha < \kappa} S_\alpha$  is eventually constant modulo  $I$ .
- (5) Any function  $f : \kappa \rightarrow \theta$  is bounded below  $\theta$  almost everywhere.

### 3. Least functions and a coloring theorem

In this section, we show that ideals  $I$  of the sort considered in Theorem 1 necessarily entail the existence of a stationary  $S^* \subseteq \kappa$  for which strong versions of  $\text{Refl}(S^*)$  (involving simultaneous reflection of stationary subsets of  $S^*$ ) hold. Once this is established, the section closes by proving a weak version of our main theorem that does not require the complicated machinery introduced later in the paper. Once again, we need to recall some terminology from the general theory of ideals.

**Definition 3.1.** Let  $I$  be an ideal on a cardinal  $\kappa$ .

- (1) A function  $f : \kappa \rightarrow \kappa$  is bounded modulo  $I$  if there is a  $\xi < \kappa$  such that  $\{\alpha < \kappa : \xi < f(\alpha)\} \in I$ .
- (2) A function  $f : \kappa \rightarrow \kappa$  is a least function modulo  $I$  if  $f$  is not bounded modulo  $I$ , but

$$g <_I f \implies g \text{ is bounded modulo } I.$$

The following proposition shows the relevance of the preceding definition to the properties that we studied in the preceding section. Shelah (Claim III.3.2A of [27]) obtains the same conclusion from a weaker hypothesis (his proof is the natural generalization of work of Kanamori and Ketonen [15,17,18] to the context of ideals that are not necessarily prime), but we include this proof in the interest of completeness and because our assumptions simplify the argument.

**Proposition 3.2** (Shelah). *Let  $I$  be an ideal on  $\kappa$ , and suppose  $I$  is  $\theta$ -indecomposable and weakly  $\theta$ -saturated for some regular  $\theta < \kappa$ . Then there is a function  $f^* : \kappa \rightarrow \kappa$  such that*

- (1) for each  $\beta < \kappa$  we have  $f^*(\alpha) > \beta$  for almost all  $\alpha < \kappa$ , and
- (2) if  $g(\alpha) < f^*(\alpha)$  for almost all  $\alpha < \kappa$ , then there is a  $\beta < \kappa$  such that  $g(\alpha) < \beta$  for almost all  $\alpha < \kappa$ .

In particular,  $f^*$  is a least function modulo  $I$ .

**Proof.** When referring to statement (1), we will be a little imprecise and say that “ $f^*$  is an upper bound for the constant functions modulo  $I$ ” and rely on the context to make it clear to which constant functions we are referring.

**Lemma 3.3.** *Under our assumptions, if  $f$  is an upper bound for the constant functions modulo  $I$  for which (2) fails, then we can find a function  $h : \kappa \rightarrow \text{Ord}$  such that*

- $h$  is an upper bound for the constant functions modulo  $I$ ,
- $h(\alpha) \leq f(\alpha)$  for all  $\alpha < \kappa$ , but
- $\{\alpha < \kappa : h(\alpha) < f(\alpha)\} \in I^+$ .

**Proof.** By our assumptions, we can find a function  $g : \kappa \rightarrow \text{Ord}$  such that  $g <_I f$ , but  $g$  is not bounded modulo  $I$ . We can freely modify  $g$  on sets in  $I$ , so we may as well assume  $g(\alpha) \leq f(\alpha)$  for all  $\alpha < \kappa$ .

For each  $\beta < \kappa$ , let us define

$$A_\beta = \{\alpha < \kappa : g(\alpha) \leq \beta\}.$$

Since the sequence  $\langle A_\beta : \beta < \kappa \rangle$  is  $\subseteq$ -increasing, we can apply Proposition 2.3 and conclude that there is a  $\beta^*$  such that

$$\beta^* \leq \beta < \kappa \rightarrow A_\beta =_I A_{\beta^*}. \tag{3.1}$$

Note that the complement of  $A_{\beta^*}$  is not in  $I$  because we assumed that  $g$  is not bounded modulo  $I$ . Furthermore, by (3.1) we see that

$$\{\alpha \in \kappa \setminus A_{\beta^*} : g(\alpha) \leq \beta\} \in I \quad \text{for all } \beta < \kappa.$$

Thus, if we define

$$h(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in A_{\beta^*}, \text{ and} \\ g(\alpha) & \text{if } \alpha \in \kappa \setminus A_{\beta^*}, \end{cases}$$

we have what we need. (Notice that we only needed the weak  $\theta$ -saturation for the proof of this lemma.)  $\square$

Assuming by way of a contradiction that there is no function  $f^*$  answering conditions (1) and (2), we use the preceding lemma to build a  $\leq_I$ -decreasing sequence of functions  $\langle f_\xi : \xi < \theta \rangle$ , each of which is an upper bound for the constant functions modulo  $I$ , using the following inductive recipe:

Let  $f_0 : \kappa \rightarrow \kappa$  be the identity function. Since  $I$  contains all bounded subsets of  $\kappa$ , it is clear that  $f_0$  is an upper bound for the constant functions modulo  $I$ .

Given the function  $f_\xi$ , we obtain  $f_{\xi+1}$  by applying the preceding claim, and for a limit ordinal  $\xi < \theta$ , we proceed as follows:

For each  $\alpha < \kappa$ , let us define

$$S_\alpha := \{f_\zeta(\alpha) : \zeta < \xi\} \cup \{\kappa\},$$

and note that  $|S_\alpha| < \theta$  for each  $\alpha < \kappa$ . Given  $\beta < \kappa$ , we can define a function  $h_\beta \in \prod_{\alpha < \kappa} S_\alpha$  by

$$h_\beta(\alpha) = \min(S_\alpha \setminus \beta).$$

One should view  $h_\beta$  as a “projection” of the function that is constant with value  $\beta$  up into the product  $\prod_{\alpha < \kappa} S_\alpha$ . Our choice of  $f_0$  guarantees that  $h_\beta(\alpha) < \kappa$  for almost all  $\alpha < \kappa$ . Finally, we note that the sequence  $\langle h_\beta : \beta < \kappa \rangle$  is  $\leq$ -increasing, and so an application of Proposition 2.8 tells us that there is an ordinal  $\beta(\xi) < \kappa$  such that  $h_\beta =_I h_{\beta(\xi)}$  whenever  $\beta(\xi) \leq \beta < \kappa$ . We now define

$$f_\xi := h_{\beta(\xi)}.$$

Notice that  $f_\xi$  is an upper bound for the constant functions modulo  $I$ , and moreover for each  $\zeta < \xi$ , if  $\beta(\xi) \leq f_\zeta(\alpha)$  (something that happens for almost all  $\alpha < \kappa$ ), then it must be the case that

$$f_\xi(\alpha) = h_{\beta(\xi)}(\alpha) = \min(S_\alpha \setminus \beta(\xi)) \leq f_\zeta(\alpha). \tag{3.2}$$



Thus  $f_\xi \leq_I f_\zeta$  for every  $\zeta < \xi$ . Since  $\xi$  is a limit ordinal, our actions at successor stages guarantee that in fact  $f_\xi \leq_I f_\zeta$  whenever  $\zeta < \xi$ .

The preceding construction generates a sequence of functions  $\langle f_\zeta : \zeta < \theta \rangle$  as well as ordinals  $\beta(\xi)$  for each limit ordinal  $\xi < \theta$ . Let us define

$$\beta^* := \sup\{\beta(\xi) : \xi < \theta\},$$

and for each limit  $\xi < \theta$ , we set

$$A_\xi := \{\alpha < \kappa : \beta^* \leq f_{\xi+1}(\alpha) < f_\xi(\alpha)\}. \tag{3.3}$$

First, note that each  $A_\xi$  is  $I$ -positive, as  $f_{\xi+1} \leq_I f_\xi$  and  $f_{\xi+1}$  is an upper bound for the constant functions modulo  $I$ . More importantly, we have the following claim:

**Claim 1.** *Suppose  $\xi < \xi^* < \theta$  are limit ordinals. If  $\alpha \in A_\xi$ , then  $f_{\xi^*}(\alpha) < f_\xi(\alpha)$ .*

**Proof.** Suppose  $\alpha \in A_\xi$ , so  $\beta^* \leq f_{\xi+1}(\alpha) < f_\xi(\alpha)$ . Since  $\xi^*$  is a limit ordinal, we know that  $\xi + 1 < \xi^*$ . Furthermore, the definition of  $\beta^*$  tells us that

$$\beta(\xi^*) \leq f_{\xi+1}(\alpha).$$

From (3.2), we conclude that

$$f_{\xi^*}(\alpha) \leq f_{\xi+1}(\alpha).$$

Thus, we have

$$f_{\xi^*}(\alpha) \leq f_{\xi+1}(\alpha) < f_\xi(\alpha),$$

and the proof is complete.  $\square$

The above claim makes it clear that no  $\alpha < \kappa$  can belong to infinitely many of the sets  $A_\xi$ . In particular, this collection of  $I$ -positive sets contradicts part (3) of [Theorem 1](#), and we are done.  $\square$

Our next move is to show that the conjunction of  $\theta$ -indecomposability and weak  $\theta$ -saturation has strong consequences for stationary reflection. The following definition will allow us to state some conclusions more precisely:

**Definition 3.4.** Let  $I$  be an ideal on the cardinal  $\kappa$ .

- (1)  $\text{Comp}(I)$  is the largest cardinal  $\tau$  for which  $I$  is  $\tau$ -complete, that is, for which  $I$  is closed under unions of fewer than  $\tau$  sets.
- (2)  $\text{Wsat}(I)$  is the least cardinal  $\theta$  for which  $I$  is weakly  $\theta$ -saturated.
- (3)  $\text{Indec}(I) = \{\tau < \kappa : I \text{ is } \tau\text{-indecomposable}\}$ .
- (4)  $S^*(I) = \{\alpha < \kappa : \text{Wsat}(I) \leq \text{cf}(\alpha) < \alpha \text{ and } \text{cf}(\alpha) \in \text{Indec}(I)\}$ .

Consider the set  $S^*(I)$  defined above for a moment. Notice that it is non-empty if and only if there is a regular  $\theta < \kappa$  for which  $I$  is both weakly  $\theta$ -saturated and  $\theta$ -indecomposable. Also, if  $S^*(I)$  is non-empty, then it is stationary. The omission of regular cardinals from  $S^*(I)$  is only relevant if  $\kappa$  happens to be Mahlo; we define the set this way so that [Theorem 2](#) applies uniformly to any cardinal.

**Theorem 2.** *Let  $I$  be an ideal on the cardinal  $\kappa$ . If there is a regular cardinal  $\theta < \kappa$  such that  $I$  is weakly  $\theta$ -saturated and  $\theta$ -indecomposable, then*

- (1)  $S^*(I)$  is stationary,
- (2) there is a least function  $f^* : \kappa \rightarrow \kappa$  modulo  $I$ , and
- (3) if  $S$  is a stationary subset of  $S^*(I)$ , then  $S \cap f^*(\alpha)$  is stationary in  $f^*(\alpha)$  for almost all  $\alpha < \kappa$ .

In particular,

- (4)  $S^*(I)$  is a stationary subset of  $\kappa$  for which  $\text{Refl}(< \text{Comp}(I), S^*(I))$  holds.

**Proof.** We have already remarked that (1) is a consequence of the given hypotheses, and (2) is the conclusion of [Proposition 3.2](#). It should also be clear that (4) follows from (3), so we will spend our time establishing (3).

Let  $S$  be a stationary subset of  $S^*(I)$ . Since (3) remains true if we establish its conclusion for a stationary subset of  $S$ , we can take advantage of the fact that the cofinality function is regressive on  $S^*(I)$  and assume that  $S$  is a subset of  $S_\tau^\kappa$  for some  $\tau$ . Notice that  $I$  is weakly  $\tau$ -saturated and  $\tau$ -indecomposable because of the definition of  $S^*(I)$ .

It suffices to prove that whenever we are given a sequence  $\langle C_\alpha : \alpha < \kappa, \alpha \text{ limit} \rangle$  with each  $C_\alpha$  closed unbounded in  $\alpha$ , we have

$$S \cap C_{f^*(\alpha)} \neq \emptyset \text{ for almost all } \alpha < \kappa. \tag{3.4}$$

Given  $\beta < \kappa$ , let us define  $f_\beta : \kappa \rightarrow \kappa$  by

$$f_\beta(\alpha) = \begin{cases} \min(C_{f^*(\alpha)} \setminus \beta) & \text{if } \beta < f^*(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Bear in mind that  $f_\beta(\alpha) < f^*(\alpha)$  for almost all  $\alpha < \kappa$ , and so our choice of  $f^*$  implies that  $f_\beta$  is bounded almost everywhere. Thus, there is a function  $g : \kappa \rightarrow \kappa$  with the property that for any  $\beta < \kappa$ ,

$$\min(C_{f^*(\alpha)} \setminus \beta) < g(\beta) \quad \text{for almost all } \alpha < \kappa. \tag{3.5}$$

Let  $E$  be the closed unbounded subset of  $\kappa$  consisting of those ordinals closed under the function  $g$ , and fix  $\delta \in E \cap S$ . Since  $\delta$  is closed under  $g$ , we know that

$$\beta < \delta \implies \beta \leq \min(C_{f^*(\alpha)} \setminus \beta) < \delta \quad \text{for almost all } \alpha < \kappa. \tag{3.6}$$

Now we define another function  $h$  by

$$h(\alpha) = \begin{cases} \sup(C_{f^*(\alpha)} \cap \delta) & \text{if } \delta \notin C_{f^*(\alpha)}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

The function  $h$  maps  $\kappa$  to  $\delta$ . Since  $\text{cf}(\delta) = \tau$  and  $I$  is  $\tau$ -indecomposable, we conclude from Corollary 2.11 that there is a  $\beta < \delta$  for which

$$h(\alpha) < \beta \quad \text{for almost all } \alpha < \kappa. \tag{3.8}$$

Thus, almost all  $\alpha < \kappa$  satisfy the following:

- $\beta \leq \min(C_{f^*(\alpha)} \setminus \beta) < \delta$ , and
- $h(\alpha) < \beta$ .

We finish the proof of (3.4) by establishing that  $\delta \in C_{f^*(\alpha)}$  for all  $\alpha < \kappa$  which satisfy both of these statements. Given such an  $\alpha$ , assume by way of a contradiction that  $\delta \notin C_{f^*(\alpha)}$ . On one hand we must have

$$h(\alpha) = \sup(C_{f^*(\alpha)} \cap \delta) < \beta, \tag{3.9}$$

while on the other, we have

$$\beta \leq \min(C_{f^*(\alpha)} \setminus \beta) < \delta. \tag{3.10}$$

This latter equation implies

$$\beta \leq \sup(C_{f^*(\alpha)} \cap \delta), \tag{3.11}$$

and clearly (3.9) and (3.11) contradict each other.  $\square$

After we noticed the above theorem, we discovered that Shelah uses essentially the same argument in a different context – it appears tucked into “Proof of 3.3 in Case  $\beta$ , Subcase (a): Second Proof” on page 149 of [27] in a result having to do with weakly inaccessible cardinals. Theorem 2 still appears to be new – one can view it as “what Shelah’s argument really shows”.

To see the power of the preceding result, we include the following theorem. This result is essentially a special case of our main theorem. It arises from combining the preceding theorem with one of the main results from [10], and its proof serves as a prototype for the argument that we employ in Section 10.

**Theorem 3.** *If  $\mu$  is a singular cardinal and  $\mu^+ \rightarrow [\mu^+]^2_{\mu^+}$  holds, then there is a regular  $\theta < \mu$  for which  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  is true.*

**Proof.** Corollary 5.2 of [10] (which itself relies on results in [8]) tells us that under our assumption, there is an ideal  $I$  on  $\mu^+$  such that

- $I$  is  $\text{cf}(\mu)$ -complete,
- $I$  is  $\tau$ -indecomposable for all regular  $\tau$  with  $\text{cf}(\mu) < \tau < \mu$ , and
- $I$  is weakly  $\theta$ -saturated for some  $\theta$  satisfying  $\text{cf}(\mu) < \theta < \mu$ .

Let  $\theta$  be the least cardinal such that  $\text{cf}(\mu) < \theta < \mu$  and  $I$  is weakly  $\theta$ -saturated. Since  $\mu^+$  is certainly not a Mahlo cardinal, it follows from the above that  $S^*(I)$  is equal to  $S_{\geq \theta}^{\mu^+}$  modulo the non-stationary ideal, and therefore the conclusion follows from Theorem 2.  $\square$

We remark that in the case where  $\text{cf}(\mu)$  is uncountable, we can use results from [8] to get the same conclusion from the failure of  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ ; we will have more to say about this later.

#### 4. Off-center club-guessing

Our focus now shifts away from the general theory of ideals to questions involving club-guessing. The proof of [Theorem 3](#) turned on properties of an ideal  $I$  whose existence will seem somewhat mysterious to those not familiar with [\[8,10\]](#). The ideal referenced in the proof of [Theorem 3](#) is related to club-guessing, and this explains why we turn our attention to this matter.

In general, a prototypical club-guessing theorem provides one with a stationary subset  $S$  of some cardinal  $\kappa$ , and a sequence  $\langle C_\delta : \delta \in S \rangle$  (called an  $S$ -club sequence) such that

- $C_\delta$  is closed and unbounded in  $\delta$  for each  $\delta \in S$ , and
- for every closed unbounded  $E \subseteq \kappa$ , there are “many”  $\delta \in S$  for which  $E \cap C_\delta$  is “large”.

One can require various conditions on the sets  $C_\delta$ , as well as varying the specific meaning of “many” and “large”.

The most well-known club-guessing theorems give us (in certain circumstances) an  $S$ -club sequence  $\langle C_\delta : \delta \in S \rangle$  for which  $\text{otp}(C_\delta) = \text{cf}(\delta)$ , and such that for each closed unbounded  $E \subseteq \kappa$ , there are stationarily many  $\delta \in S$  for which  $C_\delta \subseteq E$ . It should be clear that such theorems fit our prototype.

The techniques of [\[27,9,8,10\]](#), however, require a special sort of club-guessing that is, in a sense, both weaker and stronger than the standard results. We will give a rough explanation of this, but we need to enlarge our vocabulary a little bit first:

**Definition 4.1.** Suppose  $C$  is a closed unbounded subset of an ordinal  $\delta$ . We define

- $\text{acc}(C) = \{\alpha < \delta : \alpha = \sup(\alpha \cap C)\} \subseteq C$ , and
- $\text{nacc}(C) = C \setminus \text{acc}(C)$ .

Here “acc” stands for “accumulation points” and “nacc” for non-accumulation points. Finally,

- if  $\alpha \in \text{nacc}(C)$ , then we define  $\text{Gap}(\alpha, C)$ , the gap in  $C$  determined by  $\alpha$ , by

$$\text{Gap}(\alpha, C) = (\sup(C \cap \alpha), \alpha). \quad (4.1)$$

Returning now to our discussion, the earlier papers needed results stating that if  $\mu$  is singular and  $S \subseteq S_{\text{cf}(\mu)}^{\mu^+}$ , then there is an  $S$ -club sequence  $\langle C_\delta : \delta \in S \rangle$  such that for every closed unbounded  $E \subseteq \mu^+$ , there are stationarily many  $\delta \in S$  such that

$$(\forall \tau < \mu) [\{\alpha \in \text{nacc}(C_\delta) \cap E : \text{cf}(\alpha) > \tau\} \text{ is unbounded in } \delta].$$

This type of guessing is weaker than the standard sort in that we aren’t requiring  $C_\delta$  to be a subset  $E$ , but it is also stronger in that we do demand that  $C_\delta \cap E$  contains lots of ordinals of large cofinality. Without going into specifics, if the cofinality of  $\mu$  is uncountable, then we can prove the existence of extremely nice club-guessing sequences with the properties that we want (these are the  $S$ -good pairs of [\[8\]](#)) – sequences so nice that they can be used to generate colorings of  $[\mu^+]^2$ . The club-guessing result that we used in that paper (due originally to Shelah, but a proof can be found in [\[10\]](#)) does not seem to generalize to the case where  $\mu$  has countable cofinality. The paper [\[10\]](#) obtains a weaker result for that case, and the conclusions drawn there are correspondingly weaker – this is why [Theorem 3](#) refers to square-brackets relations instead of the stronger  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ , and it also explains the existence of the current work.

One of the main goals of this paper is to remedy the situation by moving the club-guessing “off-center” – instead of focusing on subsets of  $S_{\text{cf}(\mu)}^{\mu^+}$ , we look at stationary subsets consisting of ordinals of larger cofinality. This is done at a price, for the sets  $C_\delta$  that we construct are necessarily much more complex. Despite this added complexity, we are able to keep enough control over their structure to allow us to connect these club-guessing sequences with the existence of complicated colorings.

This section presents the club-guessing result alluded to in the preceding paragraph. The theorem is established by modifying some club-guessing arguments from [\[10\]](#) to this new context. The main difficulty in this generalization has to do with the proliferation of parameters, so we start with a list of our main assumptions and notation:

- $\lambda = \mu^+$  for  $\mu$  a singular cardinal,
- $\kappa = \text{cf}(\mu)$ ,
- $\kappa < \sigma = \text{cf}(\sigma) < \mu$ ,
- $S$  is a stationary subset of  $S_\sigma^\lambda$ ,
- $\langle \mu_i : i < \kappa \rangle$  is a continuous increasing sequence of cardinals cofinal in  $\mu$ ,
- $\langle c_\delta : \delta \in S \rangle$  is a family of functions such that
  - $c_\delta$  is an increasing and continuous function from  $\sigma$  onto a cofinal subset of  $\delta$ , and
  - for every closed unbounded  $E \subseteq \lambda$ , there are stationarily many  $\delta \in S$  for which  $\text{ran}(c_\delta) \subseteq E$ ,
- for  $\alpha < \sigma$ ,  $I_\alpha^\delta$  denotes the half-open interval  $(c_\delta(\alpha), c_\delta(\alpha + 1)]$ .

Note that if we were to define  $C_\delta$  to be  $\text{ran}(c_\delta)$  for  $\delta \in S$ , then we would end up with a standard sort of club-guessing sequence – for every closed unbounded  $E \subseteq \lambda$ , there would be stationarily many  $\delta \in S$  for which  $C_\delta \subseteq E$ .

The next definition captures some standard ideas from proofs of club-guessing; in some cases we have chosen to use more descriptive names (mostly due to Kojman [19]) for these operations as compared with the terminology prevalent in [27].

**Definition 4.2.** Suppose  $C$  and  $E$  are sets of ordinals with  $E \cap \text{sup}(C)$  closed in  $\text{sup}(C)$ . We define

$$\text{Drop}(C, E) = \{\text{sup}(\alpha \cap E) : \alpha \in C \setminus \min(E) + 1\}. \tag{4.2}$$

Furthermore, if  $C$  and  $E$  are both subsets of some cardinal  $\lambda$  and  $\langle e_\alpha : \alpha < \lambda \rangle$  is a  $C$ -sequence, then for each  $\alpha \in \text{nacc}(C) \cap \text{acc}(E)$ , we define

$$\text{Fill}(\alpha, C, E) = \text{Drop}(e_\alpha, E) \cap \text{Gap}(\alpha, C). \tag{4.3}$$

The names help one to visualize what the operations do –  $\text{Drop}(C, E)$  is the result of “dropping”  $C$  into the set  $E$ , while “Fill” gives us a reasonably canonical way of turning non-accumulation points into accumulation points. The exact choice of  $e_\alpha$  is irrelevant, and in general we do not refer to it explicitly when working with  $\text{Fill}$ . Given these two operations, we are now in a position to prove the following club-guessing theorem:

**Theorem 4.** *There is an  $S$ -club system  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  such that*

- (1)  $\text{ran}(c_\delta) \subseteq C_\delta$ ,
- (2) for each  $\epsilon < \sigma$  and  $i < \kappa$ ,  $C_\delta \cap I_{\kappa \cdot \epsilon + i}^\delta$  has cardinality  $\leq \mu_i^+$ ,
- (3) if  $\alpha \in \text{nacc}(C_\delta) \cap I_{\kappa \cdot \epsilon + i}^\delta$ , then  $\text{cf}(\alpha) > \mu_i^+$ ,
- (4) for every club  $E \subseteq \lambda$ , for stationarily many  $\delta \in S$ , for every  $\epsilon < \sigma$  and  $i < \kappa$ ,  $E \cap \text{nacc}(C_\delta) \cap I_{\kappa \cdot \epsilon + i}^\delta$  is non-empty.

**Proof.** We start by simplifying our goal somewhat, by noting that it suffices to produce  $\bar{C}$  satisfying (1), (2), and the following modified version of (4):

- (4)' for every club  $E \subseteq \lambda$ , for stationarily many  $\delta \in S$ , for every  $\epsilon < \sigma$  and  $i < \kappa$ ,  $E \cap \text{nacc}(C_\delta) \cap I_{\kappa \cdot \epsilon + i}^\delta$  contains an ordinal of cofinality  $> \mu_i^+$ .

Why does this suffice? Given such an  $S$ -club system, we simply throw away those members of  $\text{nacc}(C_\delta)$  whose cofinalities are too small for obtaining something satisfying (3), and note that the club-guessing properties that we need are not harmed by this pruning.

Moving on to the proof, let us assume by way of a contradiction that there is no  $S$ -club system  $\bar{C}$  satisfying (1), (2), and (4)'. Our aim is to exploit this assumption in order to construct a certain sequence  $\langle \bar{C}^\zeta : \zeta < \sigma^+ \rangle$  of  $S$ -club systems which will then be used to produce a countable decreasing sequence of ordinals.

Let us agree to say that an  $S$ -club system satisfies the structural requirements of *Theorem 4* if conditions (1) and (2) of the conclusion of the theorem hold. We will define objects  $E_\zeta$  and  $\bar{C}^\zeta = \langle C_\delta^\zeta : \delta \in S \rangle$  by induction on  $\zeta < \sigma^+$ . Each  $E_\zeta$  will be closed and unbounded in  $\lambda$ , while each  $\bar{C}^\zeta$  will be an  $S$ -club system satisfying the structural requirements of *Theorem 4*. Our convention is that “stage  $\zeta$ ” in our construction refers to the process of building  $E_{\zeta+1}$  and  $\bar{C}^{\zeta+1}$  from  $E_\zeta$  and  $\bar{C}^\zeta$ . Our initial set-up is to take  $E_0 = \lambda$  and  $C_\delta^0 = \text{ran}(c_\delta)$  for each  $\delta \in S$ .

Stage  $\zeta$ : Defining  $E_{\zeta+1}$  and  $\bar{C}^{\zeta+1}$

We assume that our construction furnishes us with an  $S$ -club system  $\bar{C}^\zeta$  satisfying the structural requirements of *Theorem 4*. Our assumption is that the theorem fails, and so there are closed unbounded subsets  $E_\zeta^0$  and  $E_\zeta^1$  of  $\lambda$  such that for each  $\delta \in E_\zeta^0$ , we can find  $\epsilon < \sigma$  and  $i < \kappa$  such that

$$\alpha \in E_\zeta^1 \cap \text{nacc}(C_\delta^\zeta) \cap I_{\kappa \cdot \epsilon + i}^\delta \implies \text{cf}(\alpha) \leq \mu_i^+.$$

We define

$$E_{\zeta+1} := \text{acc}(E_\zeta \cap E_\zeta^0 \cap E_\zeta^1).$$

The definition of  $\bar{C}^{\zeta+1}$  will take a bit more effort. Let us agree to call an ordinal  $\delta \in S$  active at stage  $\zeta$  if  $C_\delta^0 \subseteq \text{acc}(E_{\zeta+1})$ . Our choice of  $\langle c_\delta : \delta \in S \rangle$  ensures that at any stage, the set of active  $\delta$  is a stationary subset of  $S$ . If  $\delta \in S$  is inactive at stage  $\zeta$ , then we simply define  $C_\delta^{\zeta+1}$  to be  $C_\delta^\zeta$  and do nothing.

On the other hand, if  $\delta$  is active at stage  $\zeta$ , then  $\delta$  must be in  $E_\zeta^0$  and therefore we can find a least  $a(\delta, \zeta) < \sigma$  such that, letting

$$I(\delta, \zeta) := I_{a(\delta, \zeta)}^\delta,$$

we have

$$\alpha \in E_\zeta^1 \cap \text{nacc}(C_\delta^\zeta) \cap I(\delta, \zeta) \implies \text{cf}(\alpha) \leq \mu_{i(\delta, \zeta)}^+. \tag{4.4}$$

Our construction of  $C_\delta^{\zeta+1}$  will modify  $C_\delta^\zeta$  only on the interval  $I(\delta, \zeta)$  – everything else will be left untouched.

The ordinal  $a(\delta, \zeta)$  can be written in the form

$$a(\delta, \zeta) = \epsilon(\delta, \zeta) \cdot \kappa + i(\delta, \zeta) \tag{4.5}$$

for some unique  $\epsilon(\delta, \zeta) < \sigma$  and  $i(\delta, \zeta) < \kappa$ . These two ordinals will also play a role in our construction.

Our next move is to define

$$D_\delta^\zeta := \text{Drop}(C_\delta^\zeta \cap I(\delta, \zeta), E_{\zeta+1} \cap I(\delta, \zeta)).$$

We note the following facts about  $D_\delta^\zeta$ :

- $c_\delta(a(\delta, \zeta) + 1)$  – the top of the interval  $I(\delta, \zeta)$  – is an element of  $D_\delta^\zeta$  because  $C_\delta^0 \subseteq \text{acc}(E_{\zeta+1})$ ,
- $D_\delta^\zeta$  is a closed subset of  $E_{\zeta+1} \cap I(\delta, \zeta)$ ,
- $|D_\delta^\zeta| \leq |C_\delta^\zeta \cap I(\delta, \zeta)| \leq \mu_{i(\delta, \zeta)}^+$  (as  $\bar{C}^\zeta$  satisfies the structural requirements of Theorem 4), and
- if  $C_\delta^\zeta \cap I(\delta, \zeta)$  is unbounded in  $c_\delta(a(\delta, \zeta) + 1)$ , then so is  $D_\delta^\zeta$ .

One should picture  $D_\delta^\zeta$  as arising after “shifting”  $C_\delta^\zeta \cap I(\delta, \zeta)$  so that it lies inside of  $E_{\zeta+1}$ . Our construction will ensure that  $D_\delta^\zeta \subseteq C_\delta^{\zeta+1}$ , but we need to do more work first.

Let us say that an element  $\alpha$  of  $D_\delta^\zeta$  needs attention if

$$\alpha \in \text{acc}(E_{\zeta+1}) \cap \text{nacc}(D_\delta^\zeta), \tag{4.6}$$

and

$$\text{cf}(\alpha) \leq \mu_{i(\delta, \zeta)}^+. \tag{4.7}$$

If  $\alpha$  needs attention, then  $\text{Fill}(\alpha, C_\delta^\zeta \cap I(\delta, \zeta), E_{\zeta+1} \cap I(\delta, \zeta))$  provides us with a closed unbounded subset of  $\alpha$  lying in the interval  $\text{Gap}(\alpha, D_\delta^\zeta)$ . Notice as well that this closed unbounded subset of  $\alpha$  is of cardinality  $\text{cf}(\alpha) \leq \mu_{i(\delta, \zeta)}^+$ , and so the set

$$A_\delta^\zeta := D_\delta^\zeta \cup \{\text{Fill}(\alpha, C_\delta^\zeta \cap I(\delta, \zeta), E_{\zeta+1} \cap I(\delta, \zeta)) : \alpha \text{ needs attention}\}$$

satisfies

$$|A_\delta^\zeta| \leq |D_\delta^\zeta| \cdot \mu_{i(\delta, \zeta)}^+ \leq |C_\delta^\zeta \cap I(\delta, \zeta)| \cdot \mu_{i(\delta, \zeta)}^+ = \mu_{i(\delta, \zeta)}^+. \tag{4.8}$$

Since the instances of “Fill” needed are always closed sets lying in a “gap” of  $D_\delta^\zeta$ , the set  $A_\delta^\zeta$  is also closed in  $c_\delta(a(\delta, \zeta) + 1)$ .

Moreover,  $A_\delta^\zeta$  is also unbounded in  $c_\delta(a(\delta, \zeta) + 1)$ , for either  $D_\delta^\zeta$  is already unbounded, or it is the case that  $c_\delta(a(\delta, \zeta) + 1)$  itself needs attention.

We now define  $C_\delta^{\zeta+1}$  in piecewise fashion:

$$C_\delta^{\zeta+1} \setminus I(\delta, \zeta) = C_\delta^\zeta \setminus I(\delta, \zeta),$$

and

$$C_\delta^{\zeta+1} \cap I(\delta, \zeta) = A_\delta^\zeta.$$

The  $S$ -club system  $\bar{C}_\delta^{\zeta+1}$  satisfies the structural requirements of our theorem, and so the construction can continue.

We still need to describe how to obtain  $\bar{C}^\zeta$  and  $E_\zeta$  when  $\zeta$  is a limit. Our construction defines

$$E_\zeta = \bigcap_{\xi < \zeta} C_\xi,$$

and for each  $\delta \in S$ , we define  $C_\delta^\zeta$  to be the closure in  $\delta$  of

$$\{\alpha < \delta : \alpha \in C_\delta^\xi \text{ for all sufficiently large } \xi < \zeta\}.$$

Note that  $C_\delta^\zeta$  is closed in  $\delta$ , and it is unbounded in  $\delta$  as it contains  $C_\delta^0$ . Elementary cardinal arithmetic implies that  $\bar{C}_\zeta$  satisfies the structural requirements of Theorem 4, and so our construction can continue.

To this point in the proof, we have used the failure of Theorem 4 to produce a sequence  $\langle \bar{C}^\zeta : \zeta < \sigma^+ \rangle$  of  $S$ -club systems; our task is to show that this leads to a contradiction.

Let us define

$$E^* := \bigcap_{\zeta < \sigma^+} E_\zeta.$$

Since  $E^*$  is club in  $\lambda$ , we can find a  $\delta \in S$  for which

$$C_\delta^0 \subseteq \{\alpha < \lambda : \mu \text{ divides } \text{otp}(E^* \cap \alpha)\}, \tag{4.9}$$

and this guarantees that

$$\alpha \in \text{nacc}(C_\delta^0) \implies |E^* \cap \text{Gap}(\alpha, C_\delta^0)| = \mu. \tag{4.10}$$

We know that this  $\delta$  is active at each stage  $\zeta < \sigma^+$  because of (4.9), and therefore we can find  $\epsilon^* < \sigma$  and  $i^* < \kappa$  such that  $\epsilon(\delta, \zeta) = \epsilon^*$  and  $i(\delta, \zeta) = i^*$  for unboundedly many  $\zeta < \sigma^+$ . Letting

$$I^* := (C_\delta(\epsilon^* \cdot \kappa + i^*), C_\delta(\epsilon^* \cdot \kappa + i^* + 1)],$$

this means  $I(\delta, \zeta) = I^*$  for unboundedly many ordinals  $\zeta < \sigma^+$ . Let  $\langle \zeta_n : n < \omega \rangle$  be the increasing enumeration of the first  $\omega$  such ordinals, and let  $\zeta^* = \sup\{\zeta_n : n \in \omega\}$ .

Our construction ensures  $|I^* \cap C_\delta^\zeta| \leq \mu_{i^*}^+$  for all  $\zeta < \sigma^+$ , so an appeal to (4.10) allows us to choose

$$\beta^* \in E^* \cap I^* \setminus \bigcup_{\zeta < \sigma^+} C_\delta^\zeta. \tag{4.11}$$

Finally, define

$$\beta_n := \min(C_\delta^{\zeta_n} \setminus \beta^*).$$

Note that our choice of  $\beta^*$  ensures that  $\beta^* < \beta_n$  for all  $n$ .

**Claim 2.** For each  $n$ , we have  $\beta_{n+1} < \beta_n$ .

**Proof.** Given  $n$ , we note that

$$\beta_{n+1} = \min(C_\delta^{\zeta_{n+1}} \setminus \beta^*)$$

(notice the shift in the position of “+1” here), as the definition of  $\zeta_{n+1}$  implies  $C_\delta^\xi \cap I^* = C_\delta^{\zeta_{n+1}} \cap I^*$  whenever  $\zeta_n + 1 \leq \xi \leq \zeta_{n+1}$ . We now split the proof into two cases:

Case 1:  $\beta_n \notin \text{acc}(E_{\zeta_{n+1}})$ .

Since  $\beta^* \in E^* \subseteq E_{\zeta_{n+1}}$ , it follows that

$$\beta^* \leq \sup(\beta_n \cap E_{\zeta_{n+1}}) < \beta_n.$$

Now  $\beta^*$  is not in  $C_\delta^{\zeta_{n+1}}$  while

$$\sup(\beta_n \cap E_{\zeta_{n+1}}) \in D_\delta^{\zeta_n} \subseteq C_\delta^{\zeta_{n+1}}$$

so  $\beta^* < \beta_{n+1} < \beta_n$  as claimed.

Case 2:  $\beta_n \in \text{acc}(E_{\zeta_{n+1}})$ .

Both  $\delta$  and  $\beta_n$  are in  $E_{\zeta_{n+1}}$ , so in particular we know that  $\delta \in E_{\zeta_n}^0$  and  $\beta_n \in E_{\zeta_n}^1$ . In addition,  $\beta_n$  must lie in  $\text{nacc}(C_\delta^{\zeta_n})$  because  $\beta^* < \beta_n$ . By (4.4), we conclude that  $\text{cf}(\beta_n) \leq \mu_{i(\delta, \zeta_n)}^+$ .

We have assumed in our case hypothesis that  $\beta_n = \sup(E_{\zeta_{n+1}} \cap \beta_n)$  and this guarantees that  $\beta_n$  is an element of  $D_\delta^{\zeta_n}$ . Since  $\beta_n$  is also in  $\text{nacc}(C_\delta^{\zeta_n})$ , the set  $D_\delta^{\zeta_n}$  cannot pick up any new elements between  $\beta^*$  and  $\beta_n$  and hence

$$\beta_n = \min(D_\delta^{\zeta_n} \setminus \beta^*) > \beta^*. \tag{4.12}$$

Thus,  $\beta_n$  is in  $\text{nacc}(D_\delta^{\zeta_n})$  and we see that  $\beta_n$  needs attention during the construction of  $C_\delta^{\zeta_{n+1}}$ .

In this case, our construction makes sure that  $C_\delta^{\zeta_{n+1}}$  contains a closed unbounded subset of  $\beta_n$ . In particular,  $C_\delta^{\zeta_{n+1}} \cap (\beta^*, \beta_n) \neq \emptyset$ , and therefore

$$\beta^* < \beta_{n+1} = \min(C_\delta^{\zeta_{n+1}} \setminus \beta^*) = \min(C_\delta^{\zeta_n} \setminus \beta^*) < \beta_n,$$

as required.  $\square$

In summary, if the conclusion of our theorem fails, then our construction generates an infinite decreasing sequence of ordinals. This is absurd, and so the theorem is established.  $\square$

### 5. Organizational interlude

Our goal in this section is to lay a good foundation before proceeding to the proofs of our main theorems. A good deal of the difficulty in these proofs lies in the fact that we require so many different objects to push the argument through;



we make an attempt at organizing notation and providing a clear picture of our assumptions before proceeding. In addition, we focus only on the case where  $\text{cf}(\mu) = \aleph_0$ , as uncountable cofinalities can be handled by earlier work. We first fix our names for the various cardinals that are important for our theorems:

- $\lambda = \mu^+$  for  $\mu$  singular of cofinality  $\aleph_0$ ,
- $\aleph_0 < \sigma = \text{cf}(\sigma) < \mu$ ,
- $S$  is a stationary subset of  $\{\delta < \lambda : \text{cf}(\delta) = \sigma\}$ ,
- $\langle \mu_i : i < \omega \rangle$  is a strictly increasing sequence of regular cardinals such that
  - $\langle \mu_i : i < \omega \rangle$  is cofinal in  $\mu$ , and
  - $\sigma < \mu_0$ .

Now the club-guessing result of the previous section gives us objects  $\langle c_\delta^0 : \delta \in S \rangle$ ,  $\langle C_\delta : \delta \in S \rangle$ , and  $\{I(\delta, \epsilon, m) : \delta \in S, \epsilon < \sigma, m < \omega\}$  satisfying the following:

- $c_\delta^0$  is the increasing enumeration of a closed unbounded subset of  $\delta$  of order-type  $\sigma$ ,
- if  $E$  is a closed unbounded subset of  $\lambda$ , then the set of  $\delta \in S$  for which  $\text{ran}(c_\delta^0) \subseteq E$  is stationary,
- $C_\delta$  is a closed unbounded subset of  $\delta$  with  $\text{ran}(c_\delta^0) \subseteq C_\delta$ ,
- $I(\delta, \epsilon, m)$  denotes the half-open interval  $(c_\delta^0(\omega \cdot \epsilon + m), c_\delta^0(\omega \cdot \epsilon + m + 1)]$ ,
- $|C_\delta \cap I(\delta, \epsilon, m)| \leq \mu_m^+$ ,
- $\alpha \in \text{nacc}(C_\delta) \cap I(\delta, \epsilon, m) \implies \text{cf}(\alpha) > \mu_m^+$ , and
- if  $E$  is a closed unbounded subset of  $\lambda$ , then for stationarily many  $\delta \in S$  it is the case that for each  $\epsilon < \sigma$  and  $m < \omega$ , the set  $E \cap \text{nacc}(C_\delta) \cap I(\delta, \epsilon, m)$  is non-empty.

If  $\delta \in S$  is fixed and clear from context (as is usually the case), then we will write  $I(\epsilon, m)$  instead of  $I(\delta, \epsilon, m)$ .

It is crucial for the reader to have a good picture of the structure of the objects described above. Our notation is intended to describe something fairly simple: given  $\delta \in S$ , we use  $c_\delta^0$  (or rather, the range of  $c_\delta^0$ ) essentially to divide the ordinals less than  $\delta$  into  $\sigma$  blocks, each of which is further divided into  $\omega$  pieces. We say “essentially”, because there are a few ordinals left out for technical reasons, but the reader will be well-served by thinking of the interval  $I(\delta, \epsilon, m)$  as “the  $m$ th piece in block  $\epsilon$  built using  $c_\delta^0$ ”. The following observation tells the complete story.

**Proposition 5.1.** *Given  $\delta \in S$ , each ordinal  $\alpha < \delta$  satisfies exactly one of the following conditions:*

- (1)  $\alpha \leq c_\delta^0(0)$ ,
- (2)  $\alpha \in \text{acc}(\text{ran}(c_\delta^0))$ , or
- (3)  $\alpha \in I(\epsilon, m)$  for some unique  $\epsilon < \sigma$  and  $m < \omega$ .

*In particular, any element of  $\text{nacc}(C_\delta) \setminus \{\min(C_\delta)\}$  lies in  $I(\epsilon, m)$  for some unique  $\epsilon < \sigma$  and  $m < \omega$ .*

Our next task is to connect our club-guessing sequence to some ideals introduced by Shelah in [27]. We have already discussed club-guessing in general terms at the start of the previous section; what follows is a deeper discussion of the particular case of interest to us.

**Definition 5.2.** Given  $\delta \in S$ , let  $I_\delta$  be the ideal on  $C_\delta$  generated by the sets of the form

$$\{\gamma \in C_\delta : \gamma \in \text{acc}(C_\delta) \text{ or } \text{cf}(\gamma) < \alpha \text{ or } \gamma < \beta\} \tag{5.1}$$

for  $\alpha < \mu$  and  $\beta < \delta$ .

Shelah uses the notation  $J_{C_\delta}^{b[\mu]}$  for the ideal described above; we have gone with something a little simpler. These ideals  $I_\delta$  are natural in our given context – for example, the following two propositions show us that the ideals are compatible with the interval structure that we have imposed on the objects  $C_\delta$ .

**Proposition 5.3.** *Suppose  $\delta \in S$ . If  $\epsilon^* < \sigma$  and  $m^* < \omega$  are fixed; then  $I_\delta$ -almost all members of  $C_\delta$  are in  $\text{nacc}(C_\delta) \cap \bigcup \{I(\epsilon, m) : \epsilon^* \leq \epsilon < \sigma, m^* \leq m < \omega\}$ .*

**Proof.** This is clear from the definition of  $I_\delta$ .  $\square$

**Proposition 5.4.** *If  $E$  is a closed unbounded subset of  $\lambda$ , then there are stationarily many  $\delta$  for which  $E \cap C_\delta \notin I_\delta$ .*

**Proof.** Our assumptions give us a stationary set of  $\delta$  such that  $E \cap \text{nacc}(C_\delta) \cap I(\epsilon, m)$  is non-empty for each  $\epsilon < \sigma$  and  $m < \omega$ , so we are done if we verify that  $E \cap C_\delta \notin I_\delta$  for each such  $\delta$ . This is easily done – given  $\alpha < \mu$  and  $\beta < \delta$ , we need to show that  $E \cap C_\delta$  contains a member of  $\text{nacc}(C_\delta)$  larger than  $\alpha$  and whose cofinality is greater than  $\beta$ . Choose  $\epsilon < \sigma$  so large that  $\beta < c_\delta^0(\omega \cdot \epsilon)$  and choose  $m$  so large that  $\alpha < \mu_m$ . We know  $E \cap \text{nacc}(C_\delta) \cap I(\epsilon, m)$  is non-empty, and any ordinal in this intersection is necessarily a member of  $E \cap \text{nacc}(C_\delta)$  larger than  $\beta$  and of cofinality greater than  $\alpha$ .  $\square$

Given the preceding proposition, we can now bring in a certain club-guessing ideal which will be woven into all of our most important results. The following definition makes sense in a more general context than that which we are considering (Chapter III of [27], for example), but we will deal only with the particular case of interest to us.

**Definition 5.5.** Let  $\bar{I} = \langle I_\delta : \delta \in S \rangle$  be as in Definition 5.2. The ideal  $\text{id}_p(\bar{C}, \bar{I})$  is defined by putting  $A \in \text{id}_p(\bar{C}, \bar{I})$  if there is a closed unbounded  $E \subseteq \lambda$  such that

$$A \cap E \cap C_\delta \in I_\delta \quad \text{for all } \delta \in S \cap E.$$

The preceding definition is robust under slight modifications. For example, we get the same ideal by requiring  $A \cap E \cap C_\delta$  to be in  $I_\delta$  for all but non-stationarily many  $\delta \in S \cap E$ . Note as well that the restriction to  $\delta \in S \cap E$  is not important – if  $\delta \in S \setminus E$ , then  $E \cap C_\delta$  must be bounded and therefore  $E \cap C_\delta \in I_\delta$ . Roughly speaking, a set  $A \subseteq \lambda$  is in  $\text{id}_p(\bar{C}, \bar{I})$  if  $\bar{C}$  fails to “guess”  $E \cap A$  in the sense of Proposition 5.4. Note that in light of Proposition 5.4, the ideal  $\text{id}_p(\bar{C}, \bar{I})$  is a proper ideal on  $\lambda$  extending the non-stationary ideal.

The following fact is a specific instance of Observation 3.2 in Chapter III of Shelah – it is an elementary result establishing indecomposability properties for the club-guessing ideal under consideration. We include the proof because the result is a crucial piece of our main theorem.

**Proposition 5.6.** *Under our assumptions, the ideal  $\text{id}_p(\bar{C}, \bar{I})$  is  $\tau$ -indecomposable whenever  $\tau < \mu$  is an uncountable regular cardinal distinct from  $\sigma$ .*

**Proof.** Let  $\tau \neq \sigma$  be an uncountable regular cardinal less than  $\mu$ . Given  $\delta \in S$ , we first establish that  $I_\delta$  is  $\tau$ -indecomposable, so let  $\langle A_i : i < \tau \rangle$  be an increasing sequence of subsets of  $C_\delta$ , each of which is in  $I_\delta$ . The collection of sets of the form (5.1) is closed under finite unions, and since these sets generate  $I_\delta$ , it follows that each set in the ideal is covered by a single set of that form. Thus, we may define  $\alpha_i$  to be the least  $\alpha < \mu$  for which there is a  $\beta < \delta$  such that

$$A_i \subseteq \{\gamma \in C_\delta : \gamma \in \text{acc}(C_\delta) \text{ or } \text{cf}(\gamma) < \alpha \text{ or } \gamma < \beta\}.$$

Since the sequence  $\langle A_i : i < \tau \rangle$  is increasing, it is clear that  $\langle \alpha_i : i < \tau \rangle$  is a non-decreasing sequence of ordinals less than  $\mu$ . Since  $\text{cf}(\mu) = \aleph_0$  and  $\tau$  is an uncountable regular cardinal, it follows that this sequence is bounded by some ordinal  $\alpha^* < \mu$ .

Next, we let  $\beta_i$  be the least ordinal  $\beta < \delta$  for which

$$A_i \subseteq \{\gamma \in C_\delta : \gamma \in \text{acc}(C_\delta) \text{ or } \text{cf}(\gamma) < \alpha^* \text{ or } \gamma < \beta\}.$$

Such a  $\beta$  always exists because of our choice of  $\alpha^*$ , and since  $\langle A_i : i < \tau \rangle$  is increasing, the sequence  $\langle \beta_i : i < \tau \rangle$  is non-decreasing. Since  $\tau \neq \sigma = \text{cf}(\delta)$ , there is an ordinal  $\beta^* < \delta$  greater than all  $\beta_i$ .

Thus, we conclude that

$$\bigcup_{i < \tau} A_i \subseteq \{\gamma \in C_\delta : \gamma \in \text{acc}(C_\delta) \text{ or } \text{cf}(\gamma) < \alpha^* \text{ or } \gamma < \beta^*\}, \tag{5.2}$$

and hence this union is in  $I_\delta$ .

The rest of the argument consists in noting that  $\text{id}_p(\bar{C}, \bar{I})$  inherits the indecomposability properties of the ideals  $I_\delta$ . To see this, let  $\langle A_i : i < \tau \rangle$  be an increasing family of subsets of  $\lambda$ , each of which is in  $\text{id}_p(\bar{C}, \bar{I})$ . Set  $A$  equal to the union of these sets, and suppose by way of a contradiction that  $A \notin \text{id}_p(\bar{C}, \bar{I})$ .

Since  $A_i \in \text{id}_p(\bar{C}, \bar{I})$ , there is a closed unbounded  $E_i \subseteq \lambda$  for which

$$A_i \cap E_i \cap C_\delta \in I_\delta \quad \text{for all } \delta \in S \cap E_i. \tag{5.3}$$

If we let  $E$  be the intersection of all the sets  $E_i$ , then it is clear that  $E$  is closed and unbounded in  $\lambda$ . We have assumed that  $A \notin \text{id}_p(\bar{C}, \bar{I})$ , and thus we can fix an ordinal  $\delta \in S \cap E$  such that

$$A \cap E \cap C_\delta \notin I_\delta. \tag{5.4}$$

Define

$$B_i := A_i \cap E \cap C_\delta.$$

It should be clear that the sequence  $\langle B_i : i < \tau \rangle$  is a  $\subseteq$ -increasing sequence of subsets of  $C_\delta$ , and since  $E \subseteq E_i$ , we know that  $B_i \in I_\delta$  by (5.3). Now  $I_\delta$  is  $\tau$ -indecomposable, and so

$$B := \bigcup_{i < \tau} B_i \in I_\delta.$$

This is a contradiction, though, because we also know that

$$B = \bigcup_{i < \tau} (A_i \cap E \cap C_\delta) = \left( \bigcup_{i < \tau} A_i \right) \cap E \cap C_\delta = A \cap E \cap C_\delta,$$

which is not in  $I_\delta$  by (5.4). We conclude that  $A$  must be in  $\text{id}_p(\bar{C}, \bar{I})$ , as required.  $\square$

Now that we have established some basic results for our club-guessing sequence, our concern shifts to seeing how well some of the techniques used in [10] can be adapted to this context. We begin with an observation that the interval structure that we have placed on our sets  $C_\delta$  gives us a natural way of writing each  $C_\delta$  as a countable increasing union of approximations:

**Definition 5.7.** Given  $\delta \in S$  and  $m < \omega$ , we define

$$C_\delta[m] = \text{ran}(c_\delta^0) \cup \bigcup \{C_\delta \cap I(\epsilon, i) : \epsilon < \sigma \text{ and } i \leq m\}.$$

In more descriptive language,  $C_\delta[m]$  consists of  $\text{ran}(c_\delta^0)$  together with the parts of  $C_\delta$  lying in the first  $m + 1$  pieces of each of the  $\sigma$  blocks built using  $c_\delta^0$ . We note the following easy facts for these objects:

**Proposition 5.8.** Suppose  $\delta \in S$ . Then

- (1)  $C_\delta[m]$  is closed and unbounded in  $\delta$ ,
- (2)  $|C_\delta[m]| \leq \sigma \cdot \mu_m^+ = \mu_m^+$ ,
- (3)  $C_\delta[m] \subseteq C_\delta[m + 1]$ , and
- (4)  $C_\delta = \bigcup_{m < \omega} C_\delta[m]$ .

Our next move is to implement “Shelah’s ladder-swallowing trick” (a crucial ingredient of the proofs in [8] and [10]) in this new context. We take the following definition from [10].

**Definition 5.9.** A generalized  $C$ -sequence on  $\lambda$  is a family

$$\langle e_\alpha^m : \alpha < \lambda, m < \omega \rangle$$

such that for each  $\alpha < \lambda$  and  $m < \omega$ ,

- $e_\alpha^m$  is closed unbounded in  $\alpha$ , and
- $e_\alpha^m \subseteq e_\alpha^{m+1}$ .

We now use our collection of objects  $C_\delta[m]$  in order to construct a very special generalized  $C$ -sequence.

**Lemma 5.10.** There is a generalized  $C$ -sequence  $\langle e_\alpha^m : \alpha < \lambda, m < \omega \rangle$  such that

- (1)  $|e_\alpha^m| \leq \text{cf}(\alpha) + \mu_m^+$ , and
- (2)  $\delta \in S \cap e_\alpha^m \implies C_\delta[m] \subseteq e_\alpha^m$ .

**Proof.** Let  $e_\alpha$  be closed unbounded in  $\alpha$  of order-type  $\text{cf}(\alpha)$ . We define

$$\begin{aligned} e_\alpha^0[0] &= e_\alpha \\ e_\alpha^0[k+1] &= \text{closure in } \alpha \text{ of } e_\alpha^0[k] \cup \bigcup \{C_\delta[0] : \delta \in S \cap e_\alpha^0[k]\} \\ e_\alpha^0 &= \text{closure in } \alpha \text{ of } \bigcup \{e_\alpha^0[k] : k < \omega\} \\ e_\alpha^{m+1}[0] &= e_\alpha^m \\ e_\alpha^{m+1}[k+1] &= \text{closure in } \alpha \text{ of } e_\alpha^{m+1}[k] \cup \bigcup \{C_\delta[m+1] : \delta \in S \cap e_\alpha^{m+1}[k]\} \\ e_\alpha^{m+1} &= \text{closure in } \alpha \text{ of } \bigcup \{e_\alpha^m[k] : k < \omega\}. \end{aligned}$$

The estimate for  $|e_\alpha^m|$  holds because of the corresponding bounds on the size of each  $C_\delta[m]$ . As for the other requirement, note that since  $\sigma$  is an uncountable regular cardinal, if  $\delta \in S \cap e_\alpha^m$  then there is a  $k < \omega$  such that  $\delta \in S \cap e_\alpha^m[k]$ .  $\square$

Both conditions (1) and (2) in the preceding lemma are crucial ingredients in the proof of our main result. The name “ladder-swallowing trick” comes from condition (2), which tells us that the generalized  $C$ -sequence that we have built “swallows” the objects  $C_\delta[n]$  in a natural way. These sorts of requirements probably seem quite unmotivated at this point, but we note that it is precisely this “swallowing” behavior that allows us to lift some of Todorćević’s minimal walks machinery to successors of singular cardinals without the need for strong assumptions like  $\square$  (a technique first pioneered by Shelah in Chapter III of [27], as well as [9]).

## 6. Minimal walks and a preliminary result

We leave club-guessing behind for a bit, and turn now to Todorćević’s technique of minimal walks as it relates to our specific generalized  $C$ -sequence. The following definition presents the notation that we use to keep track of everything.

**Definition 6.1.** Given ordinals  $\alpha < \beta < \lambda$  and  $m < \omega$ , we define

$$\beta_0^m(\alpha, \beta) = \beta,$$

while

$$\beta_{i+1}^m(\alpha, \beta) = \begin{cases} \min(e_{\beta_i^m(\alpha, \beta)}^m \setminus \alpha) & \text{if } \beta_i^m(\alpha, \beta) > \alpha, \text{ and} \\ \alpha & \text{otherwise.} \end{cases}$$

Next, we define

$$\rho_2^m(\alpha, \beta) = \text{least } i \text{ for which } \beta_i^m(\alpha, \beta) = \alpha.$$

For each  $m < \omega$  and  $\alpha < \beta < \lambda$ , the sequence  $\langle \beta_i^m : i < \rho_2^m(\alpha, \beta) \rangle$  is the minimal walk from  $\beta$  to  $\alpha$  using the C-system  $\bar{e}^m$  in the sense of [30]. We will abbreviate this somewhat, and call this *the m-walk from  $\beta$  to  $\alpha$* . The use of  $\rho_2$  to stand for the length of such a walk is due to Todorćević. Minimal walks in the context of generalized C-sequences were introduced in [10]; our notation is a variant of the notation in that paper.

Now that we have the definition of an  $m$ -walk at our disposal, the statement of the following theorem makes sense. This theorem illustrates how the structure of our generalized C-system lets us establish a connection between the  $m$ -walks defined above, and the club-guessing ideal  $\text{id}_p(\bar{C}, \bar{I})$  discussed in the preceding section. This theorem will quickly be superseded by results in the next section, but it represents an important step on the way to our main theorem.

**Theorem 5.** Let  $\langle t_\alpha : \alpha < \lambda \rangle$  be a pairwise disjoint family of finite subsets of  $\lambda$ . For  $\text{id}_p(\bar{C}, \bar{I})$ -almost all  $\beta^* < \lambda$ , there are  $\alpha < \beta < \lambda$  and  $m < \omega$  such that for every  $\zeta \in t_\alpha$  and  $\xi \in t_\beta$ , the  $m$ -walk from  $\xi$  to  $\zeta$  passes through  $\beta^*$ .

The proof of Theorem 5 will fill the rest of this section. We begin with the following lemma and corollary, which are translations into the context of generalized C-sequences of some standard facts about minimal walks. Here we see our assumption that elements of  $S$  have uncountable cofinality starts to become relevant.

**Lemma 6.2.** Suppose  $\delta < \beta < \lambda$  and  $\text{cf}(\delta) > \aleph_0$ . There is an ordinal  $\gamma^*(\delta, \beta) < \delta$  such that

$$(\gamma^*(\delta, \beta), \delta) \cap e_{\beta_i^m(\delta, \beta)}^m = \emptyset \tag{6.1}$$

for all  $m < \omega$  and  $i < \rho_2^m(\delta, \beta) - 1$ .

**Proof.** If  $m < \omega$  and  $i < \rho_2^m(\delta, \beta) - 1$ , then  $\delta \notin e_{\beta_i^m(\delta, \beta)}^m$ . Since this latter set is closed, it follows that

$$\sup(\delta \cap e_{\beta_i^m(\delta, \beta)}^m) < \delta,$$

and if we define

$$\gamma^*(\delta, \beta) := \sup\{\sup(\delta \cap e_{\beta_i^m(\delta, \beta)}^m) : m < \omega, i < \rho_2^m(\delta, \beta) - 1\},$$

then  $\gamma^*(\delta, \beta) < \delta$  as  $\delta$  is of uncountable cofinality.  $\square$

The following corollary isolates the importance of  $\gamma^*(\delta, \beta)$  – if  $\alpha$  is any ordinal between  $\gamma^*(\delta, \beta)$  and  $\delta$ , then for any  $m$  the  $m$ -walk from  $\beta$  to  $\alpha$  agrees with the  $m$ -walk from  $\beta$  to  $\delta$  up to the penultimate step of the latter walk. This is a familiar pattern of argument invented by Todorćević.

**Corollary 6.3.** If  $\delta < \beta < \lambda$  and  $\text{cf}(\delta) > \aleph_0$ , then for any  $m < \omega$  and  $i < \rho_2^m(\delta, \beta)$  we have

$$\gamma^*(\delta, \beta) < \alpha < \delta \implies \beta_i^m(\alpha, \beta) = \beta_i^m(\delta, \beta).$$

**Proof.** The proof is a straightforward induction on  $i < \rho_2^m(\delta, \beta) - 1$  using the definition  $\beta_i^m(\alpha, \beta)$ .  $\square$

The next lemma takes advantage of the fact that for each  $\alpha$ ,  $\langle e_\alpha^m : m < \omega \rangle$  forms an increasing family of sets.

**Lemma 6.4.** If  $\alpha < \beta < \lambda$ , then there is an  $m(\alpha, \beta) < \omega$  such that

$$\rho_2^m(\alpha, \beta) = \rho_2^{m(\alpha, \beta)}(\alpha, \beta)$$

for all  $m \geq m(\alpha, \beta)$ . Moreover,

$$\beta_i^m(\alpha, \beta) = \beta_i^{m(\alpha, \beta)}(\alpha, \beta)$$

for all  $i \leq \rho_2^{m(\alpha, \beta)}(\alpha, \beta)$ .

**Proof.** This follows easily by induction once we note that for any ordinals  $\alpha < \beta$ , the sequence  $\langle \min(e_\alpha^m \setminus \alpha) : m < \omega \rangle$  is non-increasing and hence eventually constant.  $\square$

Armed with the last lemma, we are in a position to define another parameter needed for our construction.

**Definition 6.5.** Given  $\alpha < \beta < \lambda$ , define

$$\gamma(\alpha, \beta) := \beta_{\rho_2^{m(\alpha, \beta)}(\alpha, \beta)-1}^{m(\alpha, \beta)}. \tag{6.2}$$

In plain language, we know that the various  $m$ -walks from  $\beta$  down to  $\alpha$  are in agreement for all  $m \geq m(\alpha, \beta)$ , and so  $\gamma(\alpha, \beta)$  is simply the last ordinal visited by all of these walks before they arrive at their destination  $\alpha$ . The last two parameters that we need for the proof of [Theorem 5](#) are given in the following definition; note that we restrict the definition to the case where  $\delta \in S$ .

**Definition 6.6.** Suppose  $\delta \in S$  and  $\delta < \beta < \lambda$ . We define

$$\epsilon^*(\delta, \beta) := \text{least } \epsilon < \sigma \text{ for which } \gamma^*(\delta, \beta) \leq c_\delta^0(\omega \cdot \epsilon) \tag{6.3}$$

and

$$m^*(\delta, \beta) := \text{least } m \geq m(\delta, \beta) \text{ such that } \text{cf}(\gamma(\delta, \beta)) < \mu_m, \tag{6.4}$$

where  $c_\delta^0$  and  $(\mu_m : m < \omega)$  are as in our list of assumptions.

The following proposition captures a few facts about how the various objects that we have been considering interact; part (3) of the proposition has particular importance for us.

**Proposition 6.7.** Suppose  $\delta \in S$  and  $\delta < \beta < \lambda$ .

- (1)  $C_\delta[m] \subseteq e_{\gamma(\delta, \beta)}^m$  for all  $m \geq m^*(\delta, \beta)$ .
- (2) Assume  $\epsilon^*(\delta, \beta) \leq \epsilon < \sigma$  and  $m^*(\delta, \beta) \leq m < \omega$ . If  $\beta^* \in \text{nacc}(C_\delta) \cap I(\epsilon, m)$ , then

$$\beta^* \in \text{nacc}(e_{\gamma(\delta, \beta)}^m),$$

and

$$\gamma^*(\delta, \beta) \leq \sup(e_{\gamma(\delta, \beta)}^m \cap \beta^*).$$

- (3) Furthermore, in the situation of (2), if  $\sup(\beta^* \cap e_{\gamma(\delta, \beta)}^m) < \alpha < \beta^*$ , then

$$\beta_k^m(\alpha, \beta) = \beta_k^m(\beta^*, \beta) = \beta_k^m(\delta, \beta) \text{ for all } k < \rho_2^m(\delta, \beta), \tag{6.5}$$

and

$$\beta_{\rho_2^m(\delta, \beta)}^m(\alpha, \beta) = \beta^*. \tag{6.6}$$

In particular, the  $m$ -walk from  $\beta$  to  $\beta^*$  (including the final step!) is an initial segment of the  $m$ -walk from  $\beta$  to  $\alpha$ .

**Proof.** For (1), we know that  $\delta \in e_{\gamma(\delta, \beta)}^m$  by our definition of  $\gamma(\delta, \beta)$  and  $m^*(\delta, \beta)$ , and a glance back at [Lemma 5.10](#) gives us what we need.

Condition (2) gives us some more information — it claims that some of the structure of  $C_\delta$  survives in  $e_{\gamma(\delta, \beta)}^m$ , and it is at this point that we cash in some of our assumptions for the first time (in particular, it will become clear why we work with the somewhat awkwardly defined  $m^*(\delta, \beta)$  instead of  $m(\delta, \beta)$ ). Given  $\beta^*$  in  $\text{nacc}(C_\delta) \cap I(\epsilon, m)$ , we know that  $\beta^* \in e_{\gamma(\delta, \beta)}^m$  by (1). Condition (1) of [Lemma 5.10](#) taken with the definition of  $m^*(\delta, \beta)$  tells us that

$$|e_{\gamma(\delta, \beta)}^m| \leq \text{cf}(\gamma(\delta, \beta)) + \mu_m^+ \leq \mu_m^+.$$

However, the fact that  $\beta^* \in I(\epsilon, m)$  tells us that

$$\text{cf}(\beta^*) > \mu_m^+,$$

and therefore  $\beta^*$  cannot be an accumulation point of  $e_{\gamma(\delta, \beta)}^m$ . This, together with the definition of  $\epsilon^*(\delta, \beta)$ , implies

$$\gamma^*(\delta, \beta) \leq \sup(\beta^* \cap e_{\gamma(\delta, \beta)}^m) < \beta^*.$$

In particular, there are many  $\alpha$  satisfying the hypothesis of (3).

Given such an  $\alpha$ , the fact that  $\gamma^*(\delta, \beta) < \alpha < \beta^*$  implies

$$\beta_i^m(\alpha, \beta) = \beta_i^m(\beta^*, \beta) = \beta_i^m(\delta, \beta)$$

for all  $i < \rho_2^m(\delta, \beta)$ , and therefore

$$\beta_{\rho_2^m(\delta, \beta)-1}^m(\alpha, \beta) = \beta_{\rho_2^m(\delta, \beta)-1}^m(\delta, \beta) = \gamma(\delta, \beta).$$

Given (2) and our assumptions about  $\alpha$ , it follows that

$$\beta_{\rho_2^m(\delta, \beta)}^m(\alpha, \beta) = \min(e_{\gamma(\delta, \beta)}^m \setminus \alpha) = \beta^*,$$

and so (3) is established.  $\square$

The following corollary follows immediately, and provides a useful upgrade of the preceding proposition.

**Corollary 6.8.** *Suppose  $\delta \in S$  and  $t$  is a finite subset of  $(\delta, \lambda)$ , and define*

$$\epsilon^* := \max\{\epsilon(\delta, \xi) : \xi \in t\},$$

and

$$m^* := \max\{m^*(\delta, \xi) : \xi \in t\}.$$

Given  $m \geq m^*$  and  $\beta^* \in \text{nacc}(C_\delta) \cap I(\epsilon, m)$ , the ordinal

$$\gamma^* := \max\{\sup(e_{\gamma(\delta, \xi)}^m \cap \beta^*) : \xi \in t\} < \beta^*$$

has the property that whenever  $\gamma^* < \alpha < \beta^*$ , we have

$$\beta_k^m(\alpha, \xi) = \beta_k^m(\beta^*, \xi) = \beta_k^m(\delta, \xi) \quad \text{for all } \xi \in t \text{ and } k < \rho_2^m(\delta, \xi), \tag{6.7}$$

and

$$\beta_{\rho_2^m(\delta, \xi)}^m(\alpha, \beta) = \beta^* \quad \text{for all } \xi \in t. \tag{6.8}$$

In particular, for any  $\xi \in t$  the  $m$ -walk from  $\xi$  to  $\beta^*$  is an initial segment of the  $m$ -walk from  $\xi$  to  $\alpha$ .

**Proof.** This is an immediate consequence of Proposition 6.7.  $\square$

Let us turn now to the proof of Theorem 5, and assume that  $\langle t_\alpha : \alpha < \lambda \rangle$  is a pairwise disjoint family of finite subsets of  $\lambda$ . Define

$$E := \{\delta < \lambda : \alpha < \delta \implies t_\alpha \subseteq \delta\}.$$

Since  $E$  is closed and unbounded in  $\lambda$ , we can apply Proposition 5.4 to conclude that

$$T := \{\delta \in S : E \cap C_\delta \not\subseteq I_\delta\} \tag{6.9}$$

is stationary.

Fix  $\delta \in T$ , and choose  $\beta > \delta$  so that  $\delta < \min(t_\beta)$ . Just as in Corollary 6.8, we define

$$\epsilon^* := \max\{\epsilon(\delta, \xi) : \xi \in t_\beta\}$$

and

$$m^* := \max\{m^*(\delta, \xi) : \xi \in t_\beta\}.$$

The following claim is where the closed unbounded set  $E$  starts to become relevant.

**Claim 3.** *Given  $\epsilon^* \leq \epsilon < \sigma$ ,  $m^* \leq m < \omega$ , and  $\beta^* \in E \cap \text{nacc}(C_\delta) \cap I(\epsilon, m)$ , there is an  $\alpha$  such that*

$$\beta_{\rho_2^m(\delta, \xi)}^m(\zeta, \xi) = \beta^* \tag{6.10}$$

for all  $\zeta \in t_\alpha$  and  $\xi \in t_\beta$ .

**Proof.** Given such  $\beta^*$ , we work as in Corollary 6.8 and define

$$\gamma^* := \max\{\sup(e_{\gamma(\delta, \xi)}^m \cap \beta^*) : \xi \in t_\beta\}.$$

Since  $\beta^*$  is in  $E$ , we can choose an  $\alpha < \lambda$  so that  $t_\alpha$  is contained in the interval  $(\gamma^*, \beta^*)$ . From part (3) of Proposition 6.7, we conclude that

$$\beta_{\rho_2^m(\delta, \xi)}^m(\zeta, \xi) = \beta^*$$

for all  $\zeta \in t_\alpha$  and  $\xi \in t_\beta$ , as required.  $\square$

The proof of Theorem 5 is almost complete. Given  $\delta \in T$ , our work provides us with  $\epsilon^* < \sigma$  and  $m^* < \omega$  as in Claim 3. Let us define

$$B_\delta := \text{nacc}(C_\delta) \cap \bigcup \{I(\epsilon, m) : \epsilon^* \leq \epsilon < \sigma, m^* \leq m < \omega\}.$$

Proposition 5.3 tells us that almost all (in the sense of  $I_\delta$ ) members of  $C_\delta$  are in  $B_\delta$ , while Claim 3 establishes that for any  $\beta^* \in E \cap B_\delta$ , there are  $\alpha < \beta < \lambda$  and  $m < \omega$  such that the  $m$ -walk from  $\xi$  to  $\zeta$  passes through  $\beta^*$  for all  $\zeta \in t_\alpha$  and  $\xi \in t_\beta$ . Thus, if we define

$$A := \lambda \setminus \bigcup \{E \cap B_\delta : \delta \in T\},$$

the proof Theorem 5 will be finished if we can show

$$A \in \text{id}_p(\bar{C}, \bar{I}).$$



To establish this, we show

$$A \cap E \cap C_\delta \in I_\delta \text{ for all } \delta \in S. \tag{6.11}$$

Clearly we need only worry about those  $\delta$  for which  $E \cap C_\delta \notin I_\delta$ . Given such a  $\delta$ , we have

$$A \cap E \cap B_\delta = \emptyset \tag{6.12}$$

by the definition of  $A$ . If  $A \cap E \cap C_\delta$  were  $I_\delta$ -positive, then this set would have non-empty intersection with  $B_\delta$ , as the complement of  $B_\delta$  is in  $I_\delta$ . This cannot be the case, as we would contradict (6.12). Thus,  $A \in \text{id}_p(\bar{C}, \bar{I})$  and the proof of Theorem 5 is complete.

### 7. Scales and elementary submodels

One of our assumptions in the last two sections is that we have at our disposal a fixed increasing sequence of regular cardinals  $\langle \mu_n : n < \omega \rangle$  that is cofinal in  $\mu$ . This parameter was utilized in the definition of  $m^*$ , but otherwise it has been in the background. In the upcoming work, we are going to need to assume that this sequence of regular cardinals carries a *scale*, and since we are going to need to use quite a bit of scale combinatorics, it seems reasonable to devote some time to fixing notation and reviewing what we need.

**Definition 7.1.** Let  $\mu$  be a singular cardinal. A *scale* for  $\mu$  is a pair  $(\vec{\mu}, \vec{f})$  satisfying:

- (1)  $\vec{\mu} = \langle \mu_i : i < \text{cf}(\mu) \rangle$  is an increasing sequence of regular cardinals such that  $\sup_{i < \text{cf}(\mu)} \mu_i = \mu$  and  $\text{cf}(\mu) < \mu_0$ .
- (2)  $\vec{f} = \langle f_\alpha : \alpha < \mu^+ \rangle$  is a sequence of functions such that
  - (a)  $f_\alpha \in \prod_{i < \text{cf}(\mu)} \mu_i$ .
  - (b) If  $\gamma < \delta < \mu^+$  then  $f_\gamma <^* f_\delta$ , where the notation  $f <^* g$  means that  $\{i < \text{cf}(\mu) : g(i) \leq f(i)\}$  is bounded in  $\text{cf}(\mu)$ .
  - (c) If  $f \in \prod_{i < \text{cf}(\mu)} \mu_i$  then there is an  $\alpha < \mu^+$  such that  $f <^* f_\alpha$ .

Given an increasing sequence  $\vec{\mu}$  as above, if there is an  $\vec{f}$  such that  $(\vec{\mu}, \vec{f})$  is a scale, then we say that  $\vec{\mu}$  carries a scale.

Since we have been working in the situation where the singular cardinal  $\mu$  has countable cofinality, we will deal only with that special case in this section; we refer the reader to [8] or [7] for more information on these matters. Thus, from now on we assume

- $\langle \mu_n : n < \omega \rangle$  is an increasing sequence of regular cardinals such that
  - $\langle \mu_n : n < \omega \rangle$  is cofinal in  $\mu$ ,
  - $\sigma < \mu_0$ , and
  - $\langle \mu_n : n < \omega \rangle$  carries a scale  $\langle f_\alpha : \alpha < \lambda \rangle$ .

That such sequences exist is a very non-trivial result of Shelah (Chapter II of [27]). Interested readers can also consult [7] or [4] for expository treatments of this subject.

The function  $\Gamma$  defined below is a standard combinatorial tool associated with scales; again, we tailor the definition to the context at hand.

**Definition 7.2.** For  $\alpha < \beta < \lambda$ , we define  $\Gamma(\alpha, \beta)$  by

$$\Gamma(\alpha, \beta) = \max\{n < \omega : f_\beta(n) \leq f_\alpha(n)\}. \tag{7.1}$$

For convenience, we say that  $\Gamma(\alpha, \alpha) = \infty$ .

The function  $\Gamma$  has many nice properties – it witnesses  $\text{Pr}_1(\mu^+, \mu^+, \text{cf}(\mu), \text{cf}(\mu))$  (Conclusion 1 on page 67 of [27]), for example. We shall not make use of this fact directly, but it is certainly in the background throughout our arguments.

The next lemma is a special case of Lemma 7 in [8]. We remind the reader that notation of the form “ $(\exists^* \beta < \lambda) \psi(\beta)$ ” means that  $\{\beta < \lambda : \psi(\beta) \text{ holds}\}$  is unbounded below  $\lambda$ , while “ $(\forall^* \beta < \lambda) \psi(\beta)$ ” means that  $\{\beta < \lambda : \psi(\beta) \text{ fails}\}$  is bounded below  $\lambda$ .

**Lemma 7.3.** *There is a closed unbounded  $C \subseteq \lambda$  such that the following holds for every  $\beta \in C$ :*

$$(\forall^* n < \omega)(\forall \eta < \mu_n)(\forall v < \mu_{n+1})(\exists^* \alpha < \beta) [f_\alpha(n) > \eta \wedge f_\alpha(n+1) > v]. \tag{7.2}$$

Our conventions regarding elementary submodels are standard. We assume that  $\chi$  is a sufficiently large regular cardinal and let  $\mathfrak{A}$  denote the structure  $\langle H(\chi), \in, <_\chi \rangle$  where  $H(\chi)$  is the collection of sets hereditarily of cardinality less than  $\chi$ , and  $<_\chi$  is some well-ordering of  $H(\chi)$ . The use of  $<_\chi$  means that our structure  $\mathfrak{A}$  has definable Skolem functions, and we obtain the set of *Skolem terms* for  $\mathfrak{A}$  by closing the collection of Skolem functions under composition.

**Definition 7.4.** Let  $B \subseteq H(\chi)$ . Then  $\text{Sk}_{\mathfrak{A}}(B)$  denotes the Skolem hull of  $B$  in the structure  $\mathfrak{A}$ . More precisely,

$$\text{Sk}_{\mathfrak{A}}(B) = \{t(b_0, \dots, b_n) : t \text{ a Skolem term for } \mathfrak{A} \text{ and } b_0, \dots, b_n \in B\}.$$

The set  $\text{Sk}_{\mathfrak{A}}(B)$  is an elementary substructure of  $\mathfrak{A}$ , and it is the smallest such structure containing every element of  $B$ . The following technical lemma is due originally to Baumgartner [1]; it is a fact that is quite useful in proving things about the function  $\Gamma$  associated with a given scale. Again, we refer the reader to [8] or [7] for a proof.

**Lemma 7.5.** *Assume that  $M \prec \mathfrak{A}$  and let  $\sigma \in M$  be a cardinal. If we define  $N = \text{Sk}_{\mathfrak{A}}(M \cup \sigma)$  then for all regular cardinals  $\tau \in M$  greater than  $\sigma$ , we have*

$$\sup(M \cap \tau) = \sup(N \cap \tau).$$

As a corollary to the above, we can deduce an important fact about *characteristic functions* of models. Once again, the following definition is but a special case of a more general definition.

**Definition 7.6.** If  $M$  is an elementary submodel of  $\mathfrak{A}$  such that

- $|M| < \mu$ , and
- $\langle \mu_n : n < \omega \rangle \in M$

then the *characteristic function* of  $M$  (denoted as  $\text{Ch}_M$ ) is the function with domain  $\omega$  defined by

$$\text{Ch}_M(n) := \begin{cases} \sup(M \cap \mu_n) & \text{if } \sup(M \cap \mu_n) < \mu_n, \\ 0 & \text{otherwise.} \end{cases}$$

In the situation of **Definition 7.6**, it is clear that  $\text{Ch}_M$  is an element of the product  $\prod_{n < \omega} \mu_n$ , and furthermore,  $\text{Ch}_M(n) = \sup(M \cap \mu_n)$  for all sufficiently large  $n < \omega$ . We can now see that the following corollary follows immediately from **Lemma 7.5**.

**Corollary 7.7.** *Let  $M$  be as in **Definition 7.6**. If  $n^* < \omega$  and we define  $N$  to be  $\text{Sk}_{\mathfrak{A}}(M \cup \mu_{n^*})$ , then*

$$\text{Ch}_M \upharpoonright [n^* + 1, \omega) = \text{Ch}_N \upharpoonright [n^* + 1, \omega). \tag{7.3}$$

We end this section with one more handy bit of terminology due to Shelah [25].

**Definition 7.8.** A  $\lambda$ -approximating sequence is a continuous  $\in$ -chain  $\mathfrak{M} = \langle M_i : i < \lambda \rangle$  of elementary submodels of  $\mathfrak{A}$  such that

- (1)  $\lambda \in M_0$ ,
- (2)  $|M_i| < \lambda$ ,
- (3)  $\langle M_j : j \leq i \rangle \in M_{i+1}$ , and
- (4)  $M_i \cap \lambda$  is a proper initial segment of  $\lambda$ .

If  $x \in H(\chi)$ , then we say that  $\mathfrak{M}$  is a  $\lambda$ -approximating sequence over  $x$  if  $x \in M_0$ .

Note that if  $\mathfrak{M}$  is a  $\lambda$ -approximating sequence and  $\lambda = \mu^+$ , then  $\mu + 1 \subseteq M_0$  because of condition (4) and the fact that  $\mu$  is an element of each  $M_i$ .

### 8. Defining the coloring

In this section, we will mix ideas from [9,8] together with the proof of **Theorem 5** in order to prove a coloring theorem in ZFC. Our assumptions and notation are as in the previous two sections, and our goal is to obtain the following theorem.

**Theorem 6.** *There is a function  $c : [\lambda]^2 \rightarrow \lambda$  such that for any pairwise disjoint collection  $\langle t_\alpha : \alpha < \lambda \rangle$  of finite subsets of  $\lambda$ , it is the case that for  $\text{id}_p(\bar{C}, \bar{I})$ -almost all  $\beta^* < \lambda$ , we can find  $\alpha < \beta$  such that*

$$c(\zeta, \xi) = \beta^* \text{ for all } \zeta \in t_\alpha \text{ and } \xi \in t_\beta. \tag{8.1}$$

The proof of **Theorem 6** makes use of the function  $\Gamma$  defined from the scale  $\langle f_\alpha : \alpha < \lambda \rangle$ . We begin by defining a sequence  $\langle c_m : m < \omega \rangle$  of functions coloring the pairs from  $\lambda$ .

**Definition 8.1.** Given  $\alpha < \beta < \lambda$  and  $m < \omega$ , we define

$$c_m(\alpha, \beta) = \beta_{k_m(\alpha, \beta)}^m(\alpha, \beta),$$

where  $k_m(\alpha, \beta)$  is the least  $k \leq \rho_2^m(\alpha, \beta)$  for which

$$\Gamma(\alpha, \beta_k^m(\alpha, \beta)) \neq \Gamma(\alpha, \beta).$$

In words, the value of  $c_m(\alpha, \beta)$  is the first place on the  $m$ -walk from  $\beta$  down to  $\alpha$  where “ $\Gamma$  changes”. Except for the parameter  $m$ , this is the same coloring as we used in [8].

The following lemma contains the heart of the proof of **Theorem 6**; it shows that the sequence of colorings  $\langle c_m : m < \omega \rangle$  has many of the properties that we need.

**Lemma 8.2.** *If  $\langle t_\alpha : \alpha < \lambda \rangle$  is a pairwise disjoint collection of finite subsets of  $\lambda$ , then for  $\text{id}_p(\bar{C}, \bar{I})$ -almost all  $\beta^* < \lambda$ , there are an  $m < \omega$  and  $\beta > \beta^*$  such that*

$$(\forall^* i < \omega)(\exists^* \alpha < \beta^*)(\forall \zeta \in t_\alpha)(\forall \xi \in t_\beta) [\Gamma(\zeta, \xi) = i \wedge c_m(\zeta, \xi) = \beta^*]. \tag{8.2}$$

**Proof.** Let  $\langle t_\alpha : \alpha < \lambda \rangle$  be given. Clearly we may assume  $\alpha \leq \min(t_\alpha)$  for all  $\alpha$ , as we can pass to a subsequence of cardinality  $\lambda$  with no loss of generality. Define  $A$  to be the set of  $\beta^* < \lambda$  for which it is impossible to find an  $m < \omega$  and  $\beta > \beta^*$  with the required properties. Assume by way of a contradiction that

$$A \notin \text{id}_p(\bar{C}, \bar{I}). \tag{8.3}$$

Let  $x = \{\bar{C}, \bar{e}, S, \mu, \lambda, \langle \mu_i : i < \omega \rangle, \langle f_\alpha : \alpha < \lambda \rangle, \langle t_\alpha : \alpha < \lambda \rangle\}$  – all the parameters needed to define the sequence  $\langle c_m : m < \omega \rangle$  together with the sequence of finite sets under consideration – and let  $\mathfrak{M} = \langle M_i : i < \lambda \rangle$  be a  $\lambda$ -approximating sequence over  $x$ . The set

$$E := \{\delta < \lambda : M_\delta \cap \lambda = \delta\}$$

is closed and unbounded in  $\lambda$ , so by our assumption (8.3), there is a  $\delta \in S$  for which

$$A \cap E \cap C_\delta \notin I_\delta. \tag{8.4}$$

Choose  $\beta < \lambda$  so that  $\delta < \min(t_\beta)$ , and just as in Corollary 6.8, let

$$\epsilon^* := \max\{\epsilon(\delta, \xi) : \xi \in t_\beta\}$$

and

$$m^* := \max\{m^*(\delta, \xi) : \xi \in t_\beta\}.$$

The conclusion of Corollary 6.8 tells us that if  $\beta^* \in E \cap \text{nacc}(C_\delta) \cap I(\epsilon, m)$  for some  $\epsilon \geq \epsilon^*$  and  $m \geq m^*$ , then there is an ordinal  $\gamma^* < \beta^*$  such that whenever  $t_\alpha$  is contained in the interval  $(\gamma^*, \beta^*)$ , for every  $\zeta \in t_\alpha$  and  $\xi \in t_\beta$  we have

$$\beta_{\rho_2^m(\delta, \xi)}^m(\zeta, \xi) = \beta^* \tag{8.5}$$

and

$$\beta_k^m(\zeta, \xi) = \beta_k^m(\delta, \xi) \quad \text{for all } k < \rho_2^m(\delta, \xi). \tag{8.6}$$

Proposition 5.3 tells us that  $I_\delta$ -almost all elements of  $C_\delta$  lie in

$$B_\delta := \text{nacc}(C_\delta) \cap \bigcup \{I(\epsilon, m) : \epsilon^* \leq \epsilon < \sigma, m^* \leq m < \omega\}.$$

Since  $A \cap E \cap C_\delta$  is  $I_\delta$ -positive, it follows that

$$A \cap B_\delta \notin I_\delta.$$

In particular, we can find  $\beta^* \in A \cap E \cap C_\delta$  such that  $\beta^* \in I(\epsilon, m)$  for some  $\epsilon \geq \epsilon^*$  and  $m \geq m^*$ , and hence for which there is an ordinal  $\gamma^* < \beta^*$  with the properties promised in the discussion surrounding (8.5) and (8.6). Our goal is to get a contradiction by establishing that  $m$  and  $\beta$  witness that this  $\beta^*$  is not an element of  $A$ .

We observe that since  $\beta^* = \sup(M_{\beta^*} \cap \lambda)$ ,  $\beta^*$  must be a member of every closed unbounded subset of  $\lambda$  that is itself an element of  $M_{\beta^*}$ . We will shortly make use of this in the context of Lemma 7.3, but for now we observe the following simple fact:

$$\alpha < \beta^* \implies t_\alpha \subseteq \beta^*. \tag{8.7}$$

For each  $\alpha < \lambda$ , we define a function  $f_\alpha^{\min}$  as follows:

$$f_\alpha^{\min}(i) = \min\{f_\zeta(i) : \zeta \in t_\alpha\}.$$

Since each  $t_\alpha$  is finite, it follows that

$$(\forall^* i < \omega)[f_\alpha^{\min}(i) = f_{\min(t_\alpha)}(i)]$$

and therefore  $\langle f_\alpha^{\min} : \alpha < \lambda \rangle$  is a scale. Since this new scale is definable from parameters available in  $M_0$ , it itself must be a member of  $M_0$  as well.

As observed earlier,  $\beta^*$  is in every closed unbounded subset of  $\lambda$  that is also an element of  $M_{\beta^*}$ . Thus, an application of Lemma 7.3 inside of  $M_{\beta^*}$  to the scale  $\langle f_\alpha^{\min} : \alpha < \lambda \rangle$  tells us that (7.2) holds for  $\beta^*$ . In particular, we can fix  $i_0 < \omega$  so that whenever  $i_0 \leq i < \omega$ , it is the case that

$$(\forall \eta < \mu_i)(\forall v < \mu_{i+1})(\exists^* \alpha < \beta^*)[f_\alpha^{\min}(i) > \eta \wedge f_\alpha^{\min}(i+1) > v]. \tag{8.8}$$

The next part of our argument is going to require some Skolem hull arguments. We start by defining

$$M := \text{Sk}_{\aleph_1}(x \cup \{\beta^*\}).$$

Note that  $x \cup \{\beta^*\} \in M_{\beta^*+1}$  (as everything except  $\beta^*$  is already in  $M_0$ ) and therefore  $M \in M_\delta$  as it is definable in  $M_\delta$  by taking the Skolem hull of  $x \cup \{\beta^*\}$  inside the structure  $M_{\beta^*+1}$ . Since  $M$  is countable and  $\mu_0$  is not, it follows that

$$\text{Ch}_M(i) = \sup(M \cap \mu_i) < \mu_i \quad \text{for all } i < \omega,$$

and therefore

$$f(i) \leq \text{Ch}_M(i) \quad \text{for all } f \in M \cap \prod_{n < \omega} \mu_n \text{ and } i < \omega. \tag{8.9}$$

Since  $\delta \in E$ , it is immediate that  $\text{Ch}_M <^* f_\delta$ , and the definition of scale tells us that  $\text{Ch}_M <^* f_\gamma$  whenever  $\delta \leq \gamma < \lambda$  as well. Since  $t_\beta$  is finite, there is an  $i_1 < \omega$  such that

$$\text{Ch}_M \upharpoonright [i_1, \omega) < f_{\beta_k^m(\delta, \xi)} \upharpoonright [i_1, \omega) \quad \text{for all } \xi \in t_\beta \text{ and } k \leq \rho_2^m(\delta, \xi). \tag{8.10}$$

Finally, choose  $i_2 < \omega$  such that  $\text{cf}(\beta^*) < \mu_{i_2}$ , and define

$$i^* = \max\{i_0, i_1, i_2\}.$$

We claim that if  $i^* \leq i < \omega$ , then

$$(\exists^* \alpha < \beta^*)(\forall \zeta \in t_\alpha)(\forall \xi \in t_\beta)[c_m(\zeta, \xi) = \beta^* \wedge \Gamma(\zeta, \xi) = i]. \tag{8.11}$$

Given such an  $i$ , let  $N := \text{Sk}_\alpha(M \cup \mu_i)$ . An application of Corollary 7.7 yields

$$\text{Ch}_N \upharpoonright [i+1, \omega) = \text{Ch}_M \upharpoonright [i+1, \omega),$$

and since  $i_1 \leq i$ , we conclude from (8.9) that for any  $\zeta \in N \cap \lambda$ ,

$$f_\zeta \upharpoonright [i+1, \omega) < f_{\beta_k^m(\delta, \xi)} \upharpoonright [i+1, \omega) \quad \text{for all } \xi \in t_\beta \text{ and } k < \rho_2^m(\delta, \xi). \tag{8.12}$$

Now define

$$\eta^* := \max\{f_{\beta_k^m(\delta, \xi)}(i) : \xi \in t_\beta \text{ and } k < \rho_2^m(\delta, \xi)\},$$

and

$$v^* = f_{\beta^*}(i+1).$$

It is clear that  $\eta^* < \mu_i$  and  $v^* < \mu_{i+1}$ . Since both of these ordinals are, along with  $\beta^*$  and the scale  $\langle f_\alpha^{\min} : \alpha < \lambda \rangle$ , elements of  $N$ , we can apply (8.8) inside  $N$  and conclude that

$$N \models (\exists^* \alpha < \beta^*)[f_\alpha^{\min}(i) > \eta^* \wedge f_\alpha^{\min}(i+1) > v^*]. \tag{8.13}$$

Our choice of  $i_2$  guarantees  $\text{cf}(\beta^*) \subseteq N$ , and therefore  $N \cap \beta^*$  is unbounded in  $\beta^*$ ; when we combine this observation with (8.13), we conclude that

$$(\exists^* \alpha < \beta^*)[\alpha \in N \wedge f_\alpha^{\min}(i) > \eta^* \wedge f_\alpha^{\min}(i+1) > v^*]. \tag{8.14}$$

The next proposition will essentially finish our proof. Recall that “ $\gamma^*$ ” refers to the ordinal below  $\beta^*$  isolated in the discussion preceding (8.5) and (8.6).

**Proposition 8.3.** *Suppose  $\alpha < \beta^*$  satisfies*

- $\alpha \in N$ ,
- $t_\alpha \subseteq (\gamma^*, \beta^*)$ ,
- $f_\alpha^{\min}(i) > \eta^* \wedge f_\alpha^{\min}(i+1) > v^*$ .

*Then for any  $\zeta \in t_\alpha$  and  $\xi \in t_\beta$ , we have*

- $\Gamma(\zeta, \xi) = i$ , and
- $c_m(\zeta, \xi) = \beta^*$ .

**Proof.** Let  $\alpha$  be as hypothesized, and choose  $\zeta \in t_\alpha$  and  $\xi \in t_\beta$ . We first show

$$\Gamma(\zeta, \beta_k^m(\zeta, \xi)) = i \quad \text{for all } k < \rho_2^m(\delta, \xi). \tag{8.15}$$

Given  $k < \rho_2^m(\delta, \xi)$ , we know that

$$\beta_k^m(\zeta, \xi) = \beta_k^m(\delta, \xi)$$

by way of (8.6). Thus,

$$f_{\beta_k^m(\zeta, \xi)}(i) = f_{\beta_k^m(\delta, \xi)}(i) \leq \eta^* < f_\alpha^{\min}(i) \leq f_\zeta(i). \tag{8.16}$$

Since  $t_\alpha \in N$  and  $t_\alpha$  is finite, it follows that  $\zeta \in N$  as well, and therefore we have

$$f_\zeta \upharpoonright [i + 1, \omega) \leq \text{Ch}_M \upharpoonright [i + 1, \omega) < f_{\beta_k^m(\delta, \xi)} \upharpoonright [i + 1, \omega) = f_{\beta_k^m(\zeta, \xi)} \upharpoonright [i + 1, \omega) \tag{8.17}$$

by way of (8.12). The conjunction of (8.16) and (8.17) tells us that  $\Gamma(\zeta, \beta_k^m(\zeta, \xi)) = i$ , and this establishes (8.15).

Since  $\beta_{\rho_2^m(\delta, \xi)}^m = \beta^*$ , we are finished if we can establish

$$\Gamma(\zeta, \beta^*) \neq i, \tag{8.18}$$

but this follows easily because we have arranged that

$$f_{\beta^*}(i + 1) = v^* < f_\alpha^{\min}(i + 1) \leq f_\zeta(i + 1). \quad \square$$

We can now finish our proof of (8.2). Since  $\alpha \leq \min(t_\alpha)$ , it follows from (8.7) that  $t_\alpha \subseteq (\gamma^*, \beta^*)$  whenever  $\alpha \in (\gamma^*, \beta^*)$ . Thus, (8.14) implies that there are unboundedly many  $\alpha < \beta^*$  satisfying the assumptions of Proposition 8.3. Since  $i$  was an arbitrary element of  $(i^*, \omega)$ , it follows that  $m$  and  $\beta$  stand witness that  $\beta^*$  is not an element of  $A$ . This contradiction completes the proof of Lemma 8.2.  $\square$

We turn now to the proof of Theorem 6. Our proof requires one more parameter – we need to fix a function  $p : \omega \rightarrow \omega$  with the property that  $p^{-1}(m)$  is infinite for all  $m < \omega$ . Given this function, we define our coloring as follows:

**Definition 8.4.** Given  $\alpha < \beta < \lambda$ , we define  $c : [\lambda]^2 \rightarrow \lambda$  by

$$c(\alpha, \beta) = c_{p(\Gamma(\alpha, \beta))}(\alpha, \beta).$$

**Proof of Theorem 6.** Suppose  $\beta^*$  is as in the conclusion of Lemma 8.2, and fix  $m$  and  $\beta$  for which (8.2) is true. We can find an  $i < \omega$  sufficiently large that

- $p(i) = m$ , and
- $(\exists^* \alpha < \beta^*)(\forall \zeta \in t_\alpha)(\forall \xi \in t_\beta)[\Gamma(\zeta, \xi) = i \wedge c_m(\zeta, \xi) = \beta^*]$ .

For such  $\alpha$ , we have

$$c(\zeta, \xi) = \beta^* \quad \text{for all } \zeta \in t_\alpha \text{ and } \xi \in t_\beta, \tag{8.19}$$

just as required by Theorem 6.  $\square$

We finish this section with a corollary whose proof involves applying a well-known trick to the coloring from Theorem 6.

**Corollary 8.5.** Let  $\theta \leq \lambda$  be a cardinal. If  $\text{id}_p(\bar{C}, \bar{I})$  is not weakly  $\theta$ -saturated, then  $\text{Pr}_1(\lambda, \lambda, \theta, \aleph_0)$  holds.

**Proof.** Suppose  $\pi : \lambda \rightarrow \theta$  is a function such that  $\pi^{-1}(\gamma)$  is  $\text{id}_p(\bar{C}, \bar{I})$ -positive for each  $\gamma < \theta$ . Define a coloring  $c^* : [\lambda]^2 \rightarrow \theta$  by

$$c^*(\alpha, \beta) = \pi(c(\alpha, \beta)).$$

Suppose now that we are given a pairwise disjoint collection  $\langle t_\alpha : \alpha < \lambda \rangle$  of finite subsets of  $\lambda$  and an ordinal  $\zeta < \theta$ . Since  $\pi^{-1}(\zeta)$  is  $\text{id}_p(\bar{C}, \bar{I})$ -positive, the conclusion of Theorem 6 tells us that we can find  $\beta^* < \lambda$  and  $\alpha < \beta < \lambda$  such that

- $\pi(\beta^*) = \zeta$ , and
- $c \upharpoonright t_\alpha \times t_\beta$  is constant with value  $\beta^*$ .

Clearly  $c^* \upharpoonright t_\alpha \times t_\beta$  is constant with value  $\zeta$ , and this establishes  $\text{Pr}_1(\lambda, \lambda, \theta, \aleph_0)$ .  $\square$

In the light of the above corollary, we see that if  $\text{id}_p(\bar{C}, \bar{I})$  is not weakly  $\mu$ -saturated, then  $\text{Pr}_1(\mu^+, \mu^+, \mu, \aleph_0)$  holds. A similar situation occurred in [9,8], and in those two cases we were able to improve things to a coloring with  $\mu^+$  colors and obtain  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ . In this paper, however, the use of  $m$ -walks brings in an extra parameter that seemingly prevents the proofs from the earlier papers from being carried out. We are still able to use a “stepping-up argument” to get an upgrade to  $\lambda$  colors, but we pay a price in that the resulting coloring is defined on triples instead of pairs.

**Theorem 7.** If  $\text{id}_p(\bar{C}, \bar{I})$  is not weakly  $\mu$ -saturated, then there exists a coloring  $d : [\lambda]^3 \rightarrow \lambda$  such that whenever  $\langle t_\alpha : \alpha < \lambda \rangle$  is a pairwise disjoint collection of finite subsets of  $\lambda$ , and  $\zeta < \lambda$ , we can find  $\alpha < \beta < \gamma$  such that

$$(\forall \epsilon \in t_\alpha)(\forall \zeta \in t_\beta)(\forall \xi \in t_\gamma)[d(\epsilon, \zeta, \xi) = \zeta]. \tag{8.20}$$

**Proof.** Our main ingredients in the definition of  $d$  are the coloring  $c$  from [Theorem 6](#), and the coloring  $c^*$  from [Corollary 8.5](#). We also fix functions  $g_\beta$  mapping  $\mu$  onto  $\beta$  whenever  $\mu \leq \beta < \lambda$ . The function  $d : [\lambda]^3 \rightarrow \lambda$  is defined by

$$d(\epsilon, \zeta, \xi) = \begin{cases} g_{c(\zeta, \xi)}(c^*(\epsilon, \xi)) & \text{if } \mu \leq c(\zeta, \xi), \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{8.21}$$

Suppose now that  $\langle t_\alpha : \alpha < \lambda \rangle$  is a pairwise disjoint family of finite subsets of  $\lambda$ . Without loss of generality, we assume  $\alpha \leq \min(t_\alpha)$  and  $\max(t_\alpha) < \min(t_\beta)$  whenever  $\alpha < \beta$ ; this can easily be arranged by passing to a subfamily of size  $\lambda$ , and such a move does not interfere with our conclusion.

[Lemma 8.2](#) and the proof of [Theorem 6](#) tell us that for  $\text{id}_p(\bar{C}, \bar{I})$ -almost all  $\eta < \lambda$ , there is value  $h(\eta) > \eta$  such that

$$(\exists^* \beta < \eta)(\forall \zeta \in t_\beta)(\forall \xi \in t_{h(\eta)})[c(\zeta, \xi) = \eta]. \tag{8.22}$$

Since  $\text{id}_p(\bar{C}, \bar{I})$  contains all the non-stationary subsets of  $\lambda$ , Fodor's Lemma implies that there is a single  $\iota < \mu$  such that for stationarily many  $\eta < \lambda$ , we have both (8.22) and that  $g_\eta(\iota)$  is defined and equal to  $\zeta$ .

Let  $T$  denote the set of all  $\eta < \lambda$  satisfying the above. By tossing away a non-stationary subset of  $T$  if necessary, we can assume that  $T$  consists of limit ordinals, and that  $t_{h(\eta^*)} \subseteq \eta$  whenever  $\eta^* < \eta$  in  $T$ . Given this, if we define

$$s_\eta = \{\eta\} \cup t_{h(\eta)}, \tag{8.23}$$

then the resulting family  $\langle s_\eta : \eta \in T \rangle$  is a pairwise disjoint collection of finite subsets of  $\lambda$ .

Since  $c^*$  witnesses  $\text{Pr}_1(\lambda, \lambda, \mu, \aleph_0)$ , we can find  $\eta^* < \eta$  in  $T$  such that  $c^*$  is constant with value  $\iota$  when restricted to  $s_{\eta^*} \times s_\eta$ . If we define  $\alpha = h(\eta^*)$  and  $\gamma = h(\eta)$ , then we achieve

$$(\forall \epsilon \in t_\alpha)(\forall \xi \in t_\gamma)[c^*(\epsilon, \xi) = \iota]. \tag{8.24}$$

Our assumptions on  $T$  imply that  $\eta = \min(s_\eta)$  and  $\max(s_\eta) < \eta$ . Since  $\eta$  satisfies (8.22), we can choose  $\beta < \eta$  such that

$$\max(t_\alpha) < \beta < \eta, \tag{8.25}$$

and

$$(\forall \zeta \in t_\beta)(\forall \xi \in t_\gamma)[c(\zeta, \xi) = \eta]. \tag{8.26}$$

It is clear that  $\alpha < \beta < \gamma$ , so we need only verify (8.20). Suppose then that  $\epsilon \in t_\alpha$ ,  $\zeta \in t_\beta$ , and  $\xi \in t_\gamma$ . Then  $c^*(\epsilon, \xi) = \iota$  and  $c(\zeta, \xi) = \eta$  by (8.26) and (8.24). The definition of  $T$  implies that  $g_\eta(\iota) = \zeta$ , and so

$$d(\epsilon, \zeta, \xi) = g_{c(\zeta, \xi)}(c^*(\epsilon, \xi)) = g_\eta(\iota) = \zeta, \tag{8.27}$$

as required.  $\square$

### 9. Conclusions

In this final section, we give a proof of our main theorem. Our goal is to combine [Theorem 2](#) with the coloring theorems from the preceding section (in the case where our singular cardinal has countable cofinality) and from [\[8\]](#) (in the case where the cofinality is uncountable). We will dispense with the assumptions that have been in force for the past few sections, in order to state things in full generality. We begin with a short summary of the main results of [\[8\]](#).

**Theorem 8.** *Let  $\mu$  be a singular cardinal of uncountable cofinality, and let  $S$  be a stationary subset of  $S_{\text{cf}(\mu)}^{\mu^+}$ . There are an  $S$ -club system  $\langle C_\delta : \delta \in S \rangle$  and a coloring  $c : [\mu^+]^2 \rightarrow \mu^+$  such that, letting  $\text{id}_p(\bar{C}, \bar{I})$  be defined as in [Definition 5.5](#), the following hold:*

- (1) *The ideal  $\text{id}_p(\bar{C}, \bar{I})$  is  $\text{cf}(\mu)$ -complete, and  $\theta$ -indecomposable for every regular cardinal  $\theta$  in the interval  $(\text{cf}(\mu), \mu)$ . (Part 1 of [Observation 3.2](#) on page 139 of [\[27\]](#).)*
- (2) *Whenever  $\langle t_\alpha : \alpha < \mu^+ \rangle$  is a pairwise disjoint collection of members of  $[\mu^+]^{<\text{cf}(\mu)}$ , for  $\text{id}_p(\bar{C}, \bar{I})$ -almost all  $\beta^* < \mu^+$ , there are  $\alpha < \beta$  such that  $c \upharpoonright t_\alpha \times t_\beta$  is constant with value  $\beta^*$  ([Theorem 2](#) of [\[8\]](#)).*
- (3) *If  $\text{id}_p(\bar{C}, \bar{I})$  is not weakly  $\theta$ -saturated for  $\theta \leq \mu^+$ , then  $\text{Pr}_1(\mu^+, \mu^+, \theta, \text{cf}(\mu))$  holds ([Corollary 19](#) of [\[8\]](#)).*
- (4) *If  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$  fails, then there is an  $\text{id}_p(\bar{C}, \bar{I})$ -positive set  $A$  such that the ideal  $I$  obtained by restricting  $\text{id}_p(\bar{C}, \bar{I})$  to  $A$  is weakly  $\theta$ -saturated for some  $\theta < \mu$  ([Lemma 23](#) of [\[8\]](#)).*

Notice that our [Theorem 6](#) gives us the second conclusion of the above theorem in the case where  $\text{cf}(\mu) = \aleph_0$ . The proof in the case where  $\text{cf}(\mu) > \aleph_0$  is much simpler because we can take advantage of the stronger club-guessing theorems known for that case. We will return to the contrast between results from this paper and those from [\[8\]](#) after we state and prove our main theorem.

**Theorem 9 (Main Theorem).** *Assume  $\mu$  is a singular cardinal.*

- (1) *If  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for some  $\theta < \mu$ , then  $\text{Pr}_1(\mu^+, \mu^+, \theta, \text{cf}(\mu))$  holds.*
- (2) *If  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for arbitrarily large  $\theta < \mu$ , then we obtain both  $\text{Pr}_1(\mu^+, \mu^+, \mu, \text{cf}(\mu))$  and  $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ .*



- (3) If  $\text{cf}(\mu) > \aleph_0$  and  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for arbitrarily large  $\theta < \mu$ , then (2) can be improved to  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ .
- (4) If  $\text{cf}(\mu) = \aleph_0$  and  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for arbitrarily large  $\theta < \mu$ , then there is a function  $d : [\mu^+]^3 \rightarrow \mu^+$  such that whenever  $(t_\alpha : \alpha < \mu^+)$  is a pairwise disjoint family of finite subsets of  $\mu^+$  and  $\zeta < \mu^+$ , there are  $\alpha < \beta < \gamma$  such that  $(\forall \epsilon \in t_\alpha)(\forall \zeta \in t_\beta)(\forall \xi \in t_\gamma)[d(\epsilon, \zeta, \xi) = \zeta]$ .

**Proof.** As far as (1) is concerned, we note that the theorem is only of interest in the case where  $\text{cf}(\mu) < \theta - \text{Pr}_1(\mu^+, \mu^+, \text{cf}(\mu), \text{cf}(\mu))$  holds for any singular cardinal by a result of Shelah (Conclusion 4.1 on page 67 of [27]). Thus, we assume  $\text{cf}(\mu) < \theta$ .

Assume  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  fails for some regular  $\theta$  with  $\text{cf}(\mu) < \theta < \mu$ . If  $\mu$  has uncountable cofinality, then the ideal  $\text{id}_p(\bar{C}, \bar{I})$  mentioned in Theorem 8 cannot be weakly  $\theta$ -saturated – if it were, then part (1) of Theorem 8 taken with Theorem 2 would give us  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$ . Our result now follows by part (3) of Theorem 8.

What about the case where  $\text{cf}(\mu) = \aleph_0$ ? In this case, our assumptions give us a finite sequence  $\langle S_i : i < n \rangle$  of stationary subsets of  $S_{\geq \theta}^{\mu^+}$  which fail to reflect simultaneously. Without loss of generality, we can assume that  $S_i \subseteq S_{\tau_i}^{\mu^+}$  for some regular cardinal  $\tau_i$ . This allows us to choose a regular cardinal  $\sigma$  such that  $\max\{\tau_i : i < n\} < \sigma < \mu$ .

Let  $S = S_\sigma^{\mu^+}$ . An application of Theorem 4 gives us an  $S$ -club system  $\bar{C}$  for which Theorem 6 holds. The corresponding ideal  $\text{id}_p(\bar{C}, \bar{I})$  is  $\tau_i$ -indecomposable for  $i < n$  by Proposition 5.6, and so Theorem 2 lets us conclude that  $\text{id}_p(\bar{C}, \bar{I})$  cannot be weakly  $\theta$ -saturated – if it were, then our sets  $S_i$  would be stationary subsets of  $S^*(\text{id}_p(\bar{C}, \bar{I}))$  and thus they would reflect simultaneously. We now obtain  $\text{Pr}_1(\mu^+, \mu^+, \theta, \text{cf}(\mu))$  by way of Corollary 8.5.

As far as part (2) of our theorem goes, note that Theorem 3 gives us the square-brackets part of the conclusion already. We will prove the remainder by contrapositive, focusing only on the case where  $\mu$  is of countable cofinality because part (3) will take care of the other case.

Thus, assume  $\text{cf}(\mu) = \aleph_0$  and  $\text{Pr}_1(\mu^+, \mu^+, \mu, \text{cf}(\mu))$  fails. Apply Theorem 4 with  $S = S_{\aleph_1}^{\mu^+}$  and obtain an  $S$ -club system  $\bar{C}$  for which Theorem 6 holds. Corollary 8.5 implies that  $\text{id}_p(\bar{C}, \bar{I})$  must be weakly  $\mu$ -saturated, and Proposition 5.6 tells us that the ideal is  $\tau$ -indecomposable for every regular  $\tau$  lying between  $\aleph_1$  and  $\mu$ .

An elementary argument by contradiction establishes the existence of a regular  $\theta < \mu$  (without loss of generality greater than  $\aleph_1$ ) and an  $\text{id}_p(\bar{C}, \bar{I})$ -positive set  $A$  such that the ideal  $I = \{B \subseteq \mu^+ : A \cap B \in \text{id}_p(\bar{C}, \bar{I})\}$  is weakly  $\theta$ -saturated. This ideal  $I$  is also  $\tau$ -indecomposable for any regular  $\tau$  with  $\aleph_1 < \tau < \mu$  – it inherits this property from  $\text{id}_p(\bar{C}, \bar{I})$ . Thus,  $S^*(I)$  is equal to  $S_{\geq \theta}^{\mu^+}$  modulo the non-stationary ideal, and  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  follows by Theorem 2.

To finish, we note that parts (3) and (4) of our main theorem follow by exactly the same argument – for part (3) we take advantage of conclusion 4 of Theorem 8, and for part (4), we use Theorem 7 in the previous section.  $\square$

We conclude our paper with a general discussion of some of the issues raised by the research presented here. First and foremost, it should be clear that we obtain simultaneous reflection almost by accident – it seems that a much more important phenomenon is isolated in Theorem 6 (and Theorem 2 of [8]), where we get in ZFC a coloring theorem intertwined with an easily describable ideal. In the presence of square-brackets partition relations, the ideal in question must possess some large cardinal-type properties and it is not clear at this point if this is even possible:

**Question 2.** Suppose  $\mu$  is a singular cardinal. Is it consistent (relative to large cardinals) that ideals of the form  $\text{id}_p(\bar{C}, \bar{I})$  can be weakly  $\theta$ -saturated for some  $\theta \leq \mu$ ?

If  $2^\mu = \mu^+$ , then an old theorem of Erdős, Hajnal, and Rado [11] tells us that  $\mu^+ \not\rightarrow [\mu^+]_{\mu^+}^2$  holds. This motivates the following question, which we phrase in very specific terms:

**Question 3.** Suppose  $\mu$  is singular strong limit cardinal of countable cofinality. Is it consistent (relative to large cardinals) that  $2^\mu > \mu^+$  and there is a uniform ultrafilter  $U$  on  $\mu^+$  that is  $\theta$ -indecomposable for all uncountable regular  $\theta < \mu$ ?

The preceding question may be quite tractable – for example, if we ignore the cardinal arithmetic aspect of the question, then Ben-David and Magidor [2] have shown that such a filter can exist on  $\aleph_\omega$ . We are not sure of the extent to which the need for  $2^\mu$  to be greater than  $\mu^+$  complicates things.

Another somewhat bothersome point is the difference between parts (3) and (4) of our main theorem. This discrepancy may be resolvable by an easy argument, but as of yet we do not see how to do it. The general question is as follows:

**Question 4.** Suppose  $\mu$  is a singular cardinal for which  $\text{Pr}_1(\mu^+, \mu^+, \mu, \text{cf}(\mu))$  holds. Does  $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$  hold?

Finally, having established that square-brackets-type relations at successors of singular cardinals necessarily entail simultaneous reflection of stationary sets, we would like to have information about the cofinalities of the ordinals where the reflection takes place:

**Question 5.** Suppose  $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$  for  $\mu$  singular, and let  $\theta < \mu$  be a regular cardinal for which  $\text{Refl}(< \text{cf}(\mu), S_{\geq \theta}^{\mu^+})$  holds. Can we say anything interesting about the cofinalities of ordinals where the simultaneous reflection takes place?

The answer to the above depends on getting information about the cofinalities of ordinals in the range of  $f^*$ , where  $f^*$  is as in Theorem 2.

## 10. Final note

There has been some progress on these questions in the months since this paper was submitted. In particular, the answer to Question 4 is “yes”, and both statements are equivalent to demanding that  $\text{Pr}_1(\mu^+, \mu^+, \theta, \text{cf}(\mu))$  holds for arbitrarily large  $\theta < \mu$ . The proof will appear in [6].

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## References

- [1] James E. Baumgartner, A new class of order types, *Ann. Math. Logic* 9 (3) (1976) 187–222.
- [2] Shai Ben-David, Menachem Magidor, The weak  $\square^*$  is really weaker than the full  $\square$ , *J. Symb. Logic* 51 (4) (1986) 1029–1033.
- [3] C.C. Chang, Descendingly incomplete ultrafilters, *Trans. Amer. Math. Soc.* 126 (1967) 108–118.
- [4] James Cummings, Notes on singular cardinal combinatorics, *Notre Dame J. Formal Logic* 46 (3) (2005) 251–282 (electronic).
- [5] James Cummings, Matthew Foreman, Menachem Magidor, Squares, scales and stationary reflection, *J. Math. Log.* 1 (1) (2001) 35–98.
- [6] Todd Eisworth, A coloring theorem for successors of singular cardinals, *J. Symbolic Logic* December (2009) (submitted for publication).
- [7] Todd Eisworth, Successors of singular cardinals, in: Matt Foreman, Akihiro Kanamori (Eds.), *The Handbook of Set Theory*, Springer, ISBN: 978-1-4020-4843-2, 2010 (Chapter 15).
- [8] Todd Eisworth, A note on strong negative partition relations, *Fund. Math.* 202 (2009) 97–123.
- [9] Todd Eisworth, Saharon Shelah, Successors of singular cardinals and coloring theorems I, *Arch. Math. Logic* 44 (2005) 597–618.
- [10] Todd Eisworth, Saharon Shelah, Successors of singular cardinals and coloring theorems II, *J. Symbolic Logic* 74 (4) (2010) 1287–1309.
- [11] P. Erdős, A. Hajnal, R. Rado, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.* 16 (1965) 93–196.
- [12] Paul Erdős, András Hajnal, Attila Máté, Richard Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, in: *Studies in Logic and the Foundations of Mathematics*, vol. 106, North-Holland Publishing Co., Amsterdam, 1984.
- [13] Thomas Jech, Stationary sets, in: Matt Foreman, Akihiro Kanamori (Eds.), *The Handbook of Set Theory*, Springer, ISBN: 978-1-4020-4843-2, 2010, pp. 93–128 (Chapter 1).
- [14] Thomas Jech, *Set Theory*, in: *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.
- [15] A. Kanamori, Weakly normal filters and irregular ultrafilters, *Trans. Amer. Math. Soc.* 220 (1976) 393–399.
- [16] Akihiro Kanamori, The Higher Infinite, in: *Perspectives in Mathematical Logic*, Springer-Verlag, Berlin, 1994, Large cardinals in set theory from their beginnings.
- [17] Jussi Ketonen, Strong compactness and other cardinal sins, *Ann. Math. Logic* 5 (1972/73) 47–76.
- [18] Jussi Ketonen, Nonregular ultrafilters and large cardinals, *Trans. Amer. Math. Soc.* 224 (1976) 61–73.
- [19] Menachem Kojman, The abc of pcf (unpublished manuscript).
- [20] Kenneth Kunen, *Set Theory*, in: *Studies in Logic and the Foundations of Mathematics*, vol. 102, North-Holland Publishing Co., Amsterdam, 1980, An introduction to independence proofs.
- [21] Kenneth Kunen, Karel Prikry, On descendingly incomplete ultrafilters, *J. Symbolic Logic* 36 (1971) 650–652.
- [22] Menachem Magidor, Reflecting stationary sets, *J. Symbolic Logic* 47 (4) (1983) 755–771. 1982.
- [23] Justin Tatch Moore, A solution to the  $L$  space problem, *J. Amer. Math. Soc.* 19 (3) (2006) 717–736 (electronic).
- [24] S. Shelah, Jonsson algebras in successor cardinals, *Israel J. Math.* 30 (1–2) (1978) 57–64.
- [25] Saharon Shelah, On successors of singular cardinals, in: *Logic Colloquium'78 (Mons, 1978)*, in: *Stud. Logic Foundations Math.*, vol. 97, North-Holland, Amsterdam, 1979, pp. 357–380.
- [26] Saharon Shelah, Was Sierpiński right? I, *Israel J. Math.* 62 (3) (1988) 355–380.
- [27] Saharon Shelah, Cardinal Arithmetic, in: *Oxford Logic Guides*, vol. 29, Oxford University Press, 1994.
- [28] Saharon Shelah, Colouring and non-productivity of  $\aleph_2$ -c.c, *Ann. Pure Appl. Logic* 84 (2) (1997) 153–174.
- [29] Stevo Todorčević, Partitioning pairs of countable ordinals, *Acta Math.* 159 (3–4) (1987) 261–294.
- [30] Stevo Todorčević, Walks on Ordinals and their Characteristics, in: *Progress in Mathematics*, vol. 263, Birkhäuser Verlag, Basel, 2007.
- [31] Jan Tryba, On Jónsson cardinals with uncountable cofinality, *Israel J. Math.* 49 (4) (1984) 315–324.