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The complete superconformal index for $\mathcal{N} = 6$ Chern–Simons theory

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Abstract

We calculate the superconformal index for $\mathcal{N} = 6$ Chern–Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$ at arbitrary allowed value of the Chern–Simons level k. The calculation is based on localization of the path integral for the index. Our index counts supersymmetric gauge invariant operators containing inclusions of magnetic monopole operators, where latter operators create magnetic fluxes on 2-sphere. Through analytic and numerical calculations in various sectors, we show that our result perfectly agrees with the index over supersymmetric gravitons in $AdS_4 \times S^7/\mathbb{Z}_k$ in the large N limit. Monopole operators in nontrivial representations of $U(N) \times U(N)$ play important roles. We also comment on possible applications of our methods to other superconformal Chern–Simons theories.

1. Introduction

An important problem in AdS/CFT [1] is to understand the Hilbert spaces of both sides. The string or M-theory is put on global AdS, and the dual conformal field theory (CFT) is radially quantized. Partition function encodes the information on Hilbert space. In particular, one has to understand the spectrum of strongly interacting CFT, which is in general difficult, to use the dual string/M-theory to study various phenomena in conventional gravity.

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With supersymmetry, one can try to circumvent this difficulty by considering quantities which contain possibly less information than the partition function but do not depend on (or depend much more mildly on) the coupling constants controlling the interaction.

This has been considered in the context of AdS/CFT. If a superconformal theory has *continuous* parameters, one can construct a function called the superconformal index which does not depend on changes of them [2,3]. The general structure of the superconformal index was investigated in 4 dimension [2], and then in 3, 5, 6 dimensions [3]. See also [4]. In all these cases, the superconformal index is essentially the Witten index [5] and acquires nonzero contribution only from states preserving supersymmetry. The superconformal index was computed for a class of SCFT_4 in [2,6], including the $\mathcal{N}=4$ Yang–Mills theory. Similar quantity called the elliptic genus was also studied in 2-dimensional SCFT [7]. The latter index played a major role in understanding supersymmetric black holes in $\mathrm{AdS}_3/\mathrm{CFT}_2$ [8].

In this paper, we study the superconformal index in $\mathrm{AdS}_4/\mathrm{CFT}_3$.

Recently, based on the idea of using superconformal Chern–Simons theory [9] to describe low energy dynamics of M2-branes, and after the first discovery of a class of $\mathcal{N}=8$ superconformal Chern–Simons theories [10,11], $\mathcal{N}=6$ superconformal Chern–Simons theory with gauge group $U(N)_k \times U(N)_{-k}$ has been found and studied, where the integers k and $-k$ denote the Chern–Simons levels associated with two gauge groups [12]. This theory describes the low energy dynamics of N parallel M2-branes placed at the tip of $\mathbb{C}^4/\mathbb{Z}_k$, and is proposed to be dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. See [13–16] for further studies of this theory.

Various tests have been made for this proposal. For instance, studies of the moduli space [12] and $D2$ branes [12,17], chiral operators [12,18], higher derivative correction in the broken phase [19,20] (see also [21]), integrability and nonsupersymmetric spectrum [22] have been made. Many of these tests, perhaps except the last example above, rely on supersymmetry in some form.

One of the most refined application of supersymmetry to test this duality so far would be the calculation of the superconformal index in the type IIA limit [23]. See also [24,25]. The superconformal index was calculated in the 't Hooft limit, in which N and k are taken to be large keeping $\lambda = \frac{N}{k}$ finite. The authors of [23] argue that λ can be regarded as a continuous parameter in the 't Hooft limit. The index calculated in the free Chern–Simons theory, $\lambda \to 0$, is expected to be the same as the index in the opposite limit $\lambda \gg 1$ from the continuity argument. M-theory can be approximated by type IIA supergravity on $AdS_4 \times \mathbb{CP}^3$ in the latter limit, where $\mathbb{CP}^3$ appears as the base space in the Hopf fibration of $S^7/\mathbb{Z}_k$ [12]. The index over multiple type IIA gravitons perfectly agreed with the gauge theory result [23].

It is tempting to go beyond the 't Hooft limit and calculate the full superconformal index for any value of k (and N) for this theory. This is the main goal of this paper. The general index is expected to capture the contribution from states carrying Kaluza–Klein (KK) momenta along the fiber circle of $S^7/\mathbb{Z}_k$, or $D0$ brane charges from type IIA point of view. As k increases, the radius of the circle decreases as $\frac{1}{k}$, and the energy of the KK states would grow. So even when k is very large that weakly coupled type IIA string theory is reliable, our index captures non-perturbative correction to [23] from heavy $D0$ branes.

The gauge theory dual to the KK-momentum is argued to be appropriate magnetic flux on S^2 [12]. The gauge theory operators creating magnetic fluxes are called the monopole operators, or 't Hooft operators [12,26–28]. These operators are not completely understood to date.

Just like the ordinary partition function at finite temperature, the superconformal index for a radially quantized SCFT admits a path integral representation. In this paper we calculate the index from this path integral for the $\mathcal{N}=6$ Chern–Simons theory on Euclidean $S^2 \times S^1$. The index in the sector with nonzero KK-momentum is given by integrating over configurations carrying nonzero magnetic fluxes, which will turn out to be fairly straightforward. Therefore, lack of our understanding on 't Hooft operators will not cause a problem for us. In fact, monopole operators have been most conveniently studied in radially quantized theories [28].

Our computation is based on the fact that this path integral is 'supersymmetric,' or has a fermionic symmetry. As a Witten index, the superconformal index acquires contribution from states preserving a particular pair of supercharges which are mutually Hermitian conjugates. Calling one of them Q, which is nilpotent, $Q^2 = 0$, the fermionic symmetry of the integral is associated with Q. An integral of this kind can be computed by localization. See [29] and related references therein for a comprehensive discussion. A simple way of stating the idea is that one can deform the integrand by adding a Q-exact term QV in the measure, for any gauge invariant expression V, without changing the integral. For a given V, one can add tQV to the action $S \to S + tQV$ where t is a continuous parameter. With a favorable choice of V as will be explained later, t can be regarded as a continuous coupling constant of the deformed action admitting a 'free' theory limit as $t \to \infty$. As already mentioned in [2], it suffices for the deformed action to preserve only a subset of the full superconformal symmetry, involving Q and symmetries associated with charges with which we grade the states in the index. We only take advantage of a nilpotent symmetry rather than full superconformal symmetry.

As in [2,30–32], our result is given by an integral of appropriate unitary matrices.

We use our superconformal index to provide a nontrivial test of the $\mathcal{N} = 6$ AdS/CFT proposal. The readers may also regard it as subjecting our calculation to a test against known results from gravity, if they prefer to. In the large N limit, still keeping k finite, our index is expected to be the index over supersymmetric gravitons of M-theory at low energy. In the sectors with one, two and three units of magnetic fluxes (KK-momenta), we provide analytic calculations or evaluate the unitary matrix integral numerically, up to a fairly nontrivial order, to see that the two indices perfectly agree. Similar comparisons with gravitons in dual string theories are made in other dimensions, e.g. for the 2-dimensional elliptic genus index [33] and also for the 4-dimensional index in $\mathcal{N} = 4$ Yang–Mills theory [2]. We will find that monopole operators in nontrivial representations of $U(N) \times U(N)$, beyond those studied in [12,26,27], play crucial roles for the two indices to agree.

An interesting question would be whether our index captures contribution from supersymmetric black holes beyond the low energy limit. In AdS_3/CFT_2, the contribution to the elliptic genus index from BTZ black holes is calculated and further discussed [8]. See also [34] for a recent study of elliptic genus beyond gravitons. However, in 4 dimension, it has been found that the index does not capture the contribution from supersymmetric black holes [2] in the large N limit, possibly due to a cancelation between bosonic and fermionic states. In this paper, following [2], we consider the large N limit in which chemical potentials are set to order 1 (in the unit given by the radius of S^2). The situation here is somewhat similar to $d = 4$ in that a deconfinement phase transition at order 1 temperature like [30–32] is not found. However, more comment is given in the conclusion.

The methods developed in this paper can be applied to other superconformal Chern–Simons theories. $\mathcal{N} = 5$, $\mathcal{N} = 4$ Chern–Simons-matter theories and the gravity duals of some of them are studied in [14,16] and [35], respectively. There is an abundance of interesting superconformal Chern–Simons theories with $\mathcal{N} \leqslant 3$ supersymmetry. [36] provides a basic framework. For example, some $\mathcal{N} = 3, 2$ theories are presented in [37] with hyper-Kähler and Calabi–Yau moduli spaces. Comments on possible applications of our index to these theories are given in conclusion.

The rest of this paper is organized as follows. In Section 2 we summarize some aspects of $\mathcal{N}=6$ Chern–Simons theory and the superconformal index. We also set up the index calculation and explain our results. In Section 3 we consider a large N limit and compare our result with the index of M-theory gravitons. Section 4 concludes with comments and further directions. Most of the detailed calculation is relegated to Appendices A and B. Appendix C summarizes the index over M-theory gravitons.

2. Superconformal index for $\mathcal{N}=6$ Chern–Simons theory

2.1. The theory

The action and supersymmetry of $\mathcal{N}=6$ Chern–Simons-matter theory are presented and further studied in [12–15]. The Poincare and special supercharges form vector representations of $SO(6)$ R-symmetry, or equivalently rank 2 antisymmetric representations of $SU(4)$ with reality conditions:

$$Q_{IJ\alpha}=\frac{1}{2}\epsilon_{IJKL}\bar{Q}^{KL}_{\alpha},\qquad S^{IJ}_{\alpha}=\frac{1}{2}\epsilon^{IJKL}\bar{S}_{KL\alpha},\tag{2.1}$$

where $I,J,K,L=1,2,3,4$ and $\alpha=\pm$. Under radial quantization, the special supercharges are Hermitian conjugate to the Poincare supercharges: $S^{IJ\alpha}=(Q_{IJ\alpha})^{\dagger}$, $\bar{S}^{\alpha}_{IJ}=(\bar{Q}^{IJ}_{\alpha})^{\dagger}$. There are two (Hermitian) gauge fields A_μ, $\tilde{A}_\mu$ for $U(N)\times U(N)$. The matter fields are complex scalars and fermions in $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$, respectively. We write them as C_I and Ψ^I_α. They are all in the bifundamental representation $(\mathbf{N},\bar{\mathbf{N}})$ of $U(N)\times U(N)$.

In this paper we are interested in the superconformal index associated with a special pair of supercharges. We pick $Q\equiv Q_{34-}$ and $S\equiv S^{34-}$ without losing generality. For our purpose, it is convenient to decompose the fields in super-multiplets of $d=3$, $\mathcal{N}=2$ supersymmetry generated by $Q_\alpha\equiv Q_{34\alpha}$. Writing the matter fields as $C_I=(A_1,A_2,\bar{B}^{\dot 1},\bar{B}^{\dot 2})$ and $\Psi^I=(-\psi_2,\psi_1,-\bar{\chi}^{\dot 2},\bar{\chi}^{\dot 1})$, they group into 4 chiral multiplets as

$$(A_a,\psi_{a\alpha})\quad\text{in }(\mathbf{N},\bar{\mathbf{N}}),\qquad (B_{\dot a},\chi_{\dot a\alpha})\quad\text{in }(\bar{\mathbf{N}},\mathbf{N}),\tag{2.2}$$

where $a=1,2$ and $\dot a=\dot 1,\dot 2$ are doublet indices for $SU(2)\times SU(2)\subset SU(4)$ commuting with Q_α. The global charges of the fields and supercharges are presented in Table 1. h_1,h_2,h_3 are three Cartans of $SO(6)$ in the 'orthogonal 2-planes' basis, $\frac{1}{2}(h_1\pm h_2)$ being the Cartans of the above $SU(2)\times SU(2)$. j_3 is the Cartan of $SO(3)\subset SO(3,2)$. ϵ is the energy in radial quantization, or the scale dimension of operators. h_4 is the baryon-like charge commuting with the $\mathcal{N}=6$ superconformal group $Osp(6|4)$.

The Lagrangian is presented, among others, in [12,13]. It is convenient to introduce auxiliary fields λ_α,σ and $\tilde{\lambda}_\alpha,\tilde{\sigma}$ which form vector multiplets together with gauge fields. We closely follow the notation of [13]. The action is given by

$$\mathcal{L}=\mathcal{L}_{\rm CS}+\mathcal{L}_m,\tag{2.3}$$

where the Chern–Simons term is given by[1]

[1] Fields are related as $(A_\mu,\tilde{A}_\mu)_{\rm ours}=-(A_\mu,\hat{A}_\mu)_{\rm theirs}$, $(\sigma,D,\tilde{\sigma},\tilde{D})_{\rm ours}=(\sigma,D,\hat{\sigma},\hat{D})_{\rm theirs}$, $(\lambda_\alpha,\tilde{\lambda}_\alpha)_{\rm ours}=(\chi_\alpha,\hat{\chi}_\alpha)_{\rm theirs}$, $(A_a,B_{\dot a})=(Z^A,W_A)$, $\frac{k}{4\pi}=2K$.

Table 1
Charges of fields and supercharges.

Fields	h_1	h_2	h_3	j_3	ϵ	h_4
(A_1, A_2)	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2})$	0	$\frac{1}{2}$	$\frac{1}{2}$
$(B_{\dot{1}}, B_{\dot{2}})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2})$	0	$\frac{1}{2}$	$-\frac{1}{2}$
$(\psi_{1\pm}, \psi_{2\pm})$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$\pm\frac{1}{2}$	1	$\frac{1}{2}$
$(\chi_{\dot{1}\pm}, \chi_{\dot{2}\pm})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$\pm\frac{1}{2}$	1	$-\frac{1}{2}$
$A_\mu, \tilde{A}_\mu$	0	0	0	$(1, 0, -1)$	1	0
$\lambda_\pm, \tilde{\lambda}_\pm$	0	0	-1	$\pm\frac{1}{2}$	$\frac{3}{2}$	0
$\sigma, \tilde{\sigma}$	0	0	0	0	1	0
$Q_\pm$	0	0	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0
$S^\pm$	0	0	-1	$\mp\frac{1}{2}$	$-\frac{1}{2}$	0

$$\begin{aligned}\mathcal{L}_{\mathrm{CS}} = {}& \frac{k}{4\pi}\operatorname{tr}\left(A \wedge dA - \frac{2i}{3}A^3 + i\bar{\lambda}\lambda - 2D\sigma\right) \\ & - \frac{k}{4\pi}\operatorname{tr}\left(\tilde{A} \wedge d\tilde{A} - \frac{2i}{3}\tilde{A}^3 + i\bar{\tilde{\lambda}}\tilde{\lambda} - 2\tilde{D}\tilde{\sigma}\right)\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\mathcal{L}_m = \operatorname{tr}\big[& -D_\mu \bar{A}^a D^\mu A_a - D_\mu \bar{B}^{\dot{a}} D^\mu B_{\dot{a}} - i\bar{\psi}^a \gamma^\mu D_\mu \psi_a - i\bar{\chi}^{\dot{a}} \gamma^\mu D_\mu \chi_{\dot{a}} \\ & - (\sigma A_a - A_a\tilde{\sigma})\big(\bar{A}^a\sigma - \tilde{\sigma}\bar{A}^a\big) - (\tilde{\sigma}B_{\dot{a}} - B_{\dot{a}}\sigma)\big(\bar{B}^{\dot{a}}\tilde{\sigma} - \sigma\bar{B}^{\dot{a}}\big) \\ & + \bar{A}^a D A_a - A_a \tilde{D}\bar{A}^a - B_{\dot{a}} D \bar{B}^{\dot{a}} + \bar{B}^{\dot{a}}\tilde{D}B_{\dot{a}} \\ & - i\bar{\psi}^a \sigma \psi_a + i\psi_a \tilde{\sigma}\bar{\psi}^a + i\bar{A}^a \lambda \psi_a + i\bar{\psi}^a \bar{\lambda} A_a - i\psi_a \tilde{\lambda}\bar{A}^a - iA_a \bar{\tilde{\lambda}}\bar{\psi}^a \\ & + i\chi_{\dot{a}}\sigma\bar{\chi}^{\dot{a}} - i\bar{\chi}^{\dot{a}}\tilde{\sigma}\chi_{\dot{a}} - i\chi_{\dot{a}}\lambda\bar{B}^{\dot{a}} - iB_{\dot{a}}\bar{\lambda}\bar{\chi}^{\dot{a}} + i\bar{B}^{\dot{a}}\tilde{\lambda}\chi_{\dot{a}} + i\bar{\chi}^{\dot{a}}\bar{\tilde{\lambda}}B_{\dot{a}}\big] + \mathcal{L}_{\mathrm{sup}}.\end{aligned} \tag{2.5}$$

$\mathcal{L}_{\mathrm{sup}}$ contains scalar potential and Yukawa interaction obtained from a superpotential

$$W = -\frac{2\pi}{k}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\operatorname{tr}(A_a B_{\dot{a}} A_b B_{\dot{b}}) \tag{2.6}$$

where the fields $A_a, B_{\dot{a}}$ in the superpotential are understood as chiral superfields $A_a + \sqrt{2}\theta\psi_a + \theta^2 F_{A_a}$ and $B_{\dot{a}} + \sqrt{2}\theta\chi_{\dot{a}} + \theta^2 F_{B_{\dot{a}}}$. Integrating out the auxiliary fields, one can easily obtain the expressions for $\sigma, \tilde{\sigma}, \lambda, \tilde{\lambda}$ in terms of the matter fields.

The $\mathcal{N} = 2$ supersymmetry transformation under $Q_\alpha = Q_{34\alpha}$ can be obtained from the superfields. Most importantly,

$$\begin{aligned}& Q_\alpha \psi_{a\beta} = \sqrt{2}\epsilon_{\alpha\beta}\partial_{\bar{A}^a}\bar{W}, \qquad Q_\alpha \chi_{\dot{a}\beta} = \sqrt{2}\epsilon_{\alpha\beta}\partial_{\bar{B}^{\dot{a}}}\bar{W}, \\ & Q_\alpha \bar{\psi}^a_\beta = -\sqrt{2}i\big(\gamma^\mu\big)_{\alpha\beta} D_\mu \bar{A}^a + \sqrt{2}i\epsilon_{\alpha\beta}\big(\tilde{\sigma}\bar{A}^a - \bar{A}^a\sigma\big), \\ & Q_\alpha \bar{\chi}^{\dot{a}}_\beta = -\sqrt{2}i\big(\gamma^\mu\big)_{\alpha\beta} D_\mu \bar{B}^{\dot{a}} + \sqrt{2}i\epsilon_{\alpha\beta}\big(\sigma\bar{B}^{\dot{a}} - \bar{B}^{\dot{a}}\tilde{\sigma}\big), \\ & Q_\alpha \lambda_\beta = -\sqrt{2}i\big[\big(\gamma^\mu\big)_{\alpha\beta}(D_\mu\sigma + i\star F_\mu) + \epsilon_{\alpha\beta}D\big], \\ & Q_\alpha \bar{\lambda}_\beta = 0\end{aligned} \tag{2.7}$$

where $\epsilon^{012} = 1$. Following [13], we choose $(\gamma^\mu)_\alpha{}^\beta = (i\sigma^2, \sigma^1, \sigma^3)$ so that $(\gamma^\mu)_{\alpha\beta} = (-1, -\sigma^3, \sigma^1)$.

We will be interested in the Euclidean version of this theory. The action and supersymmetry transformation can easily be changed to the Euclidean one by Wick rotation, i.e. by replacements $x^0 = -ix_E^0$, $A_0 = i(A_E)_0$, etc. Note that D, playing the role of Lagrange multiplier in (2.4), (2.5), should be regarded as an imaginary field. After Wick rotation one obtains $\gamma_E^\mu = (-\sigma^2, \sigma^1, \sigma^3)$. The spinors, say, ψ_a and $\bar\psi^a$ are no longer complex conjugates to each other. The notation of [13] naturally lets us regard them as independent chiral and anti-chiral spinors ψ_α and $\bar\psi_{\dot\alpha}$ in Euclidean 4 dimension, reduced down to $d = 3$. Indeed, upon identifying $\sigma = A_3$, etc., the kinetic term plus the coupling to $\sigma, \tilde\sigma$ can be written as

$$i\bar\psi^{a\alpha}\left(\gamma_E^\mu\right)_\alpha{}^\beta D_\mu\psi_{a\beta} + \bar\psi^{a\alpha} D_3\psi_{a\alpha} \equiv -\bar\psi_\alpha^a\left(\bar\sigma^\mu\right)^{\alpha\beta} D_\mu\psi_{a\beta} \tag{2.8}$$

in 4-dimensional notation, where $D_3\psi_a \equiv -i\sigma\psi_a + i\psi_a\tilde\sigma$ and $(\bar\sigma^\mu)^{\alpha\beta} \equiv i\epsilon^{\alpha\gamma}(\gamma_E^\mu, -i)_\gamma{}^\beta = (1, i\sigma^3, -i\sigma^1, i\sigma^2)^{\alpha\beta}$. In our computation in appendices, it will be more convenient to choose a new $SO(3)$ frame for spinors so that

$$\bar\sigma^\mu = (1, -i\vec\sigma) = \left(1, -i\sigma^1, -i\sigma^2, -i\sigma^3\right). \tag{2.9}$$

To avoid formal manipulations in the main text being a bit nasty, this change of frame will be assumed only in appendices. Similar rearrangement can be made for $\chi_{\dot a}, \bar\chi^{\dot a}$.

In Euclidean theory, $Q_\alpha\lambda_\beta$ in (2.7) is given by

$$Q_\alpha\lambda_\beta = -\sqrt{2}i\left[\left(\gamma_E^\mu\right)_{\alpha\beta}(D_\mu\sigma - \star F_\mu) + \epsilon_{\alpha\beta}D\right]. \tag{2.10}$$

Configurations preserving two supercharges Q_α are described by the Bogomolnyi equations $(\star F)_\mu = D_\mu\sigma$ and $D = 0$. The first one is the BPS equation for magnetic monopoles in Yang–Mills theory, with a difference that σ is a composite field here. See [19] for related discussions. We shall shortly deform the theory with a Q-exact term. σ will not be a composite field then.

A conformal field theory defined on $\mathbb{R}^{d+1}$ can be radially quantized to a theory living on $S^d \times \mathbb{R}$, where $\mathbb{R}$ denotes time. The procedures of radial quantization are summarized in Appendix A. See also [38,39] for related discussions.

In the radially quantized theory, one can consider configurations in which nonzero magnetic flux is applied on spatial S^2. From the representations of matter fields under $U(N) \times U(N)$, one finds that $\operatorname{tr} F = \operatorname{tr} \tilde F$ should be satisfied. The Kaluza–Klein momentum in the dual M-theory along the fiber circle of $S^7/\mathbb{Z}_k$ is given by

$$P = \frac{k}{4\pi}\int_{S^2} \operatorname{tr} F = \frac{k}{4\pi}\int_{S^2} \operatorname{tr} \tilde F \in \frac{k}{2}\mathbb{Z} \tag{2.11}$$

in the gauge theory [12]. This, via Gauss' law constraint, turns out to be proportional to h_4 in Table 1 [12].

2.2. The superconformal index and localization

The superconformal symmetry of this theory is $Osp(6|4)$, whose bosonic generators form $SO(6) \times SO(3,2)$. Its Cartans are given by five charges: h_1, h_2, h_3 and ϵ, j_3 in $SO(2) \times SO(3) \subset SO(3,2)$. Some important algebra involving our special supercharges is

$$Q^2 = S^2 = 0, \qquad \{Q, S\} = \epsilon - h_3 - j_3. \tag{2.12}$$

The first equation says Q, S are nilpotent, while the second one implies the BPS energy bound $\epsilon \geqslant h_3 + j_3$. The special supercharges Q, S are charged under some Cartans. From Table 1, four combinations $h_1, h_2, \epsilon + j_3, \epsilon - h_3 - j_3$ commute with Q, S. The last is nothing but $\{Q, S\}$. The superconformal index for a pair of supercharges Q, S is given by [3,23]

$$I(x, y_1, y_2) = \mathrm{Tr}\big[(-1)^F e^{-\beta'\{Q,S\}} e^{-\beta(\epsilon+j_3)} e^{-\gamma_1 h_1 - \gamma_2 h_2}\big] \tag{2.13}$$

where $x \equiv e^{-\beta}$, $y_1 \equiv e^{-\gamma_1}$, $y_2 \equiv e^{-\gamma_2}$. F is the fermion number. The charges we use to grade the states in the index should commute with Q and S [2]. As a Witten index, this function does not depend on β' since it gets contribution only from states annihilated by Q and S.

The above index admits a path integral representation on $S^2 \times S^1$, where the radius of the last circle is given by the inverse temperature $\beta + \beta'$. Had the operator inserted inside the trace been $e^{-(\beta+\beta')\epsilon}$, the measure of the integral would have been given by the Euclidean action with the identification $\epsilon = -\frac{\partial}{\partial\tau}$ with Euclidean time τ. The insertion of $(-1)^F$ would also have made all fields periodic in $\tau \sim \tau + (\beta + \beta')$. The actual insertion $(-1)^F e^{-(\beta+\beta')\epsilon-(\beta-\beta')j_3+\beta' h_3-\gamma_1 h_1-\gamma_2 h_2}$ twists the boundary condition: alternatively, this twist can be undone by replacing all time derivatives in the action by

$$\partial_\tau \to \partial_\tau - \frac{\beta - \beta'}{\beta + \beta'} j_3 + \frac{\beta'}{\beta + \beta'} h_3 - \frac{\gamma_1}{\beta + \beta'} h_1 - \frac{\gamma_2}{\beta + \beta'} h_2, \tag{2.14}$$

leaving all fields periodic. The generators of Cartans assume appropriate representations depending on the field they act on. The angular momentum j_3 is given for each mode after expanding fields with spherical harmonics, or with the so-called monopole spherical harmonics [40] if nontrivial magnetic field is applied on S^2. In actual computation we will often formulate the theory on $\mathbb{R}^3$ in Cartesian coordinates, with $r = e^\tau$ (see Appendix A and also [38,39]). Change in derivatives on $\mathbb{R}^3$ due to the above twist is

$$\vec{\nabla} \to \vec{\nabla} + \frac{\vec{r}}{r^2}\left(-\frac{\beta - \beta'}{\beta + \beta'} j_3 + \frac{\beta'}{\beta + \beta'} h_3 - \frac{\gamma_1}{\beta + \beta'} h_1 - \frac{\gamma_2}{\beta + \beta'} h_2 \right). \tag{2.15}$$

From now on our derivatives are understood with this shift, hoping it will not cause confusion.

Insertion of a Q-exact operator $\{Q, V\}$, for any gauge-invariant operator V, to the superconformal index (2.13) becomes zero due to the Q-invariance of the Cartans appearing in (2.13) and the periodic boundary condition for the fields due to $(-1)^F$ [41].[2] From the nilpotency of Q, the operator $e^{-t\{Q,V\}}$ takes the form $1 + Q(\cdots)$ for a given V and a continuous parameter t. Therefore, in the path integral representation, we may add the Q-exact term to the action $S \to S + t\{Q, V\}$ without changing the integral. The parameter t can be set to a value with which the calculation is easiest.[3] In particular, by suitably choosing V, setting $t \to +\infty$ may be regarded as a semi-classical limit with t being $\hbar^{-1}$. This semi-classical or Gaussian 'approximation' then provides the exact result since the integral is t-independent.

[2] In localization calculations in different contexts, Q^2 is often zero up to a gauge transformation. In this case Q defines the so-called *equivariant* cohomology, in which case V should be gauge invariant. Although $Q^2 = 0$ in our case, we simply choose V to be gauge-invariant even if we do not seem to be forced to.

[3] There is a subtle caveat in this argument when the integration domain is non-compact. Since we integrate over the non-compact space of fields, irrelevant saddle points may 'flow in from infinity' as we change t. See [29] for more explanation. Fortunately, this problem appears to be absent with the saddle points we find below.

We choose to deform the action of the $\mathcal{N}=6$ Chern–Simons theory by a Q-exact term which looks similar to the $d=3$ $\mathcal{N}=2$ 'Yang–Mills' action as follows. Using the gaugino superfield (Euclidean)

$$\mathcal{W}_\alpha(y) \sim -\sqrt{2}i\lambda_\alpha(y) + 2D(y)\theta_\alpha + \left(\gamma^\mu\theta\right)_\alpha(D_\mu\sigma - \star F_\mu)(y) + \sqrt{2}\theta^2\left(\gamma^\mu D_\mu\bar{\lambda}(y)\right)_\alpha \tag{2.16}$$

with $y^\mu = x^\mu + i\theta\gamma^\mu\bar{\theta}$, we add

$$t\{Q,V\} = \frac{1}{g^2}\int d^3x\, r\mathcal{W}^\alpha\mathcal{W}_\alpha\bigg|_{\theta^2\bar{\theta}^0} \quad \left(\text{taking } t = \frac{1}{g^2}\right) \tag{2.17}$$

to the original action. Let us provide supplementary explanations. The multiplication of r in the integrand makes this term scale invariant: it is crucial to have this symmetry since it will be our time translation symmetry after radial quantization. Of course translation symmetry on $\mathbb{R}^3$ is broken, which does not matter to us. It is also easy to show that the above deformation is Q-exact. Taking the coefficient of θ^2 is equivalent to $\partial_{\theta^-}\partial_{\theta^+}$, which in turn is related to Q_α by $Q_\alpha = \partial_\alpha - i(\gamma^\mu\bar{\theta})_\alpha\partial_\mu$. However, since we are keeping terms with $\bar{\theta}^0$, ∂_α is effectively Q_α. Furthermore, y^μ can be replaced by x^μ for the same reason. Therefore the added term indeed takes the form $Q_-Q_+(\cdots)$ with $y^\mu \to x^\mu$ everywhere. Finally, note that we do not add a term of the form $\int d^2\bar{\theta}\ \bar{\mathcal{W}}_\alpha\bar{\mathcal{W}}^\alpha$. Expanding in components, one finds

$$\Delta S = t\{Q,V\} = \frac{1}{2g^2}\int_{1\leqslant r\leqslant e^{\beta+\beta'}} d^3x\, r\left[(\star F_\mu - D_\mu\sigma)^2 - D^2 + \lambda^\alpha\left(\sigma^\mu\right)_{\alpha\beta}D_\mu\bar{\lambda}^\beta\right] \tag{2.18}$$

where $\sigma^\mu = (1, -i\sigma^3, i\sigma^1, -i\sigma^2)$ in the basis of [13], or $(1, i\vec{\sigma})$ in the basis we use in Appendices A–C, and $D_3\bar{\lambda}^\beta = -i\sigma\bar{\lambda}^\beta + i\bar{\lambda}^\beta\sigma$. We already turned D to an imaginary field during Wick rotation, which makes $-D^2$ positive. Everything goes similarly for the other vector multiplet $\tilde{A}_\mu, \tilde{\sigma}, \tilde{\lambda}_\alpha$. Some 1-loop study has been made for 'ordinary' Yang–Mills Chern–Simons theories [42]. Due to various differences in our construction, we will obtain very different results.

2.3. Calculation of the index

Having set up the localization problem in the previous subsection, we first find saddle points in the limit $g \to 0$, and then compute the 1-loop determinants around them.

All fields are subject to the periodic boundary condition along S^1, or the radial direction: working with fields in $\mathbb{R}^3$, the equivalent boundary condition is

$$\Psi\left(r = e^\beta\right) = e^{-\beta\Delta_\Psi}\Psi(r=1), \tag{2.19}$$

where Δ_Ψ is the scale dimension of the field Ψ. See Appendix A and [38,39].

The saddle point equations, which can be deduced either from (2.18) or from the supersymmetry transformation, are given by

$$\star F_\mu = D_\mu\sigma, \quad D = 0, \qquad \star\tilde{F}_\mu = D_\mu\tilde{\sigma}, \quad \tilde{D} = 0. \tag{2.20}$$

Note $\sigma, \tilde{\sigma}$ are no longer composite fields. From the supersymmetry transformation of matter fermions, we find that only $A_a = B_{\dot{a}} = 0$ satisfies the above boundary condition. All fermions are naturally set to zero. An obvious solution for the fields $F_{\mu\nu}, \sigma$ and $\tilde{F}_{\mu\nu}, \tilde{\sigma}$ is Dirac monopoles in the diagonals $U(1)^N \times U(1)^N \subset U(N) \times U(N)$:

$$\star F_\mu = \frac{x_\mu}{2r^3}\operatorname{diag}(n_1, n_2, \ldots, n_N), \qquad \star\tilde{F}_\mu = \frac{x_\mu}{2r^3}\operatorname{diag}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N), \tag{2.21}$$

together with

$$\sigma = -\frac{1}{2r}\,\mathrm{diag}(n_1, n_2, \ldots, n_N), \qquad \tilde{\sigma} = -\frac{1}{2r}\,\mathrm{diag}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N). \tag{2.22}$$

Since we are considering the region $1 \leqslant r \leqslant e^{\beta}$ in $\mathbb{R}^3$ excluding the origin $r = 0$, this solution is regular. The spherical symmetry of the solution implies that there is no twisting in derivatives. Note that all fields satisfy the boundary condition (2.19) with $\Delta_\sigma = \Delta_{\tilde{\sigma}} = 1$ and $\Delta_F = \Delta_{\tilde{F}} = 2$. The coefficients n_i and $\tilde{n}_i$ $(i = 1, 2, \ldots, N)$ have to be integers since the diagonals of $\int_{S^2} F$ and $\int_{S^2} \tilde{F}$ are 2π times integers.

Apart from the above Abelian solutions, we could not find any other solutions satisfying the boundary condition. For instance, although the governing equation is the same, non-Abelian solutions like the embedding of $SU(2)$ 't Hooft–Polyakov monopoles are forbidden since the boundary condition is not met. There still is a possibility that deformation of derivatives from twisting might play roles, which we have not fully ruled out. Anyway, we shall only consider the above saddle points and find agreement with the graviton index, which we regard as a strong evidence that we found all relevant saddle points.

To the above solution, one can also superpose holonomy zero modes along S^1 as follows. Since the solution is diagonal, turning on constant A_τ, $\tilde{A}_\tau$ diagonal in the same basis obviously satisfies (2.20). In terms of fields normalized in $\mathbb{R}^3$, this becomes

$$A_r = \frac{1}{(\beta + \beta')r}\,\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N), \qquad \tilde{A}_r = \frac{1}{(\beta + \beta')r}\,\mathrm{diag}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_N) \tag{2.23}$$

where we insert factors of $\beta + \beta'$ for later convenience. Taking into account the large gauge transformation along S^1, the coefficients α_i, $\tilde{\alpha}_i$ are all periodic:

$$\alpha_i \sim \alpha_i + 2\pi, \qquad \tilde{\alpha}_i \sim \tilde{\alpha}_i + 2\pi \quad (i = 1, 2, \ldots, N). \tag{2.24}$$

This holonomy along the time circle is related to the Polyakov loop [31]. The full set of saddle points is parametrized by integer fluxes $\{n_i, \tilde{n}_i\}$ and the holonomy $\{\alpha_i, \tilde{\alpha}_i\}$.

The analysis around the saddle point in which all n_i, $\tilde{n}_i$ vanish was done in [23]. We provide the 1-loop analysis around all saddle points. We explain various ingredients in turn: classical contribution, gauge-fixing and Faddeev–Popov measure, 1-loop contribution, Casimir-like energy shift and finally the full answer.

We first consider the 'classical' action. Plugging in the saddle point solution to the action, the classical action proportional to g^{-2} is always zero, as expected since our final result should not depend on g. Since the classical action itself involves two classes of terms, one of order $\mathcal{O}(g^{-2})$ from Q-exact deformation and another $\mathcal{O}(g^0)$ from original action, one should keep the latter part of the classical action to do the correct 1-loop physics. This comes from the Chern–Simons term. To correctly compute it, one has to extend $S^2 \times S^1$ to a 4-manifold $\mathcal{M}_4$ bounding it, and use $\frac{1}{4\pi}\int_{S^2\times S^1} \mathrm{tr}\, A \wedge F = \frac{1}{4\pi}\int_{\mathcal{M}_4} \mathrm{tr}\, F \wedge F$. If one chooses a disk D_2 bounded by $S^1 = \partial D_2$ and take $\mathcal{M}_4 = S^2 \times D_2$, one finds

$$\frac{1}{4\pi}\,\mathrm{tr}\int\limits_{S^2\times D_2} F \wedge F = \frac{1}{2\pi}\,\mathrm{tr}\int\limits_{D_2} F \cdot \int\limits_{S^2} F = \frac{1}{2\pi}\int\limits_{S^1} A \cdot \int\limits_{S^2} F = \sum_{i=1}^{N} n_i \alpha_i. \tag{2.25}$$

Taking into account the second gauge field $\tilde{A}_\mu$ at level $-k$, the exponential of the $\mathcal{O}(g^0)$ part of the classical action is a phase given by

$$e^{ik\sum_{i=1}^{N}(n_i\alpha_i - \tilde{n}_i\tilde{\alpha}_i)}. \tag{2.26}$$

The exponent is linear in $\alpha_i, \tilde{\alpha}_i$. Apart from these linear terms, there are no quadratic terms in the holonomy (around any given value) in the rest of the classical action. Thus Gaussian approximation is not applicable and they should be treated exactly. We shall integrate over them after applying Gaussian approximation to all other degrees.

Before considering 1-loop fluctuations, we fix the gauge. Following [31] we choose the Coulomb gauge, or the background Coulomb gauge for the components of fluctuations coupled to the background. In the calculation around the saddle point where all fields are zero, the Faddeev–Popov determinant computed following [31] is that for the $U(N) \times U(N)$ unitary matrices with eigenvalues $\{e^{i\alpha_i}\}, \{e^{i\tilde{\alpha}_i}\}$

$$\prod_{i<j}\left[2\sin\left(\frac{\alpha_i - \alpha_j}{2}\right)\right]^2 \prod_{i<j}\left[2\sin\left(\frac{\tilde{\alpha}_i - \tilde{\alpha}_j}{2}\right)\right]^2. \tag{2.27}$$

In the saddle point with nonzero fluxes, the flux effectively 'breaks' $U(N) \times U(N)$ to an appropriate subgroup. For instance, with fluxes $\{3, 2, 2, 0, 0\}$ on $U(1)^5 \subset U(5)$, $U(5)$ is broken to $U(1) \times U(2) \times U(2)$. As explained in Appendix B, the Faddeev–Popov measure for the saddle point with flux is the unitary matrix measure for the *unbroken subgroup* of $U(N) \times U(N)$:

$$\prod_{i<j;\, n_i=n_j}\left[2\sin\left(\frac{\alpha_i - \alpha_j}{2}\right)\right]^2 \prod_{i<j;\, \tilde{n}_i=\tilde{n}_j}\left[2\sin\left(\frac{\tilde{\alpha}_i - \tilde{\alpha}_j}{2}\right)\right]^2 \tag{2.28}$$

where the restricted products keep a sine factor for a pair of eigenvalues in the same unbroken gauge group only.

The fluctuations of fields with nonzero quadratic terms in the action can be treated by Gaussian approximation. The result consists of several factors of determinants from matter scalars, fermions and also from fields in vector multiplets, which is schematically

$$\frac{\det_{\psi_a,\chi_a} \det_\lambda \det_{\tilde{\lambda}}}{\det_{A_a,B_a} \det_{A_\mu,\sigma} \det_{\tilde{A}_\mu,\tilde{\sigma}}}. \tag{2.29}$$

The fields in $U(N) \times U(N)$ vector multiplets will turn out to provide nontrivial contribution.

We first consider the 1-loop determinant from matter fields, namely $A_a, B_{\dot{a}}, \psi_a, \chi_{\dot{a}}$ and their conjugates. In this determinant, the index over the so-called 'letters' play important roles.[4] To start with, we consider the basic fields $A_a, \bar{B}^{\dot{a}}, \psi_a, \bar{\chi}^{\dot{a}}$ in $(N,\bar{N})$ representation of the gauge group and pick up the ijth component, where i (j) runs over $1, 2, \ldots, N$ and refers to the fundamental (anti-fundamental) index of first (second) $U(N)$. In the quadratic Lagrangian, these modes couple to the background magnetic field with charge $n_i - \tilde{n}_j$. The index over letters in this component is given by

$$\begin{aligned} f_{ij}^{+}(x, y_1, y_2) &= x^{|n_i - \tilde{n}_j|}\left[\frac{x^{1/2}}{1-x^2}\left(\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}}\right) - \frac{x^{3/2}}{1-x^2}\left(\sqrt{y_1 y_2} + \frac{1}{\sqrt{y_1 y_2}}\right)\right] \\ &\equiv x^{|n_i - \tilde{n}_j|} f^{+}(x, y_1, y_2). \end{aligned} \tag{2.30}$$

[4] Letters are defined by single basic fields with many derivatives acting on them [2,23,31]. In path integral calculation, they are simply (monopole) spherical harmonics modes of the fields.

See Appendix B.1 for the derivation. Similarly, the index over the ijth component of letters in $(\bar{N},N)$ representation is given by

$$f_{ij}^{-}(x,y_1,y_2)=x^{|n_i-\tilde{n}_j|}\left[\frac{x^{1/2}}{1-x^2}\left(\sqrt{y_1y_2}+\frac{1}{\sqrt{y_1y_2}}\right)-\frac{x^{3/2}}{1-x^2}\left(\sqrt{\frac{y_1}{y_2}}+\sqrt{\frac{y_2}{y_1}}\right)\right]$$
$$\equiv x^{|n_i-\tilde{n}_j|}f^{-}(x,y_1,y_2). \tag{2.31}$$

There is no dependence on the regulator β', as expected. Again see Appendix B.1 for details. With these letter indices, the 1-loop determinant for given $\alpha_i,\tilde{\alpha}_i$ is given by

$$\prod_{i,j=1}^{N}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(f_{ij}^{+}\left(x^n,y_1^n,y_2^n\right)e^{in(\tilde{\alpha}_j-\alpha_i)}+f_{ij}^{-}\left(x^n,y_1^n,y_2^n\right)e^{in(\alpha_i-\tilde{\alpha}_j)}\right)\right]. \tag{2.32}$$

The expression $\exp[\sum_{n=1}^{\infty}\frac{1}{n}f(x^n)]$ above, sometimes called the Plethystic exponential of a function $f(x)$, appears since we count operators made of identical letters [43]. Note that the determinant around the saddle point in which all fluxes are zero, $n_i=\tilde{n}_i=0$, reduces to the result obtained in [23] using combinatoric methods in the free theory. When all $n_i,\tilde{n}_i$ are zero, our letter indices $f_{ij}^{\pm}$ all reduces to $f^{\pm}$, which are exactly the letter indices obtained in [23].

We also consider the determinant from fields in vector multiplets. Here, the letter index over the ijth component of the adjoint fields $A_\mu,\lambda_\alpha,\sigma$ is given by

$$f_{ij}^{\rm adj}(x)=-(1-\delta_{n_in_j})x^{|n_i-n_j|}=\begin{cases}0 & \text{if } n_i=n_j,\\ -x^{|n_i-n_j|} & \text{if } n_i\neq n_j,\end{cases} \tag{2.33}$$

and similarly for the ijth component of the fields $\tilde{A}_\mu,\tilde{\lambda}_\alpha,\tilde{\sigma}$ one finds

$$\tilde{f}_{ij}^{\rm adj}(x)=-(1-\delta_{\tilde{n}_i\tilde{n}_j})x^{|\tilde{n}_i-\tilde{n}_j|}=\begin{cases}0 & \text{if } \tilde{n}_i=\tilde{n}_j,\\ -x^{|\tilde{n}_i-\tilde{n}_j|} & \text{if } \tilde{n}_i\neq\tilde{n}_j.\end{cases} \tag{2.34}$$

The full 1-loop determinant from adjoint fields is again given by the same exponential:

$$\prod_{i,j=1}^{N}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(f_{ij}^{\rm adj}\left(x^n\right)e^{-in(\alpha_i-\alpha_j)}+\tilde{f}_{ij}^{\rm adj}\left(x^n\right)e^{-in(\tilde{\alpha}_i-\tilde{\alpha}_j)}\right)\right]. \tag{2.35}$$

For the trivial vacuum with no fluxes, all $f_{ij}^{\rm adj}$ and $\tilde{f}_{ij}^{\rm adj}$ are zero that the adjoint determinant is simply 1. This is consistent with the result in [23], where the vector multiplets including gauge fields played no roles. See Appendix B.2 for the derivation.

When evaluating the determinant in Appendix B, one encounters an overall factor

$$\exp[-\beta\epsilon_0],\quad \text{where } \epsilon_0\equiv\frac{1}{2}\,{\rm tr}\left[(-1)^F(\epsilon+j_3)\right]. \tag{2.36}$$

This is similar to the ground state energy traced over all modes, twisted by j_3 basically because we are only considering charges commuting with Q,S in our index. Although this is not just energy due to j_3, we slightly abuse terminology and call this quantity Casimir energy. With appropriate regulator respecting supersymmetry, this is given by

$$\epsilon_0=\sum_{i,j=1}^{N}|n_i-\tilde{n}_j|-\sum_{i<j}|n_i-n_j|-\sum_{i<j}|\tilde{n}_i-\tilde{n}_j|. \tag{2.37}$$

See Appendix B.3 for the derivation and some properties of ϵ_0. In particular, ϵ_0 is non-negative and becomes zero if and only if the two sets of flux distributions $\{n_i\}$, $\{\tilde{n}_i\}$ are identical. Some features of this energy shift related to AdS/CFT is discussed in the next section.

Finally we integrate over the modes α_i and $\tilde{\alpha}_j$ with all factors explained above in the measure. The result for a given saddle point labeled by $\{n_i\}$, $\{\tilde{n}_i\}$ is

$$I(x, y_1, y_2) = x^{\epsilon_0} \int \frac{1}{(\text{symmetry})} \left[\frac{d\alpha_i\, d\tilde{\alpha}_i}{(2\pi)^2}\right] \prod_{\substack{i<j;\\ n_i=n_j}} \left[2\sin\left(\frac{\alpha_i-\alpha_j}{2}\right)\right]^2$$
$$\times \prod_{\substack{i<j;\\ \tilde{n}_i=\tilde{n}_j}} \left[2\sin\left(\frac{\tilde{\alpha}_i-\tilde{\alpha}_j}{2}\right)\right]^2 e^{ik\sum_{i=1}^N (n_i\alpha_i - \tilde{n}_i\tilde{\alpha}_i)}$$
$$\times \prod_{i,j=1}^N \exp\left[\sum_{n=1}^\infty \frac{1}{n}\left(f_{ij}^+\left(x^n, y_1^n, y_2^n\right)e^{in(\tilde{\alpha}_j-\alpha_i)} + f_{ij}^-\left(x^n, y_1^n, y_2^n\right)e^{in(\alpha_i-\tilde{\alpha}_j)}\right)\right]$$
$$\times \prod_{i,j=1}^N \exp\left[\sum_{n=1}^\infty \frac{1}{n}\left(f_{ij}^{\text{adj}}\left(x^n\right)e^{-in(\alpha_i-\alpha_j)} + \tilde{f}_{ij}^{\text{adj}}\left(x^n\right)e^{-in(\tilde{\alpha}_i-\tilde{\alpha}_j)}\right)\right] \tag{2.38}$$

with (2.30), (2.31), (2.33), (2.34), (2.37) for the definitions of various functions. The symmetry factor on the first line divides by the factor of identical variables among $\{\alpha_i\}$, $\{\tilde{\alpha}_i\}$ according to 'unbroken' gauge group. For the $U(5) \to U(1) \times U(2) \times U(2)$ example above, this factor is $\frac{1}{1!\times 2!\times 2!}$. The full index is the sum of (2.38) for all flux distributions $\{n_i\}$, $\{\tilde{n}_i\}$.

Apart from the first phase factor on the second line, the integrand is invariant under the overall translation of $\alpha_i, \tilde{\alpha}_i$. Therefore, the integral vanishes unless $\sum_i n_i = \sum_i \tilde{n}_i$. This is of course a consequence of a decoupled $U(1)$ as explained in [12]. The KK-momentum, or $\frac{k}{2}$ times the baryon-like charge, for states counted by the above index is given by (2.11). From the structure of this integral and the letter indices, it is also easy to infer that the energy of the states contributing to this index is bounded from below by

$$\epsilon \geqslant \frac{k}{2}\sum_{i=1}^N n_i = \frac{k}{2}\sum_{i=1}^N \tilde{n}_i \tag{2.39}$$

if the two flux distributions $\{n_i\}$, $\{\tilde{n}_i\}$ are identical. If the two distributions are different, the energy is strictly larger than this bound. We discuss this in the next section.

There is a unifying structure in the integrand of the above index if one combines the matrix integral measure on the first line to the last line. Note that the measure can be written as

$$\prod_{\substack{i<j;\\ n_i=n_j}} \left[2\sin\left(\frac{\alpha_i-\alpha_j}{2}\right)\right]^2 \prod_{\substack{i<j;\\ \tilde{n}_i=\tilde{n}_j}} \left[2\sin\left(\frac{\tilde{\alpha}_i-\tilde{\alpha}_j}{2}\right)\right]^2$$
$$= \prod_{i\neq j} \exp\left[-\sum_{n=1}^\infty \frac{1}{n}\left(\delta_{n_i n_j} e^{-in(\alpha_i-\alpha_j)} + \delta_{\tilde{n}_i\tilde{n}_j} e^{-in(\tilde{\alpha}_i-\tilde{\alpha}_j)}\right)\right].$$

As in [2,30,31], this provides a 2-body repulsive effective potentials between α_i's and $\tilde{\alpha}_i$'s in the same unbroken gauge group. From the form of adjoint letter indices in (2.33), (2.34), the above measure combines with the last line and become

$$\prod_{i\neq j}\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\left(x^{n|n_i-n_j|}e^{-in(\alpha_i-\alpha_j)}+x^{n|\tilde{n}_i-\tilde{n}_j|}e^{-in(\tilde{\alpha}_i-\tilde{\alpha}_j)}\right)\right]. \tag{2.40}$$

Therefore, in the presence of fluxes, there are repulsive 2-body potentials between all pairs with the strength $\frac{1}{n}$ weakened to $\frac{1}{n}x^{n|n_i-n_j|}$ and $\frac{1}{n}x^{n|\tilde{n}_i-\tilde{n}_j|}$, by factors of x.

3. Large N limit and index over gravitons

We further analyze the gauge theory index we obtained in the previous section in the large N limit. The limit we take is $N\to\infty$ while keeping the chemical potentials at order 1. Among other motivations, this setting lets us study the low energy spectrum which can be compared to that from supergravity.

In this limit, only $\mathcal{O}(1)$, namely $\mathcal{O}(N^0)$, numbers of $U(1)^N$'s in each $U(N)$ have nonzero magnetic flux. This is because states with more than $\mathcal{O}(1)$ U(1)'s filled with nonzero fluxes have energies bigger than $\mathcal{O}(1)$ from (2.39) and are suppressed in the large N limit we take. This implies that there always exist an $U(N-\mathcal{O}(1))\times U(N-\mathcal{O}(1))$ part in the integral over holonomies $\{\alpha_i,\tilde{\alpha}_i\}$. Let us call this unbroken gauge group $U(N_1)\times U(N_2)$, where N_1 and N_2 denote numbers of $U(1)$ with zero fluxes. In the large N limit, there is a well-known way of calculating this part of the integral [30,31]. We first introduce

$$\rho_n=\frac{1}{N_1}\sum_i e^{-in\alpha_i},\qquad \chi_n=\frac{1}{N_2}\sum_i e^{-in\tilde{\alpha}_i} \tag{3.1}$$

for nonzero integers n, where the summations are over N_1 and N_2 $U(1)$ indices, respectively. In the large N_1, N_2 limit, the integration over $\alpha_i,\tilde{\alpha}_i$ belonging to $U(N_1)\times U(N_2)$ becomes

$$\prod_{n=1}^{\infty}\left[N_1^2\,d\rho_n\,d\rho_{-n}\right]\left[N_2^2\,d\chi_n\,d\chi_{-n}\right]. \tag{3.2}$$

The integrand containing ρ_n, χ_n is given by

$$\begin{aligned}
&\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\left(N_1^2\rho_n\rho_{-n}+N_2^2\chi_n\chi_{-n}-N_1N_2f^+\left(x^n,y_1^n,y_2^n\right)\rho_n\chi_{-n}\right.\right.\\
&\quad\left.\left.-N_1N_2f^-\left(x^n,y_1^n,y_2^n\right)\rho_{-n}\chi_n\right)\right]\\
&\quad\times\exp\left[N_1\sum_{n=1}^{\infty}\frac{1}{n}\rho_n\left(\sum_{i=1}^{M_2}x^{n|\tilde{n}_i|}f^+\left(\cdot^n\right)e^{in\tilde{\alpha}_i}-\sum_{i=1}^{M_1}x^{n|n_i|}e^{in\alpha_i}\right)\right.\\
&\quad\left.+\frac{1}{n}\rho_{-n}\left(\sum_{i=1}^{M_2}x^{n|\tilde{n}_i|}f^-\left(\cdot^n\right)e^{-in\tilde{\alpha}_i}-\sum_{i=1}^{M_1}x^{n|n_i|}e^{-in\alpha_i}\right)\right]\\
&\quad\times\exp\left[N_2\sum_{n=1}^{\infty}\frac{1}{n}\chi_n\left(\sum_{i=1}^{M_1}x^{n|n_i|}f^-\left(\cdot^n\right)e^{in\alpha_i}-\sum_{i=1}^{M_2}x^{n|\tilde{n}_i|}e^{in\tilde{\alpha}_i}\right)\right.
\end{aligned}$$

$$+\frac{1}{n}\chi_{-n}\left(\sum_{i=1}^{M_1}x^{n|n_i|}f^+\left(\cdot^n\right)e^{-in\alpha_i}-\sum_{i=1}^{M_2}x^{n|\tilde{n}_i|}e^{-in\tilde{\alpha}_i}\right)\Bigg] \tag{3.3}$$

where $M_1 \equiv N - N_1$ and $M_2 \equiv N - N_2$ are numbers (of order 1) of $U(1)$'s in two gauge groups with nonzero fluxes, and $\cdot^n$ denotes taking nth powers of all arguments x, y_1, y_2. The integral of ρ_n, χ_n is Gaussian, where the first line (with $N_1 = N_2 = N$) is the one encountered in [23]. After this integration, one obtains

$$I^{(0)}\exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}V^T\left(\cdot^n\right)M\left(\cdot^n\right)V\left(\cdot^n\right)\right] \tag{3.4}$$

where

$$V=\begin{pmatrix}\sum_{i=1}^{M_2}x^{|\tilde{n}_i|}f^+e^{i\tilde{\alpha}_i}-\sum_{i=1}^{M_1}x^{|n_i|}e^{i\alpha_i}\\ \sum_{i=1}^{M_1}x^{|n_i|}f^-e^{i\alpha_i}-\sum_{i=1}^{M_2}x^{|\tilde{n}_i|}e^{i\tilde{\alpha}_i}\\ \sum_{i=1}^{M_2}x^{|\tilde{n}_i|}f^-e^{-i\tilde{\alpha}_i}-\sum_{i=1}^{M_1}x^{|n_i|}e^{-i\alpha_i}\\ \sum_{i=1}^{M_1}x^{|n_i|}f^+e^{-i\alpha_i}-\sum_{i=1}^{M_2}x^{|\tilde{n}_i|}e^{-i\tilde{\alpha}_i}\end{pmatrix},$$

$$M=\frac{1}{1-f^+f^-}\begin{pmatrix} & & 1 & f^-\\ & & f^+ & 1\\ 1 & f^+ & & \\ f^- & 1 & & \end{pmatrix} \tag{3.5}$$

and

$$I^{(0)}=\prod_{n=1}^{\infty}\det\left[M\left(x^n,y_1^n,y_2^n\right)\right]^{\frac{1}{2}}=\prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1-x^ny_1^n)(1-x^ny_1^{-n})(1-x^ny_2^n)(1-x^ny_2^{-n})}. \tag{3.6}$$

The factor $I^{(0)}$ was computed in [23]. Since the second factor becomes 1 (from $V = 0$) if there are no fluxes in the saddle point, this is a generalization of the large N result of [23].

We now turn to the remaining part of the holonomy integral in (2.38) apart from $I^{(0)}$. The integral over $M_1 + M_2$ variables $\alpha_i, \tilde{\alpha}_i$ including the second factor in (3.4) can be written as

$$\begin{aligned}&x^{\epsilon_0}\int_0^{2\pi}\frac{1}{(\text{symmetry})}\left[\frac{d\alpha}{2\pi}\right]\left[\frac{d\tilde{\alpha}}{2\pi}\right]\\ &\times\prod_{\substack{i,j;\\ n_i=n_j}}\left(2\sin\frac{\alpha_i-\alpha_j}{2}\right)^2\prod_{\substack{i,j;\\ \tilde{n}=\tilde{n}_j}}\left(2\sin\frac{\tilde{\alpha}_i-\tilde{\alpha}_j}{2}\right)^2e^{ik(\sum n_i\alpha_i-\sum\tilde{n}_i\tilde{\alpha}_i)}\\ &\times\exp\left[\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\frac{1}{n}\mathbf{f}_{ij}^{\text{bif}}\left(x^n,y_1^n,y_2^n,e^{in\alpha},e^{in\tilde{\alpha}}\right)+\sum_{i,j=1}^{M_1}\frac{1}{n}\mathbf{f}_{ij}^{\text{adj}}\left(x^n,y_1^n,y_2^n,e^{in\alpha}\right)\right.\\ &\left.+\sum_{i,j=1}^{M_2}\frac{1}{n}\tilde{\mathbf{f}}_{ij}^{\text{adj}}\left(x^n,y_1^n,y_2^n,e^{in\tilde{\alpha}}\right)\right]\end{aligned} \tag{3.7}$$

where

$$\mathbf{f}_{ij}^{\text{bif}}=\left(x^{|n_i-\tilde{n}_j|}-x^{|n_i|+|\tilde{n}_j|}\right)\left(f^+e^{i(\tilde{\alpha}_j-\alpha_i)}+f^-e^{i(\alpha_i-\tilde{\alpha}_j)}\right),$$

$$\mathbf{f}_{ij}^{\text{adj}} = \left[-(1-\delta_{n_in_j})x^{|n_i-n_j|} + x^{|n_i|+|n_j|}\right]e^{-i(\alpha_i-\alpha_j)},$$
$$\tilde{\mathbf{f}}_{ij}^{\text{adj}} = \left[-(1-\delta_{\tilde{n}_i\tilde{n}_j})x^{|\tilde{n}_i-\tilde{n}_j|} + x^{|\tilde{n}_i|+|\tilde{n}_j|}\right]e^{-i(\tilde{\alpha}_i-\tilde{\alpha}_j)}, \tag{3.8}$$

and the symmetry factor again divides by the permutation symmetry of identical variables $\alpha_i, \tilde{\alpha}_i$, depending on the gauge symmetry unbroken by fluxes. Recall that ϵ_0 is Casimir energy like quantity which can be nonzero in the background with nonzero flux.

The above integral can be factorized as follows. To explain this, we decompose nonzero fluxes $\{n_i\}$, $\{\tilde{n}_i\}$ into positive and negative ones $\{n_i^+\colon n_i^+ > 0,\ i = 1,2,\ldots,M_1^+\}$, $\{n_i^-\colon n_i^- < 0,\ i = 1,2,\ldots,M_1^-\}$ and similarly $\{\tilde{n}_i^+\colon i = 1,2,\ldots,M_2^+\}$, $\{\tilde{n}_i^-\colon i = 1,2,\ldots,M_2^-\}$. Having a look at the indices in (3.8), one can observe that none of these functions get contribution from modes connecting two $U(1)$'s with one positive and one negative flux. This simply follows from $x^{|n_i-\tilde{n}_j|} = x^{|n_i|+|\tilde{n}_j|}$, $x^{|n_i-n_j|} = x^{|n_i|+|n_j|}$ and $x^{|\tilde{n}_i-\tilde{n}_j|} = x^{|\tilde{n}_i|+|\tilde{n}_j|}$ for pairs of fluxes with different signs. Furthermore, as explained in Appendix B.3, the Casimir energy also factorizes into contributions coming from modes connecting positive fluxes or negative fluxes only. This proves a complete factorization of the integrand and the pre-factor into two pieces, each of which depending only on fluxes $\{n_i^+\}$, $\{\tilde{n}_i^+\}$ and $\{n_i^-\}$, $\{\tilde{n}_i^-\}$, respectively. Due to the overall translational invariance of α_i, $\tilde{\alpha}_i$ and factorization, the integral is nonzero only if

$$\sum_{i=1}^{M_1^+} n_i^+ = \sum_{i=1}^{M_2^+} \tilde{n}_i^+, \qquad \sum_{i=1}^{M_1^-} n_i^- = \sum_{i=1}^{M_2^-} \tilde{n}_i^-, \tag{3.9}$$

namely the total positive and negative fluxes over two gauge groups match separately.

We now write the expression for the full large N index, summing over all saddle points. Since $\frac{k}{2}$ times the total number of fluxes is the Kaluza–Klein momentum along the Hopf fiber circle of $S^7/\mathbb{Z}_k$, we grade the summation with the chemical potential y_3 as $y_3^{\frac{k}{2}\sum_{i=1}^{M_1} n_i}$ (or $y_3^{\frac{k}{2}\sum_{i=1}^{M_2} \tilde{n}_i}$). The large N index is

$$I_{N=\infty}(x, y_1, y_2, y_3) = I^{(0)}(x, y_1, y_2, y_3) I^{(+)}(x, y_1, y_2, y_3) I^{(-)}(x, y_1, y_2, y_3), \tag{3.10}$$

where $I^{(0)}$ is given by (3.6), and

$$\begin{aligned}
&I^{(+)}(x, y_1, y_2, y_3) \\
&\quad = \sum_{M_1, M_2=0}^{\infty} \sum_{\substack{n_1 \geqslant \cdots \geqslant n_{M_1} > 0 \\ \tilde{n}_1 \geqslant \cdots \geqslant \tilde{n}_{M_2} > 0}} y_3^{\frac{k}{2}\sum n_i} x^{\sum |n_i-\tilde{n}_j| - \sum_{i<j}|n_i-n_j| - \sum_{i<j}|\tilde{n}_i-\tilde{n}_j|} \\
&\qquad \times \int_0^{2\pi} \frac{1}{(\text{symmetry})} \left[\frac{d\alpha}{2\pi}\right]\left[\frac{d\tilde{\alpha}}{2\pi}\right] e^{ik(\sum n_i\alpha_i - \sum \tilde{n}_i\tilde{\alpha}_i)} \\
&\qquad \times \prod_{\substack{i,j; \\ n_i=n_j}} \left[2\sin\frac{\alpha_i-\alpha_j}{2}\right]^2 \prod_{\substack{i,j; \\ \tilde{n}_i=\tilde{n}_j}} \left[2\sin\frac{\tilde{\alpha}_i-\tilde{\alpha}_j}{2}\right]^2 \\
&\qquad \times \exp\left[\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} \frac{1}{n}\mathbf{f}_{ij}^{\text{bif}}\left(x^n, y_1^n, y_2^n, e^{in\alpha}, e^{in\tilde{\alpha}}\right) + \sum_{i,j=1}^{M_1} \mathbf{f}_{ij}^{\text{adj}}\left(x^n, y_1^n, y_2^n, e^{in\alpha}\right)\right.
\end{aligned}$$

$$+ \sum_{i,j=1}^{M_2} \tilde{\mathbf{f}}_{ij}^{\text{adj}}\big(x^n, y_1^n, y_2^n, e^{in\tilde{\alpha}}\big)\Bigg] \tag{3.11}$$

with definitions for various functions given by (3.8). The last factor $I^{(-)}(x, y_1, y_2, y_3)$, which is a summation of saddle points with negative fluxes only, takes a form similar to $I^{(+)}$ with signs of $n_i, \tilde{n}_i$ flipped. The signs of these integers appear only in $y_3^{\frac{k}{2}\sum n_i}$ and $e^{ik(\sum n_i\alpha_i - \sum \tilde{n}_i\tilde{\alpha}_i)}$. The sign flip in the first factor can be undone by replacing y_3 by $1/y_3$, and that in the second factor can be undone by changing integration variables from $\alpha, \tilde{\alpha}$ to $-\alpha, -\tilde{\alpha}$. The latter change affects $\mathbf{f}_{ij}^{\text{bif}}$ by the exchange $f^+ \leftrightarrow f^-$, which can be achieved by changing $\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}}$ and $\sqrt{y_1 y_2} + \frac{1}{\sqrt{y_1 y_2}}$. Collecting all, one finds that

$$I^{(-)}(x, y_1, y_2, y_3) = I^{(+)}(x, y_1, 1/y_2, 1/y_3) = I^{(+)}(x, 1/y_1, y_2, 1/y_3). \tag{3.12}$$

Since the knowledge of $I^{(+)}$ would be enough to obtain the full index, we will mainly consider this function in the rest of this section.

We want to compare the above result with the index over supersymmetric gravitons in $AdS_4 \times S^7/\mathbb{Z}_k$. As shown in Appendix C, the index of multiple gravitons also split into three parts, $I_{\text{mp}} = I_{\text{mp}}^{(0)} I_{\text{mp}}^{(+)} I_{\text{mp}}^{(-)}$, essentially because gravitons with positive and negative momenta do not mutually interact, even without the 'statistical interaction' for identical particles. It was shown in [23] that $I_{\text{mp}}^{(0)} = I^{(0)}$. For the gauge theory and gravity indices to agree, one has to show $I^{(+)} I^{(-)} = I_{\text{mp}}^{(+)} I_{\text{mp}}^{(-)}$, or $\frac{I^{(+)}}{I_{\text{mp}}^{(+)}} = \frac{I_{\text{mp}}^{(-)}}{I^{(-)}}$. Left-hand side and right-hand side can be Taylor-expanded in $y_3^{\frac{1}{2}}$ and $y_3^{-\frac{1}{2}}$, respectively, together with positive power expansions in x. The only way this equation can hold is both sides being a constant, which is actually 1. Thus, one only has to show

$$I^{(+)}(x, y_1, y_2, y_3) = I_{\text{mp}}^{(+)}(x, y_1, y_2, y_3) \tag{3.13}$$

to check the agreement of the indices in gauge theory and gravity.

By definition, saddle points for $I^{(+)}$ carry positive fluxes only. Since a sequence of non-decreasing positive integers can be represented by a Young diagram, we will sometimes represent positive $\{n_i\}$, $\{\tilde{n}_i\}$ by a pair of Young diagrams Y and $\tilde{Y}$, where the lengths of ith rows are n_i and $\tilde{n}_i$. We denote by $d(Y) = d(\tilde{Y})$ the total number of boxes in the Young diagram. The summation in $I^{(+)}$ can be written as

$$\begin{aligned} I^{(+)}(x, y_1, y_2, y_3) &= \sum_{Y,\tilde{Y}:d(Y)=d(\tilde{Y})} y_3^{\frac{k}{2}d(Y)} I_{Y\tilde{Y}}(x, y_1, y_2) \\ &= 1 + y_3^{\frac{k}{2}} I_{\ydiagram{1}\,\ydiagram{1}} + y_3^{k}\big(I_{\ydiagram{2}\,\ydiagram{2}} + I_{\ydiagram{1,1}\,\ydiagram{1,1}} + 2I_{\ydiagram{2}\,\ydiagram{1,1}}\big) \\ &\quad + y_3^{\frac{3k}{2}}\big(I_{\ydiagram{3}\,\ydiagram{3}} + I_{\ydiagram{2,1}\,\ydiagram{2,1}} + I_{\ydiagram{1,1,1}\,\ydiagram{1,1,1}} + 2I_{\ydiagram{3}\,\ydiagram{2,1}} + 2I_{\ydiagram{3}\,\ydiagram{1,1,1}} + 2I_{\ydiagram{2,1}\,\ydiagram{1,1,1}}\big) + \cdots \end{aligned} \tag{3.14}$$

where we used $I_{Y\tilde{Y}} = I_{\tilde{Y}Y}$, which we do not prove here but can be checked by suitable redefinitions of integration variables in (3.11).

We did not manage to analytically prove $I^{(+)} = I_{\text{mp}}^{(+)}$ generally. Below we provide nontrivial analytic and numerical checks of this claim in various sectors: we consider the sectors in which the total number of positive fluxes $\sum_i n_i^+ = \sum_i \tilde{n}^+$ is 1, 2 and 3.

3.1. One KK-momentum: Analytic tests

We analytically prove the agreement between gauge theory and gravity indices in the sector with unit KK-momentum. This amounts to comparing the coefficients of $y_3^{\frac{k}{2}}$ in $I^{(+)}$ and $I_{\text{mp}}^{(+)}$. The gravity result is simply

$$I_k^{\text{sp}}(x,y_1,y_2)=\oint\frac{d\sqrt{y_3}}{2\pi i\sqrt{y_3}}y_3^{-\frac{k}{2}}I^{\text{sp}}(x,y_1,y_2,y_3),\tag{3.15}$$

namely index over single graviton with k (i.e. minimal) units of KK-momentum. The contour for $\sqrt{y_3}$ integration is the unit circle in the complex plane. On the gauge theory side, the result comes from one saddle point with fluxes given by $n=\tilde{n}=1$: from the general formula (3.11) one obtains

$$\begin{aligned}I_{\square\square}=&\int_0^{2\pi}\frac{d\alpha\,d\tilde{\alpha}}{(2\pi)^2}e^{ik(\alpha-\tilde{\alpha})}\\&\times\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\Big(\big(1-x^{2n}\big)\big(f^+(\cdot^n)e^{in(\tilde{\alpha}-\alpha)}+f^-(\cdot^n)e^{in(\alpha-\tilde{\alpha})}\big)+2x^{2n}\Big)\right].\end{aligned}\tag{3.16}$$

The 'effective letter index' $(1-x^2)(f^+e^{i(\tilde{\alpha}-\alpha)}+f^-e^{i(\alpha-\tilde{\alpha})})+2x^2$ in the exponential is

$$\begin{aligned}&\left[x^{\frac{1}{2}}\left(\sqrt{\frac{y_1}{y_2}}+\sqrt{\frac{y_2}{y_1}}\right)-x^{\frac{3}{2}}\left(\sqrt{y_1y_2}+\frac{1}{\sqrt{y_1y_2}}\right)\right]e^{i(\tilde{\alpha}-\alpha)}\\&\quad+\left[x^{\frac{1}{2}}\left(\sqrt{y_1y_2}+\frac{1}{\sqrt{y_1y_2}}\right)-x^{\frac{3}{2}}\left(\sqrt{\frac{y_1}{y_2}}+\sqrt{\frac{y_2}{y_1}}\right)\right]e^{i(\alpha-\tilde{\alpha})}+2x^2.\end{aligned}\tag{3.17}$$

Note that $f^\pm$ have $\frac{1}{1-x^2}$ factors, coming from many derivatives acting on fields, and take the form of infinite series in x. The factor $(1-x^2)$ cancels these derivative factors and lets the effective index be a finite series. Defining an integration variable $z\equiv e^{i(\tilde{\alpha}-\alpha)}$ in (3.16), and after exponentiating (3.17), one obtains

$$I_{\square\square}=\oint\frac{dz}{(2\pi i)z}z^{-k}\frac{(1-x\sqrt{xy_1y_2}z)(1-x\sqrt{\frac{x}{y_1y_2}}z)(1-x\sqrt{\frac{xy_1}{y_2}}z^{-1})(1-x\sqrt{\frac{xy_2}{y_2}}z^{-1})}{(1-\sqrt{\frac{xy_1}{y_2}}z)(1-\sqrt{\frac{xy_2}{y_1}}z)(1-\sqrt{xy_1y_2}z^{-1})(1-\sqrt{\frac{x}{y_1y_2}}z^{-1})(1-x^2)^2}.\tag{3.18}$$

Using the relation (C.3), and identifying the integration variable as $z=\sqrt{y_3}$, the above result can be rewritten as

$$I_{\square\square}=\oint\frac{d\sqrt{y_3}}{(2\pi i)\sqrt{y_3}}y_3^{-\frac{k}{2}}\left(I^{\text{sp}}(x,y_1,y_2,y_3)+\frac{1-x^2+x^4}{(1-x^2)^2}\right).\tag{3.19}$$

Since the second term in the integrand does not survive the contour integral, this is exactly the gravity expression (3.15), proving the agreement in this sector.

Before proceeding to more nontrivial examples, let us explain a bit more on the above index. As stated in the previous section, the flux provides a lower bound to the energy of states. In the sector with unit flux, we can actually find from the integrand of (3.18) that $\#(\sqrt{x})$ in a term is always larger than or equal to $\#(z)$. We can actually arrange the terms in Taylor expansion of the integrand so that the number $\#(\sqrt{x})$–$\#(z)$ ascends. The lowest order terms come from the first two factors in the denominator containing $\sqrt{x}z$, for which this number is 0. The index for these states is

$$\oint \frac{dz}{(2\pi i)z} z^{-k} \frac{1}{(1-\sqrt{\frac{xy_1}{y_2}}z)(1-\sqrt{\frac{xy_2}{y_1}}z)} = x^{\frac{k}{2}}\left(y_1^{\frac{k}{2}} y_2^{-\frac{k}{2}} + y_1^{\frac{k}{2}-1} y_2^{-\frac{k}{2}+1} + \cdots + y_1^{-\frac{k}{2}} y_2^{\frac{k}{2}}\right). \tag{3.20}$$

From Table 1, the two factors in the integrand originate from the gauge theory letters $\bar{B}^{\dot{2}}$ and $\bar{B}^{\dot{1}}$ in s-waves. The operators made of these letters form a subset of chiral operators studied in [12,18]. The operator took the form of kth product of $\bar{B}^{\dot{a}}$, multiplied by a 't Hooft operator in the $(\mathrm{Sym}(\bar{\mathbf{N}}^k), \mathrm{Sym}(\mathbf{N}^k))$ representation to make the whole operator gauge invariant.

Since no fermionic letters can contribute in the lowest energy sector due to their larger dimensions than scalars, the above index equals to the partition function. This is not true any more as one goes beyond lowest energy as fermionic letters start to enter. This aspect in the lowest energy sector will continue to appear with more fluxes below. The full spectrum of these chiral operators, preserving specific $\mathcal{N}=2$ supersymmetry, has been studied in [12] by quantizing the moduli space [2,43,44]. We expect our result to be identical to the result in [12], which we check explicitly for the case with two fluxes in the next subsection.

3.2. Two KK-momenta: analytic and numerical tests

Monopole operators which have been studied in the context of $\mathcal{N}=6$ Chern–Simons theory are in the conjugate representations of the two $U(N)$ gauge groups, such as $(\mathrm{Sym}(\bar{\mathbf{N}}^k), \mathrm{Sym}(\mathbf{N}^k))$ in the previous subsection or more general examples studied in [26,27]. In our analysis, these are related to the saddle points in which two flux distributions $\{n_i\}$ and $\{\tilde{n}_i\}$ are the same. In the sector with two fluxes, two of the four saddle points $\Box\Box\,\Box\Box$ and $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ are in this category, while the other two $\Box\Box\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ and $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\Box\Box$ are not.Let us call the former kind of flux distributions as 'equal distributions'.

Incidently, the way of having equally distributing given amount of fluxes to many $U(1)$ factors is the same as the way of distributing same amount of momenta (in units of $\frac{k}{2}$) to multiple gravitons, each carrying positive KK-momenta. For instance, the first of the above two distributions maps to giving two units of KK-momenta to a single graviton, while the second maps to picking two gravitons and giving one unit of momentum to each. This might let one suspect that there could be some relation between the index from a saddle point with equal distribution and the multi-graviton index with the corresponding momentum distribution. What we find below empirically says that this is true up to a certain order in the x expansion. However, as we go beyond certain energy, it will turn out that only the total sum over all saddle points with equal distributions equals the total multi-graviton index. As one goes beyond an even higher energy threshold, the saddle points with unequal saddle points start to appear which add to the saddle points with equal distributions to correctly reproduce the graviton index.

The saddle points with unequal flux distributions provide examples in which monopole operators contribute to the energy, or the scaling dimension, of the whole gauge invariant operator. Monopole operators with vanishing scale dimensions are studied in [12,26,27] in $\mathcal{N}=6$ Chern–Simons theory, while in general this does not have to be the case. See [27,28] as well as recent works [45,46] on monopole operators in $\mathcal{N}=4$ and $\mathcal{N}=3$ theories.

Let us present various checks that we did in this sector. In this subsection we write $p \equiv \sqrt{y_1 y_2}$ and $r = \sqrt{\frac{y_1}{y_2}}$ to simplify the formulae. Let us define the following functions

$$\begin{aligned}
f(x,p,r,z) &= \sqrt{x}z\big(r+r^{-1}\big)+\sqrt{x}z^{-1}\big(p+p^{-1}\big)-x\sqrt{x}z\big(p+p^{-1}\big)\\
&\quad -x\sqrt{x}z^{-1}\big(r+r^{-1}\big),\\
F(x,p,r,z) &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f\big(x^n,p^n,r^n,z^n\big)\right)\\
&= \frac{(1-x\sqrt{x}pz)(1-x\sqrt{x}p^{-1}z)(1-x\sqrt{x}rz^{-1})(1-x\sqrt{x}r^{-1}z^{-1})}{(1-\sqrt{x}rz)(1-\sqrt{x}r^{-1}z)(1-\sqrt{x}pz^{-1})(1-\sqrt{x}p^{-1}z^{-1})},
\end{aligned}$$

which should be familiar from our analysis in the previous subsection, and also define

$$\begin{aligned}
f_m(x,p,r,z) &= x^m f(x,p,r,z),\\
F_m(x,p,r,z) &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_m\big(x^n,p^n,r^n,z^n\big)\right)\\
&= \frac{(1-x^{m+1}\sqrt{x}pz)(1-x^{m+1}\sqrt{x}p^{-1}z)(1-x^{m+1}\sqrt{x}rz^{-1})(1-x^{m+1}\sqrt{x}r^{-1}z^{-1})}{(1-x^m\sqrt{x}rz)(1-x^m\sqrt{x}r^{-1}z)(1-x^m\sqrt{x}pz^{-1})(1-x^m\sqrt{x}p^{-1}z^{-1})}.
\end{aligned}$$

Using these functions,

$$\begin{aligned}
&f_{|n_i-\tilde{n}_j|}\big(x,p,r,e^{i(\alpha_i-\tilde{\alpha}_j)}\big)+f_{|n_i-\tilde{n}_j|+2}\big(x,p,r,e^{i(\alpha_i-\tilde{\alpha}_j)}\big)+\cdots\\
&\quad + f_{n_i+\tilde{n}_j-2}\big(x,p,r,e^{i(\alpha_i-\tilde{\alpha}_j)}\big)
\end{aligned}\tag{3.21}$$

for $n_i \neq \tilde{n}_j$ is what we called $\mathbf{f}_{ij}^{\mathrm{bif}}$ in the previous section.

To compare $I^{(+)}$ with $I_{\mathrm{mp}}^{(+)}$ in the sector with two units of fluxes, i.e. at the order y_3^k, we write the integral expressions for indices from four saddle points. Getting rid of a trivial integral corresponding to the decoupled $U(1)$, one obtains the following expressions.

$$I_{\square\square\,\square\square} = \frac{1}{2\pi i}\oint\frac{dz}{z}z^{-2k}\frac{F(x,p,r,z)F_2(x,p,r,z)}{(1-x^4)^2}\tag{3.22}$$

where $z \equiv e^{i(\tilde{\alpha}-\alpha)}$,

$$\begin{aligned}
I_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \oint\frac{dz}{(2\pi i)z}z^{-2k}\oint\frac{da\,db}{(2\pi i)^2 ab}\left(1-\frac{a^2}{2}-\frac{1}{2a^2}\right)\left(1-\frac{b^2}{2}-\frac{1}{2b^2}\right)\\
&\quad\times\frac{F(x,p,r,zab)F(x,p,r,xab^{-1})F(x,p,r,za^{-1}b)F(x,p,r,za^{-1}b^{-1})}{(1-x^2)^4(1-x^2a^2)(1-x^2a^{-2})(1-x^2b^2)(1-x^2b^{-2})}
\end{aligned}\tag{3.23}$$

where $e^{i\alpha_1}=z^{-1/2}a$, $e^{i\alpha_2}=z^{-1/2}a^{-1}$, $e^{i\tilde{\alpha}_1}=z^{1/2}b$, $e^{i\tilde{\alpha}_2}=z^{1/2}b^{-1}$, and

$$\begin{aligned}
I_{\square\square\,\begin{smallmatrix}\square\\\square\end{smallmatrix}} = I_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\,\square\square} &= x^2\oint\frac{dz}{(2\pi i)z}z^{-2k}\oint\frac{da}{(2\pi i)a}\left(1-\frac{a^2}{2}-\frac{1}{2a^2}\right)\\
&\quad\times\frac{F_1(x,p,r,za)F_1(x,p,r,za^{-1})}{(1-x^4)^2(1-x^2)^2(1-x^2a^2)(1-x^2a^{-2})}
\end{aligned}\tag{3.24}$$

where $e^{i\alpha_1}=z^{-1/2}a$, $e^{i\alpha_2}=z^{-1/2}a^{-1}$, $e^{i\tilde{\alpha}}=z^{1/2}$ for $I_{\square\square\,\begin{smallmatrix}\square\\\square\end{smallmatrix}}$, and $e^{i\alpha}=z^{-1/2}$, $e^{i\tilde{\alpha}_1}=z^{1/2}a$, $e^{i\tilde{\alpha}_2}=z^{1/2}a^{-1}$ for $I_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\,\square\square}$. All contour integrals here and below are along the unit circle on the complex plane. The graviton index with two units of momenta is given by

$$I_{\text{mp}}^{(+)}(x,p,r)\big|_{y_3^k} = I_{2k}(x,p,r) + \frac{1}{2}I_k\left(x^2,p^2,r^2\right) + \frac{1}{2}I_k(x,p,r)^2. \tag{3.25}$$

The first term is from single graviton with two units of momenta, while the last two terms are from two identical gravitons, each with unit momentum. We expect the sum of four contributions (3.22), (3.23), (3.24) to equal (3.25).

We first study the lowest energy states in this sector analytically. As in the previous section, we arrange the Taylor expansions of the integrands in ascending orders in $\#(\sqrt{x})$–$\#(z)$. From the behaviors of the functions F, F_1, F_2, one finds that unequal distributions do not contribute to the lowest energy sector. The two equal distributions contribute with $\epsilon = k$ as

$$\begin{aligned} I_{\square\square\,\square\square} &\to \oint \frac{dz}{(2\pi i)z} z^{-2k} \frac{1}{(1-\sqrt{\frac{xy_1}{y_2}}z)(1-\sqrt{\frac{xy_2}{y_1}}z)} \\ &= x^k\left(y_1^k y_2^{-k} + y_1^{k-1}y_2^{-k+1} + \cdots + y_1^{-k}y_2^k\right) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} I_{\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}} \to \oint & \frac{dz}{(2\pi i)z} z^{-2k} \oint \frac{d\nu}{(2\pi i)\nu} \oint \frac{d\mu}{(2\pi i)\mu} (1-\mu\nu)\left(1-\mu\nu^{-1}\right) \\ &\times \left[(1-tz\mu)(1-uz\mu)\left(1-tz\mu^{-1}\right)\left(1-uz\mu^{-1}\right)(1-tz\nu)(1-uz\nu)\right. \\ &\times \left.\left(1-tz\nu^{-1}\right)\left(1-uz\nu^{-1}\right)\right]^{-1} \end{aligned} \tag{3.27}$$

where we redefined integration variables and the chemical potentials as $a^2 = \mu\nu$ and $b^2 = \mu\nu^{-1}$, $x^{\frac{1}{2}}r = t$, $x^{\frac{1}{2}}r^{-1} = u$. The two factors on the first line in the integrand come from the $U(2)\times U(2)$ measure, which we can effectively change to $(1-a^2)(1-b^2)$ since rest of the integrand is invariant under $a \to a^{-1}$ and $b \to b^{-1}$, separately.

From the results in Appendix C, the two graviton indices at the lowest energy are given by (upon identifying $z = \sqrt{y_3}$)

$$I_{2k} \to \oint \frac{dz}{(2\pi i)z} z^{-2k} \frac{1}{(1-tz)(1-uz)} \stackrel{\text{lowest}}{=} I_{\square\square\,\square\square} \tag{3.28}$$

and

$$\begin{aligned} \frac{1}{2}I_k\left(\cdot^2\right) + \frac{1}{2}I_k(\cdot)^2 \to \frac{1}{2}&\oint \frac{dz}{(2\pi i)z} z^{-2k} \frac{1}{(1-t^2z^2)(1-u^2z^2)} \\ &+ \frac{1}{2}\oint \frac{dz_1 dz_2}{(2\pi i)^2 z_1 z_2} z_1^{-k} z_2^{-k} \frac{1}{(1-tz_1)(1-uz_1)(1-tz_2)(1-uz_2)} \\ = &\oint \frac{dz}{(2\pi i)z} z^{-2k} \left[\frac{1}{2(1-t^2z^2)(1-u^2z^2)}\right. \\ &+ \left.\frac{1}{2}\oint \frac{d\nu}{(2\pi i)\nu} \frac{1}{(1-tz\nu)(1-uz\nu)(1-tz\nu^{-1})(1-uz\nu^{-1})}\right] \end{aligned} \tag{3.29}$$

where we changed to variables $z_1 = z\nu$, $z_2 = z\nu^{-1}$ on the last line. Keeping the z integral, we contour-integrate μ and ν in (3.27) and (3.29) to compare them. After some algebra, one finds

$$\frac{1}{(1-t^2z^2)(1-u^2z^2)(1-tuz^2)} + \text{const} \tag{3.30}$$

for both indices, where the last constant is 0 and -1 for gauge theory and gravity, respectively, which does not survive the remaining z integration. This shows the agreement of the indices from gauge theory and gravity at the lowest energy with two fluxes. The indices from two saddle points separately agree with two corresponding gravity indices, as explained before. This result can also be obtained by quantizing the moduli space [12], restricted to the fields $\bar{B}^{\dot{1}}$ and $\bar{B}^{\dot{2}}$.

Since we are not aware of any further way of treating the integral analytically, we compare the two indices order by order after expanding in x. Firstly, for $k=1$, we find a perfect agreement to $\mathcal{O}(x^9)$ terms that we checked, as follows. We find

$$\begin{aligned}
I_{\square\square\,\square\square} = {} & x\left(r^2+1+r^{-2}\right)+x^2\left(p+p^{-1}\right)\left(r^3+r^{-3}\right) \\
& +x^3\left[\left(p^2+p^{-2}\right)\left(r^4+r^{-4}\right)-2\left(r^2+r^{-2}\right)\right] \\
& +x^4\left[\left(p^3+p^{-3}\right)\left(r^5+r^{-5}\right)+\left(p+p^{-1}\right)\left(r+r^{-1}\right)\right] \\
& +x^5\left[\left(p^4+p^{-4}\right)\left(r^6+r^{-6}\right)+\left(r^4-2r^2-4-2r^{-2}+r^{-4}\right)\right] \\
& +x^6\left[\left(p^5+p^{-5}\right)\left(r^7+r^{-7}\right)+\left(p+p^{-1}\right)\left(-2r^3+2r\cdots\right)\right] \\
& +x^7\left[\left(p^6+p^{-6}\right)\left(r^8+r^{-8}\right)+\left(p^2+p^{-2}\right)\left(r^2+r^{-2}\right)+\left(r^4-3+r^{-4}\right)\right] \\
& +x^8\left[\left(p^7+p^{-7}\right)\left(r^9+r^{-9}\right)+\left(p+p^{-1}\right)\left(r^5-2r^3-2r\cdots\right)\right] \\
& +x^9\left[\left(p^8+p^{-8}\right)\left(r^{10}+r^{-10}\right)+\left(p^2+p^{-2}\right)\left(-2r^4+2r^2+3\cdots\right)\right. \\
& \left.+\left(-r^4+5r^2+6\cdots\right)\right]+\mathcal{O}\left(x^{10}\right),
\end{aligned}$$

$$\begin{aligned}
I_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\begin{smallmatrix}\square\\\square\end{smallmatrix}} = {} & x\left(r^2+1+r^{-2}\right)+x^2\left(p+p^{-1}\right)\left(r^3+r+r^{-1}+r^{-3}\right) \\
& +x^3\left[\left(p^2+p^{-2}\right)\left(2r^4+r^2+1+r^{-2}+2r^{-4}\right)+\left(r^4+r^{-4}\right)\right] \\
& +x^4\left[\left(p^3+p^{-3}\right)\left(2r^5+r^3+r\cdots\right)+\left(p+p^{-1}\right)\left(r^5-r^3-r\cdots\right)\right] \\
& +x^5\left[\left(p^4+p^{-4}\right)\left(3r^6+r^4+r^2+1\cdots\right)+\left(p^2+p^{-2}\right)\left(r^6-r^4\cdots\right)\right. \\
& \left.+\left(r^6-r^4+3\cdots\right)\right]+x^6\left[\left(p^5+p^{-5}\right)\left(3r^7+r^5+r^3+r\cdots\right)\right. \\
& \left.+\left(p^3+p^{-3}\right)\left(r^7-r^5\cdots\right)+\left(p+p^{-1}\right)\left(r^7+r^3\cdots\right)\right] \\
& +x^7\left[\left(p^6+p^{-6}\right)\left(4r^8+r^6+r^4+r^2+1\cdots\right)+\left(p^4+p^{-4}\right)\left(r^8-r^6\cdots\right)\right. \\
& \left.+\left(p^2+p^{-2}\right)\left(r^8-r^4-r^2-2\cdots\right)+\left(r^8-r^4-3r^2-1\cdots\right)\right] \\
& +x^8\left[\left(p^7+p^{-7}\right)\left(4r^9+r^7+r^5+r^3+r\cdots\right)+\left(p^5+p^{-5}\right)\left(r^9-r^7\cdots\right)\right. \\
& \left.+\left(p^3+p^{-3}\right)\left(r^9-r^5\cdots\right)+\left(p+p^{-1}\right)\left(r^9-r^5+r^3+4r\cdots\right)\right] \\
& +x^9\left[\left(p^8+p^{-8}\right)\left(5r^{10}+r^8+r^6+r^4+r^2+1\cdots\right)+\left(p^6+p^{-6}\right)\left(r^{10}-r^8\cdots\right)\right. \\
& +\left(p^4+p^{-4}\right)\left(r^{10}-r^6\cdots\right)+\left(p^2+p^{-2}\right)\left(r^{10}+r^4-1\cdots\right) \\
& \left.+\left(r^{10}+r^4-2r^2-3\cdots\right)\right]+\mathcal{O}\left(x^{10}\right),
\end{aligned}$$

$$\begin{aligned}
I_{\square\square\begin{smallmatrix}\square\\\square\end{smallmatrix}} = I_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\square\square} = {} & x^5-x^6\left(p+p^{-1}\right)\left(r+r^{-1}\right)+x^7\left[\left(p^2+p^{-2}\right)+\left(r^2+3+r^{-2}\right)\right] \\
& -x^8\left(p+p^{-1}\right)\left(r+r^{-1}\right)-x^9\left[\left(p^2+p^{-2}\right)\left(r^2+1+r^{-2}\right)\right. \\
& \left.+2\left(r^2+r^{-2}\right)\right]+\mathcal{O}\left(x^{10}\right)
\end{aligned}$$

and on the gravity side we find

$$
\begin{aligned}
I_2(x,p,r) = {} & x\big(r^2+1+r^{-2}\big)+x^2\big(p+p^{-1}\big)\big(r^3+r^{-3}\big) \\
& + x^3\big[\big(p^2+p^{-2}\big)\big(r^4+r^{-4}\big)-\big(r^2+r^{-2}\big)\big]+x^4\big(p^3+p^{-3}\big)\big(r^5+r^{-5}\big) \\
& + x^5\big[\big(p^4+p^{-4}\big)\big(r^6+r^{-6}\big)-\big(r^2+r^{-2}\big)\big]+x^6\big(p^5+p^{-5}\big)\big(r^7+r^{-7}\big) \\
& + x^7\big[\big(p^6+p^{-6}\big)\big(r^8+r^{-8}\big)-\big(r^2+r^{-2}\big)\big]+x^8\big(p^7+p^{-7}\big)\big(r^9+r^{-9}\big) \\
& + x^9\big[\big(p^8+p^{-8}\big)\big(r^{10}+r^{-10}\big)-\big(r^2+r^{-2}\big)\big]+\mathcal{O}\big(x^{10}\big)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2}I_1\big(x^2,p^2,r^2\big)+\frac{1}{2}I_1(x,p,r)^2 \\
& \quad = x\big(r^2+1+r^{-2}\big)+x^2\big(p+p^{-1}\big)\big(r^3+r\cdots\big)+x^3\big[\big(p^2+p^{-2}\big)\big(2r^4+r^2+1\cdots\big) \\
& \qquad + \big(r^4-r^2\cdots\big)\big]+x^4\big[\big(p^3+p^{-3}\big)\big(2r^5+r^3+r\cdots\big)+\big(p+p^{-1}\big)\big(r^5-r^3\cdots\big)\big] \\
& \qquad + x^5\big[\big(p^4+p^{-4}\big)\big(3r^6+r^4+r^2+1\cdots\big)+\big(p^2+p^{-2}\big)\big(r^6-r^4\cdots\big) \\
& \qquad + \big(r^6-r^2+1\cdots\big)\big]+x^6\big[\big(p^5+p^{-5}\big)\big(3r^7+r^5+r^3+r\cdots\big) \\
& \qquad + \big(p^3+p^{-3}\big)\big(r^7-r^5\cdots\big)+\big(p+p^{-1}\big)\big(r^7-r^3\cdots\big)\big] \\
& \qquad + x^7\big[\big(p^6+p^{-6}\big)\big(4r^8+r^6+r^4+r^2+1\cdots\big)+\big(p^4+p^{-4}\big)\big(r^8-r^6\cdots\big) \\
& \qquad + \big(p^2+p^{-2}\big)\big(r^8-r^4\cdots\big)+\big(r^8+2+r^{-8}\big)\big] \\
& \qquad + x^8\big[\big(p^7+p^{-7}\big)\big(4r^9+r^7+r^5+r^3+r\cdots\big)+\big(p^5+p^{-5}\big)\big(r^9-r^7\cdots\big) \\
& \qquad + \big(p^3+p^{-3}\big)\big(r^9-r^5\cdots\big)+\big(p+p^{-1}\big)\big(r^9-r^3\cdots\big)\big] \\
& \qquad + x^9\big[\big(p^8+p^{-8}\big)\big(5r^{10}+r^8+r^6+r^4+r^2+1\cdots\big)+\big(p^6+p^{-6}\big)\big(r^{10}-r^8\cdots\big) \\
& \qquad + \big(p^4+p^{-4}\big)\big(r^{10}-r^6\cdots\big)+\big(p^2+p^{-2}\big)\big(r^{10}-r^4\cdots\big)+\big(r^{10}+3+r^{-10}\big)\big] \\
& \qquad + \mathcal{O}\big(x^{10}\big).
\end{aligned}
$$

The terms in '$\cdots$' take negative powers of r, and can be completed from the fact that the expression in any parenthesis has $r\to r^{-1}$ invariance. From this one can check

$$
I_{\square\square\,\square\square}+I_{\substack{\square\\\square}\substack{\square\\\square}}+I_{\square\square\substack{\square\\\square}}+I_{\substack{\square\\\square}\square\square}=I_2(x,p,r)+\frac{1}{2}I_1\big(x^2,p^2,r^2\big)+\frac{1}{2}I_1(x,p,r)^2+\mathcal{O}\big(x^{10}\big)
$$

for $k=1$.

We also found perfect agreement for $k=2$ for all terms that we calculated. To reduce execution time for numerical calculation, space of presenting our result and most importantly to reduce possible typos, we set $r=p=1$ and keep x only. We find

$$
\begin{aligned}
I_{\square\square\,\square\square} = {} & 5x^2+4x^3+0x^4+8x^5-2x^6+4x^7+4x^8+4x^9-2x^{10}+8x^{11} \\
& + 5x^{12}+\mathcal{O}\big(x^{13}\big), \\
I_{\substack{\square\\\square}\substack{\square\\\square}} = {} & 6x^2+12x^3+18x^4+16x^5+29x^6+28x^7+32x^8+44x^9+29x^{10}+72x^{11} \\
& + 31x^{12}+\mathcal{O}\big(x^{13}\big), \\
I_{\square\square\substack{\square\\\square}} = {} & I_{\substack{\square\\\square}\square\square}=x^8-4x^9+10x^{10}-16x^{11}+11x^{12}+\mathcal{O}\big(x^{13}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4(x) = {} & 5x^2+4x^3+2x^4+4x^5+2x^6+4x^7+2x^8+4x^9+2x^{10}+4x^{11} \\
& + 2x^{12}+\mathcal{O}\big(x^{13}\big),
\end{aligned}
$$

$$\frac{I_2(x^2)+I_2(x)^2}{2} = 6x^2+12x^3+16x^4+20x^5+25x^6+28x^7+36x^8+36x^9+45x^{10} + 44x^{11}+56x^{12}+\mathcal{O}\big(x^{13}\big)$$

which proves

$$I_{\ydiagram{2}\,\ydiagram{2}} + I_{\ydiagram{1,1}\,\ydiagram{1,1}} + I_{\ydiagram{2}\,\ydiagram{1,1}} + I_{\ydiagram{1,1}\,\ydiagram{2}} = I_4(x) + \frac{1}{2}I_2\big(x^2\big) + \frac{1}{2}I_2(x)^2 + \mathcal{O}\big(x^{13}\big)$$

for $k=2$.

One might think that theories with $k=1,2$ are somewhat special since we expect enhancement of supersymmetry to $\mathcal{N}=8$ [12]. To ensure that the agreement has nothing to do with this property, we also check the case with $k=3$. We find

$$\begin{aligned}
I_{\ydiagram{2}\,\ydiagram{2}} &= 7x^3+4x^4+0x^5+8x^6-2x^7+4x^8+4x^9+4x^{10}-2x^{11}+8x^{12} \\
&\quad + 0x^{13}+\mathcal{O}\big(x^{14}\big), \\
I_{\ydiagram{1,1}\,\ydiagram{1,1}} &= 10x^3+16x^4+20x^5+20x^6+31x^7+32x^8+36x^9+40x^{10}+49x^{11}+52x^{12} \\
&\quad + 40x^{13}+\mathcal{O}\big(x^{14}\big), \\
I_{\ydiagram{2}\,\ydiagram{1,1}} &= I_{\ydiagram{1,1}\,\ydiagram{2}} = x^{11}-4x^{12}+10x^{13}+\mathcal{O}\big(x^{14}\big)
\end{aligned}$$

and

$$\begin{aligned}
I_6(x) &= 7x^3+4x^4+2x^5+4x^6+2x^7+4x^8+2x^9+4x^{10}+2x^{11}+4x^{12} \\
&\quad + 2x^{13}+\mathcal{O}\big(x^{14}\big), \\
\frac{I_3(x^2)+I_3(x)^2}{2} &= 10x^3+16x^4+18x^5+24x^6+27x^7+32x^8+38x^9+40x^{10}+47x^{11} \\
&\quad + 48x^{12}+58x^{13}+\mathcal{O}\big(x^{14}\big)
\end{aligned}$$

proving

$$I_{\ydiagram{2}\,\ydiagram{2}} + I_{\ydiagram{1,1}\,\ydiagram{1,1}} + I_{\ydiagram{2}\,\ydiagram{1,1}} + I_{\ydiagram{1,1}\,\ydiagram{2}} = I_6(x) + \frac{1}{2}I_3\big(x^2\big) + \frac{1}{2}I_3(x)^2 + \mathcal{O}\big(x^{14}\big)$$

for $k=3$.

In these examples, the two saddle points with equal distributions start to deviate from their 'corresponding' graviton indices at two orders higher than the lowest energy. The saddle points with unequal distributions start to enter at $2k+2$ orders higher than the lowest level. The $2k$ comes from the energy shifts in the letter indices (3.8), while 2 comes from the Casimir energy shift $\epsilon_0 = 2$ for the saddle points $\ydiagram{2}\,\ydiagram{1,1}$ and $\ydiagram{1,1}\,\ydiagram{2}$. See Table 2 in Appendix B.3.

3.3. Three KK-momenta: numerical tests

We consider the case with $k=1$ only. The gauge theory indices are given by

$$\begin{aligned}
I_{\ydiagram{3}\,\ydiagram{3}} &= x^{\frac{3}{2}}\big(4+4x+0x^2+8x^3-4x^4+8x^5+2x^6+4x^7+0x^8+\mathcal{O}\big(x^9\big)\big), \\
I_{\ydiagram{2,1}\,\ydiagram{2,1}} &= x^{\frac{3}{2}}\big(6+20x+24x^2+28x^3+64x^4+34x^5+34x^6+166x^7-32x^8+\mathcal{O}\big(x^9\big)\big), \\
I_{\ydiagram{1,1,1}\,\ydiagram{1,1,1}} &= x^{\frac{3}{2}}\big(4+12x+30x^2+52x^3+52x^4+98x^5+170x^6+130x^7+106x^8+\mathcal{O}\big(x^9\big)\big)
\end{aligned} \tag{3.31}$$

and

$$
\begin{aligned}
&I_{\ydiagram{2,1}\,\ydiagram{1,1,1}} = I_{\ydiagram{1,1,1}\,\ydiagram{2,1}} = x^{\frac{3}{2}}\left(0x^4 + 6x^5 - 10x^6 - 22x^7 + 88x^8 + \mathcal{O}\left(x^9\right)\right),\\
&I_{\ydiagram{3}\,\ydiagram{2,1}} = I_{\ydiagram{2,1}\,\ydiagram{3}} = x^{\frac{3}{2}}\left(2x^6 - 8x^7 + 16x^8 + \mathcal{O}\left(x^9\right)\right),\\
&I_{\ydiagram{3}\,\ydiagram{1,1,1}} = I_{\ydiagram{1,1,1}\,\ydiagram{3}} = x^{\frac{3}{2}}\left(\mathcal{O}\left(x^{12}\right)\right).
\end{aligned}
\tag{3.32}
$$

The last three pairs of saddle points with unequal distributions are expected to enter from $(2k+2)$th, $(4k+2)$th and $(6k+6)$th level above the lowest level for general k, respectively. Coefficients 0 written above could be accidental.[5]

From gravity, one can construct states carrying three units of momenta in the following three ways: one graviton carrying three momenta, one graviton with one momentum and another graviton with two momenta, three identical particles where each carries one momentum. Three contributions are

$$
\begin{aligned}
&I_3(x) = x^{\frac{3}{2}}\left(4 + 4x + 2x^2 + 4x^3 + 2x^4 + 4x^5 + 2x^6 + 4x^7 + 2x^8 + \mathcal{O}\left(x^9\right)\right),\\
&I_1(x)I_2(x) = x^{\frac{3}{2}}\left(6 + 20x + 26x^2 + 36x^3 + 46x^4 + 52x^5 + 66x^6 + 68x^7\right.\\
&\qquad\qquad\quad \left.+ 86x^8 + \mathcal{O}\left(x^9\right)\right),\\
&\frac{1}{3}I_1\left(x^3\right) + \frac{1}{2}I_1(x)I_1\left(x^2\right) + \frac{1}{6}I_1(x)^3\\
&\quad = x^{\frac{3}{2}}\left(4 + 12x + 26x^2 + 48x^3 + 64x^4 + 96x^5 + 122x^6 + 168x^7 + 194x^8 + \mathcal{O}\left(x^9\right)\right).
\end{aligned}
$$

From this we find

$$
\begin{aligned}
&I_{\ydiagram{3}\,\ydiagram{3}} + I_{\ydiagram{2,1}\,\ydiagram{2,1}} + I_{\ydiagram{1,1,1}\,\ydiagram{1,1,1}} + 2I_{\ydiagram{2,1}\,\ydiagram{1,1,1}} + 2I_{\ydiagram{3}\,\ydiagram{2,1}} + 2I_{\ydiagram{3}\,\ydiagram{1,1,1}}\\
&\quad = I_3(x) + I_1(x)I_2(x) + \frac{1}{3}I_1\left(x^3\right) + \frac{1}{2}I_1(x)I_1\left(x^2\right) + \frac{1}{6}I_1(x)^3 + \mathcal{O}\left(x^{\frac{3}{2}+9}\right),
\end{aligned}
\tag{3.33}
$$

which is again a perfect agreement.

4. Conclusion

In this paper we calculated the complete superconformal index for $\mathcal{N} = 6$ Chern–Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$ at level k. The low energy and large N limit shows a perfect agreement with the index over supersymmetric gravitons in M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ in all sample calculations we did.

Though we strongly suspect that the two large N indices will completely agree, it would definitely be desirable to develop a general proof of this claim. Since the two indices assume very different forms apparently, this would be a nontrivial check of the AdS_4/CFT_3 proposal based on superconformal Chern–Simons theories. Perhaps identities of unitary matrix integrals similar to those explored in [47] could be found to show this.

[5] Unfortunately, we could not go to $\mathcal{O}(x^{12})$ where one can start testing the last line of (3.32), due to the long execution time. The bottleneck was at the third line of (3.31), in which we had to integrate over the $U(3) \times U(3)$ holonomy with nine factors of F functions, etc., in the integrand. We hope we can improve our calculation in the near future. We thank Sehun Chun for his advice.

We expect our result to provide hints towards a better understanding of 't Hooft operators and their roles in AdS_4/CFT_3. In particular we found that monopole operators in nontrivial representations, beyond those considered in [12,26,27], played important roles for the agreement of gauge/gravity indices. In our calculation in the deformed theory, the degrees of freedom in the vector multiplets turned out to be important, starting from nontrivial contribution to the determinant and Casimir energy. It would be interesting to understand it directly in the physical Chern–Simons-matter theory, in which there are no propagating degrees for the fields in vector multiplets. In a preliminary study, we find that interaction between gauge fields and matters in background fluxes transmutes some of the matter scalar degrees into vector-like ones [48].

We have kept the chemical potentials to be finite and order 1 as we take the large N limit to obtain the low energy index. This setting is also partly motivated from the physics that one expects from the *partition function*, namely a deconfinement phase transition at order 1 temperature. In the context of 4-dimensional Yang–Mills theory, a first order deconfinement transition is found in [30–32]. Although the index does not see this either in the $d=4$ $\mathcal{N}=4$ Yang–Mills theory or here, possibly due to cancelations from $(-1)^F$, it is not clear to us whether this means that the trace of (supersymmetric) black holes and deconfined phase is completely absent. It is a famous fact that the situation in $d=3$ is more mysterious due to the replacement of N^2 by $N^{\frac{3}{2}}$, or $N^{\frac{3}{2}}\sqrt{k}=\frac{N^2}{\sqrt{\lambda}}$ in our case, in the 'deconfined' degrees of freedom. Another new aspect in $d=3$ compared to $d=4$ is the presence of sectors with topological charges. It might be interesting to systematically investigate finite N effects in the flux distributions.

We think the finite N and k index that we obtained can be straightforwardly extended to other superconformal Chern–Simons theories. This can be used to solidly test many interesting ideas in these theories. For example, a non-perturbative 'parity duality' and its generalization were proposed by [16] for $\mathcal{N}=5,6$ Chern–Simons theories with gauge groups $O(M)\times Sp(N)$ and $U(M)\times U(N)$, respectively, based on the study of their gravity duals. The details of duality transformation involves changes of parameters M, N and the Chern–Simons level k. The information on these parameters is of course wiped out in the index in 't Hooft limit [24], and perhaps also in the large M, N limit keeping k finite. The index with finite M, N, k should have a delicate structure for the duality to hold. It would be interesting to explore this. An analysis of finite N indices for a class of 4-dimensional SCFT was reported in [47,49].

For a class of superconformal Chern–Simons theories, the superconformal index exhibits an interesting large N phase transition. For instance, it was explained that the index for $\mathcal{N}=2,3$ $U(N)_k$ Chern–Simons theories with N_f flavors (presented in [36]) can undergo third order phase transitions [3] in the so-called Veneziano limit, which is very similar to that in the lattice gauge theory explored by Gross, Witten and Wadia [50,51]. An interesting related issue is a proposal by Giveon and Kutasov on Seiberg duality in these theories [52], based on the study of brane constructions. See also [53]. The $\mathcal{N}=3$ models were suggested to be self strong-weak dual in that $\frac{N}{k}$ cannot be small for both of the Seiberg-dual pair theories. In some cases, it seems that the calculation of [3] applies to only one of the two theories in the pair. We think that it could be important to consider the sectors with magnetic fluxes to see the dual phase transition. We hope to come back to this problem in a near future.

Finally, there have been active studies on superconformal Chern–Simons theories which could be relevant to condensed matter systems. For example, nonrelativistic versions of Chern–Simons-matter theories were studied and some properties of the superconformal indices were pointed out [54]. The study of the last references in [54] suggests that monopole operators are expected to play important roles. More recent works include Chern–Simons-matter theories with fundamental matters and their gravity duals [46,55]. It should be interesting to apply our methods to these examples.

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Appendix A. Notes on radial quantization

In this appendix we summarize the conversion between conformal field theories on $\mathbb{R}^{2+1}$ and $S^2\times\mathbb{R}$ via radial quantization. In particular we would like to obtain the action on the latter space starting from that on the former. The first procedure is analytic continuation $x^0=-ix^3_E$ to $\mathbb{R}^3$ and obtaining the action in Euclidean $S^2\times\mathbb{R}$ by setting $r=e^\tau$, where τ is the time of the latter space. One may then continue back to Lorentzian $S^2\times\mathbb{R}$ with Lorentzian time $t=-i\tau$, though in this paper we mainly consider the Euclidean theory.

We first consider kinetic terms, involving derivatives acting on fields. It suffices to consider free fields with ordinary derivatives: covariantizing with gauge fields would take obvious forms once the gauge fields are appropriately transformed into fields in $S^2\times\mathbb{R}$ [39]. Let us start from a (real) scalar field Φ on $\mathbb{R}^3$ with scale dimension $\frac{1}{2}$. The field Φ_S on $S^2\times\mathbb{R}$ is related to Φ as

$$\Phi=r^{-\frac{1}{2}}\Phi_S. \tag{A.1}$$

The kinetic term on $\mathbb{R}^3$ can be rewritten as

$$\begin{aligned}\int d^3x\,(\nabla_{\mathbb{R}^3}\Phi)^2&=\int \mathrm{vol}_{S^2}\,r^2\,dr\left[\partial_r\big(r^{-\frac{1}{2}}\Phi_S\big)^2+\frac{1}{r^2}\nabla_{S^2}\big(r^{-\frac{1}{2}}\Phi_S\big)^2\right]\\&=\int \mathrm{vol}_{S^2}\,d\tau\left[(\partial_\tau\Phi_S)^2+(\nabla_{S^2}\Phi_S)^2+\frac{1}{4}(\Phi_S)^2\right]\end{aligned} \tag{A.2}$$

after eliminating surface terms. This is nothing but the action for a scalar conformally coupled to the curvature. This result applies to all eight real scalars $A_a, B_{\dot a}$.

The gauge fields $A_\mu, \tilde A_\mu$ and scalars $\sigma,\tilde\sigma$ with dimensions 1 can also be transformed appropriately. We do not present the result here since we will mostly work directly in $\mathbb{R}^3$ when we consider these fields. One may see [39] for the details on $\mathbb{R}^4$, which is essentially the same as our case. In particular, $\sigma=r^{-1}\sigma_S$, $A_r=r^{-1}(A_S)_\tau$ and $A_a=(A_S)_a$ for $a=\theta,\phi$. We note that the Chern–Simons term takes the same form on $S^2\times\mathbb{R}$.

We also consider complex matter fermions Ψ_α with dimension 1 whose (Euclidean) kinetic term is given by

$$-i\bar\Psi^\alpha\big(\gamma^m\big)_{\alpha\beta}\nabla_m\Psi^\beta=\bar\Psi_\alpha\big(\bar\sigma^m\big)^{\alpha\beta}\nabla_m\Psi_\beta\quad(m=1,2,3\text{ for Cartesian coordi.}). \tag{A.3}$$

In the representation with $\gamma^m_E=(-\sigma^2,\sigma^1,\sigma^3)$, one obtains $\bar\sigma^m=(1,i\sigma^3,-i\sigma^1)$ or an $SO(3)$ rotated one as explained in Section 2.1. Since we are considering fermions, we should first change our dreibein frame on $\mathbb{R}^3$ from Cartesian to spherical curvilinear one:

$$e^r = dr, \qquad e^\theta = r\,d\theta, \qquad e^\phi = r\sin\theta\,d\phi. \tag{A.4}$$

This frame is related to the Cartesian frame by a local $SO(3)$ transformation which we call Λ. One obtains

$$\begin{aligned}\bar\Psi^\alpha\left(\gamma_E^m\right)_\alpha{}^\beta\nabla_m\Psi_\beta &= \bar\Psi^\alpha_{\mathrm{cur}}\left[U\left(\Lambda^\dagger\right)\gamma_E^m U(\Lambda)\right]_\alpha{}^\beta\Lambda_m{}^n E_n^\mu\nabla_\mu\Psi_{\mathrm{cur}\beta}\\ &= \bar\Psi^\alpha_{\mathrm{cur}}\left(\gamma_E^n\right)_\alpha{}^\beta E_n^\mu\nabla_\mu\Psi_{\mathrm{cur}\beta},\end{aligned} \tag{A.5}$$

where $U(\Lambda)$ is the spinor representation of Λ, $\Psi = U(\Lambda)\Psi_{\mathrm{cur}}$, and E_m is the inverse of the above frame. At the last step we used the fact that action of any local $SO(3)$ leaves $(\gamma^m)_\alpha^\beta$ invariant. Here the final derivative ∇_μ is covariantized with the following spin connection (yet with zero curvature):

$$\begin{aligned}&\omega^{\theta r} = \frac{1}{r}e^\theta = d\theta, \qquad \omega^{\phi r} = \frac{1}{r}e^\phi = \sin\theta\,d\phi,\\ &\omega^{\phi\theta} = \frac{\cot\theta}{r}e^\phi = \cos\theta\,d\phi: \quad \nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{mn}\gamma_{mn}.\end{aligned} \tag{A.6}$$

Finally we change the field Ψ_{cur} to Ψ_S living on $S^2\times\mathbb{R}$, according to its dimension 1, as

$$\Psi_{\mathrm{cur}} = r^{-1}\Psi_S. \tag{A.7}$$

This yields

$$\begin{aligned}&\int \mathrm{vol}_{S^2}\, r^2\,dr\,\bar\Psi^\alpha\left(\gamma_E^m\right)_\alpha^\beta\nabla_m\Psi_\beta\\ &\quad= \int \mathrm{vol}_{S^2}\,dr\left[\bar\Psi_S^\alpha\left(\gamma_E^n\right)_\alpha{}^\beta E_n^\mu\nabla_\mu\Psi_{S\beta} + \frac{1}{r}\bar\Psi_S^\alpha\left(\sigma^2\right)_\alpha{}^\beta\Psi_{S\beta}\right].\end{aligned} \tag{A.8}$$

The covariant derivative ∇_μ is still that on $\mathbb{R}^3$. Since we are trying to obtain an action on $S^2\times\mathbb{R}$, with metric changing from $ds^2_{\mathbb{R}^3} = dr^2 + r^2\,ds^2_{S^2} = r^2\,ds^2_{S^2\times\mathbb{R}}$ to $ds^2_{S^2\times\mathbb{R}}$, we rewrite the spin connection of $\mathbb{R}^3$ in terms of that of $S^2\times\mathbb{R}$. The covariant derivatives are related as

$$\nabla = \nabla_S + \frac{1}{2}\left(\frac{1}{r}e^\theta\gamma_{\theta r} + \frac{1}{r}e^\phi\gamma_{\phi r}\right) = \nabla_S - \frac{i}{2r}\left(e^\theta\sigma^3 - e^\phi\sigma^1\right). \tag{A.9}$$

Thus one finds that

$$\gamma_E^n E_n^\mu\nabla_\mu = \frac{1}{r}\gamma_E^n(E_S)_n^\mu(\nabla_S)_\mu - \frac{1}{r}\sigma^2, \tag{A.10}$$

where E_S denotes the inverse frame for the metric $ds^2_{S^2\times\mathbb{R}}$. The final Lagrangian for Ψ_S is

$$\int \mathrm{vol}_{S^2}\,d\tau\,\bar\Psi_S\gamma_E^m(E_S)_m^\mu(\nabla_S)_\mu\Psi_S, \tag{A.11}$$

where the second term in (A.8) is canceled by that in (A.10). Note that there is no analogue of conformal mass terms for scalars, which is well known from literatures on CFT in curved spaces [56].

Finally we consider the kinetic term for the gaugino fields λ_α, $\tilde\lambda_\alpha$ in the Q-exact deformation

$$\int d^3x\, r\lambda\gamma_E^m\nabla_m\bar\lambda, \tag{A.12}$$

and a similar term for $\tilde{\lambda}$. These fields have dimensions $\frac{3}{2}$. The analysis is almost the same as the previous paragraph, apart from the fact that the fields on $S^2 \times \mathbb{R}$ are given by

$$\lambda_{\text{cur}} = r^{-\frac{3}{2}} \lambda_S, \qquad \tilde{\lambda}_{\text{cur}} = r^{-\frac{3}{2}} \tilde{\lambda}_S. \tag{A.13}$$

Therefore the step analogous to (A.8) yields

$$\int d^3x \, r \lambda^\alpha \left(\gamma_E^m\right)_\alpha{}^\beta \nabla_m \bar{\lambda}_\beta = \int \text{vol}_{S^2} \, dr \left[\lambda_S^\alpha \left(\gamma_E^m\right)_\alpha{}^\beta E_m^\mu \nabla_\mu \bar{\lambda}_{S\beta} + \frac{3}{2r} \lambda_S^\alpha \left(\sigma^2\right)_\alpha{}^\beta \bar{\lambda}_{S\beta}\right]. \tag{A.14}$$

As in the previous paragraph, rewriting the covariant derivative on $\mathbb{R}^3$ in terms of that on $S^2 \times \mathbb{R}$ provides a term $-\frac{1}{r}\lambda_S^\alpha(\sigma^2)_\alpha{}^\beta \bar{\lambda}_{S\beta}$, which in this case *does not* completely cancel the second term in (A.14). The final kinetic term for λ_S is, in terms of $(\sigma^\mu)_{\alpha\beta} = (1, i\vec{\sigma})$,

$$\int \text{vol}_{S^2} \, d\tau \, \lambda_S^\alpha \left[\left(\sigma^m\right)_{\alpha\beta} (E_S)_m^\mu (\nabla_S)_\mu - \frac{1}{2}\delta_{\alpha\beta}\right] \bar{\lambda}_S^\beta, \tag{A.15}$$

where $\delta_{\alpha\beta}$ comes from $(\bar{\sigma}^0)_{\alpha\beta}$. Action for $\tilde{\lambda}_S$ is similar.

The terms in the action which do not involve derivatives, such as potential, Yukawa interaction, etc., transform rather obviously. Starting from $\int d^3x \, \mathcal{L}(\Phi)$ and transforming the fields according to their dimensions Δ as $\Phi = r^{-\Delta}\Phi_S$, one obtains $\int_{S^2 \times \mathbb{R}} \mathcal{L}(\Phi_S)$.

Appendix B. Details of 1-loop calculation

The 1-loop determinant comes from two contributions: firstly from the quadratic fluctuations of the matter fields and secondly from the fields in vector multiplets. We explain them in turn.

B.1. Determinant from matter fields

We first consider the matter scalar fields. The quadratic action for the scalars (on $S^2 \times S^1$) in the presence of nonzero $A_\mu, \tilde{A}_\mu, \sigma, \tilde{\sigma}$ background takes the following form:

$$\begin{aligned}\mathcal{L}_{s2} = \text{tr}\Big[&-\bar{A}^a D^\mu D_\mu A_a - \bar{B}^{\dot{a}} D^\mu D_\mu B_{\dot{a}} + \frac{1}{4}\left(A_a \bar{A}^a + B_{\dot{a}} \bar{B}^{\dot{a}}\right) \\ &+ (\sigma A_a - A_a \tilde{\sigma})\left(\bar{A}^a \sigma - \tilde{\sigma} \bar{A}^a\right) + (\tilde{\sigma} B_{\dot{a}} - \sigma B_{\dot{a}})\left(\bar{B}^{\dot{a}} \tilde{\sigma} - \bar{B}^{\dot{a}} \sigma\right)\Big]\end{aligned} \tag{B.1}$$

where derivatives D_μ are covariantized with the background gauge fields, external gauge field corresponding to the twisting and S^2 spatial connection. The fields are regarded as those on $S^2 \times \mathbb{R}$, A_{aS}, etc. $\sigma, \tilde{\sigma}$ assume their background values $(\sigma_S)_{ij} = -\frac{n_i}{2}\delta_{ij}$ and $(\tilde{\sigma}_S)_{ij} = -\frac{\tilde{n}_i}{2}\delta_{ij}$ where $i, j = 1, 2, \ldots, N$. The third term in the trace is the conformal mass term.

Since there are $U(1)^N \times U(1)^N$ background magnetic fields on S^2, each component of the scalars has to be spanned by appropriate monopole spherical harmonics on S^2 [40]. Scalars are either in $(N, \bar{N})$ or $(\bar{N}, N)$ representation of $U(N) \times U(N)$. We consider the ijth component of the scalar, where i (j) runs over $1, 2, \ldots, N$ and labels the components in the first (second) gauge group. A scalar Φ_{ij} couples to the background magnetic field whose strength is

$$\frac{s(n_i - \tilde{n}_j)}{2} \sin\theta \, d\theta \wedge d\phi \quad \big(s = \pm 1 \text{ for } (N, \bar{N}),\ (\bar{N}, N)\big), \tag{B.2}$$

corresponding to a Dirac monopole carrying $s(n_i - \tilde{n}_j)$ units of minimal charge. The monopole spherical harmonics Y_{jm} in this background, with angular momentum quantum numbers j, m given by

$$j = \frac{|n_i - \tilde{n}_j|}{2}, \frac{|n_i - \tilde{n}_j|}{2} + 1, \ldots \quad \text{and} \quad m = -j, -(j-1), \ldots, j-1, j, \tag{B.3}$$

diagonalize the spatial Laplacian on S^2 as

$$-D^a D_a Y_{jm} = \left(j(j+1) - \frac{(n_i - \tilde{n}_j)^2}{4}\right) Y_{jm} \tag{B.4}$$

where $a = 1, 2$ labels the coordinates of S^2.

Plugging this mode expansion into the quadratic action, one can easily see that the second term on the right-hand side of (B.4) is canceled by the second line of (B.1). Collecting all, the Y_{jm} mode Φ_{ij}^{jm} has a quadratic term

$$\bar{\Phi}_{ij}^{jm}\left[-(D_\tau)^2 + \left(j + \frac{1}{2}\right)^2\right]\Phi_{ij}^{jm}. \tag{B.5}$$

We hope our bad notation of using two kinds of j is not too confusing. The time derivative is

$$D_\tau = \partial_\tau - is\frac{\alpha_i - \tilde{\alpha}_j}{\beta + \beta'} - \frac{\beta - \beta'}{\beta + \beta'} m + \frac{\beta'}{\beta + \beta'} h_3 - \frac{\gamma_1}{\beta + \beta'} h_1 - \frac{\gamma_2}{\beta + \beta'} h_2, \tag{B.6}$$

where $h_{1,2,3}$ are eigenvalues of $SO(6)_R$ Cartans for the given field Φ. The determinant is evaluated for each conjugate pair of scalar fields $(\Phi, \bar{\Phi})$, where Φ may run over four complex scalars, say, A_a, $B_{\dot{a}}$. The determinant from the pair Φ, $\bar{\Phi}$ is given by

$$\begin{aligned}
&\prod_{j=\frac{|n_i-\tilde{n}_j|}{2}}^{\infty} \prod_{j_3=-j}^{j} \det\Bigg[-\left(\partial_\tau - is\frac{\alpha_i - \tilde{\alpha}_j}{\beta+\beta'} - \frac{\beta-\beta'}{\beta+\beta'} j_3 + \frac{\beta'}{\beta+\beta'} h_3 - \frac{\gamma_1 h_1 + \gamma_2 h_2}{\beta+\beta'}\right)^2 \\
&\quad + \left(j + \frac{1}{2}\right)^2\Bigg] \\
&= \prod_{n=-\infty}^{\infty} \prod_{j, j_3} \Bigg[\left(\frac{2\pi n}{\beta+\beta'} + s\frac{\alpha_i - \tilde{\alpha}_j}{\beta+\beta'} - i\frac{\beta-\beta'}{\beta+\beta'} j_3 + i\frac{\beta'}{\beta+\beta'} h_3 - i\frac{\gamma_1 h_1 + \gamma_2 h_2}{\beta+\beta'}\right)^2 \\
&\quad + \left(j + \frac{1}{2}\right)^2\Bigg].
\end{aligned}$$

Following the prescription in [31][6], we factor out a divergent constant, set it to unity, and obtain

$$\begin{aligned}
&\prod_{j, j_3} (-2i) \sin\left[\frac{1}{2}\big(s(\tilde{\alpha}_j - \alpha_i) + i\beta(\epsilon_j + j_3) + i\beta'(\epsilon_j - h_3 - j_3) + i(\gamma_1 h_1 + \gamma_2 h_2)\big)\right] \\
&\quad \times (-2i) \sin\left[\frac{1}{2}\big(-s(\tilde{\alpha}_j - \alpha_i) + i\beta(\epsilon_j - j_3) + i\beta'(\epsilon_j + h_3 + j_3) - i(\gamma_1 h_1 + \gamma_2 h_2)\big)\right],
\end{aligned} \tag{B.7}$$

[6] See Eqs. (4.23), (4.24) there and surrounding arguments.

where $\epsilon_j \equiv j + \frac{1}{2}$. Generalizing [31], the two sine factors in the final form has an obvious interpretation as contributions from a pair of particle and anti-particle modes, since all charges except 'energy' ϵ_j have different signs. Therefore, the determinant from the scalars admits a simple form

$$\begin{aligned}\det_{\text{scalar}} &= \prod_{i,j}\prod_{8\text{ scalars}}\prod_{j,j_3}\sin\Big[\frac{1}{2}\big(s(\tilde{\alpha}_j-\alpha_i)+i\beta(\epsilon_j+j_3)+i\beta'(\epsilon_j-h_3-j_3)\\&\quad+i(\gamma_1 h_1+\gamma_2 h_2)\big)\Big]\\&=\prod_{i,j}\prod_{8\text{ scalars}}\prod_{j,j_3} e^{\frac{is}{2}(\alpha_i-\tilde{\alpha}_j)+\frac{\beta}{2}(\epsilon_j+j_3)+\frac{\beta'}{2}(\epsilon_j-h_3-j_3)+\gamma_1 h_2+\gamma_2 h_2}\\&\quad\times\big(1-e^{is(\tilde{\alpha}_j-\alpha_i)}x^{\epsilon_j+j_3}(x')^{\epsilon_j-h_3-j_3}y_1^{h_1}y_2^{h_2}\big)\end{aligned}\tag{B.8}$$

where $x' \equiv e^{-\beta'}$. The product is over 8 scalars regarding conjugate pairs as independent fields.

We would like to write $(\det_{\text{scalar}})^{-1}$, appearing in the index, in terms of functions which we call the indices over 'letters,' or modes. We find that

$$\begin{aligned}&\log(\det_{\text{scalar}})^{-1}\\&\quad\equiv-\sum_{i,j}\sum_{\text{scalar}}\sum_{j,j_3}\Big[\frac{is}{2}(\alpha_i-\tilde{\alpha}_j)+\frac{\beta}{2}(\epsilon_j+j_3)+\frac{\beta'}{2}(\epsilon_j-h_3-j_3)+\gamma_1 h_2+\gamma_2 h_2\Big]\\&\qquad+\sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}\big[f_{ij}^{+B}\big(x^n,(x')^n,y_1^n,y_2^n\big)e^{in(\tilde{\alpha}_j-\alpha_i)}+f_{ij}^{-B}\big(x^n,(x')^n,y_1^n,y_2^n\big)e^{in(\alpha_i-\tilde{\alpha}_j)}\big].\end{aligned}\tag{B.9}$$

The first line provides a quantity analogous to the Casimir energy, which will be computed in Appendix B.3. The contribution from scalars to the letter index is given by

$$f_{ij}^{\pm B}(x,x',y_1,y_2)\equiv\sum_{\substack{4\text{ scalar}\\ s=\pm1}}\sum_{j=\frac{|n_i-\tilde{n}_j|}{2}}^{\infty}\sum_{j_3=-j}^{j}\big(x^{\epsilon_j+j_3}(x')^{\epsilon_j-h_3-j_3}y_1^{h_2}y_2^{h_2}\big)\tag{B.10}$$

where the first summation is restricted to fields with one of $s=\pm1$. Explicitly summing over the scalar modes, one obtains

$$\begin{aligned}&f_{ij}^{+B}(x,x',y_1,y_2)\\&\quad=\bigg(\sqrt{\frac{y_1}{y_2}}+\sqrt{\frac{y_2}{y_1}}\bigg)\sum_{j=\frac{|n_i-\tilde{n}_j|}{2}}^{\infty}x^{\frac{1}{2}}\big[(x')^{2j}+(x')^{2j-1}x+\cdots+x'x^{2j-1}+x^{2j}\big]\\&\qquad+\bigg(\sqrt{y_1y_2}+\frac{1}{\sqrt{y_1y_2}}\bigg)\sum_{j=\frac{|n_i-\tilde{n}_j|}{2}}^{\infty}x'x^{\frac{1}{2}}\big[(x')^{2j}+(x')^{2j-1}x+\cdots+x'x^{2j-1}+x^{2j}\big]\end{aligned}\tag{B.11}$$

where the two lines come from $\bar{B}^{\dot{a}}$ and A_a, respectively, and

$$f_{ij}^{-B}(x, x', y_1, y_2)$$
$$= \left(\sqrt{y_1 y_2} + \frac{1}{\sqrt{y_1 y_2}}\right) \sum_{j=\frac{|\tilde{n}_i - n_j|}{2}}^{\infty} x^{\frac{1}{2}}\left[(x')^{2j} + (x')^{2j-1}x + \cdots + x' x^{2j-1} + x^{2j}\right]$$
$$+ \left(\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}}\right) \sum_{j=\frac{|\tilde{n}_i - n_j|}{2}}^{\infty} x' x^{\frac{1}{2}}\left[(x')^{2j} + (x')^{2j-1}x + \cdots + x' x^{2j-1} + x^{2j}\right] \tag{B.12}$$

where the two lines come from $\bar{A}^a$ and $B_{\dot{a}}$, respectively. The dependence on x' is to be canceled against the contribution from fermions.

We also consider the determinant from fermions. Fermionic quadratic action is given by

$$\mathcal{L}_{f2} = \bar{\psi}^a_\alpha (\bar{\sigma}^\mu)^{\alpha\beta} D_\mu \psi^\beta_a + \bar{\psi}^{\dot{a}}_\alpha (\bar{\sigma}^\mu)^{\alpha\beta} D_\mu \chi_{\dot{a}\beta}, \tag{B.13}$$

where $D_3\psi_a = -i\sigma\psi_a + i\psi_a\tilde{\sigma}$ and $D_3\chi_{\dot{a}} = i\chi_{\dot{a}}\sigma - i\tilde{\sigma}\chi_{\dot{a}}$. As explained in Section 2.1, $\bar{\sigma}^\mu = (1, i\sigma^3, -i\sigma^1, i\sigma^2)$ is changed to $\bar{\sigma}^\mu = (1, -i\sigma^1, -i\sigma^2, -i\sigma^3)$ by an $SO(3)$ frame rotation. Since the latter basis is more convenient in that spin operator on S^2 is diagonalized, we do our computation in this basis.

Let us denote by Ψ_{ij} the ith and jth component of fermions ψ_a or $\chi_{\dot{a}}$ in the first and second gauge group, respectively. We want to compute the determinant of the matrix differential operator

$$\bar{\sigma}^\mu D_\mu = D_\tau - i\sigma^a D_a + s\sigma^3 \frac{n_i - \tilde{n}_j}{2}, \tag{B.14}$$

where $a = 1, 2$, and the last term comes from the coupling with background $\sigma, \tilde{\sigma}$. We would first like to obtain the complete basis of spinor spherical harmonics diagonalizing

$$\left(i\sigma^a D_a - s\sigma^3 \frac{n_i - \tilde{n}_j}{2}\right)\Psi = \lambda\Psi \tag{B.15}$$

with eigenvalue λ. Acting the same operator again on the above equation, one obtains

$$\left(-D^a D_a + \frac{1 - s(n_i - \tilde{n}_j)\sigma^3}{2} + \frac{|n_i - \tilde{n}_j|^2}{4}\right)\Psi = \lambda^2\Psi \tag{B.16}$$

where the second term comes from the commutator of two covariant derivatives and is the sum of the spatial curvature and the field strength. The first operator $-D^a D_a$ is a 2×2 diagonal matrix since the derivative involves $+\frac{i}{2}\omega^{\theta\phi}\sigma^3$. The spectrum of this operator is known and may be found, for instance, in [57]. For the spinor component $\alpha = \pm$, its eigenvalue is given by

$$l_\pm\left(l_\pm + \left|s(n_i - \tilde{n}_j) \mp 1\right| + 1\right) + \frac{|s(n_i - \tilde{n}_j) \mp 1|}{2} \tag{B.17}$$

where $l_\pm = 0, 1, 2, \ldots$, and $\Psi_\pm$ is given by scalar monopole harmonics with $j_\pm = l_\pm + \frac{|s(n_i - \tilde{n}_j) \mp 1|}{2}$, coupled to $s(n_i - \tilde{n}_j) \mp 1$ units of minimal Dirac monopoles. Plugging this in (B.16) and studying the upper/lower components, one obtains

$$\lambda^2 = \left(l_+ + \frac{|n_i - \tilde{n}_j|}{2}\right)^2 = \left(l_- + \frac{|n_i - \tilde{n}_j|}{2} + 1\right)^2$$
$$(l_+ = l_- + 1 = 1, 2, 3, \ldots) \quad \text{if } s(n_i - \tilde{n}_j) > 0,$$

$$\lambda^2=\left(l_+ +\frac{|n_i-\tilde n_j|}{2}+1\right)^2=\left(l_- +\frac{|n_i-\tilde n_j|}{2}\right)^2$$
$$(l_-=l_+ +1=1,2,3,\ldots)\quad \text{if } s(n_i-\tilde n_j)<0,$$
$$\lambda^2=(l_\pm+1)^2\quad (l_+=l_-=0,1,2,\ldots)\quad \text{if } n_i=\tilde n_j. \tag{B.18}$$

The eigenspinors are given as follows. In all three cases, one finds a pair of eigenspinors corresponding to $\lambda \gtrless 0$,

$$\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}=\begin{pmatrix}Y_{jm}\\ \pm Y_{jm}\end{pmatrix} \tag{B.19}$$

with $j\equiv j_+=j_-$, where the latter two are equal if one relates l_+ and l_- as explained in (B.18). $j\geqslant \frac{|n_i-\tilde n_j|+1}{2}$ is the total angular momentum of the mode.

Apart from the above modes, there is a set of exceptional modes in the complete set if $n_i\neq\tilde n_j$. For the first and second cases in (B.18), there exist nonzero modes

$$\begin{pmatrix}Y_{\frac{|n_i-\tilde n_j|-1}{2},m}\\ 0\end{pmatrix}\quad \text{if } s(n_i-\tilde n_j)>0,$$
$$\begin{pmatrix}0\\ Y_{\frac{|n_i-\tilde n_j|-1}{2},m}\end{pmatrix}\quad \text{if } s(n_i-\tilde n_j)<0. \tag{B.20}$$

These modes corresponds to $l_\pm=0$ on the first/second line of (B.18), respectively. By directly studying (B.15), one finds that the eigenvalue is always negative for both cases, i.e. $\lambda=-\frac{|n_i-\tilde n_j|}{2}=-(j+\frac{1}{2})$.

Expanding the operator (B.14) in the above basis, and following steps similar to those for the scalar determinant, one obtains

$$\begin{aligned}\det{}_{\mathrm f}=&\prod_{i,j}\prod_{8\text{ fermions}}\prod_{j\geqslant\frac{|n_i-\tilde n_j|+1}{2}}\prod_{j_3}(-2i)\sin\Big[\frac{1}{2}\big(s(\tilde\alpha_j-\alpha_i)+i\beta(\epsilon_j+j_3)\\ &+i\beta'(\epsilon_j-h_3-j_3)+i(\gamma_1h_1+\gamma_2h_2)\big)\Big]\\ &\times\prod_{i,j}\prod_{\bar\psi^a,\bar\chi^{\dot a}}\prod_{j_3=-\frac{|n_i-\tilde n_j|-1}{2}}^{\frac{|n_i-\tilde n_j|-1}{2}}(-2i)\sin\Big[\frac{1}{2}\big(s(\tilde\alpha_j-\alpha_i)+i\beta(\epsilon_j+j_3)\\ &+i\beta'(\epsilon_j-h_3-j_3)+i(\gamma_1h_1+\gamma_2h_2)\big)\Big]\end{aligned}$$

where $\epsilon_j=j+\frac{1}{2}$ for fermions as well. Let us explain how each term is derived. In the first line, 8 fermions in the product denote $\psi_a,\bar\psi^a,\chi_{\dot a},\bar\chi^{\dot a}$. This comes from the paired eigenmodes (B.19) as one evaluates the determinant of the operator (B.14). The second line is multiplied over four fields only since the modes in (B.20) do not appear in a paired form. From the fact that λ is negative when the differential operator acts on the chiral spinors $\psi_a,\chi_{\dot a}$, one can easily check that only the charges of $\bar\psi^a,\bar\chi^{\dot a}$ have to be inserted on the second line.

One can also write this determinant in terms of indices over letters as follows:

$$
\begin{aligned}
&\log(\det_{\text{fermion}}) \\
&\quad = +\sum_{i,j}\sum_{\text{fermions}}\sum_{j,j_3}\left[\frac{is}{2}(\alpha_i - \tilde{\alpha}_j) + \frac{\beta}{2}(\epsilon_j + j_3) + \frac{\beta'}{2}(\epsilon_j - h_3 - j_3) + \gamma_1 h_2 + \gamma_2 h_2\right] \\
&\qquad + \sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}\left[f_{ij}^{+B}\left(x^n, (x')^n, y_1^n, y_2^n\right)e^{in(\tilde{\alpha}_j - \alpha_i)} + f_{ij}^{-B}\left(x^n, (x')^n, y_1^n, y_2^n\right)e^{in(\alpha_i - \tilde{\alpha}_j)}\right],
\end{aligned}
\tag{B.21}
$$

where

$$
\begin{aligned}
&f_{ij}^{+F}(x, x', y_1, y_2) \\
&\quad = -\left(\sqrt{y_1 y_2} + \frac{1}{\sqrt{y_1 y_2}}\right)\sum_{j=\frac{|n_i - \tilde{n}_j|+1}{2}}^{\infty} x^{\frac{1}{2}}\left[(x')^{2j} + (x')^{2j-1}x + \cdots + x'x^{2j-1} + x^{2j}\right] \\
&\qquad - \left(\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}}\right)\sum_{j=\frac{|n_i - \tilde{n}_j|-1}{2}}^{\infty} x'x^{\frac{1}{2}}\left[(x')^{2j} + (x')^{2j-1}x + \cdots + x'x^{2j-1} + x^{2j}\right]
\end{aligned}
$$

from $\psi_{a\alpha}$ and $\bar{\chi}_{\alpha}^{\dot{a}}$, and

$$
\begin{aligned}
&f_{ij}^{-F}(x, x', y_1, y_2) \\
&\quad = -\left(\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}}\right)\sum_{j=\frac{|n_i - \tilde{n}_j|+1}{2}}^{\infty} x^{\frac{1}{2}}\left[(x')^{2j} + (x')^{2j-1}x + \cdots + x'x^{2j-1} + x^{2j}\right] \\
&\qquad - \left(\sqrt{y_1 y_2} + \frac{1}{\sqrt{y_1 y_2}}\right)\sum_{j=\frac{|n_i - \tilde{n}_j|-1}{2}}^{\infty} x'x^{\frac{1}{2}}\left[(x')^{2j} + (x')^{2j-1}x + \cdots + x'x^{2j-1} + x^{2j}\right]
\end{aligned}
$$

from $\chi_{\dot{a}\alpha}$ and $\bar{\psi}_{\alpha}^{a}$.

We combine the bosonic and fermionic determinants and obtain

$$
\begin{aligned}
&\log\left(\frac{\det_{\text{fermion}}}{\det_{\text{scalar}}}\right) \\
&\quad = -\sum_{i,j}\sum_{\text{matter}}\sum_{j,j_3}(-1)^F\left[\frac{is}{2}(\alpha_i - \tilde{\alpha}_j) + \frac{\beta}{2}(\epsilon_j + j_3) + \frac{\beta'}{2}(\epsilon_j - h_3 - j_3)\right. \\
&\qquad \left. + \gamma_1 h_2 + \gamma_2 h_2\right] \\
&\qquad + \sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}\left[f_{ij}^{+}\left(x^n, (x')^n, y_1^n, y_2^n\right)e^{in(\tilde{\alpha}_j - \alpha_i)} + f_{ij}^{-}\left(x^n, (x')^n, y_1^n, y_2^n\right)e^{in(\alpha_i - \tilde{\alpha}_j)}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
f_{ij}^{+}(x, y_1, y_2) &= f_{ij}^{+B} + f_{ij}^{+F} \\
&= x^{|n_i - \tilde{n}_j|}\left[\frac{x^{\frac{1}{2}}}{1 - x^2}\left(\sqrt{\frac{y_1}{y_2}} + \sqrt{\frac{y_2}{y_1}}\right) - \frac{x^{\frac{3}{2}}}{1 - x^2}\left(\sqrt{y_1 y_2} + \frac{1}{\sqrt{y_1 y_2}}\right)\right]
\end{aligned}
\tag{B.22}
$$

and similarly

$$f_{ij}^{-}(x, y_1, y_2) = x^{|n_i-\tilde{n}_j|}\left[\frac{x^{\frac{1}{2}}}{1-x^2}\left(\sqrt{y_1 y_2}+\frac{1}{\sqrt{y_1 y_2}}\right)-\frac{x^{\frac{3}{2}}}{1-x^2}\left(\sqrt{\frac{y_1}{y_2}}+\sqrt{\frac{y_2}{y_1}}\right)\right]. \tag{B.23}$$

This proves the assertion in Section 2.3 on determinant from matter fields.

B.2. Determinant from fields in vector multiplets

We also consider the 1-loop determinant from fields in vector multiplets. We consider the multiplet $A_\mu, \sigma, \lambda_\alpha$: the other vector multiplet can be treated in a completely same way.

We start from the bosonic part. We expand the quadratic fluctuation in the Q-exact deformation, which is dominant in the limit $g \to 0$. Denoting the fluctuation by $\delta A_\mu, \delta\sigma$, one finds the following quadratic term:

$$\left|\vec{D}\times\delta\vec{A}-\vec{D}\delta\sigma-i[\sigma,\delta\vec{A}]\right|^2. \tag{B.24}$$

We are directly working in $\mathbb{R}^3$ with $1 \leqslant r \leqslant e^\beta$ rather than going to $S^2\times S^1$. The boundary conditions are

$$\delta\vec{A}\left(r=e^\beta\right)=e^{-\beta}\delta\vec{A}(r=1), \qquad \delta\sigma\left(r=e^\beta\right)=e^{-\beta}\delta\sigma(r=1), \tag{B.25}$$

associated with their scale dimensions 1.

$\delta\sigma_{ij}$ is expanded with monopole spherical harmonics with n_i-n_j units of magnetic charge. We can also expand $\delta\vec{A}_{ij}$ using monopole vector spherical harmonics, which is nicely presented in [58]. For $j \geqslant q+1$ where $q \equiv \frac{n_i-n_j}{2} \geqslant 0$, it has three components $\vec{C}^{\lambda}_{qjm}$ (with $\lambda=+1,0,-1$) and are related to the scalar harmonics as

$$\vec{C}^{+1}_{qjm}=\frac{1}{\sqrt{2(\mathcal{J}^2+q)}}(\vec{D}+i\hat{r}\times\vec{D})Y_{qjm} \quad (j\geqslant q>0), \tag{B.26}$$

$$\vec{C}^{0}_{qjm}=\frac{\hat{r}}{r}Y_{qjm} \quad (j\geqslant q\geqslant 0), \tag{B.27}$$

$$\vec{C}^{-1}_{qjm}=\frac{1}{\sqrt{2(\mathcal{J}^2-q)}}(\vec{D}-i\hat{r}\times\vec{D})Y_{qjm} \quad (j>q\geqslant 0) \tag{B.28}$$

where $\mathcal{J}^2\equiv j(j+1)-q^2$. Knowledge on vector spherical harmonics for $q\geqslant 0$ would turn out to be enough to calculate the 1-loop determinant. For $j=q$, $\vec{C}^{-1}_{qjm}$ is absent instead of the above. For $j=q-1$, both $\vec{C}^{-1}$ and $\vec{C}^{0}$ are absent. We expand the fields as

$$\delta\vec{A}=\sum_{n=-\infty}^{\infty}\sum_{j,m}\sum_{\lambda=0,\pm1}a^{\lambda}_{njm}r^{-i\frac{2\pi n}{\beta+\beta'}}\vec{C}^{\lambda}_{jm}, \qquad \delta\sigma=\sum_{n,j,m}b_{njm}r^{-i\frac{2\pi n}{\beta+\beta'}}\frac{Y_{jm}}{r}. \tag{B.29}$$

One can expand (B.24) by inserting these expansions, using the following properties of vector harmonics,[7]

$$\vec{D}\cdot\vec{C}^{0}_{qjm}=\frac{1}{r^2}Y_{qjm}, \qquad \vec{D}\cdot\vec{C}^{\pm1}_{qjm}=-\frac{1}{r^2}\sqrt{\frac{\mathcal{J}^2\pm q}{2}}Y_{qjm}, \tag{B.30}$$

[7] We correct sign typos of [58] in some of these formulae.

$$\vec{D}\times\vec{C}^{0}_{qjm}=\frac{i}{r}\left(\sqrt{\frac{\mathcal{J}^2+q}{2}}\vec{C}^{+1}_{qjm}-\sqrt{\frac{\mathcal{J}^2-q}{2}}\vec{C}^{-1}_{qjm}\right),\tag{B.31}$$

$$\vec{D}\times\vec{C}^{\pm1}_{qjm}=\frac{i}{r}\left(\mp\sqrt{\frac{\mathcal{J}^2\pm q}{2}}\vec{C}^{0}_{qjm}\right)\tag{B.32}$$

and

$$\vec{D}\left(\frac{1}{r}Y_{qjm}\right)=\frac{1}{r}\left(-\vec{C}^{0}_{qjm}+\sqrt{\frac{\mathcal{J}^2+q}{2}}\vec{C}^{+1}_{qjm}+\sqrt{\frac{\mathcal{J}^2-q}{2}}\vec{C}^{-1}_{qjm}\right).\tag{B.33}$$

From (B.24) one finds

$$\sum_{i,j=1}^{N}\sum_{n=-\infty}^{\infty}\sum_{j,m}v^T_{-n,j,-m}\left(\mathcal{M}^{\frac{n_j-n_i}{2}}_{-n,j,-m}\right)^T\mathcal{M}^{\frac{n_i-n_j}{2}}_{njm}v_{njm}\tag{B.34}$$

where

$$\mathcal{M}^q_n=\begin{pmatrix}-\lambda+iq & 0 & is_+ & -s_+\\ 0 & \lambda+iq & -is_- & -s_-\\ -is_+ & is_- & iq & i\lambda+1\end{pmatrix},\qquad v_n=\begin{pmatrix}a_+\\ a_-\\ a_0\\ b\end{pmatrix}\tag{B.35}$$

for the modes with $j\geqslant q+1$, $\lambda=\frac{2\pi n}{\beta+\beta'}-i\frac{\beta-\beta'}{\beta+\beta'}m+(\alpha_i-\alpha_j)$, and this result is for $q=\frac{n_i-n_j}{2}\geqslant 0$.[8] We also defined $s_\pm\equiv\sqrt{\frac{\mathcal{J}^2\pm q}{2}}$. For $j=q$, there is no a_- modes and one finds

$$\mathcal{M}^q_n=\begin{pmatrix}-\lambda+iq & is_+ & -s_+\\ -is_+ & iq & i\lambda+1\end{pmatrix},\qquad v_n=\begin{pmatrix}a_+\\ a_0\\ b\end{pmatrix}\tag{B.36}$$

where $s_+=\sqrt{q}$. Finally for $j=q-1$ (possible only when $q\geqslant 1$), both a_- and a_0 modes are absent. The scalar monopole harmonics mode b is also absent. One simply finds

$$\mathcal{M}^q_n=(-\lambda+iq),\qquad v_n=(a_+)\tag{B.37}$$

where we used $s_+=0$ in this case.

Before evaluating the determinant we fix the gauge for these fluctuations. For the ijth mode for which $n_i=n_j$, we choose the Coulomb gauge $s_+a_++s_-a_-=0$ following [31]. Since they are not coupled to magnetic fields, the corresponding infinitesimal gauge transformation, call it ϵ, is expanded by ordinary spherical harmonics. The Coulomb gauge condition requires $\partial^a\partial_a\epsilon=0$ on S^2, which leaves the s-wave component of ϵ unfixed. We impose a residual gauge condition $\frac{d}{d\tau}\int_{S^2}A_\tau=0$ to fix this. The corresponding Faddeev–Popov determinant can be calculated following [31]. For the Coulomb gauge, The Faddeev–Popov determinant is that of the operator $D^a\partial_a\approx\partial^a\partial_a$ over nonzero modes. For the residual gauge, the determinant is given by

$$\prod_{\substack{i<j;\\ n_i=n_j}}\left[2\sin\frac{\alpha_i-\alpha_j}{2}\right]^2.\tag{B.38}$$

[8] The matrix $\mathcal{M}$ with $n_i<n_j$ can be easily obtained as follows. In (B.34), one of $\mathcal{M}^T$ and $\mathcal{M}$ satisfy $n_i\geqslant n_j$. Suppose $\mathcal{M}$ satisfies this condition. Firstly complex conjugate all i which are explicit in (B.35). Then change the sign of λ, which is due to the sign changes of n, j_3 and $\alpha_i-\alpha_j$. Transposing it gives the pair $\mathcal{M}^T$.

We also fix the gauge for the modes for which $n_i \neq n_j$. Analogous to the previous case, we choose the 'background Coulomb gauge' $s_+a_+ + s_-a_- = 0$. The infinitesimal gauge transformation ϵ acquires the condition $D^a D_a \epsilon = 0$. The corresponding Faddeev–Popov determinant is $\det D^a D_a$. This, and $\det \partial^a \partial_a$ above, will be canceled by a factor in the 1-loop determinant to be calculated below. See [31] for the similar results. Contrary to the operator $\partial^a \partial_a$, D^2 has no zero modes due to the absence of s-waves in monopole spherical harmonics. So we do not have a residual gauge fixing or a corresponding measure like (B.38). For $j = q$, our gauge implies $a_+ = 0$. For $j = q - 1$, there is no need to fix the gauge.

In the Coulomb gauge, we may write $a_+ = s_-a$ and $a_- = -s_+a$. Now the quadratic terms for modes with $j \geqslant q + 1$ takes the form (B.34) with $\mathcal{M}_n$ and v_n given by

$$\mathcal{M}_n^q = \begin{pmatrix} -s_-(\lambda - iq) & is_+ & -s_+ \\ -s_+(\lambda + iq) & -is_- & -s_- \\ -2is_+s_- & iq & i(\lambda - i) \end{pmatrix}, \qquad v_n = \begin{pmatrix} a \\ a_0 \\ b \end{pmatrix}. \tag{B.39}$$

The determinant of this matrix is $\det(\mathcal{M}_n^q) = -\mathcal{J}^2[(\lambda - \frac{i}{2})^2 + (j + \frac{1}{2})^2]$. $-\mathcal{J}^2$ is nothing but the eigenvalue of $D^a D_a$, whose determinant partly cancels with the Faddeev–Popov measure as claimed. The remaining determinant of bosonic fields with $j \geqslant q + 1$ is

$$\prod_{i,j=1}^{N} \prod_{n=-\infty}^{\infty} \prod_{j,j_3} \det\bigl(\mathcal{M}_{njj_3}^{\frac{|n_i - n_j|}{2}}\bigr) = \prod_{i,j} \prod_{n=-\infty}^{\infty} \prod_{j,j_3} \left[\left(j + \frac{1}{2}\right)^2 + \left(\lambda - \frac{i}{2}\right)^2\right]. \tag{B.40}$$

We can arrange the product over n to sine functions:

$$\begin{aligned} \prod_{i,j} \prod_{j=\frac{|n_i - n_j|}{2}+1}^{\infty} \prod_{j_3} & \sin\left[\frac{1}{2}\bigl(\beta(j - j_3) + \beta'(j + j_3) - i(\alpha_i - \alpha_j)\bigr)\right] \\ & \times \sin\left[\frac{1}{2}\bigl(\beta(j + 1 + j_3) + \beta'(j + 1 - j_3) + i(\alpha_i - \alpha_j)\bigr)\right]. \end{aligned} \tag{B.41}$$

Note that in each of the two sine factors, the role of energy is played by the quantities j and $j+1$, respectively. Signs of j_3 are not important since the product is symmetric under $j_3 \to -j_3$.

We also consider the modes with $j = q$ and $j = q - 1$ (for $q \geqslant 1$). For $j = q$, with the gauge choice $a_+ = 0$, one finds

$$\mathcal{M}_n^q = \begin{pmatrix} i\sqrt{q} & -\sqrt{q} \\ iq & i\lambda + 1 \end{pmatrix}, \qquad v_n = \begin{pmatrix} a_0 \\ b \end{pmatrix}. \tag{B.42}$$

From $\det(\mathcal{M}_n^q) = -\sqrt{q}(\lambda - i - iq) = -\frac{\sqrt{q}}{\beta + \beta'}[2\pi n - i\beta(q + 1 + j_3) - i\beta'(q + 1 - j_3) + (\alpha_i - \alpha_j)]$, one obtains

$$\prod_{n=-\infty}^{\infty} \prod_{j_3=-q}^{q} \sin\left[\frac{1}{2}\bigl(\beta(q + 1 + j_3) + \beta'(q + 1 - j_3) + i(\alpha_i - \alpha_j)\bigr)\right]. \tag{B.43}$$

Note that this is a contribution from modes with energy $q + 1 (= j + 1)$. For $j = q - 1$, one similarly finds

$$\prod_{n=-\infty}^{\infty} \prod_{j_3=-q}^{q} \sin\left[\frac{1}{2}\bigl(\beta(q + j_3) + \beta'(q - j_3) + i(\alpha_i - \alpha_j)\bigr)\right]. \tag{B.44}$$

This is again a contribution from modes with energy $q(= j+1)$.

Collecting all, the determinant of bosonic modes can be casted in the following form (after relabeling $j_3 \to -j_3$ for some terms)

$$\log(\det_{\text{boson}})^{-1} = -\frac{1}{2}\operatorname{tr}_B\big[\big(\beta(j+j_3)+\beta'(j-j_3)\big)+\big(\beta(j+1+j_3)+\beta'(j+1-j_3)\big)\big] + \sum_{i,j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}f_{ij}^{\text{adj},B}(x)e^{-in(\alpha_i-\alpha_j)} \tag{B.45}$$

where tr_B is trace over all bosonic modes explained above. Contribution to the adjoint letter index from modes with $j \geqslant \frac{|n_i-n_j|}{2}+1$ is

$$f_{ij}^{\text{adj},B} \leftarrow \sum_{j=\frac{|n_i-n_j|}{2}+1}^{\infty}\big[(x')^{2j}+(x')^{2j-1}x+\cdots+x^{2j}\big] + \big[(x')^{2j+1}x+(x')^{2j}x^2+\cdots+x'x^{2j+1}\big]. \tag{B.46}$$

Additional contribution from modes with $j = \frac{|n_i-n_j|}{2}$ is given by

$$f_{ij}^{\text{adj},B} \leftarrow (x')^{|n_i-n_j|+1}x+(x')^{|n_1-n_j|}x^2+\cdots+x'x^{|n_i-n_j|+1}. \tag{B.47}$$

Finally, when $\frac{|n_i-n_j|}{2} \geqslant 1$,

$$f_{ij}^{\text{adj},B} \leftarrow (x')^{|n_i-n_j|-1}x+(x')^{|n_1-n_j|-2}x^2+\cdots+x'x^{|n_i-n_j|-1} \tag{B.48}$$

from modes with $j = \frac{|n_i-n_j|}{2}-1$. A similar determinant from $\tilde{A}_\mu, \tilde{\sigma}$ is obtained with α_i, n_i replaced by $\tilde{\alpha}_i, \tilde{n}_i$.

To complete the computation we consider contribution from the fermion λ_α. The Lagrangian on $S^2 \times S^1$ is given in Appendix A, with a novel mass-like term. The calculation is similar to that in Appendix B.1 for matter fermions except for the addition of this term. The operator acting on $(\bar{\lambda}^\alpha)_{ij}$ is

$$D_\tau + i\sigma^a D_a - \frac{n_i-n_j}{2}\sigma^3 - \frac{1}{2}. \tag{B.49}$$

The eigenvalue problem for the operator consisting of second and third terms is solved, replacing $s(n_i-\tilde{n}_j)$ by n_i-n_j here. Again there appears eigenspinors (B.19) with eigenvalues $\lambda = \pm\frac{j+1}{2}$ for $j \geqslant \frac{|n_i-n_j|+1}{2}$, as well as additional modes only if $n_i \neq n_j$ with $j = \frac{|n_i-n_j|-1}{2}$ and $\lambda = -\frac{|n_i-n_j|}{2}$. The combination appearing in the determinant gets shifted by $-\frac{1}{2}$:

$$D_\tau + \lambda - \frac{1}{2} \to -\frac{i}{\beta+\beta'}\left[2\pi n + i\beta\left(\lambda - j_3 - \frac{1}{2}\right) + i\beta'\left(\lambda - \frac{1}{2} + h_3 + j_3\right) + (\alpha_i - \alpha_j)\right]. \tag{B.50}$$

For the modes with $j \geqslant \frac{|n_i-n_j|+1}{2}$, since λ appears in both signs, the 'energy' $|\lambda|$ appearing in the determinant gets shifted in two ways $|\lambda| \to |\lambda| \mp \frac{1}{2}$ where upper (lower) sign is for the positive (negative) λ. However, since λ is always negative for modes with $j = \frac{|n_i-n_j|-1}{2}$, the shifted energy is always $|\lambda| + \frac{1}{2}$ in this case. For $\bar{\lambda}_\alpha$, one inserts $h_3 = 1$. Collecting all, the fermionic determinant is ($j_3 \to -j_3$ relabeled for some terms)

$$\log(\det_{\text{fermion}}) = \frac{1}{2}\,\mathrm{tr}_F\big[\big(\beta(j+j_3)+\beta'(j+1-j_3)\big)+\big(\beta(j+1+j_3)+\beta'(j-j_3)\big)\big] \\ + \sum_{i,j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}f_{ij}^{\text{adj},F}(x)e^{-in(\alpha_i-\alpha_j)}. \tag{B.51}$$

The modes with $j \geqslant \frac{|n_i-n_j|+1}{2}$ contribute to the letter index as

$$f_{ij}^{\text{adj},F} \leftarrow -\sum_{j=\frac{|n_i-n_j|+1}{2}}^{\infty}\big[(x')^{2j+1}+(x')^{2j-1}x+\cdots+x'x^{2j}\big] \\ +\big[(x')^{2j}x+(x')^{2j-1}x^2+\cdots+x^{2j+1}\big]. \tag{B.52}$$

Additional contribution from modes with $j = \frac{|n_i-n_j|-1}{2}$ is given by

$$f_{ij}^{\text{adj},F} \leftarrow -\big[(x')^{|n_i-n_j|-1}x+(x')^{|n_1-n_j|-2}x^2+\cdots+x^{|n_i-n_j|}\big] \tag{B.53}$$

if $n_i \neq n_j$.

Comparing the determinants from bosons and fermions, one can immediately find a vast cancelation. In fact, contribution from bosonic modes with $j \geqslant \frac{|n_i-n_j|}{2}$ completely cancels with that from fermionic modes with $j \geqslant \frac{|n_i-n_j|+1}{2}$. In particular, this means that there is no net contribution from modes which do not feel the flux, i.e. $q = 0$. This is of course consistent with the result of [23], in which the authors use combinatoric methods in the free theory where the gauge fields play no role. In our case, there are exceptional modes when $n_i \neq n_j$. Contributions from fermion modes with $j = q - \frac{1}{2}$ and bosonic modes with $j = q - 1$ (if they exist) do not perfectly cancel and yield

$$f_{ij}^{\text{adj}} = -x^{|n_i-n_j|} \quad (\text{if } n_i \neq n_j). \tag{B.54}$$

Generally one can write $f_{ij}^{\text{adj}}(x) = -(1-\delta_{n_i n_j})x^{|n_i-n_j|}$. The final result is simply

$$\frac{\det_{\text{fermion}}}{\det_{\text{boson}}} = \prod_{i,j=1}^{N}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\big(f_{ij}^{\text{adj}}(x^n)e^{-in(\alpha_i-\alpha_j)}+\tilde{f}_{ij}^{\text{adj}}(x^n)e^{-in(\tilde{\alpha}_i-\tilde{\alpha}_j)}\big)\right], \tag{B.55}$$

with similarly defined $\tilde{f}_{ij}^{\text{adj}}(x)$. The evaluation of the Casimir-like energy is relegated to Appendix B.3 below.

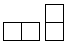
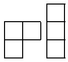
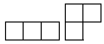
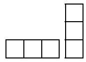
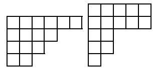
B.3. Casimir energy

We finally turn to the Casimir-energy like shift in the effective action

$$\beta\epsilon_0 \equiv \frac{1}{2}\,\mathrm{tr}\big[(-1)^F\big(\beta(\epsilon+j_3)+\beta'(\epsilon-h_3-j_3)+\gamma_1 h_1+\gamma_2 h_2\big)\big], \tag{B.56}$$

where we have dropped the holonomy variables inside the trace, $is(\alpha_i - \tilde{\alpha}_j)$ for matters and $i(\alpha_i - \alpha_j)$, etc., for adjoints, since their traces are trivially zero. To compute this formally divergent quantity, one has to correctly regularize it. A constraint is that it has to be compatible with

Table 2
Casimir energy for some positive flux distributions.

Flux					
ϵ_0	2	2	2	6	$39 - 13 - 22 = 4$

our special supersymmetry. The most general regularization would be insertion of

$$x^{\epsilon+j_3}(x')^{\epsilon-h_3-j_3}y_1^{h_1}y_2^{h_2} \tag{B.57}$$

inside the trace. The parameters x, x', y_1, y_2 are not to be confused with the chemical potentials in the rest of this paper: they are regulators and should be taken to $x, x' \to 1^-$, $y_1, y_2 \to 1$ after computing the trace. Anyway, the trace is formally very similar to the total summation of all letter indices we computed in the previous subsections. Actually the above trace, regularized as above, is

$$\begin{aligned}\beta\epsilon_0 = {} & \frac{1}{2}\lim_{x,x',y_1,y_2\to 1}(\beta\partial_x + \beta'\partial_{x'} + \gamma_1\partial_{y_1} + \gamma_2\partial_{y_2}) \\ & \times \sum_{i,j=1}^{N}\left[f_{ij}^{+}(x,y_1,y_2) + f_{ij}^{-}(x,y_1,y_2) + f_{ij}^{\text{adj}}(x) + \tilde{f}_{ij}^{\text{adj}}(x)\right].\end{aligned} \tag{B.58}$$

Since x' disappears in the letter indices, $\partial_{x'}$ is zero. Also, it is easy to see from the y_1, y_2 dependence of $f_{ij}^{\pm}$ that ∂_{y_1} and ∂_{y_2} are zero at $y_1, y_2 = 1$. Thus we only need to compute $\beta\partial_x$ acting on various functions. At $y_1 = y_2 = 1$, they are given by

$$\begin{aligned}&\lim_{x\to 1}\partial_x f_{ij}^{+} = +|n_i - \tilde{n}_j|, \qquad \lim_{x\to 1}\partial_x f_{ij}^{-} = +|n_i - \tilde{n}_j|, \\ &\lim_{x\to 1}\partial_x f_{ij}^{\text{adj}}(x) = -|n_i - n_j|, \qquad \lim_{x\to 1}\partial_x \tilde{f}_{ij}^{\text{adj}}(x) = -|\tilde{n}_i - \tilde{n}_j|.\end{aligned} \tag{B.59}$$

Therefore one finds

$$\epsilon_0 = \sum_{i,j=1}^{N}|n_i - \tilde{n}_j| - \sum_{i<j}|n_i - n_j| - \sum_{i<j}|\tilde{n}_i - \tilde{n}_j|. \tag{B.60}$$

We list a few nonzero values of ϵ_0 for some positive flux distributions in Table 2.

We explain a few useful properties of ϵ_0. The fluxes may involve positive, negative integers and zero. We first show that contributions to ϵ_0 from modes carrying $U(1)$ indices with zero fluxes cancel to zero. Then we show that contributions from modes ending on one $U(1)$ with positive flux and another $U(1)$ with negative flux also cancel.

To show the first, since modes ending on two $U(1)$'s both with zero flux is trivial, we restrict to the modes whose one end has zero flux and another nonzero. Then contribution of these modes to ϵ_0 is

$$\begin{aligned}&\left(N_2\sum_{n_i\neq 0}|n_i| + N_1\sum_{\tilde{n}_i\neq 0}|\tilde{n}_i|\right) - N_1\sum_{n_i\neq 0}|n_i| - N_2\sum_{\tilde{n}_i\neq 0}|\tilde{n}_i| \\ &\quad = (N_1 - N_2)\left(\sum_{\tilde{n}_i\neq 0}|\tilde{n}_i| - \sum_{n_i\neq 0}|n_i|\right).\end{aligned} \tag{B.61}$$

Here we use the fact that, for the index to be nonzero, total sum of positive (negative) fluxes on both gauge groups should be equal. This implies that expression in the second parenthesis is zero, proving our claim. This result implies that, to compute ϵ_0, one only has to consider contribution from modes connecting nonzero fluxes.

To show the second, note that for such modes $|n_i - \tilde{n}_j| = |n_i| + |\tilde{n}_j|$, $|n_i - n_j| = |n_i| + |n_j|$ and $|\tilde{n}_i - \tilde{n}_j| = |\tilde{n}_i| + |\tilde{n}_j|$ due to the opposite sign of the two fluxes. After an analysis similar to the previous parenthesis, their contribution to ϵ_0 is

$$\left(M_1^- - M_2^-\right)\left(\sum |\tilde{n}_i^+| - \sum |n_i^+|\right) + \left(M_1^+ - M_2^+\right)\left(\sum |\tilde{n}_i^-| - \sum |n_i^-|\right). \tag{B.62}$$

Again from the equality of total positive/negative fluxes on two gauge groups, this quantity is zero. This result implies that one can separate $\epsilon_0 = \epsilon_0^+ + \epsilon_0^-$, first one coming from modes connecting positive fluxes only and second from modes connecting negative fluxes only. This property will be important when we discuss the large N factorization in Section 3.

Finally, we show that ϵ_0 is always non-negative, and becomes zero if and only if the two sets $\{n_i\}$ and $\{\tilde{n}_i\}$ are the same. We shall actually prove a slightly more general claim. Suppose we have two decreasing functions $f(x)$ and $g(x)$ defined on $0 \leqslant x \leqslant \ell$. Then we claim that the functional $\mathcal{E}[f,g]$ defined by

$$\mathcal{E}[f,g] \equiv \int dx\,dy\left(\left|f(x) - g(y)\right| - \frac{1}{2}\left|f(x) - f(y)\right| - \frac{1}{2}\left|g(x) - g(y)\right|\right) \tag{B.63}$$

is always non-negative, and assumes its minimum at 0 if and only if $f(x) = g(x)$ everywhere.[9] To prove our claim, we vary the functional by $\delta f(x)$. Note that

$$\begin{aligned}
&\delta\left|f(x) - g(y)\right| = \delta f(x)\left[2\theta\left(f(x) - g(y)\right) - 1\right] = \delta f(x)\left[2\theta\left(y - g^{-1}f(x)\right) - 1\right],\\
&\delta\left|f(x) - f(y)\right| = \delta f(x)\left[2\theta(y - x) - 1\right] + \delta f(y)\left[2\theta(x - y) - 1\right]
\end{aligned} \tag{B.64}$$

under this variation, where $\theta(x)$ is the step function (assuming 1 for $x > 0$, and 0 for $x < 0$). From these one finds

$$\delta\mathcal{E}[f,g] = \int dx\, 2\delta f(x)\left[x - g^{-1}f(x)\right]. \tag{B.65}$$

The condition for the extremal points is $x = g^{-1}f(x)$, or $f(x) = g(x)$. Same result is obtained by the variation $\delta g(x)$. To show this is a minima, we compute the Hessian. From (B.65) and analogous result for $\delta g(x)$, one finds

$$\begin{pmatrix} \frac{\delta^2\mathcal{E}}{\delta f(x)\delta f(y)} & \frac{\delta^2\mathcal{E}}{\delta f(x)\delta g(y)} \\ \frac{\delta^2\mathcal{E}}{\delta g(x)\delta f(y)} & \frac{\delta^2\mathcal{E}}{\delta g(x)\delta g(y)} \end{pmatrix} = 2\begin{pmatrix} -\frac{\delta(x-y)}{g'(g^{-1}f(x))} & \frac{\delta(g^{-1}f(x)-y)}{g'(g^{-1}f(x))} \\ \frac{\delta(f^{-1}g(x)-y)}{f'(f^{-1}g(x))} & -\frac{\delta(x-y)}{f'(f^{-1}g(x))} \end{pmatrix} \tag{B.66}$$

where we used $\delta f^{-1}(x) = -\frac{\delta f(f^{-1}(x))}{f'(f^{-1}(x))}$ and similar formula for δg^{-1}. At the extrema $f = g$, the last matrix becomes

$$2\delta(x - y)\begin{pmatrix} -\frac{1}{f'(x)} & \frac{1}{f'(x)} \\ \frac{1}{f'(x)} & -\frac{1}{f'(x)} \end{pmatrix} = -\frac{2\delta(x - y)}{f'(x)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{B.67}$$

[9] Requiring these functions to assume integral values, admitting decreasing step function like singularities, brings us back to our original problem. We think discontinuity would not cause any problem, but if one prefers, one may slightly regularize them to smooth decreasing functions while staying arbitrarily close to our problem.

Since f is decreasing, the coefficient in front of the matrix is positive. The last 2×2 matrix has eigenvalue 0 for $\delta f(x) = \delta g(x)$ and $+2$ for $\delta f(x) = -\delta g(x)$. The first is the expected zero direction since the variation leaves the relation $f = g$ unchanged. The second shows that the extrema $f = g$ is actually a minima, proving our claim.

Appendix C. Index over gravitons in $AdS_4 \times S^7/\mathbb{Z}_k$

Index of many gravitons in $AdS_4 \times S^7/\mathbb{Z}_k$ can be obtained from the index of single graviton in $AdS_4 \times S^7$, as explained in [23]. The index of single graviton in $AdS_4 \times S^7$ is given by

$$I^{\mathrm{sp}}(x, y_1, y_2, y_3) = \frac{(\text{numerator})}{(\text{denominator})} \tag{C.1}$$

where

$$\begin{aligned}
\text{numerator} = {} & \sqrt{y_1 y_2 y_3}\,(1 + y_1 y_2 + y_2 y_3 + y_3 y_1) x^{\frac{1}{2}} \\
& - \sqrt{y_1 y_2 y_3}\,(y_1 + y_2 + y_3 + y_1 y_2 y_3) x^{\frac{7}{2}} \\
& + \big(y_1 y_2 + y_2 y_3 + y_3 y_1 + y_1 y_2 y_3 (y_1 + y_2 + y_2)\big)\big(x^3 - x\big), \\
\text{denominator} = {} & \big(1 - x^2\big)(\sqrt{y_3} - \sqrt{x y_1 y_2})(\sqrt{y_1} - \sqrt{x y_2 y_3}) \\
& \times (\sqrt{y_2} - \sqrt{x y_3 y_1})(\sqrt{y_1 y_2 y_3} - \sqrt{x}).
\end{aligned} \tag{C.2}$$

A very useful property of this function is

$$\begin{aligned}
I^{\mathrm{sp}} &= \frac{(1 - x\sqrt{x y_1 y_2 y_3})(1 - x\sqrt{\frac{x y_3}{y_1 y_2}})(1 - x\sqrt{\frac{x y_1}{y_2 y_3}})(1 - x\sqrt{\frac{x y_2}{y_1 y_3}})}{(1 - \sqrt{\frac{x y_1 y_3}{y_2}})(1 - \sqrt{\frac{x y_2 y_3}{y_1}})(1 - \sqrt{\frac{x y_1 y_2}{y_3}})(1 - \sqrt{\frac{x}{y_1 y_2 y_3}})(1 - x^2)^2} - \frac{1 - x^2 + x^4}{(1 - x^2)^2} \\
&\equiv \frac{F(x, y_1, y_2, y_3)}{(1 - x^2)^2} - \frac{1 - x^2 + x^4}{(1 - x^2)^2}
\end{aligned} \tag{C.3}$$

where the function $F(x, y_i)$ is defined in Section 3.2. The index of single gravitons in $AdS_4 \times S^7/\mathbb{Z}_k$ is obtained by expanding I^{sp} in y_3 as

$$I^{\mathrm{sp}} = \sum_{n=-\infty}^{\infty} y_3^{\frac{n}{2}} I_n^{\mathrm{sp}}(x, y_1, y_2)$$

and keeping terms in which n is a multiplet of k:

$$I_{\mathbb{Z}_k}^{\mathrm{sp}}(x, y_1, y_2, y_3) \equiv \sum_{n=-\infty}^{\infty} y_3^{\frac{kn}{2}} I_{kn}^{\mathrm{sp}}(x, y_1, y_2). \tag{C.4}$$

Each term I_{kn} represents a single particle index of gravitons carrying Kaluza–Klein momentum kn. Finally, the index of multiplet gravitons in $AdS_4 \times S^7/\mathbb{Z}_k$ is given by

$$I_{\mathrm{mp}}(x, y_1, y_2, y_3) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} I_{\mathbb{Z}_k}^{\mathrm{sp}}\big(x^n, y_1^n, y_2^n, y_3^n\big)\right]. \tag{C.5}$$

One can decompose this index into three factors, each coming from gravitons with positive/negative/zero KK-momenta, respectively, as

$$I_{\mathrm{mp}}(x, y_1, y_2, y_3) = I_{\mathrm{mp}}^{(0)} I_{\mathrm{mp}}^{(+)} I_{\mathrm{mp}}^{(-)}, \tag{C.6}$$

where

$$I_{\mathrm{mp}}^{(0)} = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}I_0^{\mathrm{sp}}\left(\cdot^n\right)\right], \qquad I_{\mathrm{mp}}^{(\pm)} = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}I_{\mathbb{Z}_k}^{\mathrm{sp}(\pm)}\left(\cdot^n\right)\right] \tag{C.7}$$

and

$$I_{\mathbb{Z}_k}^{\mathrm{sp}(\pm)} = \sum_{n=1}^{\infty} y_3^{\pm\frac{kn}{2}} I_{\pm kn}^{\mathrm{sp}}(x, y_1, y_2). \tag{C.8}$$

$I_{\mathrm{mp}}^{(\pm)}$ satisfies the property $I_{\mathrm{mp}}^{(-)}(x, y_1, y_2, y_3) = I_{\mathrm{mp}}^{(+)}(x, y_1, 1/y_2, 1/y_3)$, similar to the relation between $I^{(-)}$ and $I^{(+)}$ for the gauge theory index defined in Section 3.

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