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Irreducible Quadrangulations of the Torus

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In this paper, we find the irreducible quadrangulations of the torus. As a consequence, any two quadrangulations of the torus with the same number of vertices that are either both bipartite or both non-bipartite (except for some complete bipartite graphs) can be transformed into one another, up to homeomorphism,

1. INTRODUCTION

A *quadrangulation* G of a closed surface F^2 is a simple graph embedded in F^2 whose faces are all quadrangles. For quadrangulations, two transformations have been defined in [3], which are the *diagonal slide* and the *diagonal rotation* around a vertex of degree 2 as shown in Fig. 1. If the graph obtained by a diagonal slide is not a simple graph, then we don't carry out it. Two quadrangulations G and G' of F^2 are said to be *equivalent* to each other (under diagonal slides and diagonal rotations) and denoted by $G \approx G'$ if they are transformed into each other by a sequence of diagonal slides and diagonal rotations, up to homeomorphism. Observe that both of the transformations preserve the bipartiteness of quadrangulations. Thus, a bipartite quadrangulation and a non-bipartite quadrangulation are never equivalent to each other under diagonal slides and diagonal rotations. In [3], the author has shown the following theorem:

THEOREM 1 (A. Nakamoto [3]). *For any closed surface F^2, there exists a positive integer $N(F^2)$ such that if G_1 and G_2 are two bipartite quadrangulations of F^2 with $|V(G_1) = |V(G_2)| \geqslant N(F^2)$, then $G_1 \approx G_2$.*

Let G be a quadrangulation of a closed surface F^2 and f a face of G with its boundary cycle $abcd$. The *face contraction* of f at $\{b, d\}$ is to identify b and d along the diagonal bd of f and to eliminate f as shown in Fig. 2. However, if b and d are joined by an edge or if both b and d are adjacent with a common vertex $v \neq a, c$, then a face contraction of f at $\{b, d\}$ yields a loop or

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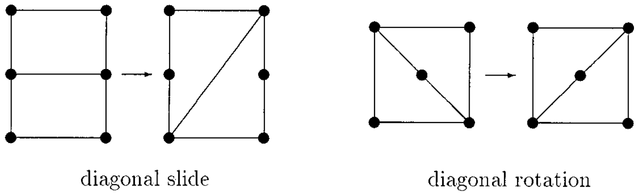


FIG. 1. The diagonal slide and the diagonal rotation.

multiple edges. If a face contraction destroys the simpleness of G, then we do not apply this deformation. Note that this deformation also preserves the bipartiteness of quadrangulations. If we can apply a face contraction of f at $\{b, d\}$, then f is said to be *contractible* at $\{b, d\}$. There are two ways to contract a face since each face has two diagonal pairs of vertices. Also, if G is obtained from a quadrangulation T by a sequence of face contractions, then G is said to be *contractible* to T. A quadrangulation G of F^2 is said to be *irreducible* if G is contractible to no other quadrangulation. It is clear that an irreducible quadrangulation has no contractible face and that any quadrangulation can be contractible to an irreducible one.

It has been shown in [4] that for a closed surface F^2 other than the sphere, an irreducible quadrangulation of F^2 has at most $186(2-\chi(F^2))-64$ vertices, where $\chi(F^2)$ denotes the Euler characteristic of F^2. This implies that for any closed surface, there exist only finitely many irreducible quadrangulations, up to homeomorphism. The finiteness of the number of irreducible ones on each closed surface played an essential role to establish Theorem 1. By our algorithm used in [3], if all the irreducible bipartite quadrangulations of F^2 are listed, then the value of $N(F^2)$ in Theorem 1 can be determined, as shown in Section 3.

We denote the sphere, the projective plane, the torus and the Klein bottle by S^2, P^2, T^2 and K^2, respectively. In [5], irreducible quadrangulations of S^2 and P^2 have been determined. The unique irreducible quadrangulation of S^2 is C_4, which is a cycle of length 4. By this result, it has been shown

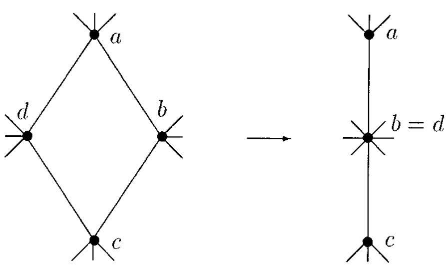


FIG. 2. Face contraction.

that any two quadrangulations of S^2 with the same number of vertices are equivalent to each other under diagonal slides and diagonal rotations. On P^2, there exist precisely two irreducible quadrangulations Q_P^1 and Q_P^2 shown in Fig. 3, which are $K_{3,4}$ and K_4, respectively. (In Fig. 3, to obtain the projective plane, identify each antipodal pair of vertices and edges of the octagon and hexagon.) By this result, it has been shown that any two quadrangulations of P^2 with the same number of vertices are equivalent to each other under diagonal slides and diagonal rotations if and only if both or neither of them are bipartite.

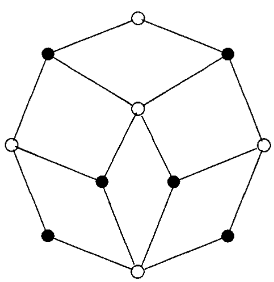
We could show that any two non-bipartite quadrangulations on P^2 with the same number of vetices are equivalent to each other. However, Theorem 1 cannot be extended to non-bipartite quadrangulations in general since there exists a pair of inequivalent non-bipartite quadrangulations of the Klein bottle with the same and arbitrarily large number of vertices [3].

In this paper, we shall determine all the irreducible quadrangulations of the torus. Moreover, by this result, we shall show Theorem 3.

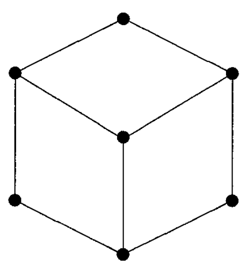
THEOREM 2. *There are exactly eight irreducible quadrangulations of the torus, up to homeomorphism.*

They are shown in Fig. 4, in which each rectangle represents the torus by identifying both pairs of opposite sides. The quadrangulations $Q_T^1, \ldots, Q_T^5$ are bipartite while Q_T^6, Q_T^7, Q_T^8 are non-bipartite.

THEOREM 3. *Any two quadrangulatons of the torus with the same number of vertices, except for complete bipartite graphs, are equivalent to each other under diagonal slides and diagonal rotations if and only if both or neither of them are bipartite.*



Q_P^1



Q_P^2

FIG. 3. Irreducible quadrangulations of the projective plane.

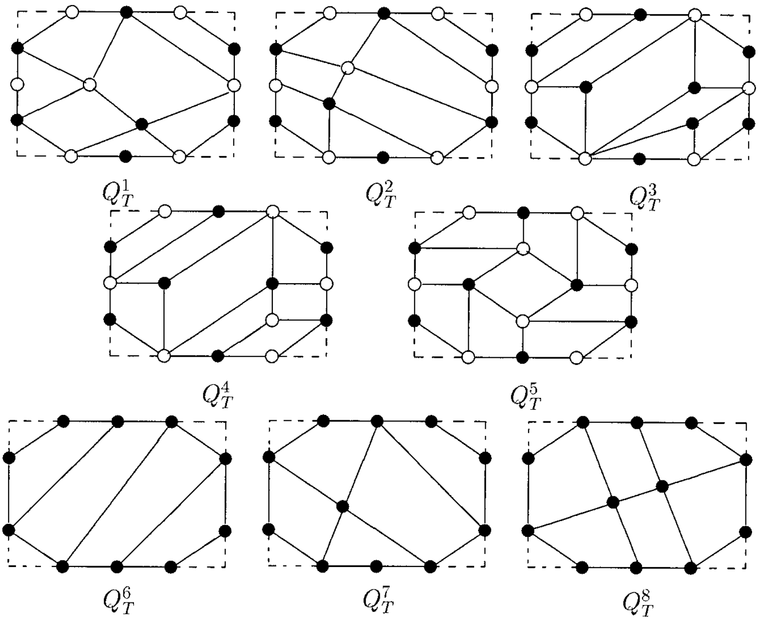


FIG. 4. Irreducible quadrangulations of the torus.

A closed curve l on a closed surface F^2 is said to be *trivial* if l bounds a 2-cell on F^2. A graph G embedded in a closed surface F^2 is said to be *n-representative* if every non-trivial closed curve on F^2 which does not intersect edges of G must contain at least n vertices of G. The *contraction* of an edge e of G is to delete e and identify its two endpoints. If G is obtained from a graph T by a sequence of deletions and contractions of edges of T, then G is said to be a *minor* of T. A graph G is said to be *minor-minimal n-representative* if G is n-representative and no minor of G is n-representative.

By the affirmative solution of Wagner's conjecture proved by Robertson and Seymour [6], it is known that any infinite sequence of graphs includes a pair of graphs in which one graph is a minor of the other. Also, it is known that this argument can be replaced with surface minor, which implies that there exist finitely many minor-minimal n-representative graphs on any closed surface F^2. In particular, Schrijver [7] has given the number of equivalence classes of minor-minimal n-representative graphs on the torus with respect to the Y–Δ transformations. More concretely, Vitray [8] has determined the minor-minimal 2-representative and 3-representative graphs on the projective plane. In Section 4, we shall list up all the minor-minimal 2-representative graphs on the torus, which are obtained from irreducible bipartite quadrangulations of the torus determined in Section 2.

2. Proof of Theorem 2

In this section, we shall determine irreducible quadrangulations of T^2. If a quadrangulation has a vertex v of degree 2, then a face incident with v is contractible. Hence an irreducible quadrangulation of a closed surface other than the sphere has no vertex of degree less than 3.

Proposition 4. *The average degree of quadrangulations of the torus is precisely* 4.

By Proposition 4, we can see that an irreducible quadrangulation of the torus is either 4-regular or one which contains a vertex of degree 3. Thus, the classification of 4-regular quadrangulations of the torus, mentioned later, will play an important role for our purpose.

The following lemma is Lemma 3 in [3].

Lemma 5. *Let G be a quadrangulation of a closed surface F^2. If there is a 2-cell region of G which is bounded by a 4-cycle but not a face of G, then there is a contractible face in this 2-cell.*

Lemma 6. *Let G be an irreducible quadrangulation of a closed surface F^2 and F a hexagonal 2-cell region of G. Then, inside F, there is either a single edge or a single vertex of degree* 3.

Proof. Let G be an irreducible quadrangulation of F^2. Let F be a hexagonal region of G bounded by a closed walk $v_1v_2v_3v_4v_5v_6$. Here, v_i and v_{i+3} may coincide. Since F is hexagonal, F contains at least two faces. Since G is irreducible, each diagonal pair of a face is joined or adjacent with a common vertex.

We first consider the case that F contains a face f such that $\partial f \cap \partial F \supseteq \{v_{i-1}, v_i, v_{i+1}\}$. Suppose that f is bounded by a cycle $v_1v_2v_3x$, where $x \in V(G)$ is in F. Notice that a 2-cell region bounded by an odd cycle is never quadrangulated. Since f is not contractible at $\{x, v_2\}$, x coincides with v_4 or v_6, or there is an edge xv_5. If $x = v_4$ or $x = v_6$, then F contains one edge, by Lemma 5. If x is joined with v_5, then F contains a vertex of degree 3.

Next, we suppose that F does not contain a face f such that $\partial f \cap \partial F \supseteq \{v_{i-1}, v_i, v_{i+1}\}$, and focus on the face h in F containing an edge v_1v_2. Suppose that h is bounded by v_1v_2xy, where $x, y \in V(G)$ are in F. Since h is not contractible at $\{x, v_1\}$, x coincides with v_3 or v_5, or is adjacent with v_4 or v_6. We may suppose that $x \neq v_3$, by the assumption on F. If $x = v_5$, then a face $v_2v_3v_4v_5$ occurs, by Lemma 5. And if x is adjacent with v_4 or v_6, then, by Lemma 5, we obtain a face $v_2v_3v_4x$ or $v_6v_1v_2x$, respectively. Thus, in any case, F contains a face f such that $\partial f \cap \partial F \supseteq \{v_{v-1}, v_i, v_{i+1}\}$. Therefore, F contains one edge or a vertex of degree 3. ∎

Let H denote a complete bipartite graph $K_{3,4}$ with partite sets $\{x, a, b, c\}$ and $\{1, 2, 3\}$ embedded in a closed surface so that $a1x3$, $b2x1$ and $c3x2$ bound faces, respectively. Let R denote the hexagonal 2-cell region of H which is the union of $a1x3$, $b2x1$ and $c3x2$. (See Fig. 5 (1).)

LEMMA 7. *Let G be an irreducible bipartite quadrangulation of a closed surface other than the sphere. If G contains a vertex of degree* 3, *then G contains H as a subgraph.*

Proof. Let G be an irreducible bipartite quadrangulation of a closed surface F^2 and x a vertex of G of degree 3. The union of three faces incident with x form a hexagonal 2-cell region R. Let a, 1, b, 2, c and 3 be vertices of G lying along ∂R in this order. Suppose that x is adjacent with 1, 2 and 3. Notice that the seven vertices are all distinct. Otherwise, an odd cycle or multiple edges arise, a contradiction. Since the face $x3a1$ is not contractible at $\{a, x\}$, there is an edge $a2$ outside R. Similarly there are edges $b3$ and $c1$ unless F^2 is the sphere. (That is, if F^2 is the sphere, then the vertices b and 3 are separated by the cycle $a1x2$. Since the cycle $a1x2$ must bound a 2-cell on the sphere, b and 3 cannot be joined by an edge. Similarly, by the cycle $a3x2$, two vertices c and 1 cannot be joined by an edge. Actually, the

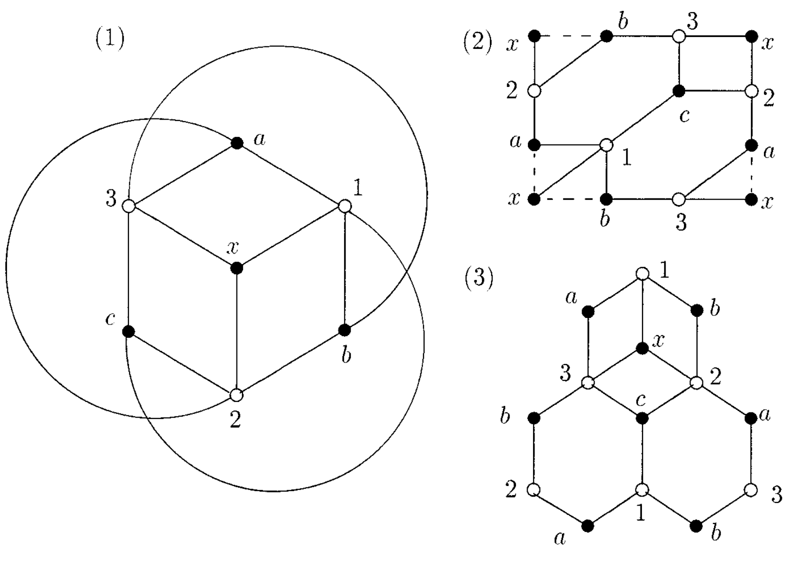


FIG. 5. H embedded in T^2.

complete bipartite graph $K_{3,4}$ is not embeddable in the sphere.) Thus, G contains H as a subgraph. ∎

LEMMA 8. *There exist pecisely two irreducible quadrangulations of the torus with a vertex of degree* 3 *up to homeomorphism, which are* Q_T^3 *and* Q_T^4 *in Figure* 4.

Proof. Let G be an irreducible bipartite quadrangulation of T^2 with a vertex of degree 3. Suppose that G is 2-colored. By Lemma 7, G must contain H as a subgraph. Cut open the torus in which H is embedded along the following two simple closed curves crossing at x. One is along $x2a$, the other is along $x3b$. Since two closed curves on T^2 crossing at a single point must be meridian and longitude on the torus up to homeomorphism, we can obtain the rectangle from the torus as shown in Fig. 5 (2), up to symmetry. We can re-draw Fig. 5 (2) symmetrically, and obtain Fig. 5 (3).

We have only to quadrangulate the two hexagonal regions $b3c1a2$ and $c2a3b1$ in Fig. 5 (3). By Lemma 6, we do put vertices y and z of degree 3 into $b3c1a2$ and $c2a3b1$ respectively, since a multiple edge arises if we add an edge. There are two ways to add y and z, up to symmetry and coloring;

- All of x, y and z have the same color. A complete bipartite graph $K_{3,6}$ is obtained. Denote this quadrangulation by Q_T^3.
- One of x, y and z has a different color from the other two. Denote this quadrangulation by Q_T^4.

Thus, there exist two irreducible bipartite quadrangulations of the torus containing a vertex of degree 3.

Now we consider the case that G is non-bipartite. Let x be a vertex of degree 3 of G and R' the union of faces incident with x bounded by a closed walk $v_1v_2v_3v_4v_5v_6$. Here, we suppose that x is adjacent with v_1, v_3 and v_5. Since G is irreducible, v_i and v_{i+3} coincide or are joined by an edge. There are three diagonal pairs of vertices $\{v_1, v_4\}$, $\{v_2, v_5\}$ and $\{v_3, v_6\}$ in R'. If all of v_i's are distinct, then G must be bipartite by the above argument. And if at least two diagonal pairs coincide, say $v_1 = v_4$ and $v_2 = v_5$, then pasting the edge v_1v_2 with v_4v_5 in R' yields a Möbius band and breaks the orientability of the torus, a contradiction. Thus, we consider the case when only one diagonal pair coincides, say $v_1 = v_4$. Similarly to the bipartite case, we cut open the torus along the following two closed curves crossing at x. One is along xv_2v_5 and the other is along xv_3v_6. The obtained rectangle is nothing but the one obtained from Fig. 5 (2) by contracting the edge $1c$. This rectangle has a 2-cell region bounded by an odd cycle. Since a 2-cell region bounded by an odd cycle can never be quadrangulated, this is a contradiction. Therefore, an irreducible non-bipartite quadrangulation of the torus does not contain a vertex of degree 3. ▮

We shall consider the *universal covering space* of the torus. The universal covering space of the torus is homeomorphic to the x–y plane in $\mathbf{R}^2$. Let $\tilde{G}$ be the union of horizontal and vertical lines through integral points in $\mathbf{R}^2$ so that

$$V(\tilde{G}) = \{(x, y) \in \mathbf{R}^2 \mid x, y \in \mathbf{Z}\}.$$

The graph $\tilde{G}$ is a 4-regular infinite one and quadrangulates $\mathbf{R}^2$. We call $\tilde{G}$ the *universal* 4-*regular quadrangulation*. Let $\Gamma(p, q, r)$ denote the collection of all transformations

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ p \end{pmatrix} + \beta \begin{pmatrix} r \\ -q \end{pmatrix} \qquad (\alpha, \beta \in \mathbf{Z})$$

over $\mathbf{R}^2$, for non-negative integers p, q and r with $pr \neq 0$ and $q \geqslant 0$. Clearly, $\Gamma(p, q, r)$ is a group with respect to the composition of transformations. Since $pr \neq 0$, the group $\Gamma(p, q, r)$ freely acts on $\mathbf{R}^2$ and any element of $\Gamma(p, q, r)$ leaves $\tilde{G}$ invariant. (See Fig. 6.) The orbit space $\mathbf{R}^2/\Gamma(p, q, r)$ of the group action is homeomorphic to the torus and the projection $\tilde{G}/\Gamma(p, q, r)$ of $\tilde{G}$ is a 4-regular quadragulation of the torus. Then we define the 4-regular graph $\tilde{G}/\Gamma(p, q, r)$ on the torus $\mathbf{R}^2/\Gamma(p, q, r)$ as the standard form $T(p, q, r)$ of 4-regular quadrangulations of the torus.

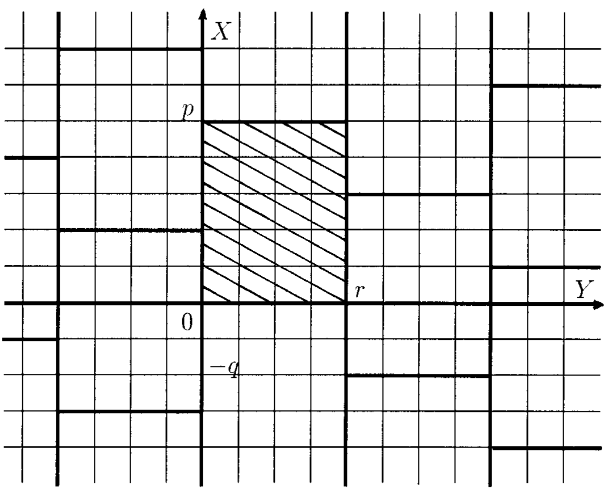


FIG. 6. $\tilde{G}$ with $\Gamma(p, q, r)$.

To describe $T(p, q, r)$ more concretely, consider the fundamental domain of the group action

$$\{(x, y) \in \mathbf{R}^2: 0 \leqslant x \leqslant r, 0 \leqslant y \leqslant p\}.$$

First, identifying the upper and lower sides of the rectangle, we obtain an annulus, which is called a (p, r)-*annulus*. Second glue the two boundaries of the (p, r)-annulus so that the point $(0, y)$ coincides with the point (r, y') $(0 \leqslant y, y' \leqslant p)$ if $y - y' \equiv q \pmod{p}$. By this procedure, we obtain the torus $\mathbf{R}^2/\Gamma(p, q, r)$ and the quadrangulation $T(p, q, r)$.

Let G be a 4-regular quadrangulation of T^2 and v a vertex of G. Suppose that v is adjacent with v_1, v_2, v_3 and v_4 in this cyclic order. A path P in G is said to be *locally straight* at v if P passes through v from v_i to v_{i+2}. A cycle C in G is called *geodesic* if C is locally straight at each vertex of C. The following fact was the key to determine the standard form of 4-regular quadrangulations of the torus in [1].

LEMMA 9. *Let G be a 4-regular quadrangulation of the torus. For any edge e of G, there exists the unique geodesic cycle in G containing e.*

THEOREM 10 (A. Altshuler [1]). *A 4-regular quadrangulation of the torus can be represented as a standard form $T(p, q, r)$ with three integers p, q and r with $p, r > 0$ and $q \geqslant 0$.*

By Theorem 10, if we fix a geodesic cycle C in $T(p, q, r)$, suppose that C is a boundary cycle of the (p, r)-annulus, then we can take r parallel geodesic cycles with C in $T(p, q, r)$, which form a spanning subgraph of $T(p, q, r)$. We call it a *geodesic 2-factor* of $T(p, q, r)$. With respect to $\tilde{T}(p, q, r) \subset \mathbf{R}^2$, the geodesic 2-factor of $T(p, q, r)$ with C corresponds to the set of straight lines $x = \alpha$ $(\alpha \in \mathbf{Z})$ in $\mathbf{R}^2$. We call the set of straight lines in $\mathbf{R}^2$ which corresponds to a geodesic 2 factor an *universal geodesic 2-factor*.

For a geodesic cycle of a 4-regular quadrangulation of T^2, the fundamental domain can be uniquely determined and also the standard form $T(p, q, r)$ is determined. So, if we take another geodesic 2-factor of $T(p, q, r)$, that is, we take the set of horizontal lines in $\tilde{T}(p, q, r) \subset \mathbf{R}^2$ as the universal geodesic 2-factor, then the fundamental domain and the standard form also change.

Let G_1 and G_2 be two graphs embedded in closed surfaces F_1^2 and F_2^2, respectively. Tow graphs G_1 and G_2 are said to be *homeomorphic* to each other if there is a homeomorphism $h: F_1^2 \to F_2^2$ with $h(G_1) = G_2$ which induces an isomorphism from G_1 to G_2.

PROPOSITION 11. *Let $T(p, q, r)$ and $T(p', q', r')$ denote two 4-regular quadrangulations of the torus. $T(p, q, r)$ and $T(p', q', r')$ are homeomorphic* *to each other if and only if either of* (0) *and* (*I*) *holds, where* (a, b) *denotes the greatest common devisor of two integers* a *and* b.

$$(0)\begin{cases} p' = p. \\ q' \equiv \pm q \pmod{p'}, \\ r' = r. \end{cases} \quad (I)\begin{cases} p' = pr/(p, q), \\ q' \equiv \pm \beta r \pmod{p'}, \quad -\beta q \equiv (p, q) \pmod{p}, \\ r' = (p, q). \end{cases}$$

Remark that in each equation, (p', q', r') can uniquely be determined by (p, q, r) under the restriction $0 \leqslant q' \leqslant p'/2$.

Proof. Let $T(p, q, r)$ and $T(p', q', r')$ be two 4-regular quadrangulations of the torus. Suppose that $\tilde{T}(p, q, r)$ and $\tilde{T}(p', q', r')$ are the universal 4-regular quadrangulations of $T(p, q, r)$ and $T(p', q', r')$, respectively.

First, suppose that $0 \leqslant q \leqslant p$. Then, it is easy to see that if $p' = p$, $q' = p - q$, $r' = r$, then $T(p', q', r')$ is the mirror image of $T(p, q, r)$. This translation from $T(p, q, r)$ to $T(p', q', r')$ corresponds to a linear transformation $f: \mathbf{R}^2 \to \mathbf{R}^2$ such that $f(x, y) = (x, -y)$ or $f(x, y) = (-x, y)$ in the universal 4-regular quadrangulation. Second, if $p' = p, q' = q + \alpha p, r' = r$, then $T(p, q, r)$ and $T(p', q', r')$ are homeomorphic to each other, that is, an α-fold twist along the boundary of the (p, r)-annulus in $T(p, q, r)$ yields $T(p', q', r')$. Also, the universal 4-regular quadrangulation is invariant under this translation. Thus, by the above argument, we can see that if $p' = p$, $q' \equiv \pm q \pmod{p'}$, $r' = r$, then $T(p, q, r)$ and $T(p', q', r')$ are homeomorphic to each other. The translation (0) does not change an universal geodesic 2-factor of $\tilde{T}(p, q, r)$.

Now, we shall consider a translation which changes an universal geodesic 2-factor in $\tilde{T}(p, q, r)$. So, we shall take a geodesic cycle C of $T(p, q, r)$ containing the edge between $(0, 0)$ and $(1, 0)$ in $\mathbf{R}^2$. This is possible, by Lemma 9. By the definition of $\Gamma(p, q, r)$, the point (0,0) is carried on the point $(\beta r, \alpha p - \beta q)$ by $\Gamma(p, q, r)$, where $\alpha, \beta \in \mathbf{Z}$. If such a point lies on the x-axis, then the integral equation $\alpha p - \beta q = 0$ holds and the minimum positive β is equal to $pr/(p, q)$. Thus, the path on the x-axis connecting $(0, 0)$ and $(pr/(p, q), 0)$ is nothing but the required geodesic cycle of $T(p, q, r)$, and we have $p' = pr/(p, q)$. Also, since $T(p, q, r)$ and $T(p', q', r')$ have the same number of vertices, we have $pr = p'r'$ and hence $r' = (p, q)$.

We have determined the fundamental domain of $T(p', q', r')$ of $\tilde{T}(p, q, r)$ to be

$$\{(x, y) \in \mathbf{R}^2 \colon 0 \leqslant x \leqslant pr/(p, q), 0 \leqslant y \leqslant (p, q)\}.$$

So, in order to determine q', it suffices to see where $(0, 0)$ is carried on the segment between $(0, (p, q))$ and $(pr/(p, q), (p, q))$. Since $(0, 0)$ is carried on

$(\beta r, \alpha p - \beta q)$, we have $\alpha p - \beta q = (p, q)$ and hence $-\beta q \equiv (p, q) \pmod{p}$. Thus, we obtain $q' \equiv \beta r \pmod{p'}$ for β satisfying $-\beta q \equiv (p, q) \pmod{p}$. Talking the translation (0) into consideration, we obtain (I).

Since there are only two different universal geodesic 2-factors in an universal 4-regular quadrangulation, we need the only two translations (0) and (I). Therefore, the lemma follows ∎.

LEMMA 12. *There exist precisely six irreducible* 4-*regular quadrangulations of the torus up to homeomorphism, which are* $Q_T^1, Q_T^2, Q_T^5, \ldots, Q_T^8$ *in Fig.* 4.

Proof. Let G be an irreducible 4-regular quadrangulation of T^2. By Theorem 10, we suppose that G can be represented as $T(p, q, r)$ with three integers p, q and r. Figure 7 shows a local structure of G. The face bounded by $fgkj$ is supposed to be F. Since G is irreducible, F is not contractible at both diagonal pairs of F. We focus on the diagonal pair $\{f, k\}$. Since F is not contractible at $\{f, k\}$, f and k are either adjacent with a common vertex $v(\neq g, j)$ or joined by an edge, and hence we have one of the following cases: In case that f and k are adjacent with a common vertex, (1) $e = l$. (2) $b = o$. (3) $b = l$. (4) $e = o$. In case that f and k are joined by an edge, (5) $f = l$. (6) $f = o$. (7) $k = b$. (8) $k = e$.

CLAIM 1. *If* $b = o$ *or if* $e = l$, *then* G *is represented as* $T(p, 3, 1)$ *for some* p.

Proof. We regard Fig. 7 as a part of the universal covering of G. In case of $b = o$, by Lemma 9, we can take the straight line through b, f, j, n as a geodesic cycle of G, which is denoted by C. In this case, all the vertical lines form the universal goedesic 2-factor of G. Then, since $b = o$, the straight line through c, g, k, o has to be identified with C so that b and o coincide when we construct the torus from the universal 4-regular quadrangulation.

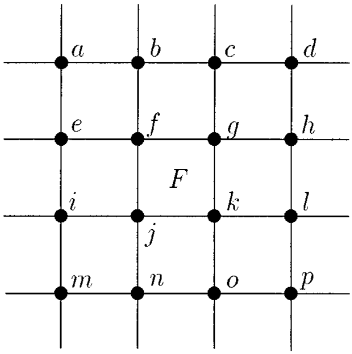


FIG. 7. The local structure of G.

Thus, G can be represented as $T(p, 3, 1)$ for some positive integer p. In case of $e=l$, by taking each horizontal line as a geodesic cycle, we obtain the same argument as above. Therefore, the claim follows. ▮

We omit the proof of the following claims since their proof is very similar to that of Claim 1.

CLAIM 2. *If $b=l$ or if $e=o$, then G is represented as $T(p, 2, 2)$ for some p.*

CLAIM 3. *If either of $f=l, f=o, k=b$ and $k=e$ holds, then G is represented as $T(p, 2, 1)$ for some p.*

Now, we shall determine p in each case, depending on the structure of another diagonal pair $\{g, j\}$ of F. We demonstrate only case of $T(p, 3, 1)$ to avoid the repetitions of the similar process. We label vertices of $T(p, 3, 1)$ as shown in the left of Fig. 8. And we place it on $\mathbf{R}^2$ so that the vertex labeled p coincides with (0,0), and consider the universal covering of $T(p, 3, 1)$ as shown in the right of Fig. 8, which is denoted by $\tilde{T}(p, 3, 1)$. Note that the local situation of $T(p, 3, 1)$ is lifted to $\tilde{T}(p, 3, 1)$ and that each face of $\tilde{T}(p, 3, 1)$ bounded by 4 vertices whose coodinates are (i, j), $(i+1, j)$, $(i+1, j+1)$ and $(i, j+1)$ on $\mathbf{R}^2$ is not contractible at $\{(i, j+1), (i+1, j)\}$.

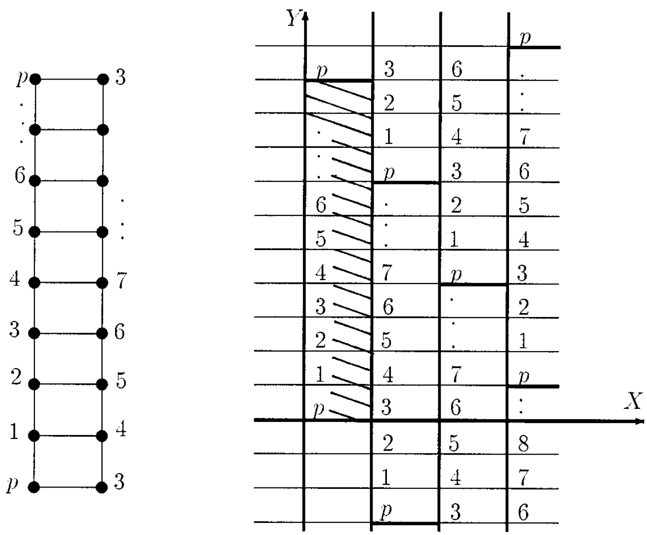


FIG. 8. $T(p, 3, 1)$ and $\tilde{T}(p, 3, 1)$.

We claim that the two cases $b=o$ and $e=l$ are essentially same, up to direction of the universal geodesic 2-factor. So, it suffices to consider case of $b=o$. Since F is not contractible at $\{g, j\}$, we have one of the following cases: (i) $c=n$. (ii) $i=h$. (iii) $c=i$ or $h=n$. (iv) $g=i$ or $h=j$. (v) $c=j$ or $g=n$. Only in case (ii), it might happen that $b=o=i=h$, and hence we treat this case as the case (vi). However, such a phenomena does not happen in other cases.

(i) $b=o$ and $c=n$ means that G contains a Möbius Band or G contains multiple edges. This is contrary to the orientability of the torus or the simpleness of G.

(ii) $i=h$ induces the identification of two points whose coordinates are (i, j) and $(i+3, j+1)$ on $\tilde{T}(p, q, r) \subset \mathbf{R}^2$. By this, the vertex labeled p is identified with the one labeled 10, and hence we obtain that $p=10$.

(iii) Both of $c=i$ and $h=n$ induce the identification of (i, j) and $(i+2, j+2)$ on $\tilde{T}(p, q, r) \subset \mathbf{R}^2$. By this, we obtain that $p=8$, similarly to the case (ii).

(iv) Both $g=i$ and $h=j$ induce the identification of (i, j) and $(i+2, j+1)$ on $\tilde{T}(p, q, r) \subset \mathbf{R}^2$. Similarly to the case (ii), we obtain that $p=7$.

(v) Both $c=j$ and $g=n$ induce the identification of (i, j) and $(i+1, j+2)$ on $\tilde{T}(p, q, r) \subset \mathbf{R}^2$. By this, we obtain that $p=5$ similarly to the case (ii).

(vi) $b=o=i=h$ induce the identification of (i, j) and $(i+1, j+2)$ on $\tilde{T}(p, q, r) \subset \mathbf{R}^2$. However, this case has been considered in the case (v).

Therefore, by the above procedure, we obtain $T(10, 3, 1)$, $T(8, 3, 1)$, $T(7, 3, 1)$ and $T(5, 3, 1)$ from $T(p, 3, 1)$. By the similar process, we obtain $T(4, 2, 2)$ from $T(p, 2, 2)$, and we obtain $T(5, 2, 1)$, $T(6, 2, 1)$ and $T(7, 2, 1)$ from $T(p, 2, 1)$. We can see that $T(7, 3, 1)$ and $T(7, 2, 1)$ are homeomorphic and that $T(5, 3, 1)$ and $T(5, 2, 1)$ are homeomorphic, by Lemma 11. Thus, we can obtain Table I.

Though Q_T^1 and Q_T^2 are $K_{4,4}$ as graphs, they have the different standard forms and hence they are not homeomorphic. So, we treat them separately in this paper. Here, unifying the eight irreducible quadrangulations of T^2 obtained as above into the same appearance by homeomorphism, we obtain Fig. 4. ∎

Proof of Theorem 2. By Lemmas 8 and 12, we can see that the theorem follows. ∎

TABLE I

Irreducible 4-Regular Quadrangulations of T^2

Standard form	Notation	Graph
$T(4, 2, 2)$	Q_T^1	$K_{4,4}$
$T(8, 3, 1)$	Q_T^2	$K_{4,4}$
$T(10, 3, 1)$	Q_T^5	$K_{5,5}-$ 1-factor
$T(5, 2,1)$	Q_T^6	K_5
$T(6, 2, 1)$	Q_T^7	K_6- 1-factor
$T(7, 2, 1)$	Q_T^8	K_7- hamilton cycle

3. Proof of Theorem 3

By the obtained complete list of irreducible quadrangulations of the torus, we shall prove Theorem 3. The following two lemmas have been shown in [3].

Lemma 13. *Any vertex of degree 2 of a quadrangulation of F^2 can be moved into any face of G by a sequence of diagonal slides.*

Here, Γ_n denotes a quadrilateral region which contains n vertices of degree 2 as shown in Fig. 9. Let T be a quadrangulation of a closed surface F^2 and $T+\Gamma_n$ a quadrangulation of F^2 obtained from T by adding Γ_n to a face of T. The quadrangulation $T+\Gamma_n$ represents various quadrangulations depending on our choice of a face to add Γ_n. However, by Lemma 13, $T+\Gamma_n$ denotes an unique quadrangulation, up to equivalence.

Lemma 14. *Let G and T be two quadrangulations of a closed surface. If T is obtained from G by a sequence of face contractions, then $G \approx T+\Gamma_m$ with $m=|V(G)|-|V(T)|$.*

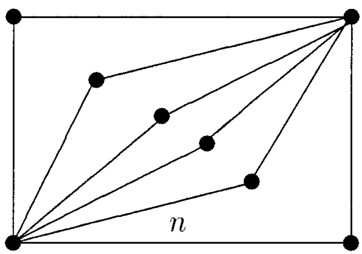


Fig. 9. Γ_n.

LEMMA 15. *Let G_1 and G_2 be two quadrangulations of a closed surface and m a non-negative integer. If $G_1 \approx G_2$, then $G_1 + \Gamma_m \approx G_2 + \Gamma_m$.*

Proof. It is easy to see that the lemma follows since a vertex of degree 2 can be moved into any face by diagonal slides, by Lemma 13. ▮

THEOREM 16. *Any two bipartite quadrangulations G_1 and G_2 of the torus are equivalent to each other under diagonal slides and diagonal rotations if $|V(G_1)| = |V(G_2)| \geqslant N(T_2) = 10$. Here, $N(T^2) = 10$ is sharp.*

Proof. Let G_1 and G_2 be two bipartite quadrangulations of the torus with $|V(G_1)| = |V(G_2)| = m \geqslant 10$. Any bipartite quadrangulation of T^2 is contractible to one of $Q_T^1, ..., Q_T^5$ by Theorem 2. By Lemma 14, each of G_1 and G_2 is equivalent to one of $Q_T^1 + \Gamma_{m-8}$, $Q_T^2 + \Gamma_{m-8}$, $Q_T^3 + \Gamma_{m-9}$, $Q_T^4 + \Gamma_{m-9}$ and $Q_T^5 + \Gamma_{m-10}$.

So, in order to show that $G_1 \approx G_2$, we shall prove that

$$Q_T^1 + \Gamma_{m-8} \approx Q_T^2 + \Gamma_{m-8} \approx Q_T^3 + \Gamma_{m-9} \approx Q_T^4 + \Gamma_{m-9} \approx Q_T^5 + \Gamma_{m-10}.$$

Since we can move a vertex of degree 2 freely, by Lemma 13, and since $m \geqslant 10$, we have

$$\begin{aligned} Q_T^1 + \Gamma_2 + \Gamma_{m-10} &\approx Q_T^2 + \Gamma_2 + \Gamma_{m-10} \approx Q_T^3 + \Gamma_1 + \Gamma_{m-10} \\ &\approx Q_T^4 + \Gamma_1 + \Gamma_{m-10} \approx Q_T^5 + \Gamma_{m-10}. \end{aligned}$$

Thus, by Lemma 15, it suffices to show that $Q_T^1 + \Gamma_2 \approx Q_T^2 + \Gamma_2 \approx Q_T^3 + \Gamma_1 \approx Q_T^4 + \Gamma_1 \approx Q_T^5$. We shall demonstrate only $Q_T^1 + \Gamma_2 \approx Q_T^5$ by Fig. 10. Similarly, others can be shown easily via Q_T^5. Therefore, the theorem follows.

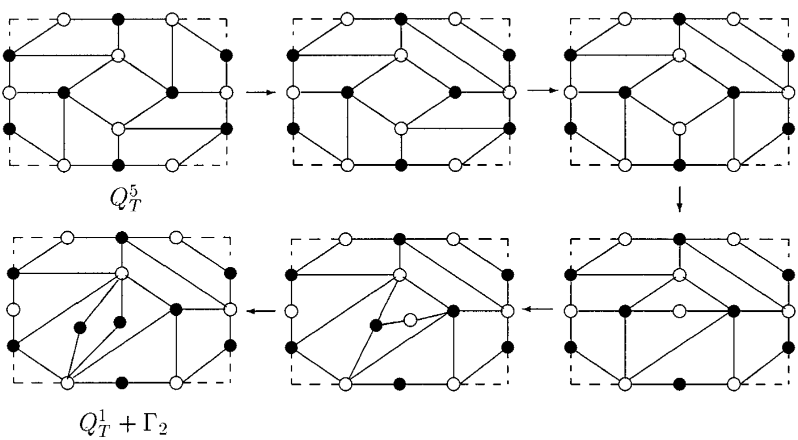


FIG. 10. $Q_T^1 + \Gamma_2 \approx Q_T^5$.

If $m < 10$, the theorem does not follow. Since Q_T^3 is a complete bipartite graph, we can not move any edge of it and hence it is not equivalent to any other one. Thus, Q_T^3 and $Q_T^1 + \Gamma_1$ are a pair of inequivalent quadrangulations with $m = 9$. Thus, $N(T^2) = 10$ is sharp. ∎

By this theorem, the value of $N(T^2)$ in Theorem 1 has been determined to be 10. In the above proof, if we exclude complete bipartite graphs Q_T^1, Q_T^2 and Q_T^3, we can also show the equivalence of any two bipartite quadrangulations of T^2 with the same number of vertices, without restricting the lower bound of the number of vertices.

THEOREM 17. *Any two bipartite quadrangulations of the torus with the same number of vertices, except for complete bipartite graphs, are equivalent to each other under diagonal slides and diagonal rotations.*

Proof. Let G_1 and G_2 be two bipartite quadrangulations of T^2 with $|V(G_1)| = |V(G_2)| = m$ which are not complete bipartite graphs. If $m \geqslant 10$, this theorem holds, by Theorem 16. Also, in case of $m \leqslant 8$, there exists no biparite quadrangulation of T^2 which is not a complete bipartite graph, by Theorem 2. So, we shall consider case of $m = 9$. A bipartite quadrangulation of T^2 with 9 vertices which is not a complete bipartite graph is isomorphic to either of the one contractible to Q_T^1, the one contractible to Q_T^2 or Q_T^4, by Theorem 2. Thus, it suffices to show that $Q_T^1 + \Gamma_1 \approx Q_T^2 + \Gamma_1 \approx Q_T^4$. This can be shown similarly to the above procedure shown in Fig. 10. Therefore, the theorem follows. ∎

THEOREM 18. *Any two non-bipartite quadrangulations of the torus with the same number of vertices are equivalent to each other using only diagonal slides.*

Proof. In non-bipartite quadrangulations, it has been already shown that a diagonal rotation can be realized by a sequence of diagonal slides [5]. By Theorem 2 and Lemma 14, any non-bipartite quadrangulation of T^2 with $m \geqslant 5$ vertices is contractible to one of Q_T^6, Q_T^7 and Q_T^8 and equivalent to one of $Q_T^6 + \Gamma_{m-5}$, $Q_T^7 + \Gamma_{m-6}$ and $Q_T^8 + \Gamma_{m-7}$. Now we show $Q_T^7 \approx Q_T^6 + \Gamma_1$ and $Q_T^8 \approx Q_T^6 + \Gamma_2$. Similarly to the above case, they can be easily shown. Thus, any non-bipartite quadrangulation of T^2 with m vertices can be transformed into $Q_T^6 + \Gamma_{m-5}$ by diagonal slides, and hence the theorem follows. ∎

Proof of Theorem 3. The two transformations, a diagonal slide and a diagonal rotation, preserve the bipartiteness of quadrangulations. Therefore, Theorem 3 follows just as a corollary of Theorems 17 and 18. ∎

4. The 2-Representative Graphs on the Torus

How do 2-representative graphs have connections with quadrangulations? The following two propositions are the keys and easy to see.

Proposition 19. *Let G be a 2-connected graph 2-cell embedded in a closed surface F^2. Then, G is 2-representative if and only if each face of G is bounded by a cycle.*

Let G be a graph with black vertices 2-cell embedded in a closed surface F^2. Put a white vertex into each face of G and join it with the black vertices of G lying along the boundary walk of the face. And delete all edges of G. The resulting graph is called the *radial graph* $R(G)$ of G [2]. It is easy to see that $R(G)$ is bipartite and each face of $R(G)$ is quadrilateral, but $R(G)$ is not always a quadrangulation. If there is a face whose boundary walk is not a cycle, then $R(G)$ has multiple edges.

Proposition 20. *A graph G is embedded in a closed surface F^2 so that each face of G is bounded by a cycle if and only if $R(G)$ is a bipartite quadrangulation of F^2.*

Observe that a face contraction at white vertices in $R(G)$ corresponds to a deletion of an edge in G, and that a face contraction at black vertices in $R(G)$ corresponds to a contraction of an edge in G. Thus, a face contraction of $R(G)$ corresponds to one of two operations which produce a minor of G. Therefore, we can see the following proposition immediately, from Proposition 19 and 20.

Proposition 21. *Let G be a graph embedded in a closed surface F^2 and $R(G)$ its radial graph. Then, G is minor-minimal 2-representative on F^2 if and only if $R(G)$ is an irreducible bipartite quadrangulation of F^2.*

Proof. See the above comment. ∎

By Proposition 21, we can translate Theorem 2 to the following theorem by regarding each Q_T^i as a radial graph. Actually, the two minor-minimal 2-representative graphs on the projective plane, determined by Vitray [8], are obtained from Q_P^1 in Fig. 3 by regarding it as a radial graph.

Theorem 22. *If a graph G embedded in the torus is minor-minimal 2-representative, then G is isomorphic to one of $T_1, \ldots, T_7$ shown in Fig. 11, up to homeomorphism.*

In Fig. 11, each rectangle represents the torus by identifying each pair of opposite lines. In particular, T_3 and T_7 are isomorphic to $K_{3,3}$ and K_5 as

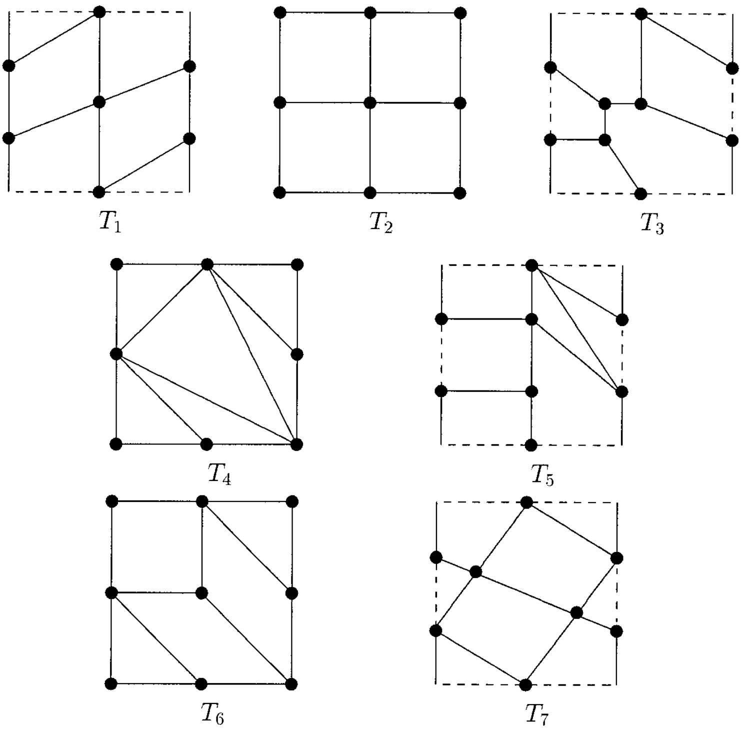


FIG. 11. Minor-minimal 2-representative graphs on the torus.

graphs, respectively. (T_3, T_4) and (T_5, T_6) are dual pairs of graphs. Note that T_3, T_4, T_5 and T_6 are transformed into each other by Y–Δ transformations. A Y–Δ transformation in a graph G corresponds to the diagonal rotation around a vertex of degree 3 in $R(G)$. On the other hand, T_1, T_2 and T_7 are 4-regular, that is, they contain neither a vertex of degree 3 nor a triangular face.

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