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Coexistence and asymptotic periodicity in a competitor–competitor–mutualist model ☆

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Abstract

In this paper, the competitor–competitor–mutualist three-species Lotka–Volterra model is discussed. Firstly, by Schauder fixed point theory, the coexistence state of the strongly coupled system is given. Applying the method of upper and lower solutions and its associated monotone iterations, the true solutions are constructed. Our results show that this system possesses at least one coexistence state if cross-diffusions and cross-reactions are weak. Secondly, the existence and asymptotic behavior of T-periodic solutions for the periodic reaction–diffusion system under homogeneous Dirichlet boundary conditions are investigated. Sufficient conditions which guarantee the existence of T-periodic solution are also obtained.

Keywords: Competition; Mutualism; Coexistence; T-periodic solution; Asymptotic behavior

1. Introduction

In this paper, we consider the strongly coupled elliptic system with Dirichlet boundary condition:

$$\begin{cases} -\Delta[(d_1+\alpha_{11}u_1+\alpha_{12}u_2)u_1]=u_1(a_1-b_{11}u_1-b_{12}u_2), \quad x\in\Omega,\\ -\Delta[(d_2+\alpha_{21}u_1+\alpha_{22}u_2+\frac{\alpha_{23}}{\beta+u_3})u_2]=u_2(a_2-b_{21}u_1-b_{22}u_2+b_{23}u_3), \quad x\in\Omega,\\ -\Delta[(d_3+\frac{\alpha_{32}}{\gamma+u_2}+\alpha_{33}u_3)u_3]=u_3(a_3+b_{32}u_2-b_{33}u_3), \quad x\in\Omega,\\ u_i(x)=0, \quad i=1,2,3,\ x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian operator, Ω is a bounded domain in $\mathcal{R}^N$ with a smooth boundary $\partial\Omega$ and $d_i,\beta,\gamma,a_i,b_{ij}$, $i,j=1,2,3$ are positive constants except for α_{ij} which may be nonnegative constants. The system represents a model which involves interacting and migrating in the same habitat Ω among a competitor, a competitor–mutualist and a mutualist. Here u_i, $i=1,2,3$ denotes the density of competitor, competitor–mutualist and mutualist, respectively.

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The boundary condition means that the habitat Ω is surrounded by a hostile environment. The diffusion terms can be written as

$$\begin{aligned}
&\operatorname{div}\left\{(d_1 + 2\alpha_{11}u_1 + \alpha_{12}u_2)\nabla u_1 + \alpha_{12}u_1\nabla u_2\right\},\\
&\operatorname{div}\left\{\alpha_{21}u_2\nabla u_1 + \left(d_2 + \alpha_{21}u_1 + 2\alpha_{22}u_2 + \frac{\alpha_{23}}{\beta + u_3}\right)\nabla u_2 + \frac{-\alpha_{23}u_2}{(\beta + u_3)^2}\nabla u_3\right\},\\
&\operatorname{div}\left\{\frac{-\alpha_{32}u_3}{(\gamma + u_2)^2}\nabla u_2 + \left(d_3 + \frac{\alpha_{32}}{\gamma + u_2} + 2\alpha_{33}u_3\right)\nabla u_3\right\}.
\end{aligned}$$

The terms

$$d_1 + 2\alpha_{11}u_1 + \alpha_{12}u_2, \quad d_2 + \alpha_{21}u_1 + 2\alpha_{22}u_2 + \frac{\alpha_{23}}{\beta + u_3}, \quad d_3 + \frac{\alpha_{32}}{\gamma + u_2} + 2\alpha_{33}u_3$$

represent the "self-diffusion" and the terms

$$\alpha_{12}u_1, \quad \alpha_{21}u_2, \quad \frac{-\alpha_{23}u_2}{(\beta + u_3)^2}, \quad \frac{-\alpha_{32}u_3}{(\gamma + u_2)^2}$$

represent the "cross-diffusion." Here $\alpha_{12}u_1 > 0$ and $\alpha_{21}u_2 > 0$ imply that the flux of u_1 and u_2 in x-direction are directed toward decreasing population of u_2 and u_1 respectively, i.e. the two competitors avoid each other. While $\frac{-\alpha_{23}u_2}{(\beta+u_3)^2} < 0$ and $\frac{-\alpha_{32}u_3}{(\gamma+u_2)^2} < 0$ imply that the flux of u_2 and u_3 in x-direction are directed toward increasing population of u_3 and u_2 respectively, i.e. the two mutualists chase each other. The above model means that, in addition to the dispersive force, the diffusion also depends on population pressure from other species. Here a solution (u_1, u_2, u_3) to system (1.1) is said to be positive if $u_i(x) > 0$, $i = 1, 2, 3$ for all $x \in \Omega$, the existence of a positive solution (u_1, u_2, u_3) to system (1.1) is also called a coexistence. We are mainly concerned with the coexistence states of system (1.1).

In the case when $\alpha_{ij} = 0$ for $i, j = 1, 2, 3$, the above system is the classic competitor–competitor–mutualist model, while if $\alpha_{ij} \neq 0$ for some i or j the system becomes a strongly coupled elliptic system. The strongly coupled systems of elliptic equations have been extensively studied by many mathematicians [1–8]. In an attempt to investigate the spatial segregation under self- and cross-population pressure, Shigesada et al. [1] proposed the strongly coupled elliptic system describing two species Lotka–Volterra competition model. For the Dirichlet boundary value problem of the system, positive solutions are found when birth rates lie in certain range, or when cross-diffusion are sufficiently large by [2]. For the homogeneous Neumann boundary value problem of the system, the effects of diffusion, self-diffusion and cross-diffusion were investigated by Lou and Ni [3]. Applying the bifurcation theory and Lyapunov–Schmidt procedure, the multiple coexistence states for a prey–predator system with cross-diffusion was proved by Kuto and Yamada [4].

Recently, the method of construction of solutions for a general class of strongly coupled elliptic systems was developed by Pao [8] and was based on upper and lower solutions and its associated monotone iterations.

For the related parabolic systems, reader can see [9–12] and references therein.

We will also consider the competitior–competitior–mutualist model with time delays and diffusions under Neumann boundary conditions

$$\begin{cases}
\frac{\partial u_1}{\partial t} - d_1(x,t)\Delta u_1 = u_1[a_1(x,t) - b_{11}(x,t)u_1(x,t) - b_{12}(x,t)u_2(x,t-\tau_2)],\\
\quad \text{in } \Omega \times (0,\infty),\\
\frac{\partial u_2}{\partial t} - d_2(x,t)\Delta u_2 = u_2[a_2(x,t) - b_{21}(x,t)u_1(x,t-\tau_1) - b_{22}(x,t)u_2(x,t) + b_{23}(x,t)u_3(x,t-\tau_3)],\\
\quad \text{in } \Omega \times (0,\infty),\\
\frac{\partial u_3}{\partial t} - d_3(x,t)\Delta u_3 = u_3[a_3(x,t) + b_{32}(x,t)u_2(x,t-\tau_2) - b_{33}(x,t)u_3],\\
\quad \text{in } \Omega \times (0,\infty),\\
\frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = \frac{\partial u_3}{\partial \eta} = 0, \quad \text{on } \partial\Omega \times (0,\infty),
\end{cases} \tag{1.2}$$

with the periodic condition

$$u_i(x,t) = u_i(x,t+T), \quad i = 1,2,3, \ (x,t) \in \Omega \times [-\tau_i, 0], \tag{1.3}$$

and under the initial condition

$$u_i(x,t)=\eta_i(x,t), \quad i=1,2,3,\ (x,t)\in\Omega\times[-\tau_i,0], \tag{1.4}$$

where $d_i \equiv d_i(x,t), a_i \equiv a_i(x,t), b_{ij} \equiv b_{ij}(x,t), (i=1,2,3)$ are smooth positive T-periodic functions on $\Omega\times(0,\infty)$. Ω is a bounded domain in $\mathcal{R}^N$ with boundary $\partial\Omega$, η denotes the outward normal derivative on $\partial\Omega$. It is assumed that the boundary $\partial\Omega$ is of a class $C^{1+\alpha}$ and $\eta_i \in C^{\alpha/2,\alpha}(D_0^{(i)})$ and satisfies the compatibility condition, where $D_0^{(i)} = \Omega\times[-\tau_i,0], i=1,2,3$. We are interested in the existence of the T-periodic solution as well as the asymptotic behavior of (1.2), (1.4) in relation to the maximal and minimal T-periodic solution of systems (1.2), (1.3).

Periodic solutions of parabolic boundary value problems have been investigated by many researchers, and various methods have been proposed for the existence and qualitative properties of the solution. The logistic delay differential equation as a model of single-species population growth has been considered in [13,14]. Nonlinear periodic diffusion equations arise naturally in population models [15] where the birth and death rates, rates of diffusion, rates of interactions and environmental carrying capacities are periodic on seasonal scale. The existence and global stability of a T-periodic solution of periodic boundary-value problem of the logistic model has been studied by Hess [16]. A coupled system of parabolic equations with time delays has been investigated by the method of upper and lower solutions in [17,18]. The monotone iterative scheme associated with this method leads to various computation algorithms for numerical solutions of the periodic boundary problem [19]. The stability and attractivity analysis which are for quasimonotone nondecreasing and mixed quasimonotone reaction functions by the monotone iteration scheme were given in [20,21].

The paper is organized as follows: based on the idea introduced by Pao [8], we try to obtain sufficient conditions which guarantee the coexistence state of system (1.1) in Section 2, and the true solutions of (1.1) are constructed in the same section. In Section 3, we get the sufficient conditions for the existence of T-periodic solutions of systems (1.2), (1.3), the stability and attractivity of the maximal and minimal T-periodic solutions, a global attractor of the system relative to a sector are established provided that $-\lambda_1(a_1)\geqslant b_{12}M_2$, $-\lambda_2(a_2)>b_{21}M_1$ and $-\lambda_3(a_3)>0$.

2. Coexistence

We will give a sufficient condition for that system (1.1) has a positive solution by constructing a coupled upper and lower solutions as in [8]. We first give an equivalent form of the problem (1.1):

$$\begin{cases} -\Delta[D_1(u_1,u_2)] = f_1(u_1,u_2), & x\in\Omega,\\ -\Delta[D_2(u_1,u_2,u_3)] = f_2(u_1,u_2,u_3), & x\in\Omega,\\ -\Delta[D_3(u_2,u_3)] = f_3(u_2,u_3), & x\in\Omega,\\ u_1(x)=u_2(x)=u_3(x)=0, & x\in\partial\Omega, \end{cases} \tag{2.1}$$

where

$$\begin{aligned}
&D_1(u_1,u_2) = (d_1+\alpha_{11}u_1+\alpha_{12}u_2)u_1,\\
&f_1(u_1,u_2) = u_1(a_1-b_{11}u_1-b_{12}u_2),\\
&D_2(u_1,u_2,u_3) = \left(d_2+\alpha_{21}u_1+\alpha_{22}u_2+\frac{\alpha_{23}}{\beta+u_3}\right)u_2,\\
&f_2(u_1,u_2,u_3) = u_2(a_2-b_{21}u_1-b_{22}u_2+b_{23}u_3),\\
&D_3(u_2,u_3) = \left(d_3+\frac{\alpha_{32}}{\gamma+u_2}+\alpha_{33}u_3\right)u_3,\\
&f_3(u_2,u_3) = u_3(a_3+b_{32}u_2-b_{33}u_3).
\end{aligned}$$

Define

$$w_1 = D_1(u_1,u_2), \qquad w_2 = D_2(u_1,u_2,u_3), \qquad w_3 = D_3(u_2,u_3).$$

A direct calculation shows that the Jacobian J of the transformation w_1, w_2, w_3 is given by

$$J = \frac{\partial(w_1,w_2,w_3)}{\partial(u_1,u_2,u_3)} \geqslant d_1d_2d_3 > 0 \quad \text{for } (u_1,u_2,u_3)\geqslant(0,0,0).$$

Then the inverse $u_1 = g_1(w_1, w_2, w_3)$, $u_2 = g_2(w_1, w_2, w_3)$, $u_3 = g_3(w_1, w_2, w_3)$ exists whenever $(u_1, u_2, u_3) \geqslant (0, 0, 0)$. Hence the corresponding equivalent of (2.1) becomes

$$\begin{cases} -\Delta w_1 + k_1 w_1 = F_1(u_1, u_2), & x \in \Omega, \\ -\Delta w_2 + k_2 w_2 = F_2(u_1, u_2, u_3), & x \in \Omega, \\ -\Delta w_3 + k_3 w_3 = F_3(u_2, u_3), & x \in \Omega, \\ u_i = g_i(w_1, w_2, w_3), & i = 1, 2, 3,\ x \in \Omega, \\ w_i(x) = 0, & i = 1, 2, 3,\ x \in \partial\Omega, \end{cases} \tag{2.2}$$

where $F_i(u_1, u_2, u_3) = k_i D_i(u_1, u_2, u_3) + f_i(u_1, u_2, u_3)$, $i = 1, 2, 3$.

Now we consider the monotonicity of F_i with respect to u_j and also the monotonicity of g_i with respect to w_j for $i, j = 1, 2, 3$. First it is easy to see that

$$\frac{\partial u_1}{\partial w_1} > 0, \qquad \frac{\partial u_1}{\partial w_2} \leqslant 0, \qquad \frac{\partial u_1}{\partial w_3} \leqslant 0, \qquad \frac{\partial u_2}{\partial w_1} \leqslant 0, \qquad \frac{\partial u_2}{\partial w_2} > 0,$$
$$\frac{\partial u_2}{\partial w_3} \geqslant 0, \qquad \frac{\partial u_3}{\partial w_1} \leqslant 0, \qquad \frac{\partial u_3}{\partial w_2} \geqslant 0, \qquad \frac{\partial u_3}{\partial w_3} > 0$$

from direct calculations. This shows that $u_1 = g_1(w_1, w_2, w_3)$ is nondecreasing in w_1 and nonincreasing in w_2, w_3 and $u_2 = g_2(w_1, w_2, w_3)$ is nondecreasing in w_2, w_3 and nonincreasing in w_1, while $u_3 = g_3(w_1, w_2, w_3)$ is nondecreasing in w_2, w_3 and nonincreasing in w_1 for all $(w_1, w_2, w_3) \geqslant (0, 0, 0)$.

Secondly if we choose $k_i = \frac{b_{ii}}{\alpha_{ii}}$, $i = 1, 2, 3$. Then

$$\frac{\partial F_1}{\partial u_1} = \frac{b_{11}}{\alpha_{11}} d_1 + a_1 + \left(\frac{b_{11}}{\alpha_{11}}\alpha_{12} - b_{12}\right) u_2,$$
$$\frac{\partial F_1}{\partial u_2} = \left(\frac{b_{11}}{\alpha_{11}}\alpha_{12} - b_{12}\right) u_2, \qquad \frac{\partial F_1}{\partial u_3} = 0;$$
$$\frac{\partial F_2}{\partial u_2} = \frac{b_{22}}{\alpha_{22}} d_2 + a_2 + \left(\frac{b_{22}}{\alpha_{22}}\alpha_{21} - b_{21}\right) u_1 + \frac{b_{22}\alpha_{23}}{(\beta + u_3)\alpha_{22}} + b_{23} u_3,$$
$$\frac{\partial F_2}{\partial u_1} = \left(\frac{b_{22}}{\alpha_{22}}\alpha_{21} - b_{21}\right) u_2, \qquad \frac{\partial F_2}{\partial u_3} = \left[b_{23} - \frac{b_{22}\alpha_{23}}{\alpha_{22}(\beta + u_3)^2}\right] u_2;$$
$$\frac{\partial F_3}{\partial u_3} = \frac{b_{33}}{\alpha_{33}} d_3 + a_3 + \frac{b_{33}\alpha_{32}}{\alpha_{33}(\gamma + u_2)} + b_{32} u_2,$$
$$\frac{\partial F_3}{\partial u_2} = \left[b_{32} - \frac{b_{33}\alpha_{32}}{\alpha_{33}(\gamma + u_2)^2}\right] u_3, \qquad \frac{\partial F_3}{\partial u_1} = 0.$$

Assume that

$$\frac{b_{11}}{\alpha_{11}} < \frac{b_{12}}{\alpha_{12}}, \qquad \frac{b_{22}}{\alpha_{22}} < \frac{b_{21}}{\alpha_{21}}, \qquad \frac{b_{22}}{\alpha_{22}} < \frac{b_{23}\beta^2}{\alpha_{23}}, \qquad \frac{b_{33}}{\alpha_{33}} < \frac{b_{32}\gamma^2}{\alpha_{32}}, \tag{2.3}$$

Obviously, we can get $\frac{\partial F_1}{\partial u_2} \leqslant 0$, $\frac{\partial F_2}{\partial u_1} \leqslant 0$, $\frac{\partial F_2}{\partial u_3} \geqslant 0$, $\frac{\partial F_3}{\partial u_2} \geqslant 0$, $\frac{\partial F_3}{\partial u_3} \geqslant 0$ for every $(u_1, u_2, u_3) \geqslant (0, 0, 0)$. Furthermore, choose

$$\overline{M}_1 = \frac{b_{22} d_2 + a_2 \alpha_{22}}{b_{21}\alpha_{22} - b_{22}\alpha_{21}}, \qquad \overline{M}_2 = \frac{b_{11} d_1 + a_1 \alpha_{11}}{b_{12}\alpha_{11} - b_{11}\alpha_{12}},$$

we can obtain that $\frac{\partial F_1}{\partial u_1} \geqslant 0$, $\frac{\partial F_2}{\partial u_2} \geqslant 0$ when $(u_1, u_2, u_3) \in [0, \overline{M}_1] \times [0, \overline{M}_2] \times [0, \infty)$. Therefore, when $(u_1, u_2, u_3) \in [0, \overline{M}_1] \times [0, \overline{M}_2] \times [0, \infty)$, the function F_1 is nonincreasing in u_2 and nondecreasing in u_1; F_2 is nonincreasing in u_1 and nondecreasing in u_2, u_3; F_3 is nondecreasing in u_2, u_3.

Next we give the definition of coupled upper and lower solutions of (2.2) as the following:

Definition 2.1. A pair of 6-vector functions $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$, $(\hat{\mathbf{u}}, \hat{\mathbf{w}}) = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{w}_1, \hat{w}_2, \hat{w}_3)$ in $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are called coupled upper and lower solutions of (2.2), if $\tilde{u}_1 \leqslant \overline{M}_1$, $\tilde{u}_2 \leqslant \overline{M}_2$, $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \geqslant (\hat{\mathbf{u}}, \hat{\mathbf{w}})$ and if their components satisfy the relation

$$\begin{cases} -\Delta\tilde{w}_1+k_1\tilde{w}_1\geqslant F_1(\tilde{u}_1,\hat{u}_2), & -\Delta\hat{w}_1+k_1\hat{w}_1\leqslant F_1(\hat{u}_1,\tilde{u}_2), & x\in\Omega,\\ -\Delta\tilde{w}_2+k_2\tilde{w}_2\geqslant F_2(\hat{u}_1,\tilde{u}_2,\tilde{u}_3), & -\Delta\hat{w}_2+k_2\hat{w}_2\leqslant F_2(\tilde{u}_1,\hat{u}_2,\hat{u}_3), & x\in\Omega,\\ -\Delta\tilde{w}_3+k_3\tilde{w}_3\geqslant F_3(\tilde{u}_2,\tilde{u}_3), & -\Delta\hat{w}_3+k_3\hat{w}_3\leqslant F_3(\hat{u}_2,\hat{u}_3), & x\in\Omega,\\ \tilde{u}_1\geqslant g_1(\tilde{w}_1,\hat{w}_2,\hat{w}_3), & \hat{u}_1\leqslant g_1(\hat{w}_1,\tilde{w}_2,\tilde{w}_3), & x\in\Omega,\\ \tilde{u}_2\geqslant g_2(\hat{w}_1,\tilde{w}_2,\tilde{w}_3), & \hat{u}_2\leqslant g_2(\tilde{w}_1,\hat{w}_2,\hat{w}_3), & x\in\Omega,\\ \tilde{u}_3\geqslant g_3(\hat{w}_1,\tilde{w}_2,\tilde{w}_3), & \hat{u}_3\leqslant g_3(\tilde{w}_1,\hat{w}_2,\hat{w}_3), & x\in\Omega,\\ \tilde{w}_i(x)\geqslant 0\geqslant\hat{w}_i(x), & & i=1,2,3,\ x\in\partial\Omega. \end{cases} \tag{2.4}$$

We set

$$S=\{\mathbf{u}\in\mathcal{C}^\alpha(\overline{\Omega});\ \hat{\mathbf{u}}\leqslant\mathbf{u}\leqslant\tilde{\mathbf{u}}\};\qquad S^*=\{\mathbf{w}\in\mathcal{C}^\alpha(\overline{\Omega});\ \hat{\mathbf{w}}\leqslant\mathbf{w}\leqslant\tilde{\mathbf{w}}\}$$

where $\mathbf{u}=(u_1,u_2,u_3)$, $\mathbf{w}=(w_1,w_2,w_3)$, $\tilde{\mathbf{u}}=(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$, $\hat{\mathbf{u}}=(\hat{u}_1,\hat{u}_2,\hat{u}_3)$ and $\tilde{\mathbf{w}}=(\tilde{w}_1,\tilde{w}_2,\tilde{w}_3)$, $\hat{\mathbf{w}}=(\hat{w}_1,\hat{w}_2,\hat{w}_3)$.

For definiteness, we choose

$$\begin{aligned} &\tilde{u}_1=g_1(\tilde{w}_1,\hat{w}_2,\hat{w}_3), && \tilde{u}_2=g_2(\hat{w}_1,\tilde{w}_2,\tilde{w}_3), && \tilde{u}_3=g_3(\hat{w}_1,\tilde{w}_2,\tilde{w}_3),\\ &\hat{u}_1=g_1(\hat{w}_1,\tilde{w}_2,\tilde{w}_3), && \hat{u}_2=g_2(\tilde{w}_1,\hat{w}_2,\hat{w}_3), && \hat{u_3}=g_3(\tilde{w}_1,\hat{w_2},\hat{w_3}), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\tilde{w}_1=D_1(\tilde{u}_1,\hat{u}_2), && \tilde{w}_2=D_2(\hat{u}_1,\tilde{u}_2,\tilde{u}_3), && \tilde{w}_3=D_3(\tilde{u}_2,\tilde{u}_3),\\ &\hat{w}_1=D_1(\hat{u}_1,\tilde{u}_2), && \hat{w}_2=D_2(\tilde{u}_1,\hat{u}_2,\hat{u}_3), && \hat{w}_3=D_3(\hat{u}_2,\hat{u}_3). \end{aligned}$$

Then the requirements of $(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$, $(\hat{u}_1,\hat{u}_2,\hat{u}_3)$ in (2.4) are satisfied and those of $(\tilde{w}_1,\tilde{w}_2,\tilde{w}_3)$, $(\hat{w}_1,\hat{w}_2,\hat{w}_3)$ are reduced to

$$\begin{cases} -\Delta[D_1(\tilde{u}_1,\hat{u}_2)]+k_1D_1(\tilde{u}_1,\hat{u}_2)\geqslant F_1(\tilde{u}_1,\hat{u}_2), & x\in\Omega,\\ -\Delta[D_2(\hat{u}_1,\tilde{u}_2,\tilde{u}_3)]+k_2D_2(\hat{u}_1,\tilde{u}_2,\tilde{u}_3)\geqslant F_2(\hat{u}_1,\tilde{u}_2,\tilde{u}_3), & x\in\Omega,\\ -\Delta[D_3(\tilde{u}_2,\tilde{u}_3)]+k_3D_3(\tilde{u}_2,\tilde{u}_3)\geqslant F_3(\tilde{u}_2,\tilde{u}_3), & x\in\Omega,\\ -\Delta[D_1(\hat{u}_1,\tilde{u}_2)]+k_1D_1(\hat{u}_1,\tilde{u}_2)\leqslant F_1(\hat{u}_1,\tilde{u}_2), & x\in\Omega,\\ -\Delta[D_2(\tilde{u}_1,\hat{u}_2,\hat{u}_3)]+k_2D_2(\tilde{u}_1,\hat{u}_2,\hat{u}_3)\leqslant F_2(\tilde{u}_1,\hat{u}_2,\hat{u}_3), & x\in\Omega,\\ -\Delta[D_3(\hat{u}_2,\hat{u}_3)]+k_3D_3(\hat{u}_2,\hat{u}_3)\leqslant F_3(\hat{u}_2,\hat{u}_3), & x\in\Omega,\\ \tilde{u}_i(x)\geqslant 0\geqslant\hat{u}_i(x), & i=1,2,3,\ x\in\partial\Omega. \end{cases} \tag{2.5}$$

We call the pair $(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$, $(\hat{u}_1,\hat{u}_2,\hat{u}_3)$ satisfying (2.5) and $\tilde{u}_1\leqslant\overline{M}_1$, $\tilde{u}_2\leqslant\overline{M}_2$, $(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)\geqslant(\hat{u}_1,\hat{u}_2,\hat{u}_3)$ are coupled upper and lower solutions of (1.1).

Now we seek a pair of coupled upper and lower solutions of (1.1) in the form

$$(\tilde{u}_1,\ \tilde{u}_2,\ \tilde{u}_3)=(M_1,\ M_2,\ M_3),\qquad(\hat{u}_1,\ \hat{u}_2,\ \hat{u}_3)=(\delta_1\phi,\ \delta_2\phi,\ \delta_3\phi)$$

where M_i and δ_i $(i=1,2,3)$ are some positive constants with δ_i sufficiently small, and $\phi\equiv\phi(x)$ is the (normalized) positive eigenfunction corresponding to λ_0, where λ_0 is the smallest eigenvalue of the Laplacian $(-\Delta)$ under Dirichlet boundary condition. Indeed (M_1,M_2,M_3), $(\delta_1\phi,\delta_2\phi,\delta_3\phi)$ satisfy the inequalities in (2.5) if

$$\begin{cases} -\Delta[(d_1+\alpha_{11}M_1+\alpha_{12}\delta_2\phi)M_1]\geqslant M_1(a_1-b_{11}M_1-b_{12}\delta_2\phi),\\ -\Delta[(d_2+\alpha_{21}\delta_1\phi+\alpha_{22}M_2+\frac{\alpha_{23}}{\beta+M_3})M_2]\geqslant M_2(a_2-b_{21}\delta_1\phi-b_{22}M_2+b_{23}M_3),\\ -\Delta[(d_3+\frac{\alpha_{32}}{\gamma+M_2}+\alpha_{33}M_3)M_3]\geqslant M_3(a_3+b_{32}M_2-b_{33}M_3),\\ -\Delta[(d_1+\alpha_{11}\delta_1\phi+\alpha_{12}M_2)\delta_1\phi]\leqslant\delta_1\phi(a_1-b_{11}\delta_1\phi-b_{12}M_2),\\ -\Delta[(d_2+\alpha_{21}M_1+\alpha_{22}\delta_2\phi+\frac{\alpha_{23}}{\beta+\delta_3\phi})\delta_2\phi]\leqslant\delta_2\phi(a_2-b_{21}M_1-b_{22}\delta_2\phi+b_{23}\delta_3\phi),\\ -\Delta[(d_3+\frac{\alpha_{32}}{\gamma+\delta_2\phi}+\alpha_{33}\delta_3\phi)\delta_3\phi]\leqslant\delta_3\phi(a_3+b_{32}\delta_2\phi-b_{33}\delta_3\phi). \end{cases} \tag{2.6}$$

Since that δ_i, $i=1,2,3$ is sufficiently small and $-\Delta\phi=\lambda_0\phi$, the inequalities in (2.6) are equivalent to

$$\begin{cases} a_1 - b_{11}M_1 \leqslant 0, \\ a_2 - b_{22}M_2 + b_{23}M_3 \leqslant 0, \\ a_3 + b_{32}M_2 - b_{33}M_3 \leqslant 0, \\ (d_1 + \alpha_{12}M_2)\lambda_0 < a_1 - b_{12}M_2, \\ (d_2 + \frac{\alpha_{23}}{\beta} + \alpha_{21}M_1)\lambda_0 < a_2 - b_{21}M_1, \\ (d_3 + \frac{\alpha_{32}}{\gamma})\lambda_0 < a_3. \end{cases} \tag{2.7}$$

Assume that

$$b_{23}b_{32} < b_{22}b_{33} \tag{2.8}$$

and

$$\begin{cases} \frac{a_1}{b_{11}} \leqslant M_1 < \frac{a_2 - (d_2 + \frac{\alpha_{23}}{\beta})\lambda_0}{\lambda_0\alpha_{21} + b_{21}}, \\ \frac{a_2b_{33} + a_3b_{23}}{b_{22}b_{33} - b_{23}b_{32}} \leqslant M_2 < \frac{a_1 - \lambda_0 d_1}{\lambda_0\alpha_{12} + b_{12}}, \\ M_3 \geqslant \frac{b_{22}a_3 + b_{32}a_2}{b_{22}b_{33} - b_{23}b_{32}}, \qquad (d_3 + \frac{\alpha_{32}}{\gamma})\lambda_0 < a_3. \end{cases} \tag{2.9}$$

Then the requirements in (2.7) are fulfilled and also the inequalities $M_1 \leqslant \overline{M}_1$, $M_2 \leqslant \overline{M}_2$ hold. In all, assume that

$$\begin{gathered} \frac{b_{11}}{\alpha_{11}} < \frac{b_{12}}{\alpha_{12}}, \qquad \frac{b_{22}}{\alpha_{22}} < \frac{b_{21}}{\alpha_{21}}, \qquad \frac{b_{22}}{\alpha_{22}} < \frac{b_{23}\beta^2}{\alpha_{23}}, \qquad \frac{b_{33}}{\alpha_{33}} < \frac{b_{32}\gamma^2}{\alpha_{32}}, \qquad b_{23}b_{32} < b_{22}b_{33}, \\ \frac{a_1}{b_{11}} < \frac{a_2 - (d_2 + \frac{\alpha_{23}}{\beta})\lambda_0}{\lambda_0\alpha_{21} + b_{21}}, \qquad \frac{a_2b_{33} + a_3b_{23}}{b_{22}b_{33} - b_{23}b_{32}} < \frac{a_1 - \lambda_0 d_1}{\lambda_0\alpha_{12} + b_{12}}, \qquad \left(d_3 + \frac{\alpha_{32}}{\gamma}\right)\lambda_0 < a_3, \end{gathered} \tag{2.10}$$

there exist positive constants M_i, δ_i $(i = 1, 2, 3)$ and ϕ such that the pair $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (M_1, M_2, M_3)$, $(\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\delta_1\phi, \delta_2\phi, \delta_3\phi)$ are coupled upper and lower solutions of problem (1.1).

Using Theorem 2.1 of [8] yields the following existence result:

Theorem 2.1. *The problem* (1.1) *admits at least one positive solution* $\mathbf{u} = (u_1, u_2, u_3)$ *under the condition* (2.10)*.*

Remark 2.1. It is easy to see that if $\lambda_0 d_1 \geqslant a_1$ or $\lambda_0 d_2 \geqslant a_2$ or $\lambda_0 d_3 \geqslant a_3$, then problem (1.1) has no positive solution, see [2]. Our result shows that if $\lambda_0 d_1 < a_1$, $\lambda_0 d_2 < a_2$ and $\lambda_0 d_3 < a_3$, then problem (1.1) has at least one coexistence state provided that cross-diffusions $\alpha_{12}, \alpha_{21}, \alpha_{23}, \alpha_{32}$ and cross-reactions $b_{12}, b_{21}, b_{23}, b_{32}$ are sufficiently small.

In what follows, we will construct the true solutions of (1.1) based on monotone iterative schemes. Under the condition (2.10), we know that (M_1, M_2, M_3), $(\delta_1\phi, \delta_2\phi, \delta_3\phi)$ are coupled upper and lower solutions of problem (1.1). Now we use $(\bar{u}_1^{(0)}, \bar{u}_2^{(0)}, \bar{u}_3^{(0)}) = (M_1, M_2, M_3)$, $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \underline{u}_3^{(0)}) = (\delta_1\phi, \delta_2\phi, \delta_3\phi)$ as an initial iteration in the iteration process

$$\begin{cases} -\Delta\overline{w}_1^{(m)} + k_1\overline{w}_1^{(m)} = F_1(\bar{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}), & x \in \Omega, \\ -\Delta\overline{w}_2^{(m)} + k_2\overline{w}_2^{(m)} = F_2(\underline{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}, \bar{u}_3^{(m-1)}), & x \in \Omega, \\ -\Delta\overline{w}_3^{(m)} + k_3\overline{w}_3^{(m)} = F_3(\bar{u}_2^{(m-1)}, \bar{u}_3^{(m-1)}), & x \in \Omega, \\ -\Delta\underline{w}_1^{(m)} + k_1\underline{w}_1^{(m)} = F_1(\underline{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}), & x \in \Omega, \\ -\Delta\underline{w}_2^{(m)} + k_2\underline{w}_2^{(m)} == F_2(\bar{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}, \underline{u}_3^{(m-1)}), & x \in \Omega, \\ -\Delta\underline{w}_3^{(m)} + k_3\underline{w}_3^{(m)} = F_3(\underline{u}_2^{(m-1)}, \underline{u}_3^{(m-1)}), & x \in \Omega, \\ \bar{u}_1^{(m)} = g_1(\overline{w}_1^{(m)}, \underline{w}_2^{(m)}, \underline{w}_3^{(m)}), \qquad \underline{u}_1^{(m)} = g_1(\underline{w}_1^{(m)}, \overline{w}_2^{(m)}, \overline{w}_3^{(m)}), & x \in \Omega, \\ \bar{u}_2^{(m)} = g_2(\underline{w}_1^{(m)}, \overline{w}_2^{(m)}, \overline{w}_3^{(m)}), \qquad \underline{u}_2^{(m)} = g_1(\overline{w}_1^{(m)}, \underline{w}_2^{(m)}, \underline{w}_3^{(m)}), & x \in \Omega, \\ \bar{u}_3^{(m)} = g_3(\underline{w}_1^{(m)}, \overline{w}_2^{(m)}, \overline{w}_3^{(m)}), \qquad \underline{u}_3^{(m)} = g_3(\overline{w}_1^{(m)}, \underline{w}_2^{(m)}, \underline{w}_3^{(m)}), & x \in \Omega, \\ \overline{w}_i^{(m)}(x) = \underline{w}_i^{(m)}(x) = 0, & i = 1, 2, 3, \ x \in \partial\Omega, \end{cases} \tag{2.11}$$

where $m = 1, 2, \ldots$. Using the Lemma 3.1 of [8], we know that the sequences $\{(\bar{\mathbf{u}}^{(m)}, \bar{\mathbf{w}}^{(m)})\}$, $\{(\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}))\}$ governed by (2.11) are well defined and possess the monotone property

$$\begin{aligned}(\hat{\mathbf{u}}, \hat{\mathbf{w}}) &\leqslant \left(\underline{\mathbf{u}}^{(m-1)}, \underline{\mathbf{w}}^{(m-1)}\right) \leqslant \left(\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\right) \leqslant \left(\bar{\mathbf{u}}^{(m)}, \bar{\mathbf{w}}^{(m)}\right) \\ &\leqslant \left(\bar{\mathbf{u}}^{(m-1)}, \bar{\mathbf{w}}^{(m-1)}\right) \leqslant (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}),\end{aligned}$$

for every $m = 1, 2, \ldots$.

Therefore the pointwise limits

$$\lim_{m\to\infty}\left(\bar{\mathbf{u}}^{(m)}, \bar{\mathbf{w}}^{(m)}\right) = (\bar{\mathbf{u}}, \bar{\mathbf{w}}), \qquad \lim_{m\to\infty}\left(\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\right) = (\underline{\mathbf{u}}, \underline{\mathbf{w}})$$

exist and satisfy the relation

$$(\hat{\mathbf{u}}, \hat{\mathbf{w}}) \leqslant \left(\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\right) \leqslant (\underline{\mathbf{u}}, \underline{\mathbf{w}}) \leqslant (\bar{\mathbf{u}}, \bar{\mathbf{w}}) \leqslant \left(\bar{\mathbf{u}}^{(m)}, \bar{\mathbf{w}}^{(m)}\right) \leqslant (\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$$

for every $m = 1, 2, \ldots$.

From the last four equations in the iteration process (2.11), we obtain

$$\begin{aligned}
&\bar{u}_1^{(m)} = g_1\left(\bar{w}_1^{(m)}, \underline{w}_2^{(m)}, \underline{w}_3^{(m)}\right), &&\underline{u}_1^{(m)} = g_1\left(\underline{w}_1^{(m)}, \bar{w}_2^{(m)}, \bar{w}_3^{(m)}\right),\\
&\bar{u}_2^{(m)} = g_2\left(\underline{w}_1^{(m)}, \bar{w}_2^{(m)}, \bar{w}_3^{(m)}\right), &&\underline{u}_2^{(m)} = g_2\left(\bar{w}_1^{(m)}, \underline{w}_2^{(m)}, \underline{w}_3^{(m)}\right),\\
&\bar{u}_3^{(m)} = g_3\left(\underline{w}_1^{(m)}, \bar{w}_2^{(m)}, \bar{w}_3^{(m)}\right), &&\underline{u}_3^{(m)} = g_3\left(\bar{w}_1^{(m)}, \underline{w}_2^{(m)}, \underline{w}_3^{(m)}\right),
\end{aligned}$$

which is equivalent to

$$\begin{cases}
\bar{w}_1^{(m)} = D_1(\bar{u}_1^{(m)}, \underline{u}_2^{(m)}), & \underline{w}_1^{(m)} = D_1(\underline{u}_1^{(m)}, \bar{u}_2^{(m)}),\\
\bar{w}_2^{(m)} = D_2(\underline{u}_1^{(m)}, \bar{u}_2^{(m)}, \bar{u}_3^{(m)}), & \underline{w}_2^{(m)} = D_2(\bar{u}_1^{(m)}, \underline{u}_2^{(m)}, \underline{u}_3^{(m)}),\\
\bar{w}_3^{(m)} = D_3(\bar{u}_2^{(m)}, \bar{u}_3^{(m)}), & \underline{w}_3^{(m)} = D_3(\underline{u}_2^{(m)}, \underline{u}_3^{(m)}).
\end{cases} \tag{2.12}$$

By the relation in (2.12), let $m \to \infty$ and using the standard regularity argument for elliptic boundary problems show that $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ and $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ satisfy the equations

$$\begin{cases}
-\Delta[D_1(\bar{u}_1, \underline{u}_2)] + k_1 D_1(\bar{u}_1, \underline{u}_2) = F_1(\bar{u}_1, \underline{u}_2), & x \in \Omega,\\
-\Delta[D_2(\underline{u}_1, \bar{u}_2, \bar{u}_3)] + k_2 D_2(\underline{u}_1, \bar{u}_2, \bar{u}_3) = F_2(\underline{u}_1, \bar{u}_2, \bar{u}_3), & x \in \Omega,\\
-\Delta[D_3(\bar{u}_2, \bar{u}_3)] + k_3 D_3(\bar{u}_2, \bar{u}_3) = F_3(\bar{u}_2, \bar{u}_3), & x \in \Omega,\\
-\Delta[D_1(\underline{u}_1, \bar{u}_2)] + k_1 D_1(\underline{u}_1, \bar{u}_2) = F_1(\underline{u}_1, \bar{u}_2), & x \in \Omega,\\
-\Delta[D_2(\bar{u}_1, \underline{u}_2, \underline{u}_3)] + k_2 D_2(\bar{u}_1, \underline{u}_2, \underline{u}_3) = F_2(\bar{u}_1, \underline{u}_2, \underline{u}_3), & x \in \Omega,\\
-\Delta[D_3(\underline{u}_2, \underline{u}_3)] + k_3 D_3(\underline{u}_2, \underline{u}_3) = F_3(\underline{u}_2, \underline{u}_3), & x \in \Omega,\\
\bar{u}_i(x) = \underline{u}_i(x) = 0, & i = 1, 2, 3,\ x \in \partial\Omega.
\end{cases} \tag{2.13}$$

By virtue of the monotonicity of the functions $F_i(u_1, u_2, u_3)$ and $D_i(u_1, u_2, u_3)$ $i = 1, 2, 3$, the functions F_i, D_i possess the following property

$$\begin{cases}
F_1(\bar{u}_1, \underline{u}_2) = f_1(\bar{u}_1, \underline{u}_2) + k_1 D_1(\bar{u}_1, \underline{u}_2), & x \in \Omega,\\
F_2(\underline{u}_1, \bar{u}_2, \bar{u}_3) = f_2(\underline{u}_1, \bar{u}_2, \bar{u}_3) + k_2 D_2(\underline{u}_1, \bar{u}_2, \bar{u}_3), & x \in \Omega,\\
F_3(\bar{u}_2, \bar{u}_3) = f_3(\bar{u}_2, \bar{u}_3) + k_3 D_3(\bar{u}_2, \bar{u}_3), & x \in \Omega,\\
F_1(\underline{u}_1, \bar{u}_2) = f_1(\underline{u}_1, \bar{u}_2) + k_1 D_1(\underline{u}_1, \bar{u}_2), & x \in \Omega,\\
F_2(\bar{u}_1, \underline{u}_2, \underline{u}_3) = f_2(\bar{u}_1, \underline{u}_2, \underline{u}_3) + k_2 D_2(\bar{u}_1, \underline{u}_2, \underline{u}_3), & x \in \Omega,\\
F_3(\underline{u}_2, \underline{u}_3) = f_3(\underline{u}_2, \underline{u}_3) + k_3 D_3(\underline{u}_2, \underline{u}_3), & x \in \Omega.
\end{cases} \tag{2.14}$$

Therefore

$$\begin{cases} -\Delta[D_1(\bar{u}_1, \underline{u}_2)] = f_1(\bar{u}_1, \underline{u}_2), & x \in \Omega, \\ -\Delta[D_2(\underline{u}_1, \bar{u}_2, \bar{u}_3)] = f_2(\underline{u}_1, \bar{u}_2, \bar{u}_3), & x \in \Omega, \\ -\Delta[D_3(\bar{u}_2, \bar{u}_3)] = f_3(\bar{u}_2, \bar{u}_3), & x \in \Omega, \\ -\Delta[D_1(\underline{u}_1, \bar{u}_2)] = f_1(\underline{u}_1, \bar{u}_2), & x \in \Omega, \\ -\Delta[D_2(\bar{u}_1, \underline{u}_2, \underline{u}_3)] = f_2(\bar{u}_1, \underline{u}_2, \underline{u}_3), & x \in \Omega, \\ -\Delta[D_3(\underline{u}_2, \underline{u}_3)] = f_3(\underline{u}_2, \underline{u}_3), & x \in \Omega, \\ \bar{u}_i(x) = \underline{u}_i(x) = 0, & i = 1, 2, 3,\ x \in \partial\Omega. \end{cases} \tag{2.15}$$

Then $(\underline{u}_1, \bar{u}_2, \bar{u}_3)$ and $(\bar{u}_1, \underline{u}_2, \underline{u}_3)$ are true solutions of (1.1).

If $\bar{u}_1 = \underline{u}_1$ or $\bar{u}_2 = \underline{u}_2$ or $\bar{u}_3 = \underline{u}_3$, then $(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\underline{u}_1, \underline{u}_2, \underline{u}_3)$ $(\equiv (u_1^*, u_2^*, u_3^*))$ and (u_1^*, u_2^*, u_3^*) is the unique solution of (1.1). To see this, let us consider the case $\bar{u}_1 = \underline{u}_1 \equiv u_1^*$. By a subtraction of the first equation from the fourth equation in (2.15) and

$$D_1(\bar{u}_1, \underline{u}_2) - D_1(\underline{u}_1, \bar{u}_2) = -\alpha_{12} u_1^*(\bar{u}_2 - \underline{u}_2),$$

we obtain

$$\Delta\big[\alpha_{12} u_1^*(\bar{u}_2 - \underline{u}_2)\big] = -u_1^* b_{12}(\underline{u}_2 - \bar{u}_2), \quad \text{in } \Omega.$$

In view of $u_1^* > 0$, $\alpha_{12} > 0$, $b_{12} > 0$, and $\bar{u}_2 - \underline{u}_2 = 0$ on $\partial\Omega$, the above equation yields $\bar{u}_2 = \underline{u}_2$. We can take use of the similar method to obtain $\bar{u}_3 = \underline{u}_3$. This shows that $(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\underline{u}_1, \underline{u}_2, \underline{u}_3)$. Then (u_1^*, u_2^*, u_3^*) is the unique solution.

To summarize the above conclusions we have the following theorem:

Theorem 2.2. *Under the condition* (2.10), *the sequences* $\{\bar{u}_1^{(m)}, \bar{u}_2^{(m)}, \bar{u}_3^{(m)}\}$, $\{\underline{u}_1^{(m)}, \underline{u}_2^{(m)}, \underline{u}_3^{(m)}\}$ *obtained from* (2.11) *with* $(\bar{u}_1^{(0)}, \bar{u}_2^{(0)}, \bar{u}_3^{(0)}) = (M_1, M_2, M_3)$, $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \underline{u}_3^{(0)}) = (\delta_1\phi, \delta_2\phi, \delta_3\phi)$ *and* $k_1 = \frac{b_{11}}{\alpha_{11}}$, $k_2 = \frac{b_{22}}{\alpha_{22}}$, $k_3 = \frac{b_{33}}{\alpha_{33}}$, *converge monotonically to some limits* $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ *and* $(\underline{u}_1, \bar{u}_2, \bar{u}_3)$, $(\bar{u}_1, \underline{u}_2, \underline{u}_3)$ *are true solutions of* (1.1); *if either* $\underline{u}_1 = \bar{u}_1$ *or* $\underline{u}_2 = \bar{u}_2$ *or* $\underline{u}_3 = \bar{u}_3$, *then* $(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\underline{u}_1, \underline{u}_2, \underline{u}_3)(\equiv (u_1^*, u_2^*, u_3^*))$ *and* (u_1^*, u_2^*, u_3^*) *is the unique solution of problem* (1.1) *in* $\mathcal{S}$.

3. Existence of periodic solution

In this section, we study the periodic solution of the problem (1.2), (1.3) and we first consider the periodic eigenvalue problem

$$\begin{cases} \partial\phi/\partial t - L\phi - a\phi = \lambda\phi, & (x,t) \in \Omega \times (0,\infty), \\ B\phi = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \phi(x,0) = \phi(x,T), & x \in \Omega, \end{cases} \tag{3.1}$$

where

$$L = \sum_{j,k=1}^{n} a_{jk}(x,t)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial}{\partial x_j},$$

$$B = \alpha(x,t)\frac{\partial}{\partial \nu} + \beta(x,t).$$

It follows from [16] that for any T-periodic function $a \equiv a(x,t)$ the principle eigenvalue of (3.1) denoted by $\lambda(a)$, is real and its corresponding eigenfunction $\phi \equiv \phi(x,t)$ may be chosen positive in $\Omega \times (0,\infty)$.

For the convenience, we let $D = \Omega \times [0,\infty)$, $\overline{D} = \overline{\Omega} \times [0,\infty)$, $\Gamma = \partial\Omega \times [0,\infty)$, and for each $i = 1,2,3$ we set $D_0^{(i)} = \Omega \times [-\tau_i, 0]$, $Q^{(i)} = \overline{\Omega} \times [-\tau_i, \infty)$, $D_0 = D_0^{(1)} \times D_0^{(2)} \times D_0^{(3)}$, $Q = Q^{(1)} \times Q^{(2)} \times Q^{(3)}$.

To show the existence problem we make a transformation by letting $w_1 = M - u_1$ for a sufficiently large constant $M > 0$. Then the problem (1.2) with (1.3), (1.4) become the following problem:

$$\begin{cases} \dfrac{\partial w_1}{\partial t} - d_1(x,t)\Delta w_1 = -(M-w_1)[a_1(x,t) - b_{11}(M-w_1) - b_{12}u_2(x,t-\tau_2)], \\ \quad (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_2}{\partial t} - d_2(x,t)\Delta u_2 = u_2[a_2(x,t) - b_{22}u_2 - b_{21}(M-w_1)(x,t-\tau_1) + b_{23}u_3(x,t-\tau_3)], \\ \quad (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u_3}{\partial t} - d_3(x,t)\Delta u_3 = u_3[a_3(x,t) + b_{32}u_2(x,t-\tau_2) - b_{33}u_3], \quad (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial w_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = \frac{\partial u_3}{\partial \eta} = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{3.2}$$

with the periodic condition

$$w_1(x,t) = w_1(x,t+T), \qquad u_i(x,t) = u_i(x,t+T), \quad i = 2,3, \ (x,t) \in \Omega \times [-\tau_i, 0], \tag{3.3}$$

and under the initial condition

$$w_1(x,t) = M - \eta_1(x,t), \qquad u_i(x,t) = \eta_i(x,t), \quad i = 2,3, \ (x,t) \in \Omega \times [-\tau_i, 0]. \tag{3.4}$$

Denote the reaction functions of (3.2) by F_1, F_2, F_3. It is easily to see that F_i is quasimonotone nondecreasing in $S \times S_\tau$ where $S = S_\tau = [0, M] \times R^+ \times R^+$. Next we give the definition of ordered upper and lower solutions of (3.2):

Definition 3.1. Let $\mathbf{u} \in S$, $\mathbf{v} \in S_\tau$, a pair of 3-vector functions $\tilde{\mathbf{u}} = (\tilde{w}_1, \tilde{u}_2, \tilde{u}_3)$, $\hat{\mathbf{u}} = (\hat{w}_1, \hat{u}_2, \hat{u}_3)$ in $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are called ordered upper and lower solutions of (3.2) (3.3), if $(\tilde{w}_1, \tilde{u}_2, \tilde{u}_3) \geqslant (\hat{w}_1, \hat{u}_2, \hat{u}_3)$ and if their components satisfy the relation

$$\begin{cases} \frac{\partial \tilde{w}_1}{\partial t} - d_1\Delta\tilde{w}_1 \geqslant F_1(x,t,\tilde{\mathbf{u}},\tilde{\mathbf{v}}), & (x,t) \in D, \\ \frac{\partial \tilde{u}_2}{\partial t} - d_2\Delta\tilde{u}_2 \geqslant F_2(x,t,\tilde{\mathbf{u}},\tilde{\mathbf{v}}), & (x,t) \in D, \\ \frac{\partial \tilde{u}_3}{\partial t} - d_3\Delta\tilde{u}_3 \geqslant F_3(x,t,\tilde{\mathbf{u}},\tilde{\mathbf{v}}), & (x,t) \in D, \\ \frac{\partial \hat{w}_1}{\partial t} - d_1\Delta\hat{w}_1 \leqslant F_1(x,t,\hat{\mathbf{u}},\hat{\mathbf{v}}), & (x,t) \in D, \\ \frac{\partial \hat{u}_2}{\partial t} - d_2\Delta\hat{u}_2 \leqslant F_2(x,t,\hat{\mathbf{u}},\hat{\mathbf{v}}), & (x,t) \in D, \\ \frac{\partial \hat{u}_3}{\partial t} - d_2\Delta\hat{u}_3 \leqslant F_3(x,t,\hat{\mathbf{u}},\hat{\mathbf{v}}), & (x,t) \in D, \\ \frac{\partial \tilde{w}_1}{\partial \eta} \geqslant 0 \geqslant \frac{\partial \hat{w}_1}{\partial \eta}, \qquad \frac{\partial \tilde{u}_i}{\partial \eta} \geqslant 0 \geqslant \frac{\partial \hat{u}_i}{\partial \eta}, & i = 2,3, \ (x,t) \in \Gamma, \\ \tilde{w}_1(x,t) \geqslant \tilde{w}_1(x,t+T), \qquad \hat{w}_1(x,t) \leqslant \hat{w}_1(x,t+T), & (x,t) \in D_0^{(1)}, \\ \tilde{u}_i(x,t) \geqslant \tilde{u}_i(x,t+T), \qquad \hat{u}_i(x,t) \leqslant \hat{u}_i(x,t+T), & i = 2,3, \ (x,t) \in D_0^{(i)}, \end{cases} \tag{3.5}$$

where

$$S \equiv \left\{\mathbf{u} \in \mathcal{C}(\overline{\Omega});\ \hat{\mathbf{u}} \leqslant \mathbf{u} \leqslant \tilde{\mathbf{u}}, \text{ on } \overline{D}\right\}, \qquad S_\tau \equiv \left\{\mathbf{v} \in \mathcal{C}(Q);\ \hat{\mathbf{v}} \leqslant \mathbf{v} \leqslant \tilde{\mathbf{v}}, \text{ on } \overline{D}\right\},$$

and $\mathbf{u} = (w_1, u_2, u_3)$, $\mathbf{v} = (w_{1\tau_1}, u_{2\tau_2}, u_{3\tau_3})$, $\tilde{\mathbf{u}} = (\tilde{w}_1, \tilde{u}_2, \tilde{u}_3)$, $\hat{\mathbf{u}} = (\hat{w}_1, \hat{u}_2, \hat{u}_3)$ and $\tilde{\mathbf{v}} = (\tilde{w}_{1\tau_1}, \tilde{u}_{2\tau_2}, \tilde{u}_{3\tau_3})$, $\hat{\mathbf{v}} = (\hat{w}_{1\tau_1}, \hat{u}_{2\tau_2}, \hat{u}_{3\tau_3})$.

Let $\lambda_i(a_i)$ and $\phi_i(x,t)$ be the principle eigenvalue and its corresponding positive eigenfunction of problem (3.1) with $L = -d_i(x,t)\Delta$, $B = \frac{\partial}{\partial \eta}$ and $a = a_i(x,t)$ $(i = 1,2,3)$.

Next we seek a pair of ordered upper and lower solution of problem (3.2), (3.3) in the form $(\tilde{w}_1, \tilde{u}_2, \tilde{u}_3) = (M - \delta_1\phi_1, \rho_2, \rho_3)$, $(\hat{w}_1, \hat{u}_2, \hat{u}_3) = (M - \rho_1, \delta_2\phi_2, \delta_3\phi_3)$ where ρ_i, δ_i are some positive constants with δ_i sufficiently small and $\rho_1 < M$, $\phi_i \equiv \phi_i(x,t)$ $(i = 1,2,3)$. Then it is easy to verify that $(\tilde{w}_1, \tilde{u}_2, \tilde{u}_3)$, $(\hat{w}_1, \hat{u}_2, \hat{u}_3)$ satisfy all the requirement of upper and lower solutions if

$$\begin{cases} \frac{\partial(M-\delta_1\phi_1)}{\partial t} - d_1\Delta(M-\delta_1\phi_1) \geqslant -\delta_1\phi_1(a_1 - b_{11}\delta_1\phi_1 - b_{12}\rho_2), \\ \frac{\partial\rho_2}{\partial t} - d_2\Delta\rho_2 \geqslant \rho_2(a_2 - b_{21}(\delta_1\phi_1)_{\tau_1} - b_{22}\rho_2 + b_{23}\rho_3), \\ \frac{\partial\rho_3}{\partial t} - d_3\Delta\rho_3 \geqslant \rho_2(a_3 + b_{32}\rho_2 - b_{33}\rho_3), \\ \frac{\partial(M-\rho_1)}{\partial t} - d_1\Delta(M-\rho_1) \leqslant -\rho_1(a_1 - b_{11}\rho_1 - b_{12}(\delta_2\phi_2)_{\tau_2}), \\ \delta_2[\frac{\partial\phi_2}{\partial t} - d_2\Delta\phi_2] \leqslant \delta_2\phi_2(a_2 - b_{21}\rho_1 - b_{22}\delta_2\phi_2 + b_{23}(\delta_3\phi_3)_{\tau_3}), \\ \delta_3[\frac{\partial\phi_3}{\partial t} - d_3\Delta\phi_3] \leqslant \delta_3\phi_3(a_3 + b_{32}(\delta_2\phi_2)_{\tau_2} - b_{33}\delta_3\phi_3). \end{cases} \tag{3.6}$$

In view of (3.1) the above inequalities are satisfied by some sufficiently small δ_1, δ_2 if

$$\begin{cases} a_1 - \rho_1 b_{11} \leqslant 0, \qquad a_2 - b_{22}\rho_2 + b_{23}\rho_3 \leqslant 0, \qquad a_3 + b_{32}\rho_2 - b_{33}\rho_3 \leqslant 0; \\ \lambda_1(a_1) < -b_{12}\rho_2, \qquad \lambda_2(a_2) < -b_{21}\rho_1, \qquad \lambda_3(a_3) < 0. \end{cases} \tag{3.7}$$

Assuming that

$$b_{23}b_{32} < b_{22}b_{33} \tag{3.8}$$

and setting

$$M_1 = \max_{\overline{D}}\left[\frac{a_1(x,t)}{b_{11}(x,t)}\right], \tag{3.9}$$

$$M_2 = \max_{\overline{D}}\left[\frac{a_3(x,t)b_{23}(x,t) + a_2(x,t)b_{33}(x,t)}{b_{22}(x,t)b_{33}(x,t) - b_{23}(x,t)b_{32}(x,t)}\right], \tag{3.10}$$

$$M_3 = \max_{\overline{D}}\left[\frac{a_3(x,t)b_{22}(x,t) + a_2(x,t)b_{32}(x,t)}{b_{22}(x,t)b_{33}(x,t) - b_{23}(x,t)b_{32}(x,t)}\right]. \tag{3.11}$$

Then the requirements in (3.7) are fulfilled by some $\rho_i > M_i$ $(i = 1, 2, 3)$ if (3.8) holds and

$$-\lambda_1(a_1) > b_{12}M_2, \qquad -\lambda_2(a_2) > b_{21}M_1, \qquad -\lambda_3(a_3) > 0. \tag{3.12}$$

From Theorem A of [20] we have that under conditions (3.8), (3.12), the problem (3.2), (3.3) has a maximal T-periodic solution $(\overline{w}_1, \bar{u}_2, \bar{u}_3)$ and a minimal T-periodic solution $(\underline{w}_1, \underline{u}_2, \underline{u}_3)$ such that

$$(M-\rho_1, \delta_2\phi_2, \delta_3\phi_3) \leqslant (\underline{w}_1, \underline{u}_2, \underline{u}_3) \leqslant (\overline{w}_1, \bar{u}_2, \bar{u}_3) \leqslant (M-\delta_1\phi_1, \rho_2, \rho_3).$$

Moreover, by Theorem 3.1 of [20] the solution $\mathbf{u} = (w_1, u_2, u_3)$ of the initial boundary problem (3.2), (3.4) possesses the following convergence:

$$\lim_{m\to\infty} \mathbf{u}(x, t+mT; \eta) = \begin{cases} \underline{\mathbf{u}}(x,t) & \text{if } \hat{\mathbf{u}} \leqslant \eta \leqslant \underline{\mathbf{u}} \text{ in } D_0, \\ \bar{\mathbf{u}}(x,t) & \text{if } \bar{\mathbf{u}} \leqslant \eta \leqslant \tilde{\mathbf{u}} \text{ in } D_0 \end{cases} \tag{3.13}$$

and

$$\underline{\mathbf{u}}(x,t) \leqslant \mathbf{u}(x, t+mT; \eta) \leqslant \bar{\mathbf{u}}(x,t) \quad \text{on } \overline{D} \text{ as } m \to \infty. \tag{3.14}$$

Now by the transformation $u_1 = M - w_1$, the pair $(\underline{u}_1, \bar{u}_2, \bar{u}_3)$ and $(\bar{u}_1, \underline{u}_2, \underline{u}_3)$ where $\underline{u}_1 = M - \overline{w}_1$, $\bar{u}_1 = M - \underline{w}_1$ are positive T-periodic solutions of the problem (1.2), (1.3) and satisfy the relation $\delta_i\phi_i \leqslant \underline{u}_i \leqslant \bar{u}_i \leqslant \rho_i$ on $\overline{D}$.

Furthermore for any $\delta_i\phi_i \leqslant \eta_i \leqslant \rho_i$ in $D_0^{(i)}$, $i = 1, 2, 3$, the solution of the initial boundary problem (1.2), (1.4) is given by $(u_1, u_2, u_3) = (M - w_1, u_2, u_3)$ and satisfies the relation $\delta_1\phi_1 \leqslant u_i \leqslant \rho_i$, $i = 1, 2, 3$ on $\overline{D}$.

According to (3.13), (3.14) and Theorem 3.1 of [20], the solution of (1.2), (1.4) with the initial condition $\delta_i\phi_i \leqslant \eta_i \leqslant \rho_i$ in $D_0^{(i)}$, $i = 1, 2, 3$ possesses the convergence

$$\lim_{m\to\infty} (u_1, u_2, u_3)(x, t+mT; \eta) = \begin{cases} (\bar{u}_1, \underline{u}_2, \underline{u}_3), & \text{if } \eta_1 \geqslant \bar{u}_1,\ 0 \leqslant \eta_i \leqslant \underline{u}_i,\ i = 2, 3, \\ (\underline{u}_1, \bar{u}_2, \bar{u}_3), & \text{if } 0 \leqslant \eta_1 \leqslant \underline{u}_1,\ \eta_i \geqslant \bar{u}_i,\ i = 2, 3, \end{cases} \tag{3.15}$$

and

$$(\underline{u}_1, \underline{u}_2, \underline{u}_3) \leqslant (u_1, u_2, u_3)(x, t+mT; \eta) \leqslant (\bar{u}_1, \bar{u}_2, \bar{u}_3) \quad \text{on } \overline{D} \text{ as } m \to \infty. \tag{3.16}$$

To summarize the above conclusions we have the following theorem.

Theorem 3.1. *Let* $(u_1(x,t;\eta_1), u_2(x,t;\eta_2), u_3(x,t;\eta_3))$ *be the solution of* (1.2), (1.4) *for* (η_1,η_2,η_3) *with* $0 < \eta_i \leqslant \rho_i$, $i = 1,2,3$ *and let conditions* (3.8), (3.12) *be satisfied. Then we have*

(i) *problem* (1.2), (1.3) *has positive T-periodic solutions* $(\underline{u}_1, \bar{u}_2, \bar{u}_3)$, $(\bar{u}_1, \underline{u}_2, \underline{u}_3)$ *such that* $\underline{u}_i \leqslant \bar{u}_i$, $i = 1,2,3$ *on* $\overline{D}$;

(ii) *the solution* $(u_1(x,t;\eta_1), u_2(x,t;\eta_2), u_3(x,t;\eta_3))$ *of* (1.2), (1.4) *possesses the convergence properties* (3.15) *and* (3.16);

(iii) *if* $(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\underline{u}_1, \underline{u}_2, \underline{u}_3) = (u_1^*, u_2^*, u_3^*)$, *then*

$$\lim_{m\to\infty}\big(u_1(x,t+mT;\eta_1), u_2(x,t+mT;\eta_2), u_3(x,t+mT;\eta_3)\big) = \big(u_1^*(x,t), u_2^*(x,t), u_3^*(x,t)\big),$$
$$t > 0,\ x \in \overline{\Omega}.$$

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