Coexistence and asymptotic periodicity in a competitor–competitor–mutualist model ✩

Wenzhen Gan a,b,*, Zhigui Lin b

a Department of Basic Courses, Jiangsu Teachers University of Technology, Changzhou 213001, China
b School of Mathematical Science, Yangzhou University, Yangzhou 225002, China

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Abstract
In this paper, the competitor–competitor–mutualist three-species Lotka–Volterra model is discussed. Firstly, by Schauder fixed point theory, the coexistence state of the strongly coupled system is given. Applying the method of upper and lower solutions and its associated monotone iterations, the true solutions are constructed. Our results show that this system possesses at least one coexistence state if cross-diffusions and cross-reactions are weak. Secondly, the existence and asymptotic behavior of T-periodic solutions for the periodic reaction–diffusion system under homogeneous Dirichlet boundary conditions are investigated. Sufficient conditions which guarantee the existence of T-periodic solution are also obtained.

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1. Introduction

In this paper, we consider the strongly coupled elliptic system with Dirichlet boundary condition:

\[
\begin{align*}
- \Delta [(d_1 + a_{11} u_1 + a_{12} u_2) u_1] &= a_1 (a_1 - b_{11} u_1 - b_{12} u_2), \quad x \in \Omega, \\
- \Delta [(d_2 + a_{21} u_1 + a_{22} u_2 + \frac{\alpha_{23}}{\beta + \gamma} u_3) u_2] &= a_2 (a_2 - b_{21} u_1 - b_{22} u_2 + b_{23} u_3), \quad x \in \Omega, \\
- \Delta [(d_3 + \frac{\alpha_{32}}{\gamma + \alpha_{33}} + \alpha_{33} u_3) u_3] &= a_3 (a_3 + b_{32} u_2 - b_{33} u_3), \quad x \in \Omega, \\
\end{align*}
\]

\[u_i(x) = 0, \quad i = 1, 2, 3, \quad x \in \partial \Omega,\]

where \(\Delta\) is the Laplacian operator, \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with a smooth boundary \(\partial \Omega\) and \(d_i, \beta, \gamma, a_i, b_{ij}, i, j = 1, 2, 3\) are positive constants except for \(a_{ij}\) which may be nonnegative constants. The system represents a model which involves interacting and migrating in the same habitat \(\Omega\) among a competitor, a competitor–mutualist and a mutualist. Here \(u_i, i = 1, 2, 3\) denotes the density of competitor, competitor–mutualist and mutualist, respectively.

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* Corresponding author.

E-mail address: ganwenzhen@yahoo.com.cn (W. Gan).

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The boundary condition means that the habitat $\Omega$ is surrounded by a hostile environment. The diffusion terms can be written as

$$\text{div}\left\{ (d_1 + 2\alpha_{11} u_1 + \alpha_{12} u_2) \nabla u_1 + \alpha_{12} u_1 \nabla u_2 \right\},$$

$$\text{div}\left\{ \alpha_{21} u_2 \nabla u_1 + \left( d_2 + \alpha_{21} u_1 + 2\alpha_{22} u_2 + \frac{\alpha_{23}}{\beta + u_3} \right) \nabla u_2 + \frac{-\alpha_{23} u_2}{(\beta + u_3)^2} \nabla u_3 \right\},$$

$$\text{div}\left\{ \frac{-\alpha_{32} u_3}{(\gamma + u_2)^2} \nabla u_2 + \left( d_3 + \frac{\alpha_{32}}{\gamma + u_2} + 2\alpha_{33} u_3 \right) \nabla u_3 \right\}.$$  

The terms

$$d_1 + 2\alpha_{11} u_1 + \alpha_{12} u_2, \quad d_2 + \alpha_{21} u_1 + 2\alpha_{22} u_2 + \frac{\alpha_{23}}{\beta + u_3}, \quad d_3 + \frac{\alpha_{32}}{\gamma + u_2} + 2\alpha_{33} u_3$$

represent the “self-diffusion” and the terms

$$\alpha_{12} u_1, \quad \alpha_{21} u_2, \quad \frac{-\alpha_{23}}{(\beta + u_3)^2}, \quad \frac{-\alpha_{32}}{(\gamma + u_2)^2}$$

represent the “cross-diffusion.” Here $\alpha_{ij} > 0$ and $\alpha_{21} u_2 > 0$ imply that the flux of $u_1$ and $u_2$ in $x$-direction are directed toward decreasing population of $u_2$ and $u_1$ respectively, i.e. the two competitors avoid each other. While $-\frac{\alpha_{23} u_2}{(\beta + u_3)^2} < 0$ and $-\frac{\alpha_{32} u_2}{(\gamma + u_2)^2} < 0$ imply that the flux of $u_2$ and $u_3$ in $x$-direction are directed toward increasing population of $u_3$ and $u_2$ respectively, i.e. the two mutualists chase each other. The above model means that, in addition to the dispersive force, the diffusion also depends on population pressure from other species. Here a solution $(u_1, u_2, u_3)$ to system (1.1) is said to be positive if $u_i(x) > 0, i = 1, 2, 3$ for all $x \in \Omega$, the existence of a positive solution $(u_1, u_2, u_3)$ to system (1.1) is also called a coexistence. We are mainly concerned with the coexistence states of system (1.1).

In the case when $\alpha_{ij} = 0$ for $i, j = 1, 2, 3$, the above system is the classic competitor–competitor–mutualist model, while if $\alpha_{ij} \neq 0$ for some $i$ or $j$ the system becomes a strongly coupled elliptic system. The strongly coupled systems of elliptic equations have been extensively studied by many mathematicians [1–8]. In an attempt to investigate the spatial segregation under self- and cross-population pressure, Shigesada et al. [1] proposed the strongly coupled elliptic system describing two species Lotka–Volterra competition model. For the Dirichlet boundary value problem of the system, positive solutions are found when birth rates lie in certain range, or when cross-diffusion are sufficiently large by [2]. For the homogeneous Neumann boundary value problem of the system, the effects of diffusion, self-diffusion and cross-diffusion were investigated by Lou and Ni [3]. Applying the bifurcation theory and Lyapunov–Schmidt procedure, the multiple coexistence states for a prey–predator system with cross-diffusion was proved by Kuto and Yamada [4].

Recently, the method of construction of solutions for a general class of strongly coupled elliptic systems was developed by Pao [8] and was based on upper and lower solutions and its associated monotone iterations.

For the related parabolic systems, reader can see [9–12] and references therein.

We will also consider the competitor–competitor–mutualist model with time delays and diffusions under Neumann boundary conditions

$$\begin{align*}
\frac{\partial u_1}{\partial \tau} - d_1(x, t) \Delta u_1 &= u_1[a_1(x, t) - b_{11}(x, t) u_1(x, t) - b_{12}(x, t) u_2(x, t - \tau_2)], \\
&\quad \text{in } \Omega \times (0, \infty),\\
\frac{\partial u_2}{\partial \tau} - d_2(x, t) \Delta u_2 &= u_2[a_2(x, t) - b_{21}(x, t) u_1(x, t - \tau_1) - b_{22}(x, t) u_2(x, t) + b_{23}(x, t) u_3(x, t - \tau_3)], \\
&\quad \text{in } \Omega \times (0, \infty),\\
\frac{\partial u_3}{\partial \tau} - d_3(x, t) \Delta u_3 &= u_3[a_3(x, t) + b_{32}(x, t) u_2(x, t - \tau_2) - b_{33}(x, t) u_3], \\
&\quad \text{in } \Omega \times (0, \infty),\\
\frac{\partial u_1}{\partial \eta} &= \frac{\partial u_2}{\partial \eta} = \frac{\partial u_3}{\partial \eta} = 0, \quad \text{on } \partial \Omega \times (0, \infty),
\end{align*}$$

with the periodic condition

$$u_i(x, t) = u_i(x, t + T), \quad i = 1, 2, 3, \quad (x, t) \in \Omega \times [-\tau_i, 0],$$

and under the initial condition
where $d_i \equiv d_i(x, t), a_i \equiv a_i(x, t), b_{ij} \equiv b_{ij}(x, t), (i = 1, 2, 3)$ are smooth positive T-periodic functions on $\Omega \times (0, \infty)$. $\Omega$ is a bounded domain in $\mathbb{R}^N$ with boundary $\partial \Omega$. $\eta_i$ denotes the outward normal derivative on $\partial \Omega$. It is assumed that the boundary $\partial \Omega$ is of a class $C^{1+\alpha}$ and $\eta_i \in C^{\alpha/2, \alpha}(\partial \Omega)$ and satisfies the compatibility condition, where $D^{(i)}_0 = \Omega \times [-\tau_i, 0], i = 1, 2, 3$. We are interested in the existence of the T-periodic solution as well as the asymptotic behavior of (1.2), (1.4) in relation to the maximal and minimal T-periodic solution of systems (1.2), (1.3).

Periodic solutions of parabolic boundary value problems have been investigated by many researchers, and various methods have been proposed for the existence and qualitative properties of the solution. The logistic delay differential equation as a model of single-species population growth has been considered in [13,14]. Nonlinear periodic diffusion equations arise naturally in population models [15] where the birth and death rates, rates of diffusion, rates of interactions and environmental carrying capacities are periodic on seasonal scale. The existence and global stability of a T-periodic solution of periodic boundary-value problem of the logistic model has been studied by Hess [16]. A coupled system of parabolic equations with time delays has been investigated by the method of upper and lower solutions in [17,18]. The monotone iterative scheme associated with this method leads to various computation algorithms for numerical solutions of the periodic boundary problem [19]. The stability and attractivity analysis which are for quasimonotone nondecreasing and mixed quasimonotone reaction functions by the monotone iteration scheme were given in [20,21].

The paper is organized as follows: based on the idea introduced by Pao [8], we try to obtain sufficient conditions which guarantee the coexistence state of system (1.1) in Section 2, and the true solutions of (1.1) are constructed in sections 2, 3. We are interested in the existence of the T-periodic solution as well as the asymptotic behavior of (1.2), (1.4) in relation to the maximal and minimal T-periodic solution of systems (1.2), (1.3).

2. Coexistence

We will give a sufficient condition for that system (1.1) has a positive solution by constructing a coupled upper and lower solutions as in [8]. We first give an equivalent form of the problem (1.1):

$$
\begin{align*}
-\Delta [D_1(u_1, u_2)] &= f_1(u_1, u_2), & x \in \Omega, \\
-\Delta [D_2(u_1, u_2, u_3)] &= f_2(u_1, u_2, u_3), & x \in \Omega, \\
-\Delta [D_3(u_2, u_3)] &= f_3(u_2, u_3), & x \in \Omega, \\
u_1(x) &= u_2(x) = u_3(x) = 0, & x \in \partial \Omega,
\end{align*}
$$

(2.1)

where

$$
D_1(u_1, u_2) = (d_1 + a_{11}u_1 + a_{12}u_2)u_1, \\
f_1(u_1, u_2) = u_1(a_1 - b_{11}u_1 - b_{12}u_2), \\
D_2(u_1, u_2, u_3) = \left(d_2 + a_{21}u_1 + a_{22}u_2 + \frac{a_{23}}{\beta + u_3}\right)u_2, \\
f_2(u_1, u_2, u_3) = u_2(a_2 - b_{21}u_1 - b_{22}u_2 + b_{23}u_3), \\
D_3(u_2, u_3) = \left(d_3 + \frac{a_{32}}{\gamma + u_2} + a_{33}u_3\right)u_3, \\
f_3(u_2, u_3) = u_3(a_3 + b_{32}u_2 - b_{33}u_3).
$$

Define

$$
w_1 = D_1(u_1, u_2), \quad w_2 = D_2(u_1, u_2, u_3), \quad w_3 = D_3(u_2, u_3).
$$

A direct calculation shows that the Jacobian $J$ of the transformation $w_1, w_2, w_3$ is given by

$$
J = \frac{\partial (w_1, w_2, w_3)}{\partial (u_1, u_2, u_3)} \geq d_1d_2d_3 > 0 \quad \text{for} \ (u_1, u_2, u_3) \geq (0, 0, 0).
$$
Then the inverse $u_1 = g_1(w_1, w_2, w_3)$, $u_2 = g_2(w_1, w_2, w_3)$, $u_3 = g_3(w_1, w_2, w_3)$ exists whenever $(u_1, u_2, u_3) \geq (0, 0, 0)$. Hence the corresponding equivalent of (2.1) becomes

$$
\begin{align*}
-\Delta w_1 + k_1 w_1 &= F_1(u_1, u_2), & x \in \Omega, \\
-\Delta w_2 + k_2 w_2 &= F_2(u_1, u_2, u_3), & x \in \Omega, \\
-\Delta w_3 + k_3 w_3 &= F_3(u_2, u_3), & x \in \Omega, \\
u_i = g_i(w_1, w_2, w_3), & i = 1, 2, 3, x \in \Omega, \\
w_j(x) = 0, & i = 1, 2, 3, x \in \partial \Omega,
\end{align*}
$$

(2.2)

where $F_i(u_1, u_2, u_3) = k_i D_i(u_1, u_2, u_3) + f_i(u_1, u_2, u_3), i = 1, 2, 3$.

Now we consider the monotonicity of $F_i$ with respect to $u_j$ and also the monotonicity of $g_i$ with respect to $w_j$ for $i, j = 1, 2, 3$. First it is easy to see that

$$
\frac{\partial u_1}{\partial w_1} > 0, \quad \frac{\partial u_1}{\partial w_2} < 0, \quad \frac{\partial u_2}{\partial w_3} < 0, \quad \frac{\partial u_2}{\partial w_1} < 0, \quad \frac{\partial u_2}{\partial w_2} > 0, \quad \frac{\partial u_3}{\partial w_3} > 0
$$

from direct calculations. This shows that $u_1 = g_1(w_1, w_2, w_3)$ is nondecreasing in $w_1$ and nonincreasing in $w_2, w_3$ and $u_2 = g_2(w_1, w_2, w_3)$ is nondecreasing in $w_2, w_3$ and nonincreasing in $w_1$, while $u_3 = g_3(w_1, w_2, w_3)$ is nondecreasing in $u_2, w_3$ and nonincreasing in $w_1$ for all $(w_1, w_2, w_3) \geq (0, 0, 0)$.

Secondly if we choose $k_i = \frac{b_{ii}}{a_{ii}}, i = 1, 2, 3$. Then

$$
\begin{align*}
\frac{\partial F_1}{\partial u_1} &= \frac{b_{11}}{a_{11}} d_1 + a_1 + \left(\frac{b_{11}}{a_{11}} a_{12} - b_{12}\right) u_2, \\
\frac{\partial F_1}{\partial u_2} &= \left(\frac{b_{11}}{a_{11}} a_{12} - b_{12}\right) u_2, \quad \frac{\partial F_1}{\partial u_3} = 0; \\
\frac{\partial F_2}{\partial u_2} &= \frac{b_{22}}{a_{22}} d_2 + a_2 + \left(\frac{b_{22}}{a_{22}} a_{21} - b_{21}\right) u_1 + \frac{b_{22} \alpha_{23}}{(\beta + u_3) a_{22}} + b_{23} u_3, \\
\frac{\partial F_2}{\partial u_1} &= \left(\frac{b_{22}}{a_{22}} a_{21} - b_{21}\right) u_2, \quad \frac{\partial F_2}{\partial u_3} = \left[b_{23} - \frac{b_{22} \alpha_{23}}{a_{22} (\beta + u_3)^2}\right] u_2; \\
\frac{\partial F_3}{\partial u_3} &= \frac{b_{33}}{a_{33}} d_3 + a_3 + \frac{b_{33} \alpha_{32}}{a_{33} (\gamma + u_2)} + b_{32} u_2, \\
\frac{\partial F_3}{\partial u_2} &= \left[b_{32} - \frac{b_{33} \alpha_{32}}{a_{33} (\gamma + u_2)^2}\right] u_3, \quad \frac{\partial F_3}{\partial u_1} = 0.
\end{align*}
$$

Assume that

$$
\frac{b_{11}}{a_{11}} < \frac{b_{12}}{a_{12}}, \quad \frac{b_{22}}{a_{22}} < \frac{b_{21}}{a_{21}}, \quad \frac{b_{22}}{a_{22}} < \frac{b_{23} \beta^2}{a_{23}}, \quad \frac{b_{33}}{a_{33}} < \frac{b_{32} \gamma^2}{a_{32}},
$$

(2.3)

Obviously, we can get $\frac{\partial F_1}{\partial u_1} \leq 0$, $\frac{\partial F_2}{\partial u_1} \leq 0$, $\frac{\partial F_2}{\partial u_2} \geq 0$, $\frac{\partial F_2}{\partial u_3} \geq 0$, $\frac{\partial F_3}{\partial u_2} \geq 0$ for every $(u_1, u_2, u_3) \geq (0, 0, 0)$. Furthermore, choose

$$
\bar{M}_1 = \frac{b_{22} d_2 + a_2 \alpha_{22}}{b_{21} a_{22} - b_{22} \alpha_{21}}, \quad \bar{M}_2 = \frac{b_{11} d_1 + a_1 \alpha_{11}}{b_{12} a_{11} - b_{11} \alpha_{12}},
$$

we can obtain that $\frac{\partial F_1}{\partial u_1} \geq 0$, $\frac{\partial F_2}{\partial u_2} \geq 0$ when $(u_1, u_2, u_3) \in [0, \bar{M}_1] \times [0, \bar{M}_2] \times [0, \infty)$. Therefore, when $(u_1, u_2, u_3) \in [0, \bar{M}_1] \times [0, \bar{M}_2] \times [0, \infty)$, the function $F_1$ is nonincreasing in $u_2$ and nondecreasing in $u_1$; $F_2$ is nonincreasing in $u_1$ and nondecreasing in $u_2, u_3$; $F_3$ is nondecreasing in $u_2, u_3$.

Next we give the definition of coupled upper and lower solutions of (2.2) as the following:

**Definition 2.1.** A pair of 6-vector functions $(\tilde{u}, \tilde{w}) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$, $(\tilde{\tilde{u}}, \tilde{\tilde{w}}) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ in $C^2(\Omega) \cap C(\bar{\Omega})$ are called coupled upper and lower solutions of (2.2), if $\tilde{u}_1 \leq \bar{M}_1$, $\tilde{u}_2 \leq \bar{M}_2$, $(\tilde{\tilde{u}}, \tilde{\tilde{w}}) \geq (\tilde{u}, \tilde{w})$ and if their components satisfy the relation
\[
\begin{align*}
-\Delta \hat{w}_1 + k_1 \hat{w}_1 & \geq F_1(\hat{u}_1, \hat{u}_2), \\
-\Delta \hat{w}_2 + k_2 \hat{w}_2 & \geq F_2(\hat{u}_1, \hat{u}_2, \hat{u}_3), \\
-\Delta \hat{w}_3 + k_3 \hat{w}_3 & \geq F_3(\hat{u}_2, \hat{u}_3), \\
\hat{u}_1 & \geq g_1(\hat{w}_1, \hat{w}_2, \hat{w}_3), \\
\hat{u}_2 & \geq g_2(\hat{w}_1, \hat{w}_2, \hat{w}_3), \\
\hat{u}_3 & \geq g_3(\hat{w}_1, \hat{w}_2, \hat{w}_3), \\
\hat{w}_1(x) & \geq 0 \geq \hat{w}_i(x), i = 1, 2, 3, x \in \partial \Omega.
\end{align*}
\]

We set
\[
S = \{ u \in C^a(\Omega); \ \hat{u} \leq u \leq \bar{u} \}; \quad S^a = \{ w \in C^a(\Omega); \ \hat{w} \leq w \leq \bar{w} \}
\]

where \( u = (u_1, u_2, u_3), w = (w_1, w_2, w_3), \hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \) and \( \bar{w} = (\bar{w}_1, \bar{w}_2, \bar{w}_3), \hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3) \).

For definiteness, we choose
\[
\begin{align*}
\hat{u}_1 &= g_1(\hat{w}_1, \hat{w}_2, \hat{w}_3), \\
\hat{u}_2 &= g_2(\hat{w}_1, \hat{w}_2, \hat{w}_3), \\
\hat{u}_3 &= g_3(\hat{w}_1, \hat{w}_2, \hat{w}_3), \\
\end{align*}
\]

which is equivalent to
\[
\begin{align*}
\hat{w}_1 &= D_1(\hat{u}_1, \hat{u}_2), \\
\hat{w}_2 &= D_2(\hat{u}_1, \hat{u}_2, \hat{u}_3), \\
\hat{w}_3 &= D_3(\hat{u}_2, \hat{u}_3), \\
\end{align*}
\]

Then the requirements of \((\bar{u}_1, \bar{u}_2, \bar{u}_3), (\hat{u}_1, \hat{u}_2, \hat{u}_3)\) in (2.4) are satisfied and those of \((\hat{w}_1, \hat{w}_2, \hat{w}_3), (\bar{w}_1, \bar{w}_2, \bar{w}_3)\) are reduced to
\[
\begin{align*}
-\Delta[D_1(\bar{u}_1, \bar{u}_2)] + k_1 D_1(\bar{u}_1, \bar{u}_2) & \geq F_1(\bar{u}_1, \bar{u}_2), \quad x \in \Omega, \\
-\Delta[D_2(\bar{u}_1, \bar{u}_2, \bar{u}_3)] + k_2 D_2(\bar{u}_1, \bar{u}_2, \bar{u}_3) & \geq F_2(\bar{u}_1, \bar{u}_2, \bar{u}_3), \quad x \in \Omega, \\
-\Delta[D_3(\bar{u}_2, \bar{u}_3)] + k_3 D_3(\bar{u}_2, \bar{u}_3) & \geq F_3(\bar{u}_2, \bar{u}_3), \quad x \in \Omega, \\
-\Delta[D_1(\hat{u}_1, \hat{u}_2)] + k_1 D_1(\hat{u}_1, \hat{u}_2) & \leq F_1(\hat{u}_1, \hat{u}_2), \quad x \in \Omega, \\
-\Delta[D_2(\hat{u}_1, \hat{u}_2, \hat{u}_3)] + k_2 D_2(\hat{u}_1, \hat{u}_2, \hat{u}_3) & \leq F_2(\hat{u}_1, \hat{u}_2, \hat{u}_3), \quad x \in \Omega, \\
-\Delta[D_3(\hat{u}_2, \hat{u}_3)] + k_3 D_3(\hat{u}_2, \hat{u}_3) & \leq F_3(\hat{u}_2, \hat{u}_3), \quad x \in \Omega, \\
\hat{u}_i(x) & \geq 0 \geq \hat{u}_i(x), i = 1, 2, 3, x \in \partial \Omega.
\end{align*}
\]

We call the pair \((\hat{u}_1, \hat{u}_2, \hat{u}_3), (\bar{u}_1, \bar{u}_2, \bar{u}_3)\) satisfying (2.5) and \(\hat{u}_1 \leq \bar{M}_1, \hat{u}_2 \leq \bar{M}_2, (\hat{u}_1, \hat{u}_2, \hat{u}_3) \geq (\bar{u}_1, \bar{u}_2, \bar{u}_3)\) are coupled upper and lower solutions of (1.1).

Now we seek a pair of coupled upper and lower solutions of (1.1) in the form
\[
(\hat{u}_1, \hat{u}_2, \hat{u}_3) = (M_1, M_2, M_3), \quad (\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\delta_1 \phi, \delta_2 \phi, \delta_3 \phi)
\]

where \( M_i \) and \( \delta_i \) \((i = 1, 2, 3)\) are some positive constants with \( \delta_i \) sufficiently small, and \( \phi \equiv \phi(x) \) is the (normalized) positive eigenfunction corresponding to \( \lambda_0 \), where \( \lambda_0 \) is the smallest eigenvalue of the Laplacian \((-\Delta)\) under Dirichlet boundary condition. Indeed \((M_1, M_2, M_3), (\delta_1 \phi, \delta_2 \phi, \delta_3 \phi)\) satisfy the inequalities in (2.5) if
\[
\begin{align*}
-\Delta[(d_1 + \alpha_1 M_1 + \alpha_2 \delta_3 \phi) M_1] & \geq M_1(a_1 - b_{11} M_1 - b_{21} \delta_2 \phi), \\
-\Delta[(d_2 + \alpha_2 \delta_1 \phi + \alpha_2 M_2 + \frac{\alpha_3}{\beta + \delta_3 \phi} M_2)] & \geq M_2(a_2 - b_{21} \delta_1 \phi - b_{22} M_2 + b_{23} M_3), \\
-\Delta[(d_3 + \frac{\alpha_3}{\beta + \delta_3 \phi} + \alpha_3 M_3) M_3] & \geq M_3(a_3 + b_{23} M_2 - b_{33} M_3), \\
-\Delta[(d_1 + \alpha_1 \delta_1 \phi + \alpha_2 M_2) \delta_1 \phi] & \leq \delta_1 \phi(a_1 - b_{11} \delta_1 \phi - b_{12} M_2), \\
-\Delta[(d_2 + \alpha_2 \delta_2 \phi + \frac{\alpha_3}{\beta + \delta_3 \phi} \delta_3 \phi)] & \leq \delta_2 \phi(a_2 - b_{21} \delta_2 \phi + b_{22} \delta_2 \phi + b_{23} \delta_3 \phi), \\
-\Delta[(d_3 + \frac{\alpha_3}{\beta + \delta_3 \phi} \delta_3 \phi) \delta_3 \phi] & \leq \delta_3 \phi(a_3 + b_{23} \delta_2 \phi - b_{33} \delta_3 \phi).
\end{align*}
\]

Since that \( \delta_i, i = 1, 2, 3 \) is sufficiently small and \(-\Delta \phi = \lambda_0 \phi\), the inequalities in (2.6) are equivalent to
Theorem 2.1. The problem (1.1) admits at least one positive solution \( u = (u_1, u_2, u_3) \) under the condition (2.10).

Remark 2.1. It is easy to see that if \( \lambda_0d_1 \geq a_1 \) or \( \lambda_0d_2 \geq a_2 \) or \( \lambda_0d_3 \geq a_3 \), then problem (1.1) has no positive solution, see [2]. Our result shows that if \( \lambda_0d_1 < a_1 \), \( \lambda_0d_2 < a_2 \) and \( \lambda_0d_3 < a_3 \), then problem (1.1) has at least one coexistence state provided that cross-diffusions \( \alpha_{12}, \alpha_{21}, \alpha_{23}, \alpha_{32} \) and cross-reactions \( b_{12}, b_{21}, b_{23}, b_{32} \) are sufficiently small.

In what follows, we will construct the true solutions of (1.1) based on monotone iterative schemes. Under the condition (2.10), we know that \( (M_1, M_2, M_3), (\delta_1\phi, \delta_2\phi, \delta_3\phi) \) are coupled upper and lower solutions of problem (1.1). Now we use \( (\hat{u}_1, \hat{u}_2, \hat{u}_3) = (M_1, M_2, M_3), (u_1^{(0)}, u_2^{(0)}, u_3^{(0)}) = (\delta_1\phi, \delta_2\phi, \delta_3\phi) \) as an initial iteration in the iteration process

\[
\begin{align*}
-\Delta \bar{u}_1^{(m)} + k_1 \bar{u}_1^{(m)} &= F_1(\bar{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}), \quad x \in \Omega, \\
-\Delta \bar{u}_2^{(m)} + k_2 \bar{u}_2^{(m)} &= F_2(\bar{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}, \bar{u}_3^{(m-1)}), \quad x \in \Omega, \\
-\Delta \bar{u}_3^{(m)} + k_3 \bar{u}_3^{(m)} &= F_3(\bar{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}), \quad x \in \Omega, \\
-\Delta \bar{u}_1^{(m)} + k_1 \bar{u}_1^{(m)} &= F_1(\bar{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}), \quad x \in \Omega, \\
-\Delta \bar{u}_2^{(m)} + k_2 \bar{u}_2^{(m)} &= F_2(\bar{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}), \quad x \in \Omega, \\
-\Delta \bar{u}_3^{(m)} + k_3 \bar{u}_3^{(m)} &= F_3(\bar{u}_1^{(m-1)}, \bar{u}_2^{(m-1)}), \quad x \in \Omega, \\
\hat{u}_1^{(m)} &= g_1(\bar{u}_1^{(m)}, \bar{u}_2^{(m)}, \bar{u}_3^{(m)}), \quad x \in \Omega, \\
\hat{u}_2^{(m)} &= g_2(\bar{u}_1^{(m)}, \bar{u}_2^{(m)}, \bar{u}_3^{(m)}), \quad x \in \Omega, \\
\hat{u}_3^{(m)} &= g_3(\bar{u}_1^{(m)}, \bar{u}_2^{(m)}), \quad x \in \Omega, \\
\bar{u}_i^{(m)}(x) &= \bar{u}_i^{(m)}(x) = 0, \quad i = 1, 2, 3, \quad x \in \partial \Omega,
\end{align*}
\]
where \( m = 1, 2, \ldots \). Using the Lemma 3.1 of [8], we know that the sequences \( \{(\tilde{u}^{(m)}, \tilde{w}^{(m)})\}, \{(\bar{u}^{(m)}, \bar{w}^{(m)})\} \) governed by (2.11) are well defined and possess the monotone property

\[
(\tilde{u}, \tilde{w}) \leq (u^{(m-1)}, w^{(m-1)}) \leq (u^{(m)}, w^{(m)}) \leq (\tilde{u}^{(m)}, \tilde{w}^{(m)})
\]

for every \( m = 1, 2, \ldots \).

Therefore the pointwise limits

\[
\lim_{m \to \infty} (\tilde{u}^{(m)}, \tilde{w}^{(m)}) = (\tilde{u}, \tilde{w}), \quad \lim_{m \to \infty} (u^{(m)}, w^{(m)}) = (u, w)
\]

exist and satisfy the relation

\[
(\tilde{u}, \tilde{w}) \leq (u^{(m)}, w^{(m)}) \leq (u, w) \leq (\tilde{u}^{(m)}, \tilde{w}^{(m)}) \leq (\tilde{u}, \tilde{w})
\]

for every \( m = 1, 2, \ldots \).

From the last four equations in the iteration process (2.11), we obtain

\[
\begin{align*}
\tilde{u}^{(m)}_1 &= g_1(\tilde{w}^{(m)}_1, \tilde{w}^{(m)}_2, \tilde{w}^{(m)}_3), \\
\tilde{u}^{(m)}_2 &= g_2(\tilde{w}^{(m)}_1, \tilde{w}^{(m)}_2, \tilde{w}^{(m)}_3), \\
\tilde{u}^{(m)}_3 &= g_3(\tilde{w}^{(m)}_1, \tilde{w}^{(m)}_2, \tilde{w}^{(m)}_3), \\
\tilde{w}^{(m)}_1 &= D_1(\bar{u}^{(m)}_1, \bar{u}^{(m)}_2), \\
\tilde{w}^{(m)}_2 &= D_2(\bar{u}^{(m)}_1, \bar{u}^{(m)}_2, \bar{u}^{(m)}_3), \\
\tilde{w}^{(m)}_3 &= D_3(\bar{u}^{(m)}_2, \bar{u}^{(m)}_3)
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\tilde{w}^{(m)}_1 &= D_1(\bar{u}^{(m)}_1, \bar{u}^{(m)}_2), \\
\tilde{w}^{(m)}_2 &= D_2(\bar{u}^{(m)}_1, \bar{u}^{(m)}_2, \bar{u}^{(m)}_3), \\
\tilde{w}^{(m)}_3 &= D_3(\bar{u}^{(m)}_2, \bar{u}^{(m)}_3), \\
\tilde{w}^{(m)}_1 &= D_1(\bar{u}^{(m)}_1, \bar{u}^{(m)}_2), \\
\tilde{w}^{(m)}_2 &= D_2(\bar{u}^{(m)}_1, \bar{u}^{(m)}_2, \bar{u}^{(m)}_3), \\
\tilde{w}^{(m)}_3 &= D_3(\bar{u}^{(m)}_2, \bar{u}^{(m)}_3).
\end{align*}
\]

By the relation in (2.12), let \( m \to \infty \) and using the standard regularity argument for elliptic boundary problems show that \( (\bar{u}_1, \bar{u}_2, \bar{u}_3) \) and \( (u_1, u_2, u_3) \) satisfy the equations

\[
\begin{align*}
-\Delta[D_1(\bar{u}_1, \bar{u}_2)] + k_1 D_1(\bar{u}_1, \bar{u}_2) &= F_1(\bar{u}_1, \bar{u}_2), & x \in \Omega, \\
-\Delta[D_2(u_1, u_2, u_3)] + k_2 D_2(u_1, u_2, u_3) &= F_2(u_1, u_2, u_3), & x \in \Omega, \\
-\Delta[D_3(\bar{u}_2, \bar{u}_3)] + k_3 D_3(\bar{u}_2, \bar{u}_3) &= F_3(\bar{u}_2, \bar{u}_3), & x \in \Omega, \\
-\Delta[D_1(u_1, \bar{u}_2)] + k_1 D_1(u_1, \bar{u}_2) &= F_1(u_1, \bar{u}_2), & x \in \Omega, \\
-\Delta[D_2(\bar{u}_1, u_2, u_3)] + k_2 D_2(\bar{u}_1, u_2, u_3) &= F_2(\bar{u}_1, u_2, u_3), & x \in \Omega, \\
-\Delta[D_3(u_2, u_3)] + k_3 D_3(u_2, u_3) &= F_3(u_2, u_3), & x \in \Omega, \\
\bar{u}_i(x) = u_i(x) &= 0, & i = 1, 2, 3, \ x \in \partial \Omega.
\end{align*}
\]

By virtue of the monotonicity of the functions \( F_i(u_1, u_2, u_3) \) and \( D_i(u_1, u_2, u_3) \) \( i = 1, 2, 3 \), the functions \( F_i, D_i \) possess the following property

\[
\begin{align*}
F_1(\bar{u}_1, \bar{u}_2) &= f_1(\bar{u}_1, \bar{u}_2) + k_1 D_1(\bar{u}_1, \bar{u}_2), & x \in \Omega, \\
F_2(u_1, u_2, u_3) &= f_2(u_1, u_2, u_3) + k_2 D_2(u_1, u_2, u_3), & x \in \Omega, \\
F_3(\bar{u}_2, \bar{u}_3) &= f_3(\bar{u}_2, \bar{u}_3) + k_3 D_3(\bar{u}_2, \bar{u}_3), & x \in \Omega, \\
F_1(u_1, \bar{u}_2) &= f_1(u_1, \bar{u}_2) + k_1 D_1(u_1, \bar{u}_2), & x \in \Omega, \\
F_2(\bar{u}_1, u_2, u_3) &= f_2(\bar{u}_1, u_2, u_3) + k_2 D_2(\bar{u}_1, u_2, u_3), & x \in \Omega, \\
F_3(u_2, u_3) &= f_3(u_2, u_3) + k_3 D_3(u_2, u_3), & x \in \Omega.
\end{align*}
\]
Therefore
\[
\begin{align*}
-\Delta [D_1(\bar{u}_1, u_2)] &= f_1(\bar{u}_1, u_2), \quad x \in \Omega, \\
-\Delta [D_2(u_1, \bar{u}_2, \bar{u}_3)] &= f_2(u_1, \bar{u}_2, \bar{u}_3), \quad x \in \Omega, \\
-\Delta [D_3(\bar{u}_2, \bar{u}_3)] &= f_3(\bar{u}_2, \bar{u}_3), \quad x \in \Omega, \\
-\Delta [D_1(u_1, \bar{u}_2)] &= f_1(u_1, \bar{u}_2), \quad x \in \Omega, \\
-\Delta [D_2(\bar{u}_1, u_2, u_3)] &= f_2(\bar{u}_1, u_2, u_3), \quad x \in \Omega, \\
-\Delta [D_3(u_2, u_3)] &= f_3(u_2, u_3), \quad x \in \Omega, \\
\bar{u}_i(x) &= u_i(x) = 0, \quad i = 1, 2, 3, \ x \in \partial \Omega.
\end{align*}
\] (2.15)

Then \((u_1, \bar{u}_2, \bar{u}_3)\) and \((\bar{u}_1, u_2, u_3)\) are true solutions of (1.1).

If \(\bar{u}_1 = u_1\) or \(\bar{u}_2 = u_2\) or \(\bar{u}_3 = u_3\), then \((\bar{u}_1, u_2, u_3) = (u_1, u_2, u_3)\) \((\equiv (u^*_1, u^*_2, u^*_3))\) and \((u^*_1, u^*_2, u^*_3)\) is the unique solution of (1.1). To see this, let us consider the case \(\bar{u}_1 = u_1 \equiv u^*_1\). By a subtraction of the first equation from the fourth equation in (2.15) and
\[D_1(\bar{u}_1, u_2) - D_1(u_1, \bar{u}_2) = -\alpha_1 u^*_1(\bar{u}_2 - u_2),\]
we obtain
\[\Delta [\alpha_1 u^*_1(\bar{u}_2 - u_2)] = -u^*_1 b_{12}(u_2 - \bar{u}_2), \quad \text{in } \Omega.\]

In view of \(u^*_1 > 0\), \(\alpha_1 > 0\), \(b_{12} > 0\), and \(\bar{u}_2 - u_2 = 0\) on \(\partial \Omega\), the above equation yields \(\bar{u}_2 = u_2\). We can take use of the similar method to obtain \(\bar{u}_3 = u_3\). This shows that \((\bar{u}_1, \bar{u}_2, \bar{u}_3) = (u_1, u_2, u_3)\). Then \((u^*_1, u^*_2, u^*_3)\) is the unique solution.

To summarize the above conclusions we have the following theorem:

**Theorem 2.2.** Under the condition (2.10), the sequences \(\{\bar{u}^{(m)}_1, \bar{u}^{(m)}_2, u^{(m)}_3\}, \{u^{(m)}_1, u^{(m)}_2, u^{(m)}_3\}\) obtained from (2.11) with \((\bar{u}^{(0)}_1, \bar{u}^{(0)}_2, u^{(0)}_3) = (M_1, M_2, M_3), (u^{(0)}_1, u^{(0)}_2, u^{(0)}_3) = (\delta_1, \delta_2, \delta_3)\) and \(k_1 = \frac{b_{11}}{\alpha_1}, k_2 = \frac{b_{22}}{\alpha_2}, k_3 = \frac{b_{33}}{\alpha_3}\), converge monotonically to some limits \((\bar{u}_1, \bar{u}_2, \bar{u}_3), (u_1, u_2, u_3)\) and \((u_1, \bar{u}_2, \bar{u}_3), (\bar{u}_1, u_2, u_3)\) are true solutions of (1.1); if either \(u_1 = \bar{u}_1\) or \(u_2 = \bar{u}_2\) or \(u_3 = \bar{u}_3\), then \((\bar{u}_1, \bar{u}_2, \bar{u}_3) = (u_1, u_2, u_3)\) \((\equiv (u^*_1, u^*_2, u^*_3))\) and \((u^*_1, u^*_2, u^*_3)\) is the unique solution of problem (1.1) in \(S\).

3. Existence of periodic solution

In this section, we study the periodic solution of the problem (1.2), (1.3) and we first consider the periodic eigenvalue problem
\[
\begin{align*}
\partial \phi/\partial t - L\phi - a^*\phi &= \lambda \phi, \quad (x, t) \in \Omega \times (0, \infty), \\
B\phi &= 0, \quad (x, t) \in \partial \Omega \times (0, \infty), \\
\phi(x, 0) &= \phi(x, T), \quad x \in \Omega,
\end{align*}
\] (3.1)

where
\[
\begin{align*}
L &= \sum_{j,k=1}^n a_{jk}(x, t) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j(x, t) \frac{\partial}{\partial x_j}, \\
B &= \alpha(x, t) \frac{\partial}{\partial \nu} + \beta(x, t).
\end{align*}
\]

It follows from [16] that for any T-periodic function \(a \equiv a(x, t)\) the principle eigenvalue of (3.1) denoted by \(\lambda(a)\), is real and its corresponding eigenfunction \(\phi \equiv \phi(x, t)\) may be chosen positive in \(\Omega \times (0, \infty)\).

For the convenience, we let \(D = \Omega \times [0, \infty), \ D = \Omega \times [0, \infty), \ \Gamma = \partial \Omega \times [0, \infty), \) and for each \(i = 1, 2, 3\) we set \(D^{(i)}_0 = \Omega \times [-\tau_i, 0], \ Q^{(i)} = \Omega \times [-\tau_i, \infty), \ D_0 = D^{(1)}_0 \times D^{(2)}_0 \times D^{(3)}_0, \ Q = Q^{(1)} \times Q^{(2)} \times Q^{(3)}\).

To show the existence problem we make a transformation by letting \(w_1 = M - u_1\) for a sufficiently large constant \(M > 0\). Then the problem (1.2) with (1.3), (1.4) become the following problem:
\[
\begin{align*}
\partial_w - d_1(x,t) \Delta w & = -(M - w)(a_1(x,t) - b_{11}(M - w) + b_{12}(x,t, t - \tau_2)), \\
(x,t) & \in \Omega \times (0, \infty), \\
\partial u_2 - d_2(x,t) \Delta u_2 & = u_2[a_2(x,t) - b_{22}u_2 - b_{21}(M - w)(x,t, t - \tau_1) + b_{23}u_3(x,t, t - \tau_3)], \\
(x,t) & \in \Omega \times (0, \infty), \\
\partial u_3 - d_3(x,t) \Delta u_3 & = u_3[a_3(x,t) + b_{32}u_2(x,t, t - \tau_2) - b_{33}u_3], \\
(x,t) & \in \Omega \times (0, \infty), \\
\partial u_i - d_i \Delta u_i & = \frac{\partial u_i}{\partial \eta} = \frac{\partial u_i}{\partial \eta} = 0, \\
(x,t) & \in \partial \Omega \times (0, \infty),
\end{align*}
\]

with the periodic condition
\[
w_1(x,t) = w_1(x, t + T), \quad u_i(x,t) = u_i(x, t + T), \quad i = 2, 3, \quad (x,t) \in \Omega \times [-\tau_i, 0],
\]
and under the initial condition
\[
w_1(x,t) = M - \eta_1(x,t), \quad u_i(x,t) = \eta_i(x, t), \quad i = 2, 3, \quad (x,t) \in \Omega \times [-\tau_i, 0].
\]

Denote the reaction functions of (3.2) by $F_1$, $F_2$, $F_3$. It is easily to see that $F_i$ is quasimonotone nondecreasing in $S \times S$, where $S = S_\tau = [0, M] \times R^+ \times R^+$. Next we give the definition of ordered upper and lower solutions of (3.2):

**Definition 3.1.** Let $u \in S$, $v \in S_\tau$, a pair of 3-vector functions $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$, $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ in $C^2(\Omega) \cap C(\overline{\Omega})$ are called ordered upper and lower solutions of (3.2) (3.3), if $(\hat{w}_1, \hat{u}_2, \hat{u}_3) \geq (\hat{w}_1, \hat{u}_2, \hat{u}_3)$ and if their components satisfy the relation

\[
\begin{align*}
\partial \hat{w}_1 - d_1 \Delta \hat{w}_1 & \geq F_1(x,t, \hat{u}, \hat{v}), \quad (x,t) \in D, \\
\partial \hat{u}_2 - d_2 \Delta \hat{u}_2 & \geq F_2(x,t, \hat{u}, \hat{v}), \quad (x,t) \in D, \\
\partial \hat{u}_3 - d_3 \Delta \hat{u}_3 & \geq F_3(x,t, \hat{u}, \hat{v}), \quad (x,t) \in D, \\
\partial \hat{w}_1 - d_1 \Delta \hat{w}_1 & \leq F_1(x,t, \hat{u}, \hat{v}), \quad (x,t) \in D, \\
\partial \hat{u}_2 - d_2 \Delta \hat{u}_2 & \leq F_2(x,t, \hat{u}, \hat{v}), \quad (x,t) \in D, \\
\partial \hat{u}_3 - d_3 \Delta \hat{u}_3 & \leq F_3(x,t, \hat{u}, \hat{v}), \quad (x,t) \in D, \\
\partial \hat{u}_i - d_i \Delta \hat{u}_i & \geq 0 \geq \frac{\partial \hat{u}_i}{\partial \eta}, \quad \partial \hat{u}_i - d_i \Delta \hat{u}_i \geq 0 \geq \frac{\partial \hat{u}_i}{\partial \eta}, \quad i = 2, 3, \quad (x,t) \in \Gamma, \\
\hat{w}_1(x,t) & \geq \hat{w}_1(x, t + T), \quad \hat{w}_1(x,t) \leq \hat{w}_1(x, t + T), \quad (x,t) \in D_0^{(1)}(x), \\
\hat{u}_i(x,t) & \geq \hat{u}_i(x, t + T), \quad \hat{u}_i(x,t) \leq \hat{u}_i(x, t + T), \quad i = 2, 3, \quad (x,t) \in D_0^{(i)}(x),
\end{align*}
\]

where

\[
S = \{ u \in C(\overline{\Omega}); \hat{u} \leq u \leq \hat{v}, \text{ on } \overline{D} \}, \quad S_\tau = \{ v \in C(Q); \hat{v} \leq v \leq \hat{v}, \text{ on } \overline{D} \},
\]

and $u = (u_1, u_2, u_3)$, $v = (w_1, u_2, u_3)$, $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$, $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ and $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$. Let $\hat{u}_i(x,t)$ be the principle eigenvalue and its corresponding positive eigenfunction of problem (3.1) with $L = -d_1(x,t)\Delta$, $B = \frac{d}{\partial \eta}$ and $a = a(x,t) (i = 1, 2, 3)$.

Next we seek a pair of ordered upper and lower solution of problem (3.2), (3.3) in the form $(\hat{w}_1, \hat{u}_2, \hat{u}_3) = (M - \delta_1 \phi_1, \rho_2, \rho_3)$, $(\hat{w}_1, \hat{u}_2, \hat{u}_3) = (M - \rho_1, \delta_2 \phi_2, \phi_3)$ where $\rho_1, \delta_1$ are some positive constants with $\delta_1$ sufficiently small and $\rho_1 < M$. Then it is easy to verify that $(\hat{w}_1, \hat{u}_2, \hat{u}_3)$ satisfy all the requirement of upper and lower solutions if
In view of \((3.1)\) the above inequalities are satisfied by some sufficiently small \(\delta_1, \delta_2\) if
\[
\begin{align*}
\begin{cases}
a_1 - \rho_1 b_{11} & \leq 0, \\
a_2 - b_{22} \rho_2 b_{12} + b_{22} \rho_3 & \leq 0, \\
\lambda_1 (a_1) & < -b_{12} \rho_2, \\
\lambda_2 (a_2) & < -b_{21} \rho_1, \\
\lambda_3 (a_3) & < 0.
\end{cases}
\end{align*}
\]
Assuming that
\[
b_{22} b_{32} < b_{22} b_{33}
\]  
and setting
\[
M_1 = \max_D \left[ \frac{a_1(x, t)}{b_{11}(x, t)} \right],
\]
\[
M_2 = \max_D \left[ \frac{a_3(x, t) b_{23}(x, t) + a_2(x, t) b_{33}(x, t)}{b_{22}(x, t) b_{33}(x, t) - b_{23}(x, t) b_{32}(x, t)} \right],
\]
\[
M_3 = \max_D \left[ \frac{a_3(x, t) b_{22}(x, t) + a_2(x, t) b_{32}(x, t)}{b_{22}(x, t) b_{33}(x, t) - b_{23}(x, t) b_{32}(x, t)} \right].
\]
Then the requirements in \((3.7)\) are fulfilled by some \(\rho_i > M_i\) \((i = 1, 2, 3)\) if \((3.8)\) holds and
\[
-\lambda_1 (a_1) > b_{12} M_2, \quad -\lambda_2 (a_2) > b_{21} M_1, \quad -\lambda_3 (a_3) > 0.
\]
From Theorem A of \([20]\) we have that under conditions \((3.8), (3.12)\), the problem \((3.2), (3.3)\) has a maximal \(T\)-periodic solution \((\bar{w}_1, \bar{u}_2, \bar{u}_3)\) and a minimal \(T\)-periodic solution \((\bar{w}_1, \bar{u}_2, \bar{u}_3)\) such that
\[
(M - \rho_1, \delta_2 \rho_2, \delta_3 \rho_3) \leq (\bar{w}_1, \bar{u}_2, \bar{u}_3) \leq (M, \bar{w}_1, \bar{u}_3) \leq (M - \delta_1 \rho_1, \rho_2, \rho_3).
\]
Moreover, by Theorem 3.1 of \([20]\) the solution \(u = (w_1, u_2, u_3)\) of the initial boundary problem \((3.2), (3.4)\) possesses the following convergence:
\[
\lim_{m \to \infty} u(x, t + mT; \eta) = \begin{cases} u(x, t) & \text{if } \hat{u} \leq \eta \leq \bar{u} \text{ in } D_0, \\
\bar{u}(x, t) & \text{if } \bar{u} \leq \eta \leq \bar{u} \text{ in } D_0 \end{cases}
\]
and
\[
u(x, t) \leq u(x, t + mT; \eta) \leq \bar{u}(x, t) \quad \text{on } D \text{ as } m \to \infty.
\]
Now by the transformation \(u_1 = M - w_1\), the pair \((u_1, \bar{u}_2, \bar{u}_3)\) and \((\bar{u}_1, u_2, u_3)\) where \(u_1 = M - \bar{w}_1, \bar{u}_1 = M - \bar{w}_1\) are positive \(T\)-periodic solutions of the problem \((1.2), (1.3)\) and satisfy the relation \(\delta_i \phi_i \leq u_i \leq \bar{u}_i \leq \rho_i\) on \(D\).

Furthermore for any \(\delta_i \phi_i \leq \eta_i \leq \rho_i\) in \(D_0^{(i)}, i = 1, 2, 3\), the solution of the initial boundary problem \((1.2), (1.4)\) is given by \((u_1, u_2, u_3) = (M - w_1, u_2, u_3)\) and satisfies the relation \(\delta_i \phi_i \leq u_i \leq \rho_i, i = 1, 2, 3\) on \(D\).

According to \((3.13), (3.14)\) and Theorem 3.1 of \([20]\), the solution of \((1.2), (1.4)\) with the initial condition \(\delta_i \phi_i \leq \eta_i \leq \rho_i\) in \(D_0^{(i)}, i = 1, 2, 3\), possesses the convergence
\[
\lim_{m \to \infty} (u_1, u_2, u_3)(x, t + mT; \eta) = \begin{cases} (\bar{u}_1, \bar{u}_2, \bar{u}_3), & \text{if } \eta_1 \geq \bar{u}_1, 0 \leq \eta_i \leq u_i, i = 2, 3, \\
(u_1, \bar{u}_2, \bar{u}_3), & \text{if } 0 \leq \eta_i \leq u_i, \eta_1 \geq \bar{u}_1, i = 2, 3,
\end{cases}
\]
and
\[
(u_1, u_2, u_3) \leq (u_1, u_2, u_3)(x, t + mT; \eta) \leq (\bar{u}_1, \bar{u}_2, \bar{u}_3) \quad \text{on } D \text{ as } m \to \infty.
\]
To summarize the above conclusions we have the following theorem.
Theorem 3.1. Let \((u_1(x,t;\eta_1), u_2(x,t;\eta_2), u_3(x,t;\eta_3))\) be the solution of (1.2), (1.4) for \((\eta_1, \eta_2, \eta_3)\) with \(0 < \eta_i \leq \rho_i, i = 1, 2, 3\) and let conditions (3.8), (3.12) be satisfied. Then we have

(i) problem (1.2), (1.3) has positive T-periodic solutions \((\bar{u}_1, \bar{u}_2, \bar{u}_3)\) such that \(u_j \leq \bar{u}_i, i = 1, 2, 3\) on \(\overline{D}\);

(ii) the solution \((u_1(x,t;\eta_1), u_2(x,t;\eta_2), u_3(x,t;\eta_3))\) of (1.2), (1.4) possesses the convergence properties (3.15) and (3.16);

(iii) if \((\bar{u}_1, \bar{u}_2, \bar{u}_3) = (u_1^*, u_2^*, u_3^*)\), then

\[
\lim_{m \to \infty} \left( u_1(x,t+mT;\eta_1), u_2(x,t+mT;\eta_2), u_3(x,t+mT;\eta_3) \right) = \left( u_1^*(x,t), u_2^*(x,t), u_3^*(x,t) \right),
\]

\(t > 0, \; x \in \Omega\).

References