Stability of Volterra Integro-differential Equations with Impulsive Effect

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1. INTRODUCTION

The stability analysis of ordinary differential equations with impulsive effect has been the subject of many investigations [1–3, 5] in recent years and various interesting results have been reported. However, not much has been developed in the direction of integro-differential equations with impulsive effect except a few [4, 6] in which impulsive integral inequalities are used. The purpose of this paper is to investigate sufficient conditions for uniform stability and uniform asymptotic stability of integro-differential equations with impulsive effect by employing a class of piecewise continuous Liapunov functional without decrescent property. It is also proved that every solution of an integro-differential system meets any given surface at most once and thus there exist no pulse phenomena in the system.

2. PRELIMINARIES AND BASIC RESULTS

Let the hypersurfaces $\sigma_k$ be defined by the equations

$\sigma_k: t = \tau_k(x), \quad 0 < \tau_1(x) < \tau_2(x) < \cdots, \quad \sigma_k: R^n \to (0, \infty)$ for each $k = 1, 2, \ldots$
where \( \tau_k \to \infty \) as \( k \to \infty \). \( PC^+ [A, B] \) denotes the class of piecewise continuous functions from \( A \to B \), where \( A \) and \( B \) are finite dimensional vector spaces, with discontinuities of the first kind at \( t = \tau_k(x), \ k = 1, 2, ... \), and left continuous at \( t = \sigma_k(x) \).

Let \( \tau_0(x) \equiv 0 \) for \( x \in \mathbb{R}^n \) and

\[
G_k = \{(t, x) \in J \times S(\rho), \quad \tau_{k-1}(x) < t < \tau_k(x)\}, \ k = 1, 2, ...
\]

where \( J: [t_0, \infty), \ t_0 \geq 0 \).

\[
S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho, \ \rho > 0\} \quad \text{and} \quad \overline{S}(\rho) = \{x \in \mathbb{R}^n : \|x\| \leq \rho, \ \rho > 0\}.
\]

The function \( V: J \times \mathbb{R}^n \to \mathbb{R}_+ \) belongs to class \( V_0 \) if

(i) the function \( V \) is continuous on each of the set \( G_k \) and \( V(t, 0) \equiv 0 \),

(ii) for each \( k = 1, 2, ... \) and \( (t_0, x_0) \in \sigma_k \) there exist finite limits

\[
V(t_0 - 0, x_0) = \lim_{(t, x) \to (t_0, x_0)} V(t, x)
\]

and

\[
V(t_0 + 0, x_0) = \lim_{(t, x) \to (t_0, x_0)} V(t, x),
\]

with \( V(t_0 - 0, x_0) = V(t_0, x_0) \) satisfied.

Also if \( (t_0, x_0) \in G_k \), then \( V(t_0 + 0, x_0) = V(t_0, x_0) \).

Let \( V \in V_0 \). For \( (t, x) \in \bigcup_{k=1}^{\infty} G_k \), \( D^+ V \) is defined as

\[
D^+ V(t, x(t)) = \lim_{h \to 0^+} \sup_{h} \left[ \frac{V(t + h, x(t + h)) - V(t, x(t))}{h} \right].
\]

Consider the integro-differential system

\[
x'(t) = A(t)x(t) + \int_{t_0}^{t} K(t, s)x(s) \, ds, \quad t \neq \tau_k(x), \quad k = 1, 2, ...
\]

\[
\Delta x |_{t = \tau_k(x)} = I_k(x), \quad x(t_k^+) = x_0, \quad t \in J,
\]

where \( A \in PC^+ [J, \mathbb{R}^n], \ K \in PC^+ [J \times J, \mathbb{R}^n], \ I_k \in C[\mathbb{R}^n, \mathbb{R}^n], \ \tau_k \in C^1[\mathbb{R}^n, \mathbb{R}_+], \) and \( I_k(0) = 0 \).

In the following result we shall give sufficient conditions for the absence of beating, that is, the system (2.1) has no pulse phenomena.
THEOREM 2.1.

Let the following conditions be satisfied:

(i) $\|A(t)\| \leq \beta$ for $t \in J$;

(ii) $\|K(t, s)\| \leq M e^{-\alpha(t-s)}$ for $t_0 \leq s \leq t < \infty$;

(iii) there exists a positive number $h$ such that

$$\sup_{0 \leq s \leq 1} \left\langle \frac{\partial \tau_k}{\partial x} (x + sI_k(x)), I_k(x) \right\rangle \leq 0$$

and

$$\sup_{\|x\| < h} \left\| \frac{\partial \tau_k(x)}{\partial x} \right\| \leq N, \quad k = 1, 2, \ldots;$$

(iv) there exists a positive number $\rho \leq h$ such that

$$\left[ \frac{\beta + M}{\alpha} \right] \rho N < 1,$$

where $\alpha$, $\beta$, $M$, and $N$ are positive constants and $\langle \cdot, \cdot \rangle$ is the usual inner product of two vectors.

Then the integral curve $\{(t, x(t)) : t \in [t_0, T]\}$ of every solution of (2.1) which lies in the ball $\bar{S}(\rho)$, meets the hypersurface $\sigma_k : t = \tau_k(x)$ at most once.

Proof. Let $x(t)$ be a solution of (2.1) which exists for all $t \geq t_0$ and lies in the ball $\bar{S}(\rho)$ for $t_0 \leq t \leq T$, $t_0 > 0$, where $\rho$ being the same as in condition (iv). Let

$$F(t, x(t)) = A(t) x(t) + \int_{t_0}^t K(t, s) x(s) \, ds.$$

Then from assumption (i) and (ii) we obtain

$$\|F(t, x(t))\| \leq \|A(t) x(t)\| + \int_{t_0}^t \|K(t, s)\| \|x(s)\| \, ds$$

$$\leq \beta \rho + M \int_{t_0}^t e^{-\alpha(t-s)} \sup_{0 \leq s \leq T} \|x(s)\| \, ds$$

$$\leq \beta \rho + M \rho \int_{t_0}^t e^{-\alpha(t-s)} \, ds$$
and hence
\[
\|F(t, x(t))\| \leq \left[ \beta + \frac{M}{\alpha} \right] \rho \quad \text{for all } t_0 \leq t \leq T. \tag{2.2}
\]

Suppose that there is a solution \( x(t) \) of (2.1) which meets some surface \( \sigma_k \) more than once. Let \( t = t_j > t_0 \) be the point at which the solution \( x(t) \) of (2.1) first meets the surface \( \sigma_k : t = \tau_k(x) \) for some \( j \) and again another closest hit at \( t = t^* \) such that \( t^* - t_j > 0 \). Then we have
\[
t_j = \tau_k(x(t_j)) \quad \text{and} \quad t^* = \tau_k(x(t^*)),
\]
where \( t_0 < t_j < t^* \).

From (2.1) it follows that
\[
x(t) = x_j + I_k(x_j) + \int_{t_j}^t F(s, x(s)) \, ds, \quad t > t_j,
\]
where \( F(s, x(s)) = A(s) x(s) + \int_{t_0}^s K(s, \sigma) x(\sigma) \, d\sigma \). Therefore \( \tau_k(x(t^*)) = \tau_k(x_j + I_k(x_j) + \int_{t_j}^{t^*} F(s, x(s)) \, ds) \).

Define a function
\[
\Psi(s) = \tau_k(x_j + I_k(x_j) + sh) + \tau_k(x_j + sI_k(x_j))
\]
for \( s \in [0, 1] \) where \( h = \int_{t_j}^{t^*} F(s, x(s)) \, ds \). Then by the mean-value theorem, we have
\[
\Psi(1) - \Psi(0) = \int_0^1 \Psi'(s) \, ds
\]
and hence
\[
t^* - t_j = \int_0^1 \frac{\partial \tau_k}{\partial x} (x_j + I_k(x_j) + sh), \quad h > ds
\]
\[+ \int_0^1 \frac{\partial I_k}{\partial x} (x_j + sI_k(x_j)), \quad I_k(x_j) > ds. \tag{2.3}
\]

From the inequality (2.2) and the second part of assumption (iii) it follows by using the Cauchy–Schwartz inequality that
\[
\int_0^1 \frac{\partial \tau_k}{\partial x} (x_j + I_k(x_j) + sh), \quad h > ds
\]
\[\leq N \rho \left[ \beta + \frac{M}{\alpha} \right] (t^* - t_j). \tag{2.4}
\]
Thus from (2.3) and (2.4) we obtain

$$\left[ 1 - N_\rho \left( \beta + \frac{M}{\lambda} \right) \right] (t^* - t_j) \leq \int_0^{t^*} \frac{\partial \tau_k}{\partial x} (x_j + sI_k(x_j)), \quad I_k(x_j) > ds.$$ 

Since \((\beta + (M/\lambda)) N_\rho < 1\), in view of assumption (iii) this leads to a contradiction and hence the proof is complete.

In view of Theorem 2.1, we shall investigate in the next two sections the stability behavior of solutions of integro-differential equations with fixed moments of impulse effect (that is, the system (2.1) has no pulse phenomena). Therefore, we assume in Sections 3 and 4 that the hypersurfaces \(\sigma_k : t = \tau_k(x)\) take a simple shape of hyperplanes \(\tau_k(x) = t_k\), where \(t_k (k = 1, 2, ...)\) are fixed times such that \(0 < t_0 < t_1 < t_2 < \cdots; t_k \to \infty\) as \(k \to \infty\).

**3. Linear Systems**

Consider the linear integro-differential system with fixed moments of impulse effect

$$x'(t) = A(t)x(t) + \int_{t_0}^{t} K(t, s)x(s)ds, \quad t \neq t_k, \quad k = 1, 2, ...,$$

$$Ax = I_k(x), \quad t = t_k, \quad x(t_0) = x_0, \quad t \in J,$$

where \(A \in PC^+[J, R^n]\), \(K \in PC^+\{J \times J, R^n\}\), \(I_k \in C[R^n, R^n]\), and \(I_k(0) = 0\).

Let us also consider the linear impulsive ordinary differential system

$$x' = A(t)x, \quad t \neq t_k,$$

$$Ax = B_kx, \quad t = t_k, \quad x(t_0^+) = x_0,$$

where \(B_k (k = 1, 2, ...)\) are \(n \times n\) constant matrices such that \(\det(I + B_k) \neq 0\). \(I\) being the identity matrix.

Let \(\varphi_k(t, s)\) be a fundamental matrix solution of the linear system

$$x' = A(t)x, \quad t_{k-1} < t \leq t_k.$$

Then the solution of (3.2) can be written (see [2, p. 74]) in the form

$$x(t, t_0, x_0) = \mathcal{P}(t, t_0 + 0) x_0,$$
where

\[
\Psi(t, s) = \begin{cases} 
\varphi_k(t, s) & \text{for } t, s \in (t_{k-1}, t_k] \\
\varphi_{k+1}(t, t_k)(I + B_k) \varphi_k(t_k, s) & \text{for } t_{k-1} < s \leq t_k < t \leq t_{k+1} \\
\varphi_{k+1}(t, t_k) \left[ \prod_{j=k}^{i+1} (I + B_j) \varphi_j(t_j, t_{j-1}) \right] (I + B_i) \varphi_i(t_i, s) & \text{for } t_{i-1} < s \leq t_i < t_k < t \leq t_{k+1}.
\end{cases}
\]

Define the matrix \( G(t) \) as

\[
G(t) = \int_t^\infty \Psi^T(s, t) \Psi(s, t) \, ds
\]

where \( \Psi^T \) is the transpose of \( \Psi \). Clearly \( G(t) \) is symmetric.

**Theorem 3.1.** Assume that the following conditions hold for \( x \in S(\rho) \):

(a) \( L \|x\| \leq \langle G(t)x, x \rangle^{1/2} \leq 1/2 \hat{M} \|x\| \),

(b) \( \|G(t)x\| \leq \hat{K} \langle G(t)x, x \rangle^{1/2} \),

(c) \( -\hat{M} + \hat{\beta} \int_t^\infty \|K(u, t)\| \, du \leq 0 \), \( \hat{\beta} \geq \hat{K} \),

(d) \( \|x\| > \|x + I_k(x)\| \) \ and \( \langle G(t)x, x \rangle^{1/2} > \langle G(t)(x + I_k(x)), (x + I_k(x)) \rangle^{1/2} \),

where \( L, \hat{M}, \hat{K}, \) and \( \hat{\beta} \) are positive real numbers. Then the zero solution of (3.1) is uniformly stable.

**Proof.** Let \( W(t, x) = \langle G(t)x, x \rangle^{1/2} \). Then we have

\[
W'(t, x) = \frac{\langle G'(t)x, x \rangle}{2\langle G(t)x, x \rangle^{1/2}} + \frac{\langle 2G(t)x', x \rangle}{2\langle G(t)x, x \rangle^{1/2}} \tag{3.3}
\]

for

\[
t \neq t_k, \quad (t, x) \in \bigcup_{k=1}^\infty G_k.
\]

From the fact that

\[
\frac{\partial \Psi}{\partial t}(s, t) = -\Psi(s, t) A(t) \quad \text{and} \quad \frac{\partial \Psi^T}{\partial t}(s, t) = -A^T(t) \Psi^T(s, t) \quad \text{for } t \neq t_k, (t, x) \in \bigcup_{k=1}^\infty G_k
\]

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it follows that
\[
G'(t) = -I + \int_t^\infty \left[ \frac{\partial \Psi_T}{\partial t}(s, t) \Psi(s, t) + \Psi_T(s, t) \frac{\partial \Psi}{\partial t}(s, t) \right] ds
\]
\[
= -I - A^T(t) G(t) - G(t) A(t), \quad t \neq t_k, \quad (t, x) \in \bigcup_{k=1}^\infty G_k.
\]
Hence (3.3) gives
\[
W_{(3.1)}'(t, x) = \frac{-\langle x, x \rangle}{2\langle G(t)x, x \rangle^{1/2}} + \frac{\langle G(t)x, \int_t^\infty K(t, s)x(s) ds \rangle}{\langle G(t)x, x \rangle^{1/2}}
\]
for \( t \neq t_k, \quad (t, x) \in \bigcup_{k=1}^\infty G_k. \) (3.4)

Now define a Liapunov function \( V \) such that \( V \in V_0 \) and for \( (t, x) \in \bigcup_{k=1}^\infty G_k \)
\[
V(t, x) = W(t, x) + \hat{\beta} \int_t^\infty \int_t^\infty \| K(u, s) \| du \| x(s) \| ds.
\]
From (3.4) and assumptions (a) and (b), we obtain
\[
V_{(3.1)}'(t, x) \leq -\hat{M} \| x \| + \hat{K} \int_t^\infty \| K(t, s) \| \| x(s) \| ds
\]
\[
+ \hat{\beta} \int_t^\infty \| K(u, t) \| du \| x(t) \|
\]
\[
- \hat{\beta} \int_t^\infty \| K(t, s) \| \| x(s) \| ds
\]
for \( t \neq t_k \) and \( (t, x) \in \bigcup_{k=1}^\infty G_k. \)
Hence in view of assumption (c) it follows that
\[
V_{(3.1)}'(t, x) \leq 0 \quad \text{for} \quad t \neq t_k, \quad (t, x) \in \bigcup_{k=1}^\infty G_k. \quad (3.5)
\]
Further from assumption (d) it is clear that
\[
V(t_k^+, x_k + I_k(x_k)) \leq V(t_k, x_k), \quad (3.6)
\]
where $x_k = x(t_k)$. From assumption (a), (3.5), (3.6), and the definition of $V$, we obtain

$$L \|x(t)\| \leq V(t, x(t)) \leq V(t_0, x_0) \leq \frac{1}{2M} \|x_0\| \quad \text{for all} \quad t \geq t_0,$$

where $x(t)$ is any solution of (3.1).

This implies that the zero solution of (3.1) is uniformly stable and hence the proof of the theorem is complete.

**Remark 3.1.** Theorem 3.1 indicates that the decrescent property on the Liapunov function $V$ can be relaxed in proving the uniform stability of the zero solution of (3.1).

**Theorem 3.2.** Assume that all the conditions of Theorem 3.1 hold except that condition (c) is replaced by

(C) $\dot{\gamma} \leq \tilde{M} - \tilde{\beta} \int_t^\infty \|K(u, t)\| du$ for some $\dot{\gamma} > 0$ and $\tilde{\beta} > \tilde{K}.$

Then the zero solution of (3.1) is uniformly asymptotically stable.

**Proof.** By Theorem 3.1, the zero solution of (3.1) is uniformly stable. Following the proof of Theorem 3.1 we obtain

$$V'(t, x) \leq -\dot{\gamma} \|x\| \quad \text{for} \quad t \neq t_k, \quad (t, x) \in \bigcup_{k=1}^{\infty} G_k \quad (3.7)$$

and

$$V(t_k^+, x_k + I_k(x_k)) \leq V(t_k, x_k), \quad (3.8)$$

where $x_k = x(t_k)$.

Let $\delta > 0$ be the number corresponding to $\varepsilon (0 < \varepsilon < \rho)$ in the definition of uniform stability. Let $\delta_0 = \delta(\rho)$ and choose $T = T(\varepsilon) > 0$ as

$$T(\varepsilon) = \frac{1}{2\dot{\gamma} \delta} \|\delta_0\|.$$

We now claim that

$$\|x(t^*, t_0, x_0)\| < \delta \quad \text{for some} \quad t^* \in [t_0, t_0 + T], \quad T > 0.$$
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Suppose not. Then we have

$$\|x(t, t_0, x_0)\| \geq \delta \quad \text{for all} \quad t \in [t_0, t_0 + T]. \quad (3.9)$$

From assumption (a), (3.7), (3.8), and (3.9) it follows that

$$0 < L \delta \leq L \|x(t_0, t_0, x_0)\| \leq V(t, x(t))$$

$$\leq V(t_0, x_0) - \gamma \int_{t_0}^{t} \|x(s)\| \, ds.$$

For \( t = t_0 + T \) we get

$$0 < L \delta \leq L \|x(t_0 + T, t_0, x_0)\| \leq V(t_0 + T, x(t_0 + T))$$

$$\leq V(t_0, x_0) - \gamma \int_{t_0}^{t_0 + T} \|x(s)\| \, ds.$$  

Again using (3.9) for \( t = t_0 + T \), we obtain

$$0 < L \delta \leq L \|x(t_0 + T, t_0, x_0)\| \leq V(t_0 + T, x(t_0 + T))$$

$$\leq V(t_0, x_0) - \gamma \delta T$$

$$\leq \frac{1}{2M} \delta_0 - \gamma \delta T$$

$$= \frac{1}{2M} \delta_0 - \gamma \delta - \frac{1}{2M \gamma \delta} \delta_0$$

$$= 0.$$  

This is a contradiction to (3.9). Hence there exists a \( t^* \in [t_0, t_0 + T] \) such that

$$\|x(t^*, t_0, x_0)\| < \delta.$$  

Therefore by uniform stability it follows that

$$\|x(t, t_0, x_0)\| < \varepsilon \quad \text{for all} \quad t > t^*.$$  

In particular for \( t^* = t_0 + T \), we have

$$\|x(t, t_0, x_0)\| < \varepsilon \quad \text{for all} \quad t \geq t_0 + T$$

and thus the proof of the theorem is complete.
4. NONLINEAR SYSTEMS

Consider the nonlinear integro-differential system with fixed moments of impulsive effect

\[ x' = f(t, x) + \int_{t_0}^{t} g(t, s, x(s)) \, ds, \quad t \neq t_k, \quad k = 1, 2, \ldots \]

\[ \Delta x = I_k(x), \quad t = t_k, \quad x(t_0^-) = x_0, \quad t \in J \]

in which \( f \in PC^+[J \times \mathbb{R}^n, \mathbb{R}^n] \), \( g \in PC^+[J \times J \times \mathbb{R}^n, \mathbb{R}^n] \), \( g(t, s, 0) \equiv 0 \), \( I_k \in C^1[\mathbb{R}^n, \mathbb{R}^n] \), \( f_x = \partial f/\partial x \) exist, and \( f_x \in PC^+[J \times \mathbb{R}^n, \mathbb{R}^n] \) with \( f(t, 0) = 0 \) and \( I_k(0) = 0 \).

Consider the impulsive ordinary differential system

\[ x' = f(t, x), \quad t \neq t_k, \quad k = 1, 2, \ldots \]

\[ \Delta x = I_k(x), \quad t = t_k, \quad x(t_0) = x_0. \]

Let \( \Psi(t, s) \) be a fundamental matrix solution of the linear system

\[ y' = f_x(t, 0) y, \quad t \neq t_k \]

\[ \Delta y = B_k y, \quad t = t_k, \]

where

\[ B_k = \frac{\partial I_k}{\partial x} \bigg|_{x=0} \quad \text{and} \quad f(t, x) = f_x(t, 0)x + F(t, x). \]

Define the matrix \( G(t) \) as

\[ G(t) = \int_{t_0}^{t} \Psi^T(s, t) \Psi(s, t) \, ds, \]

where \( \Psi^T \) is the transpose of \( \Psi \), and a scalar function

\[ W(t, x) = \langle G(t)x, x \rangle^{1/2}. \]

**Theorem 4.1.** Assume that the following conditions hold for \( x \in S(\rho) \):

(i) there exists a positive number \( M \) such that

\[ |W(t, x) - W(t, y)| \leq M \| x - y \|, \]

(ii) \( \alpha \| x \| \leq \langle G(t)x, x \rangle^{1/2} \leq L \| x \| \), \( \alpha, L > 0 \),

(iii) \( \| G(t)F(t, x) \| \leq \| x \| / A, \quad A > 0 \),
(iv) \( \|g(u, t, x(t))\| \leq R(u, t) \|x(t)\| \), where
\[
\sup_{t \geq t_0} \int_{t}^{\infty} R(u, t) \, du < \infty,
\]

(v) \( \beta - K \int_{t}^{\infty} R(u, t) \, du \geq 0 \), where \( \beta = (Ax - 2L/2ALx) > 0 \) and \( K \geq M \),

(vi) \( \|x\| > \|x + I_k(x)\| \) and
\[
\langle G(t)x, x \rangle^{1/2} > \langle G(t)(x + I_k(x)), x + I_k(x) \rangle^{1/2}.
\]

Then the zero solution of (4.1) is uniformly stable.

Proof. From the definition \( W(t, x) = \langle G(t)x, x \rangle^{1/2} \) and \( f(t, x) = f_s(t, 0)x + F(t, x) \) it follows for \( t \neq t_k \) and \( (t, x) \in \bigcup_{k=1}^{\infty} G_k \) that
\[
W'(t, x) = \frac{\langle [G'(t) + 2G(t)f_s(t, 0)]x, x \rangle}{2\langle G(t)x, x \rangle^{1/2}}
+ \frac{\langle 2G(t)F(t, x), x \rangle}{2\langle G(t)x, x \rangle^{1/2}}.
\]

From the fact
\[
\frac{\partial \Psi}{\partial t} (s, t) = -\Psi(s, t) f_s(t, 0) \quad \text{and} \quad \frac{\partial \Psi^T}{\partial t} (s, t) = -f_s^T(t, 0) \Psi^T(s, t)
\]
for \( t \neq t_k \), \( (t, x) \in \bigcup_{k=1}^{\infty} G_k \), we have
\[
G'(t) = -I + \int_{t}^{\infty} \left[ \frac{\partial \Psi^T}{\partial t} (s, t) \Psi(s, t) + \Psi^T(s, t) \frac{\partial \Psi}{\partial t} (s, t) \right] ds
= -I - f_s^T(t, 0) G(t) - G(t) f_s(t, 0).
\]

Therefore
\[
\langle [G'(t) + 2G(t)f_s(t, 0)]x, x \rangle = -\langle x, x \rangle
\]
and hence
\[
W'(t, x) = \frac{-\langle x, x \rangle}{2\langle G(t)x, x \rangle^{1/2}} + \frac{2\langle G(t)F(t, x), x \rangle}{2\langle G(t)x, x \rangle^{1/2}}.
\]

This, together with assumptions (ii) and (iii), gives
\[
W'(t, x) \leq \frac{-\|x\|}{2L} + \frac{\|x\|}{Ax}
= -\beta \|x\| \quad \text{for} \quad t \neq t_k \quad \text{and} \quad (t, x) \in \bigcup_{k=1}^{\infty} G_k. \quad (4.3)
\]
We now define a Liapunov function $V$ such that $V \in V_0$ and for $(t, x) \in \bigcup_{k=1}^{\infty} G_k$

$$V(t, x) = W(t, x) + K \int_{t_0}^{t} \int_{t}^{\infty} \| g(u, s, x(s)) \| \, du \, ds.$$ 

Thus in view of assumption (i) we obtain

$$V'_{(4.1)}(t, x) = W'_{(4.2)}(t, x) + M \int_{t_0}^{t} \| g(t, s, x(s)) \| \, ds$$

$$+ K \int_{t_0}^{\infty} \| g(u, t, x(t)) \| \, du$$

$$- K \int_{t_0}^{t} \| g(t, s, x(s)) \| \, ds.$$ 

Using hypothesis (iv) and (4.3), we get

$$V'_{(4.1)}(t, x) \leq - \left( \beta - K \int_{t}^{\infty} R(u, t) \, du \right) \| x \|$$

$$+ (M - K) \int_{t_0}^{t} \| g(t, s, x(s)) \| \, ds$$ 

for all $t \neq t_k$, $(t, x) \in \bigcup_{k=1}^{\infty} G_k$. \hspace{1cm} (4.4)

From assumption (v) it follows that

$$V'_{(4.1)}(t, x) \leq 0 \text{ for } t \neq t_k, \quad (t, x) \in \bigcup_{k=1}^{\infty} G_k.$$

Further from hypothesis (vi), we obtain

$$V(t_k^+, x_k + I_k(x_k)) \leq V(t_k, x_k),$$

where $x_k = x(t_k)$.

Therefore for all $t \geq t_0$, we have

$$\alpha \| x(t) \| \leq V(t, x(t)) \leq V(t_0, x_0) = W(t_0, x_0) \leq L \| x_0 \|,$$

which in turn implies that the zero solution of (4.1) is uniformly stable and hence the proof is complete.
THEOREM 4.2. Suppose all the conditions of Theorem 4.1 are satisfied except that condition (v) is replaced by

\[(\bar{v})\gamma \leq -K \int_t^\infty R(u, t) \, du,\]

where \(\beta = (Ax - 2L/2ALx) > 0, K > M, \) and \(\gamma > 0.\) Then the zero solution of (4.1) is uniformly asymptotically stable.

Proof. As in the proof of Theorem 4.1, by (4.4) it follows that

\[V''(t, x) \leq -\gamma \|x\| \quad \text{for} \quad t \neq t_k, \quad (t, x) \in \bigcup_{k=1}^\infty G_k.\]

The rest of the proof is similar to that of Theorem 3.2 and hence omitted.

REFERENCES